J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
34 (1987), 741-761.

Topology of the moduli space of $SU(2)$-instantons with instanton number 2

Dedicated to Professor Itiro Tamura on his sixtieth birthday

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§1. Introduction and statement of results.

We shall denote by $\mathcal{M}_k$ an $\tilde{\mathcal{M}}_k$ the moduli space and the framed moduli space of $SU(2)$-instantons with instanton number $-c_2=k$ respectively. Precisely, $\mathcal{M}_k$ and $\tilde{\mathcal{M}}_k$ are the quotients of the space of self-dual $SU(2)$-connections on the standard 4-sphere S^4 with $c_2=-k$ divided by the gauge group and the restricted gauge group (consisting automorphisms which are the identity on the base point $\infty \in S^4$) respectively.

In this note we shall give a topological description of $\mathcal{M}_2$ and $\tilde{\mathcal{M}}_2$. There is an algebraic description of $\mathcal{M}_k$ and $\tilde{\mathcal{M}}_k$ due to Atiyah-Drinfeld-Hitchin-Manin [ADHM]. However the topology of the moduli spaces $\mathcal{M}_k$ and $\tilde{\mathcal{M}}_k$ are not easily seen from their description. Donaldson [D] and Taubes [T] proved that $\mathcal{M}_k$ and $\tilde{\mathcal{M}}_k$ are connected. Recently Hurtubise [H] showed that $\pi_1(\tilde{\mathcal{M}}_k)=\boldsymbol{Z}_2$ and $\pi_1(\mathcal{M}_k)=\boldsymbol{Z}_2$ or 0 according as k is even or k is odd.

It is known by [AHS] that the space $\mathcal{M}_k$ and $\tilde{\mathcal{M}}_k$ are smooth manifolds of dimension $8k-3$ and $8k$ respectively. From the ADHM description we shall first deduce the following.

THEOREM 1. *There is a diffeomorphism*

$$\varphi : \mathcal{M}_2 \longrightarrow \tau\backslash(\boldsymbol{C}P^4-\boldsymbol{R}P^4)\times \boldsymbol{R}_+\times \boldsymbol{H}$$

where $\boldsymbol{C}P^4$ is the 4-dimensional complex projective space, τ is the complex conjugation and $\boldsymbol{R}P^4$ is the 4-dimensional real projective space regarded as the real part of $\boldsymbol{C}P^4$; moreover $\boldsymbol{R}_+$ and $\boldsymbol{H}$ denote the positive real numbers and the quaternion numbers respectively.

Let Q_3 be the hypersurface in $\boldsymbol{C}P^4$ defined by the equation

$$z_0^2+z_1^2+\cdots+z_4^2=0\,.$$

It can be shown that $\boldsymbol{C}P^4-\boldsymbol{R}P^4$ is deformed into Q_3 equivariantly with respect to τ. It is well-known that Q_3 can be identified with the Grassmann manifold $\tilde{G}_{5,2}$ of oriented 2-planes in $\boldsymbol{R}^5$. Moreover it is easy to see that τ can be identified with the involution on $\tilde{G}_{5,2}$ which sends 2 plane W into $-W$. Hence the quotient $\tau\backslash Q_3$ is identified with the Grassmann manifold $G_{5,2}$ of 2-planes in $\boldsymbol{R}^5$. Thus we obtain

COROLLARY 2. *The moduli space $\mathcal{M}_2$ is diffeomorphic to the total space of a vector bundle over $G_{5,2}$. The bundle is isomorphic to Whitney sum of a non-orientable 2-plane bundle and the trivial 5-plane bundle. In particular, $\mathcal{M}_2$ has the same homotopy type as $G_{5,2}$.*

Let $V_{5,2}$ denote the Stiefel manifold of orthonormal 2-frames in $\boldsymbol{R}^5$ and ξ the canonical $O(2)$-bundle $V_{5,2}\to G_{5,2}$. Set

$$\alpha=w_1(\xi)\,,\quad \bar{x}=w_2(\xi)\quad\text{and}\quad z=p_1(\xi)\,.$$

We also denote by the same letters the corresponding cohomology classes of $\mathcal{M}_2$. Then the cohomology of $\mathcal{M}_2$ is given by the following

PROPOSITION 3. *The non-trivial cohomology groups of $\mathcal{M}_2$ with coefficients in $\boldsymbol{Z}_2$ and $\boldsymbol{Z}$ are given by the following table.*

q	0	1	2	3	4	5	6
$H^q(\mathcal{M}_2;\boldsymbol{Z}_2)$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$	$\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$
generators	1	α	$\alpha^2, \bar{x}$	$\alpha^3, \bar{x}\alpha$	$\alpha^4, \bar{x}\alpha^2$	α^5	α^6

Moreover we have the relations.

$$\bar{x}\alpha^3=0\,,\quad \bar{x}^2=\bar{x}\alpha^2+\alpha^4\quad (\textit{hence }\ \bar{x}^3=\alpha^6)\,.$$

q	0	2	3	4	6
$H^q(\mathcal{M}_2;\boldsymbol{Z})$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}\oplus\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$
generators	1	β	x'	z, β^2	β^3

where β $\operatorname{mod} 2=\alpha^2$, x' $\operatorname{mod} 2=\bar{x}\alpha$ and z satisfies

$$z \operatorname{mod} 2=\bar{x}^2=\bar{x}\alpha^2+\alpha^4\,.$$

Note. We also have the relations

$$S_q^1\bar{x}=\bar{x}\alpha\,,\quad x'^2=\beta^3 \quad \text{and} \quad \beta z=0\,.$$

We turn to the framed moduli space $\tilde{\mathcal{M}}_2$. There is a principal fibration

$$p:\tilde{\mathcal{M}}_2 \longrightarrow \mathcal{M}_2$$

with structure group $SO(3)=SU(2)/\{\pm 1\}$. The following theorem gives a grasp to the topology of $\tilde{\mathcal{M}}_2$.

THEOREM 4. *The Stiefel-Whitney class and the Pontrjagin class of the* $SO(3)$*-bundle* $p:\tilde{\mathcal{M}}_2\to\mathcal{M}_2$ *are given by*

$$w_2=\bar{x}+\alpha^2 \quad \textit{and} \quad p_1=3z+\beta^2\,.$$

Using Theorem 4 we obtain

PROPOSITION 5. *The non-trivial cohomology groups of* $\tilde{\mathcal{M}}_2$ *are given by the following table.*

q	0	2	4	5	6	7	9
$H^q(\tilde{\mathcal{M}}_2;\boldsymbol{Z})$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_6$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$

The ring structure will be stated in § 6.

The orientation preserving isometry group $SO(5)$ and the conformal transformation group $GL(2,\boldsymbol{H})/\boldsymbol{R}^*$ of S^4 acts in a natural manner on $\mathcal{M}_2$. The action can be described via the diffeomorphism φ. The precise statements about the action will be given in § 3 after the detailed description of φ.

