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Topological entropy and Thurston's norm of atoroidal surface bundles over the circle

To Professor Itiro Tamura on his 60th birthday

By Shigenori MATSUMOTO

§ 1. Introduction.

Given an irreducible oriented closed 3-manifold M, Thurston [3] has defined for $a \in H_2(M:\boldsymbol{R})$, an invariant $\|a\|$ known as Thurston's norm as follows. For a closed surface $S=\coprod_i S_i$ embedded in M, S_i a connected component, define

$$\chi_-(S)=\sum_i \max\{-\chi(S_i), 0\},$$

where χ denotes the Euler number. For $a \in H_2(M:\boldsymbol{Z})$, let

$$\|a\|=\min_S \{\chi_-(S)\},$$

where S ranges over all the embedded surfaces in M which represent a. It then turns out that $\|\cdot\|$ extends homogeneously and continuously to a pseudonorm on $H_2(M:\boldsymbol{R})$, also denoted by $\|\cdot\|$.

Throughout this paper, unless otherwise specified, we shall work under the following hypothesis.

HYPOTHESIS (1.1). *M is an atoroidal, irreducible, oriented, closed 3-manifold which fibers over S^1.*

Then, since M is atoroidal, the pseudonorm becomes a norm, of which the unit sphere $\mathcal{S}$ is known to be a convex polyhedral sphere in $H_2(M:\boldsymbol{R})$. Identify $H_2(M:\boldsymbol{R})$ with $H^1(M:\boldsymbol{R})$ via the Poincaré duality. Let $\mathcal{N}$ be the subset of those classes which are representable by *nonsingular* closed 1-forms. $\mathcal{N}$ is nonempty because M fibers over S^1. In [T], the structure of $\mathcal{N}$ has been investigated as follows. See also Fried, exposé 14 of [FLP].

THEOREM (1.2). *There is a collection of open top cells $e_1, \cdots, e_r$ of $\mathcal{S}$ such that $\mathcal{N}=\coprod_i C(e_i)$, where $C(e_i)$ is an open cone over e_i.*

Let e be one of the above e_i's and let $C=C(e)$. Denote by C_Z the set of all the integral points of C. Each point a of C_Z corresponds to a fibration of M over S^1 and, viewed as a point in $H_2(M:\boldsymbol{R})$, a is represented by its fiber S. S is connected if and only if a is a prime integral class. In any case, the monodromy $[\phi]$ of the fibration is a well defined isotopy class of homeomorphisms of S. Since M is atoroidal, $[\phi]$ is irreducible in the sense of Nielsen, that is, has, as its representative, a pseudo Anosov diffeomorphism ϕ. Then its topological entropy $h(\phi)$, or the logarithm of the dilatation of ϕ, depends solely upon the class $a\in C_Z$. So denote it by $h(a)$. For a positive integer n, we have $h(na)=n^{-1}h(a)$. Thus h^{-1} is a function on C_Z which is homogeneous of degree one. Therefore it is extended by a standard fashion to C_Q, the set of rational points of C.

The following fact is established by Fried [F].

THEOREM (1.3). *There is an extension of h^{-1} to a concave function of C.*

The purpose of the present paper is to show the following stronger result.

THEOREM (1.4). *h^{-1} is strictly concave on C.*

Fried [F] has also shown that if $\{a_n\}\subset C$ tends toward a point in $\partial C\backslash\{0\}$, then $h(a_n)$ tends toward the infinity. Therefore Theorem (1.4) has the following immediate corollary. Note that the function $\|a\|h(a)$ is homogeneous of degree zero.

COROLLARY (1.5). *$\|a\|h(a)$ takes its minimum at a unique ray in C.*

The proof of Theorem (1.4) is an elaboration of the argument of Long-Oertel [LO], where a new proof of Theorem (1.3) is given. However their proof of Theorem 4.1 in [LO] is rather intuitive, simply indicating the idea. In order to obtain a refined result, it is absolutely necessary to provide a fundamental facts upon which the argument is based. By this reason and for the convenience of the reader, we aim to give the proof in full details. However some familiarity is assumed about train tracks and pseudo Anosov diffeomorphisms. Fundamental facts concerning them are found in [C], [FLP], [HP] or [P].

The contents are as follows. In § 2, fundamental properties of train tracks and pseudo Anosov diffeomorphisms are prepared, which enable one to construct a branched surface L in M. In § 3, graded measures on L are constructed. § 4 and § 5 are devoted to the proof of Theorem (1.4).

With pleasure the author expresses his gratitude to S. Kojima for many stimulating conversations.

§ 2. Train tracks and branched surfaces.

Let F be an oriented closed surface of genus$\geqq 2$. Consider a train track τ on F. Let N be an adapted neighbourhood of τ. N has a transverse foliation $\mathcal{J}$, a foliation by intervals which is transverse to τ. Let τ' be another train track.