It would be appropriate to point out here the following observation. Donaldson gave a presentation of $\tilde{\mathcal{M}}_k$ and $\mathcal{M}_k$ in [D]. His presentation is easier to handle than the ADHM presentation as given in this paper. However the latter is suitable for the description of the action of $GL(2,\boldsymbol{H})/\boldsymbol{R}^*$ on the moduli spaces of $Sp(n)$ instantons.

The author is grateful to M. Furuta for useful conversations.

§ 2. Proof of Theorem 1.

We follow the description of $\mathcal{M}_2$ given by Atiyah in [A]. Let $\mathcal{E}$ denote the set of quaternion matrices

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \Lambda \\ B \end{pmatrix}, \qquad \begin{matrix} \Lambda = (\lambda_1, \lambda_2) \\ B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \end{matrix}$$

satisfying the following conditions (2.1) and (2.2).

(2.1) $\Lambda^*\Lambda + B^*B$ is real ($*$=transpose conjugate).

(2.2) For any $x \in \boldsymbol{H}$ the rank of the matrix

$$\begin{pmatrix} \Lambda \\ B - xE \end{pmatrix}$$

is equal to 2 where E is the identity matrix.

Here the rank means the maximum number of independent column vectors with respect to right multiplication by quaternions or equivalently the maximum number of independent row vectors with respect to left multiplication.

We let act the group $Sp(1) \times O(2)$ on $\mathcal{E}$ (from the left) by the formula

$$(q, T) \cdot \begin{pmatrix} \Lambda \\ B \end{pmatrix} = \begin{pmatrix} q\Lambda T^{-1} \\ TBT^{-1} \end{pmatrix}, \qquad (q, T) \in Sp(1) \times O(2) .$$

Then, according to [A], we can state

PROPOSITION (2.3). *The moduli space $\mathcal{M} = \mathcal{M}_2$ and the framed moduli $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_2$ are the quotients*

$$\mathcal{M} = Sp(1) \times O(2) \backslash \mathcal{E} \quad \textit{and} \quad \tilde{\mathcal{M}} = O(2) \backslash \mathcal{E}$$

respectively.

Note. $(-1, -E) \in Sp(1) \times O(2)$ acts trivially on $\mathcal{E}$ so that we get the fibration by $SO(3) = SU(2)/\{\pm 1\}$

$$p : \tilde{\mathcal{M}} \longrightarrow \mathcal{M} .$$

We now make the following convention to represent points of $\boldsymbol{C}P^3 = S^7/S^1$ and those of the hyperplane section bundle $\xi = S^7 \times \boldsymbol{C}/S^1$ where $S^1 \subset \boldsymbol{C}$ acts on $S^7 \times \boldsymbol{C} \subset \boldsymbol{C}^4 \times \boldsymbol{C}$ by scalar multiplication. We identify the point $(z_0, z_1, z_2, z_3) \in \boldsymbol{C}^4$ with the point $(q_1, q_2) \in \boldsymbol{H}^2 = \boldsymbol{R}^4 \times \boldsymbol{R}^4$ where q_1 and q_2 are the real and imaginary parts of the vector (z_0, z_1, z_2, z_3) respectively. Similarly $(z_0, z_1, z_2, z_3, w) \in \boldsymbol{C}^4 \times \boldsymbol{C}$ will be identified with $((q_1, q_2), (r_1, r_2)) \in \boldsymbol{H}^2 \times \boldsymbol{R}^2$ where $w = r_1 + r_2 i$. Note that the scalar multiplication $e^{i\theta}(z_0, z_1, z_2, z_3, w)$ by $e^{i\theta}$ is transformed into

(2.4) $$((q_1, q_2)R(\theta), (r_1, r_2)R(\theta))$$

where $R(\theta)=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

With these understood we define the map $\varphi: \mathcal{E}\to S^7\times \boldsymbol{C}\times \boldsymbol{R}_+\times \boldsymbol{H}$ by the formula

(2.5) $$\varphi\begin{pmatrix} \lambda_1 & \lambda_2 \\ a & b \\ b & c \end{pmatrix}=\left(\frac{\left(\frac{a-c}{\sqrt{2}}, \sqrt{2}b\right)}{\sqrt{2|b|^2+\frac{|a-c|^2}{2}}}, \left(\frac{|\lambda_1|^2-|\lambda_2|^2}{2}, \operatorname{Re}\bar{\lambda}_1\lambda_2\right), \sqrt{2|b|^2+\frac{|a-c|^2}{2}}, a+c\right).$$

If we identify $e^{i\theta}\in S^1=U(1)$ with ${}^tR(\theta)=R(-\theta)\in SO(2)$ as usual, the following equivariance can be checked by calculation.

(2.6) $$\varphi(q, e^{i\theta})\begin{pmatrix} A \\ B \end{pmatrix}=e^{2i\theta}\varphi\begin{pmatrix} A \\ B \end{pmatrix}, \quad (q, e^{i\theta})\in Sp(1)\times SO(2),$$

where $SO(2)$ acts on the factor $S^7\times \boldsymbol{C}$ via (2.4) and it acts trivially on the factor $\boldsymbol{R}_+\times \boldsymbol{H}$. Of course we let act $Sp(1)$ trivially on the right hand side.

Setting

(2.7) $$\tilde{\mathcal{M}}^\vee=SO(2)\backslash\mathcal{E} \quad \text{and} \quad \mathcal{M}^\vee=Sp(1)\times SO(2)\backslash\mathcal{E}$$

we see that φ induces a map (which we shall denote by the same letter φ)

(2.8) $$\varphi: \mathcal{M}^\vee \longrightarrow \xi\times \boldsymbol{R}_+\times \boldsymbol{H}.$$

REMARK. If $\mathcal{G}_k$ denotes the gauge group of the $SU(2)$-bundle over S^4 with instanton number k then it is known that $\mathcal{G}_k$ has two connected components. Let $\mathcal{G}_k^0$ denote the component of $\mathcal{G}_k$ containing 1. Then it can be shown that the center $\{\pm 1\}$ of $SU(2)$ is contained in $\mathcal{G}_k^0$ when k is even. Thus, in this case, the moduli space $\mathcal{M}_k$ admits a double covering corresponding to $\mathcal{G}_k/\mathcal{G}_k^0$ which can be identified with $\mathcal{M}^\vee$ for $k=2$.

Let τ denote the complex conjugation on ξ and ξ^τ the fixed point set of τ. The involution τ is transformed into

(2.9) $$\tau((q_1, q_2), (r_1, r_2))=((q_1, -q_2), (r_1, -r_2))$$

as an action on $\boldsymbol{H}^2\times \boldsymbol{R}^2$. We also identify τ with the non-trivial element of $O(2)/SO(2)$.

THEOREM (2.10). *The map φ in* (2.8) *is equivariant with respect to τ and its image is precisely equal to*

$$(\xi-\xi^\tau)\times \boldsymbol{R}_+\times \boldsymbol{H}.$$

An immediate corollary of (2.10) is the following

COROLLARY (2.11). *The map φ induces a diffeomorphism*

$$\varphi: \mathcal{M}=\tau\backslash\mathcal{M}^{\vee} \longrightarrow \tau\backslash(\xi-\xi^{\tau})\times \boldsymbol{R}_{+}\times \boldsymbol{H}\,.$$

Corollary (2.11) implies Theorem 1 in Introduction since ξ can be naturally embedded in $\boldsymbol{C}P^4$ and $\xi^{\tau}=\xi\cap \boldsymbol{R}P^4$.

The rest of this section is devoted to the proof of (2.10). Since it is quite computational we shall only give indications of main steps.

(a) The condition (2.1) is equivalent to

$$\mathrm{Im}\,\bar{\lambda}_1\lambda_2=\mathrm{Im}\,\bar{b}(a-c)\,. \tag{2.12}$$

This implies that $\mathrm{Re}\,\bar{\lambda}_1\lambda_2$ determines $\bar{\lambda}_1\lambda_2$.

(b) The map $Sp(1)\backslash\boldsymbol{H}^2\to\boldsymbol{R}\times\boldsymbol{H}$ defined by

$$(\lambda_1, \lambda_2)\longmapsto(|\lambda_1|^2-|\lambda_2|^2, \bar{\lambda}_1\lambda_2)$$

is a bijection.

(c) Let $\mathcal{B}$ denote the set of quaternion matrices

$$B=\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \qquad \text{with } |a-c|^2+|b|^2\neq 0\,.$$

Then from the condition (2.2) it follows that $\begin{pmatrix} \Lambda \\ B \end{pmatrix}\in\mathcal{E}$ implies $B\in\mathcal{B}$. Hence the obvious projection $\mathcal{E}\to\mathcal{B}$ is defined. Moreover this projection is equivariant with respect to $O(2)$-actions.

(d) The projection $\mathcal{E}\to\mathcal{B}$ induces the projection $r:\mathcal{M}^{\vee}=Sp(1)\times SO(2)\backslash\mathcal{E}\to\mathcal{B}^{\vee}=SO(2)\backslash\mathcal{B}$ which is compatible with the action of $\tau\in O(2)/SO(2)$.

(e) The fixed point set $(\mathcal{B}^{\vee})^{\tau}$ coincides with

$$\{[B]\,;\,\mathrm{Im}\,\bar{b}(a-c)=0\}$$

where $[B]$ denotes the image of B in $\mathcal{B}^{\vee}$.

(f) Let $\mathcal{F}$ denote the set of $\begin{pmatrix} \Lambda \\ B \end{pmatrix}$ such that $B\in\mathcal{B}$ and Λ satisfies (2.1) or, equivalently (2.12), and set $\mathcal{N}^{\vee}=Sp(1)\times SO(2)\backslash\mathcal{F}$. We can also consider the projection $r':\mathcal{N}^{\vee}\to\mathcal{B}^{\vee}$. Then, regarding $\mathcal{M}^{\vee}$ as a subset of $\mathcal{N}^{\vee}$, we have that

$$r^{-1}([B])=r'^{-1}([B]) \qquad \text{if } [B]\notin(\mathcal{B}^{\vee})^{\tau}$$

and

$$r^{-1}([B])=r'^{-1}([B])-(r'^{-1}([B]))^{\tau} \qquad \text{if } [B]\in(\mathcal{B}^{\vee})^{\tau}\,.$$

Thus we see that

(2.13) $$\mathcal{M}^{\vee}=\mathcal{N}^{\vee}-(\mathcal{N}^{\vee})^{\tau}\,.$$

The facts (c), (e) and (f) are based on the following lemma whose proof is also computational and is omitted.

LEMMA (2.14). 1) $B=\begin{pmatrix} a & b \\ b & c\end{pmatrix}$ *belongs to* $\mathcal{B}$ *if and only if* $\operatorname{rank}(B-xE)\geqq 1$ *for any* $x\in \boldsymbol{H}$.

2) Let $B\in\mathcal{B}$. Then $\begin{pmatrix} A \\ B\end{pmatrix}$ satisfies (2.2) if and only if A is not of the form

(*) $$A=\begin{cases} \mu\left(1, \dfrac{-d\pm\sqrt{d^2+4}}{2}\right), & \text{in case } b\neq 0, \text{ where } d=b^{-1}(a-c) \\ \mu(1,0) \quad\text{or}\quad \mu(0,1), & \text{in case } b=0\,, \end{cases}$$

for any $\mu\in\boldsymbol{H}$. It should be noticed that the roots $\pm\sqrt{d^2+4}$ occupy a subset of $\boldsymbol{H}$ homeomorphic to S^2 if d^2+4 is a negative real number.

3) Assume $B\in\mathcal{B}$. Then $[B]^{\tau}=[B]$ if and only if there exists a reflection T such that $TBT^{-1}=B$. Such T is unique if $b\neq 0$; if $b=0$ then $T=\pm\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$. Moreover, in this case, the right hand side of (*) in 2) is a scalar multiple of the axis of the reflection T and its normal.

4) Assume $B\in\mathcal{B}$ and A satisfies (2.12). If $[B]\notin(\mathcal{B}^{\vee})^{\tau}$ then A satisfies (2.2) automatically.

(g) The map φ in (2.8) is naturally extended to $\varphi:\mathcal{N}^{\vee}\to\xi\times\boldsymbol{R}_{+}\times\boldsymbol{H}$. The map φ is compatible with τ.

Now we can proceed to the proof of Theorem (2.10). From the facts (a) and (b) it follows that φ is a bijection. Since it commutes with the action of τ, the image of $\mathcal{M}^{\vee}$ (which equals $\mathcal{N}^{\vee}-(\mathcal{N}^{\vee})^{\tau}$ by (2.13)) coincides with $(\xi-\xi^{\tau})\times\boldsymbol{R}_{+}\times\boldsymbol{H}$. This completes the proof.

§ 3. Action of the isometry group.

The group of all orientation preserving isometries of the standard S^4 is $SO(5)$. It acts on $\mathcal{M}_k$ in the following way. We regard $\mathcal{M}_k$ as the set of all equivalence classes of (P,ω) where P is an $SU(2)$-bundle over S^4 with $c_2(p)=-k$ and ω is a self-dual connection on P. If $g\in SO(5)$ then we define a right action of $SO(5)$ on such pairs by

(3.1) $$(P,\omega)g=(g^*P, g^*\omega)\,.$$

This induces the desired action of $SO(5)$ on $\mathcal{M}_k$.

In a similar way the group of all orientation preserving conformal transformations of S^4 acts on $\mathcal{M}_k$ since such transformations preserve self-duality.

The framed moduli space $\tilde{\mathcal{M}}_k$ can be considered as the set of all equivalence class (P, κ, ω) where P, ω are as above and κ is a trivialization of P at the base point $\infty \in S^4$. Therefore the subgroup $SO(4)$ fixing the point ∞ acts on $\tilde{\mathcal{M}}_k$ via a formula similar to (3.1).

The universal covering group Spin(5) of $SO(5)$ can be identified with $Sp(2)$ in the following way. We first identify S^4 with the quaternion projective line $\boldsymbol{H}P^1 = P(\boldsymbol{H}^2)$. Here we regard elements of $\boldsymbol{H}^2$ as column vectors

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

acted on by $\boldsymbol{H}^*$ from the right. Then the action of $g \in Sp(2)$ on $\boldsymbol{H}P^1$ is induced from the obvious one on $\boldsymbol{H}^2$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad g = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}. \tag{3.2}$$

Note that the isotropy subgroup at $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in S^4$ is precisely

$$\mathrm{Spin}(4) = Sp(1) \times Sp(1) = \left\{ \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \right\}.$$

We take $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the base point ∞.

Similarly the group of all orientation preserving conformal transformations is identified with $GL(2, \boldsymbol{H})/\boldsymbol{R}^*$ where the action of $g \in GL(2, \boldsymbol{H})$ is given by the same formula (3.2). The isotropy subgroup U of $GL(2, \boldsymbol{H})$ at ∞ is the upper triangular group

$$\left\{ \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} \right\}.$$

We let act U on $\tau \backslash (\boldsymbol{C}P^4 - \boldsymbol{R}P^4) \times \boldsymbol{R}_+ \times \boldsymbol{H}$ from the right by the formula

$$\begin{aligned} &([(q_1, q_2), (r_1, r_2)], r, q)g \\ &\quad = ([(g_{11}^{-1} q_1 g_{22}, g_{11}^{-1} q_2 g_{22}), (r_1, r_2)], r, g_{11}^{-1} q g_{22} - 2 g_{11}^{-1} g_{12}) \end{aligned} \tag{3.3}$$

where

$$g = \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} \in U.$$

Note that $Sp(1) \times Sp(1)$ is the subgroup of U with $g_{12} = 0$ and above formula defines an action of $SO(4)$.

With these preliminary understood we can state the following

THEOREM (3.4). *The diffeomorphism φ in Corollary* (2.11) *is equivariant with respect to the action of $Sp(1)\times Sp(1)$ and of U.*

REMARK. The action of $Sp(2)$ or $GL(2, \boldsymbol{H})$ on $\mathcal{M}=\mathcal{M}_2$ could also be described in principle using the description given in Proposition (2.4). However the parametrization used in (2.4) depends on a particular choice of the base point ∞ and the case of $Sp(1)\times Sp(1)$ or U is much easier to compute. We thus content ourselves with this case.

Proof of (3.4). Set

$$C=\begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad D=\begin{pmatrix} \lambda_1 & \lambda_2 \\ a & b \\ b & c \end{pmatrix}$$

where $D\in\mathcal{E}$ as in § 2. If $g=\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}\in GL(2, \boldsymbol{H})$ then we consider the pair (C', D') given by

$$(C', D')=(C, D)g=(Cg_{11}+Dg_{21}, Cg_{12}+Dg_{22})\,.$$

Take $R\in Sp(3)$ and $S\in GL(2, \boldsymbol{R})$ such that $RC'S=C$. Then we have the following lemma which follows from the geometric meaning of the pair (C, D) as it is explained in [A].

LEMMA (3.5). *If $R\in Sp(3)$ and $S\in GL(2, \boldsymbol{R})$ is as above then $RD'S\in\mathcal{E}$ represents $[D]g\in\mathcal{M}$ where $[D]$ denotes the class of $D\in\mathcal{E}$ in $\mathcal{M}$.*

We apply this to $g=\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}\in U$. Then, in this case,

$$(C', D')=(Cg_{11}, Cg_{12}+Dg_{22})\,.$$

Therefore we can take g_{11}^{-1} as R and 1 as S. Hence we see that $g_{11}^{-1}Cg_{12}+g_{11}^{-1}Dg_{22}$ represents $[D]g$.

We now compare $\varphi(g_{11}^{-1}Cg_{12}+g_{11}^{-1}Dg_{22})$ with $\varphi(D)$ and see that the former is of the form (3.3) if we write

$$\varphi(D)=((q_1, g_2), (r_1, r_2), r, q)\,,$$

cf. (2.5). This completes the proof of Theorem (3.4).

§4. Cohomology of the moduli space $\mathcal{M}_2$.

For the sake of completeness we first give an explicit deformation retraction of $\boldsymbol{C}P^4-\boldsymbol{R}P^4$ onto Q_3.

More generally we shall give a deformation of $\boldsymbol{C}P^{n+1}-\boldsymbol{R}P^{n+1}$ onto the complex quadric Q_n. For that purpose we regard elements of $\boldsymbol{C}^{n+2}$ or $\boldsymbol{R}^{n+2}$ as column vectors and identify $z=x+iy\in\boldsymbol{C}^{n+2}$ with the real $(n+2)\times 2$ matrix $Z=(x, y)$. Note that z lies in $\boldsymbol{C}^{n+2}-0$ so that it represents a point of $\boldsymbol{C}P^{n+1}$ iff $\operatorname{rk} Z\geqq 1$ and z represents a point of $\boldsymbol{C}P^{n+1}-\boldsymbol{R}P^{n+1}$ iff $\operatorname{rk} Z=2$ as is easily proved. In the sequel we shall regard $\boldsymbol{C}P^{n+1}$ as the quotient of the sphere of radius $\sqrt{2}$ in $\boldsymbol{C}^{n+2}$ devided by S^1. Here the action of $e^{i\theta}\in S^1$ is translated to the right multiplication by $R(\theta)$ as in (2.4). Then the equation $z_0^2+\cdots+z_{n+1}^2=0$ defining Q_n corresponds to

$$ {}^tZZ=E \qquad \text{(unit matrix)}. \tag{4.1}$$

Hereafter we shall assume $\operatorname{rk} Z=2$ and $\|Z\|^2=\|x\|^2+\|y\|^2=2$. If we set $\sigma={}^tZZ$ then, by the assumption, σ is a positive definite symmetric matrix. We set $u=\sigma^{-1/2}$, then the following assertion can be proved without difficulty.

ASSERTION (4.2). *${}^t(Zu)Zu=E$ and*

$$r_t(Z)=tZ+(1-t)Zu$$

has rank 2 *for any* $t\in[0, 1]$. *Moreover we have*

$$r_t(ZR(\theta))=r_t(Z)R(\theta).$$

From (4.2) it follows that r_0 induces a deformation retract r: $\boldsymbol{C}P^{n+1}-\boldsymbol{R}P^{n+1}\to Q_n$.

The following assertions can also be proved by straightforward calculation.

ASSERTION (4.3). *The deformation r_t is equivariant with respect to the complex conjugation τ. Thus $\tau\backslash Q_n$ is a deformation retract of $\tau\backslash(\boldsymbol{C}P^{n+1}-\boldsymbol{R}P^{n+1})$.*

ASSERTION (4.4). *In case $n=3$ the deformation r_t is equivariant with respect to the action of $Sp(1)\times Sp(1)$ (or $U\subset GL(2, \boldsymbol{H})$) which is defined like* (3.3).

We now turn to the cohomology of $\mathcal{M}_2$. Let η denote the canonical

$O(3)$-bundle

$$O(5)/O(2) \longrightarrow O(5)/O(2)\times O(3)=G_{5,2}\,.$$

Then it is known (see e. g. [MS, § 7]) that the cohomology algebra $H^*(G_{5,2}\,;\,\boldsymbol{Z}_2)$ is generated by $w_1(\xi), w_2(\xi), w_1(\eta), w_2(\eta), w_3(\eta)$ with the relations

$$(1+w_1(\xi)+w_2(\xi))(1+w_1(\eta)+w_2(\eta)+w_3(\eta))=1\,.$$

This is equivalent to the description in Proposition 3. The cohomology algebra $H^*(G_{5,2}\,;\,\boldsymbol{Z})$ is also known (see e. g. [MS, § 15]) as is stated in Proposition 3.

§ 5. The bundle $\tilde{\mathcal{M}}_2\to\mathcal{M}_2$.

We consider the commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{M}}^{\check{}} & \longrightarrow & \tilde{\mathcal{M}}=\tilde{\mathcal{M}}_2 \\ \downarrow p^{\check{}} & & \downarrow p \\ \mathcal{M}^{\check{}} & \longrightarrow & \mathcal{M}=\mathcal{M}_2 \end{array} \tag{5.1}$$

where $\tilde{\mathcal{M}}^{\check{}}$ and $\mathcal{M}^{\check{}}$ is defined as in (2.7). Note that $\pi:\mathcal{M}^{\check{}}\to\mathcal{M}$ is homotopically equivalent to $\pi: Q_3=\tilde{G}_{5,2}\to G_{5,2}$.

The non-trivial cohomology groups of $Q_3=\tilde{G}_{5,2}$ are given by the following table (see e. g. [BH]).

q	0	2	4	6
$H^q(Q_3\,;\,\boldsymbol{Z})$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$
generator	1	x	y	xy

Moreover we have the relation $x^2=2y$. Here x is the Euler class of canonical oriented 2-plane bundle $\tilde{\xi}$ over $Q_3=\tilde{G}_{5,2}$ and $2y=p_1(\tilde{\xi})$. Note that $\tilde{\xi}=\pi^*(\xi)$ as 2-plane bundle so that

$$x \bmod 2=\pi^*(\bar{x}) \quad \text{and} \quad 2y=\pi^*(z)\,.$$

The oriented 2-plane bundle $\tilde{\xi}$ is also the restriction to Q_3 of the Hopf bundle over $\boldsymbol{CP}^4$.

The aim of this section is to prove the following

PROPOSITION (5.2). *The Stiefel-Whitney class and the Pontrjagin* *class of the $SO(3)$-bundle $\tilde{\mathcal{M}}^{\vee}\to\mathcal{M}^{\vee}$ are given by*

$$w_2=x \bmod 2 \quad and \quad p_1=6y .$$

Before giving the proof we shall derive the following consequence of (5.2).

COROLLARY (5.3). *The Stiefel-Whitney and the Pontrjagin class of $\tilde{\mathcal{M}}\to\mathcal{M}$ must be one of the following two possibilities;*

$$(5.4) \qquad \begin{cases} w_2=\bar{x} \\ p_1=3z \end{cases} \quad or \quad \begin{cases} w_2=\bar{x}+\alpha^2 \\ p_1=3z+\beta^2 . \end{cases}$$

In fact from (5.2) it follows that

$$\pi^*(w_2(\tilde{\mathcal{M}}))=x \bmod 2 \quad \text{and} \quad \pi^*(p_1(\tilde{\mathcal{M}}))=6y .$$

Hence we should have $w_2\equiv\bar{x} \bmod \alpha^2$ and $p_1\equiv 3z \bmod \beta^2$. Moreover $p_1 \bmod 2$ equals w_2^2. But $z \bmod 2=\bar{x}^2=\bar{x}\alpha^2+\alpha^4$. Therefore $p_1=3z$ if $w_2=\bar{x}$ and $p_1=3z+\beta^2$ if $w_2=\bar{x}+\alpha^2$.

This completes the proof of (5.3).

The former possibility in (5.4) is excluded by Theorem 4. But we postpone its proof until § 6.

The rest of this section will be devoted to the proof of (5.2). Since the inclusion $Q_2\subset Q_3\simeq\mathcal{M}^{\vee}$ induces the monomorphism in the cohomology up to dimension 4 the determination of w_2 and p_1 of the bundle $\tilde{\mathcal{M}}^{\vee}\to\mathcal{M}^{\vee}$ reduces to that of the bundle restricted on Q_2.

Consider the subset

$$\left\{([q_1, q_2, 0, 0], 2, 0)\,;\ \mathrm{Re}\,\bar{q}_1q_2=0,\ |q_1|^2=|q_2|^2=\frac{1}{2}\right\}$$

of $(CP^4-RP^4)\times \boldsymbol{R}_+\times \boldsymbol{H}$. It is easy to see that this subset can be identified with Q_2; cf. (4.1). This corresponds, via φ, to the quotient image in $\mathcal{M}^{\vee}$ of the subset $\mathcal{E}'$ of $\mathcal{E}$ consisting of matrices

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ a & b \\ b & c \end{pmatrix} \text{ such that } c=-a,\ \mathrm{Re}\,\bar{a}b=0,\ |a|^2=|b|^2=1,\quad \mathrm{Re}\,\bar{\lambda}_1\lambda_2=0,\ |\lambda_1|^2=|\lambda_2|^2 .$$

Thus we shall identify Q_2 with $Sp(1)\times SO(2)\backslash\mathcal{E}'\subset\mathcal{M}^{\vee}$ and consider the $SO(3)$-bundle

$$P=SO(2)\backslash\mathcal{E}' \longrightarrow Q_2 .$$

This bundle is nothing but the restriction of $\tilde{\mathcal{M}}^{\vee}\to\mathcal{M}^{\vee}$ to Q_2.

Now if the matrix

$$\begin{pmatrix}\lambda_1 & \lambda_2\\ a & b\\ b & -a\end{pmatrix}$$

belongs to $\mathcal{E}'$ then (λ_1, λ_2) is of the form $(\lambda_1, \lambda_2)=\sqrt{2}\lambda(1, \bar{b}a)$ with $\lambda\in Sp(1)$ in view of the definition of $\mathcal{E}'$ and the condition (2.12). Note that $\bar{b}a$ is pure imaginary and has norm 1. We set

$$V=\{(a, b)\in Sp(1)\times Sp(1)\ ;\ \bar{a}b \text{ pure imaginary}\}.$$

Then we can summarize the above observation in the following

LEMMA (5.5). *The space $\mathcal{E}'$ is identified with $V\times Sp(1)$.*

Moreover we have the following

LEMMA (5.6). *The action of $SO(2)$ on $\mathcal{E}'$ is translated to the following action on $V\times Sp(1)$:*

$$e^{i\theta}(a, b, \lambda)=(a\cos 2\theta-b\sin 2\theta,\ a\sin 2\theta+b\cos 2\theta,\ \lambda(\cos\theta+\bar{a}b\sin\theta)). \tag{5.7}$$

PROOF. We noted in (2.6) that the action on the V-factor is as above. On the other hand $\Lambda=(\lambda_1, \lambda_2)=\sqrt{2}\lambda(1, \bar{b}a)$ is transformed by $e^{i\theta}$ into

$$\begin{aligned}\Lambda R(\theta)&=(\lambda_1\cos\theta-\lambda_2\sin\theta,\ \lambda_1\sin\theta+\lambda_2\cos\theta)\\ &=\sqrt{2}\lambda(\cos\theta-\bar{b}a\sin\theta,\ \sin\theta+\bar{b}a\cos\theta)\\ &=\sqrt{2}\lambda(\cos\theta+\bar{a}b\sin\theta)(1, \bar{b}a).\end{aligned}$$

Hence λ is transformed into $\lambda(\cos\theta+\bar{a}b\sin\theta)$.

In the sequel we regard $S^3=Sp(1)$ as the quaternions with norm 1 and S^1 as the complex numbers with norm 1. Moreover we shall identify the coset space S^3/S^1 with S^2 by identifying $qS^1\in S^3/S^1$ with $qiq^{-1}\in S^2$. Dividing by $\{\pm 1\}$ this identification also induces an identification of $SO(3)/S^1$ with S^2. Here $S^1\subset SO(3)$ is doubly covered by $S^1\subset S^3$.

With this identification we have the map $S^1\times S^2\to S^3$ and $S^1\times S^2\to SO(3)$ both defined by the same formula

$$(h, gS^1)\longmapsto ghg^{-1} \qquad (g\in S^3 \text{ or } SO(3)).$$

Note that if $gS^1\in S^3/S^1$ corresponds to $q\in S^2$ then

$$ge^{i\theta}g^{-1}=\cos\theta+gig^{-1}\sin\theta=\cos\theta+q\sin\theta\ .$$

Therefore (5.7) can also be written as

$$(5.7)' \qquad e^{i\theta}(a, b, \lambda)=(a\cos 2\theta-b\sin 2\theta, a\sin 2\theta+b\cos 2\theta, \lambda ge^{i\theta}g^{-1})$$

where $gS^1\in S^3/S^1$ corresponds to $\bar{a}b\in S^2$.

Since ± 1 acts trivially on the V-factor in (5.7)′ the action of $SO(2)$ on $V\times Sp(1)$ can be factered through the action on $V\times SO(3)$. Explicitly it is given by the formula

$$(5.8) \qquad e^{i\theta}(a, b, k)=(a\cos\theta-b\sin\theta, a\sin\theta+b\cos\theta, kge^{i\theta}g^{-1})$$

where $k\in SO(3)$ and $gS^1\in SO(3)/S^1$ corresponds to $\bar{a}b\in S^2$.

To sum up we have the following characterization of the bundle $P\rightarrow Q_2$.

ASSERTION (5.9). *The $SO(3)$-bundle $P\rightarrow Q_2$ can be identified with the bundle*

$$SO(2)\diagdown V\times SO(3)\longrightarrow SO(2)\diagdown V$$

where the action of $SO(2)$ is given by (5.8).

Here we note that we are regarding $\tilde{\mathscr{M}}^{\vee}$ and P as the principal bundles with left principal action of $SO(3)$. Moreover, with this identification, we can regard the projection $V\rightarrow SO(2)\diagdown V$ as a principal S^1-bundle $V\rightarrow Q_2$ with left S^1 action.

Hereafter we shall consider principal bundles with right group action rather than left action. To convert a left action to right action we shall obey to the usual convention that $gu=ug^{-1}$ for any group element g. For example, with this convention, (5.8) can be rewritten in the form

$$(5.8)' \qquad (a, b, k)h^{-1}=((a, b)h^{-1}, kghg^{-1})$$

where we put $h=e^{i\theta}$.

If we regard the projection $S^3\rightarrow S^3/S^1=S^2$ as a right S^1-bundle where S^1 acts on S^3 as its subgroup then it is easy to see that the bundle is the hyperplane section bundle ζ. Note that the usual action of S^1 on $S^3\subset \boldsymbol{C}^2$ by scalar multiplication is different from the above action and it gives the dual bundle ζ^*, i.e. the Hopf bundle.

We set $\mu=c_1(\zeta)\in H^2(S^2; \boldsymbol{Z})$. Note that the bundle $SO(3)\rightarrow SO(3)/S^1=S^2$ coincides with the 2-fold tensor product ζ^2 so that $c_1(\zeta^2)=2\mu$.

Let now $p: V\rightarrow X$ be a principal S^1 bundle over a CW complex X in general. We construct an $SO(3)$-bundle $\tilde{V}$ over $X\times S^2$ as follows. We fix an integer l and let S^1 act on $V\times S^2\times SO(3)$ by the formula

(5.10) $$(u, gS^1, k)h=(uh, gS^1, gh^{-l}g^{-1}k)\,, \qquad h\in S^1\,,$$

where S^2 is identified with $SO(3)/S^1$ as before. Then we set $\tilde{V}=(V\times S^2\times SO(3))/S^1$ with the projection $\tilde{p}: \tilde{V}\to X\times S^2$ defined by

(5.11) $$\tilde{p}([u, gS^1, k])=(p(u), gS^1)\,,$$

where $[u, gS^1, k]$ denotes the image of (u, gS^1, k) in $\tilde{V}$.

The structure group $SO(3)$ acts on $\tilde{V}$ on the right. To obtain the bundle with left action we need to use the following formula (5.10)′ and (5.11)′ instead of (5.10) and (5.11). The bundle $\tilde{V}$ will be called l-fold twisted $SO(3)$-extension of the S^1-bundle $V\times S^2\to X\times S^2$.

(5.10)′ $$(u, gS^1, k)h=(uh, gS^1, kgh^{l}g^{-1})$$

(5.11)′ $$\tilde{p}(u, gS^1, k)=(p(u), gS^1)\,.$$

The following Proposition (5.12) is a main key to the proof of (5.2).

PROPOSITION (5.12). *The $SO(3)$-bundle $\tilde{V}\to X\times S^2$ is isomorphic to the $SO(3)$-extension of the S^1-bundle $V^l\otimes\zeta^2$ where V^l denotes the l-fold tensor product of V. In particular we have*

$$w_2(\tilde{V})=lc_1(V)\times 1 \qquad \mathrm{mod}\ 2$$

$$p_1(\tilde{V})=l^2c_1(V)^2\times 1+4lc_1(V)\times\mu\,.$$

PROOF. Let $W\to Y$ be a principal S^1-bundle. Then the tensor product $V^l\otimes W\to X\times Y$ can be described as the quotient of $V\times W$ by the relation

$$(v, w)\sim(vh, wh^{-l})\,, \qquad h\in S^1\,.$$

Then S^1 acts on the quotient by the formula

(5.13) $$[v, w]h=[vh, w]=[v, wh^l]\,.$$

This action is not effective if $|l|>1$, since $\boldsymbol{Z}_{|l|}\subset S^1$ acts trivially. Therefore the $SO(3)$ extension of $V^l\otimes W$ can be described as the quotient of $V^l\otimes W\times SO(3)$ by the action of S^1 given by

(5.14) $$([v, w], k)h=([v, w]h, h^{-l}k)\,.$$

Taking $SO(3)\to S^2$ as $W\to Y$ we construct a map

$$\varphi: V^l\otimes SO(3)\underset{S^1}{\times}SO(3) \longrightarrow \tilde{V}$$

by

$$\varphi([[v, g], k])=[v, gS^1, gk].$$

Using (5.13) and (5.14) we see easily this map φ is well-defined and equivariant with respect to the right action of $SO(3)$. Hence φ gives an $SO(3)$-bundle isomorphism from the $SO(3)$-extension of $V^l\otimes\zeta^2$ to $\tilde{V}$. This proves Proposition (5.12).

Suppose now that there is given a map $\varphi: X\to S^2$. Let $\check{\varphi}: X\to X\times S^2$ be defined by

$$\check{\varphi}(x)=(x, \varphi(x)).$$

As an easy consequence of Proposition (5.12) we have

COROLLARY (5.15). *Let φ and $\check{\varphi}$ be as above. If $V\to X$ is an S^1-bundle and $\tilde{V}\to X\times S^2$ is the l-fold twisted $SO(3)$-extension of $V\times S^2\to X\times S^2$, then we have*

$$w_2(\check{\varphi}^*\tilde{V})=lc_1(V) \bmod 2$$

$$p_1(\check{\varphi}^*\tilde{V})=l^2c_1(V)^2+4lc_1(V)\varphi^*(\mu).$$

We now return to the S^1-bundle $V\to Q_2$ introduced in the beginning of this section. Recall that V is the submanifold of $S^3\times S^3$ consisting of elements (a, b) such that $\bar{a}b\in S^2=S^3/S^1$. The correspondence $(a, b)\leftrightarrow(a, \bar{a}b)$ gives the identification $V\leftrightarrow S^3\times S^2$. The S^1-action on V given as before (cf. (5.8)) is transported to the action on $S^3\times S^2$ given by

$$(u, gS^1)h^{-1}=(ugh^{-1}g^{-1}, gS^1), \qquad g\in S^3,\ h\in S^1, \tag{5.16}$$

as is easily checked.

Since the map $S^3\times S^2\to S^2\times S^2$ defined by $(u, gS^1)\mapsto(ugS^1, gS^1)$ is compatible with the action (5.16) the identification $V=S^3\times S^2$ induces the identification $Q_2=S^2\times S^2$.

LEMMA (5.17). *With the above identification, we have*

$$c_1(V)=\mu\times 1-1\times\mu.$$

PROOF. Consider the restriction $V|S^2\times *\to S^2\times *$. Taking $g=1$ in (5.16) we see that $V|S^2\times *=S^3\times *$ on which $h\in S^1$ acts by $u\mapsto uh$. Therefore $V|S^2\times *$ is isomorphic to $S^3\to S^2$, i.e. to ζ. Hence

$$c_1(V|S^2\times *)=\mu\times 1. \tag{5.18}$$

Next we consider $V|*\times S^2$. Since $(u, gS^1)\in S^3\times S^2$ lies over $*\times S^2$ iff $ugS^1=S^1$, $V|*\times S^2$ is identified with the space $\{(u, u^{-1}S^1) ; u\in S^3\}$. Thus we can define a bundle map $V|*\times S^2\to S^3$ by $(u, u^{-1}S^1)\mapsto u^{-1}$. In fact from (5.16) it follows that $(u, u^{-1}S^1)h^{-1}\mapsto u^{-1}h$. Therefore the S^1-bundle $V|*\times S^2$ is isomorphic to the dual of the bundle $S^3\to S^2$ and we have

$$c_1(V|*\times S^2)=-1\times\mu\,.$$

Combining this with (5.18) we see that

$$c_1(V)=\mu\times 1-1\times\mu\,.$$

This completes the proof of (5.17).

PROPOSITION (5.19). *The Stiefel-Whitney class and the Pontrjagin class of the $SO(3)$-bundle $P=SO(2)\diagdown\mathcal{E}'\to Q_2$ are given by*

$$w_2(P)=\mu\times 1+\mu\times 1 \mod 2$$

$$p_1(P)=-6\mu\times\mu\,.$$

PROOF. Let $\varphi: Q_2=S^2\times S^2\to S^2$ denote the projection onto the second factor. Combining (5.8)′, (5.9), (5.10)′ we see that the bundle P is isomorphic to $\check{\varphi}^*\tilde{V}$ where $\tilde{V}$ is the (-1)-fold twisted extension of $V\times S^2$. Thus we can apply (5.15); using (5.17) we deduce

$$w_2(P)=\mu\times 1+1\times\mu \mod 2$$

and

$$p_1(P)=(\mu\times 1-1\times\mu)^2-4(\mu\times 1-1\times\mu)(1\times\mu)$$
$$=-6\mu\times\mu\,.$$

This proves (5.19).

If $i: Q_2\to Q_3$ denotes the inclusion then it is known that $H^q(Q_3)\to H^q(Q_2)$ $(q=2, 4)$ is an isomorphism onto a direct factor. Moreover the canonical $SO(2)$-bundle $\tilde{\xi}$ over $Q_3=\tilde{G}_{5,2}$ restricted to Q_2 coincides with $V\to Q_2$. In fact the restriction $\tilde{\xi}|Q_2$ also coincides with the restriction of the Hopf bundle $S^7\to \boldsymbol{CP}^3$ to $Q_2\subset \boldsymbol{CP}^3$. If $x+iy\in S^7$ corresponds to $(x, y)\in \boldsymbol{H}^2$ then the transform $e^{i\theta}(x+iy)$ corresponds to

$$(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta)\,.$$

Comparing this with (5.8) we see that $\tilde{\xi}|Q_2$ coincides with $V\to Q_2$.

From the above observation and Lemma (5.17) it follows that

$$i^*(x)=i^*(c_1(\tilde{\xi}))=\mu\times 1-1\times\mu$$

and

$$i^*(y)=\frac{1}{2}i^*(p_1(\tilde{\xi}))=\frac{1}{2}(\mu\times 1-1\times\mu)^2=-\mu\times\mu\ .$$

Since $SO(3)$-bundle $\tilde{\mathcal{M}}^{\vee}\to\mathcal{M}^{\vee}$ restricted to Q_2 is $P\to Q_2$ we have by Proposition (5.19) that

$$i^*(w_2(\tilde{\mathcal{M}}^{\vee}))=\mu\times 1+1\times\mu \mod 2$$

$$i^*(p_1(\tilde{\mathcal{M}}^{\vee}))=-6\mu\times\mu\ .$$

Hence we obtain

$$w_2(\tilde{\mathcal{M}}^{\vee})=x \mod 2 \quad\text{and}\quad p_1(\tilde{\mathcal{M}}^{\vee})=6y\ .$$

This establishes Proposition (5.2).

§ 6. Cohomology of $\tilde{\mathcal{M}}_2$.

The aim of this section is to complete the proof of Theorem 4 and also to prove Proposition 5 together with a description of multiplicative structure of $H^*(\tilde{\mathcal{M}}_2\,;\boldsymbol{Z})$.

We first prove Theorem 4. We consider the inclusion $i: G_{4,2}=Q_2/\tau\to G_{5,2}=Q_3/\tau$. We shall identify $H^2(G_{4,2}\,;\boldsymbol{Z}_2)$ with $H^2(G_{5,2}\,;\boldsymbol{Z}_2)$ via i^*. By virtue of Corollary (5.3) it clearly suffices to show that

(6.1) $$w_2(P/\tau)=\bar{x}+\alpha^2\in H^2(G_{4,2}\,;\boldsymbol{Z}_2)\ .$$

The diagonal map $\Delta: S^2\to S^2\times S^2=Q_2$ induces $\bar{\Delta}: \boldsymbol{R}P^2=S^2/\tau\to Q_2/\tau=G_{4,2}$.

LEMMA (6.2). $\bar{\Delta}^*(\bar{x})=\bar{\Delta}^*(\alpha^2)=\alpha^2\in H^2(\boldsymbol{R}P^2\,;\boldsymbol{Z}_2)$.

PROOF. Since the projection $\tilde{\xi}=V=S^3\times S^2\to S^2\times S^2$ is given by $(a, gS^1)\mapsto(agS^1, gS^1)$, the induced bundle $\Delta^*(\tilde{\xi})$ can be identified with

$$\{(a, gS^1)\,;\,g^{-1}ag\in S^1\}\ .$$

We define a trivialization $\psi: \Delta^*(\tilde{\xi})\to S^2\times S^1$ by

$$\psi(a, gS^1)=(gS^1, g^{-1}ag)\ .$$

On the other hand it is easy to see that the involution τ on $\tilde{\xi}=V$ is given by

$$\tau(a, gS^1)=(a, gjS^1)\ .$$

Since $(gj)^{-1}agj=j^{-1}g^{-1}agj=\overline{g^{-1}ag}\in S^1$, ψ transports τ to the involution on $S^2\times S^1$ given by $(x, h)\mapsto(-x, \bar{h})$. It follows easily that

$$\bar{\Delta}^*(w_1(\xi))=\bar{\Delta}^*(w_1(\tilde{\xi}/\tau))=\alpha$$

$$\bar{\Delta}^*(w_2(\xi))=\bar{\Delta}^*(w_2(\tilde{\xi}/\tau))=\alpha^2 .$$

Recalling that $\alpha=w_1(\xi)$ and $\bar{x}=w_2(\xi)$ we obtain the desired result.

LEMMA (6.3). *The SO(3)-bundle $\bar{\Delta}^*(P/\tau)$ over $\boldsymbol{R}P^2$ is trivial. In particular we have*

$$\bar{\Delta}^*(w_2(P/\tau))=0 .$$

PROOF. $\Delta^*(P)$ can be identified with

$$\{[a, gS^1, k]\,;\, a\in Sp(1),\ g, k\in SO(3), g^{-1}\bar{a}g\in S^1\}$$

where $\bar{a}$ denotes the image of a in $SO(3)$. Then the map $\psi: \Delta^*(P)\to S^2\times SO(3)$ given by

$$\psi[a, gS^1, k]=(gS^1, \bar{a}gJg^{-1}k^{-1}gJg^{-1})$$

is a well-defined $SO(3)$-bundle equivalence, where

$$J=\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix}\in SO(3) .$$

Moreover the involution on $\Delta^*(P)$ is given by

$$\tau[a, gS^1, k]=[a, gJS^1, k]$$

and is transported to the involution on $S^2\times SO(3)$ given by

$$(x, k)\longmapsto(-x, k) .$$

Hence the $SO(3)$-bundle $\Delta^*(P/\tau)$ is trivial.

Now we can prove (6.1). By (5.4) we know that $w_2(P/\tau)=\bar{x}+\alpha^2$ or $\bar{x}$, and by (6.3)

$$\Delta^*w_2(P/\tau)=0 .$$

But this occurs only if $w_2(P/\tau)=\bar{x}+\alpha^2$ by (6.2). This completes the proof of Theorem 4.

Hereafter to make the contents concise we shall omit most of the details and content ourselves with indicating the ideas of main steps.

Step 1. We consider the $SO(3)$-bundle $p^\vee: \tilde{\mathcal{M}}_2^\vee\to\mathcal{M}_2^\vee$. By using the Serre spectral sequences of this bundle with coefficients in $\boldsymbol{Z}_2$ and $\boldsymbol{Z}$ simultaneously with Proposition (5.2) in hand we obtain the following

LEMMA (6.4). *The non-trivial integral cohomology groups of* $\tilde{\mathcal{M}}_2^{\vee}$ *are given by the following table.*

q	0	2	4	6	7	9
$H^q(\tilde{\mathcal{M}}_2^{\vee}; \boldsymbol{Z})$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{Z}_6$	$\boldsymbol{Z}_6$	$\boldsymbol{Z}$	$\boldsymbol{Z}$
generator	1	u	$\bar{y}$	$u\bar{y}$	v	uv

where u *is such that* $p^{\vee *}(x)=2u$ *and* $p^{\vee *}(y)=\bar{y}$.

Then Proposition 5 can be restated in the following form.

PROPOSITION (6.5). *The non-trivial integral cohomology groups of* $\tilde{\mathcal{M}}_2$ *and their generators are given by the following table.*

	0	2	4	5	6	7	9
$H^q(\tilde{\mathcal{M}}_2; \boldsymbol{Z})$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2+\boldsymbol{Z}_3$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}$	$\boldsymbol{Z}_2$
generators	1	β	$\gamma, \bar{z}$	δ	$\beta\gamma$	v'	$\beta v'$

where β comes from $\mathcal{M}_2$ via p^* (cf. Proposition 4) and satisfies $\beta^2=0$; $\bar{z}=p^*(z)$ and v' is such that $\tilde{\pi}^*(v')=v$ where $\tilde{\pi}: \tilde{\mathcal{M}}_2^{\vee}\to\tilde{\mathcal{M}}_2$.

Step 2. We consider the mod 2 Gysin sequence of the double covering $\tilde{\mathcal{M}}_2^{\vee}\to\tilde{\mathcal{M}}_2$. Using Theorem 4 we see, among other things, that

$$\dim H^2(\tilde{\mathcal{M}}_2; \boldsymbol{Z}_2)=\dim H^3(\tilde{\mathcal{M}}_2; \boldsymbol{Z}_2)=1 .$$

Step 3. We consider the Serre spectral sequences of the $SO(3)$-bundle $\tilde{\mathcal{M}}_2\to\mathcal{M}_2$ with $\boldsymbol{Z}_2$ and $\boldsymbol{Z}$ coefficients simultaneously using Theorem 4. Combining with the results obtained in Step 2 we deduce Proposition (6.5). The details are cumbersome but rather routine and will be left to the reader.

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(Received January 13, 1987)

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Added in proof. The following reference was brought to the author's attention.

H. Aupetit, d'après R. Hartshorne, Fibrés stables de rang 2 sur $\boldsymbol{P}^3\boldsymbol{C}$ avec $c_1=0$, $c_2=2$. Les equations de Yang-Mills, Astérisques **71-72** (1980), 171-195.

In this paper Theorem 1 and Corollary 2 of the present article were already proved. Cf. also Hartshorne's paper, Stable vector-bundles and instantons, Comm. Math. Phys. **59** (1978), 1-15.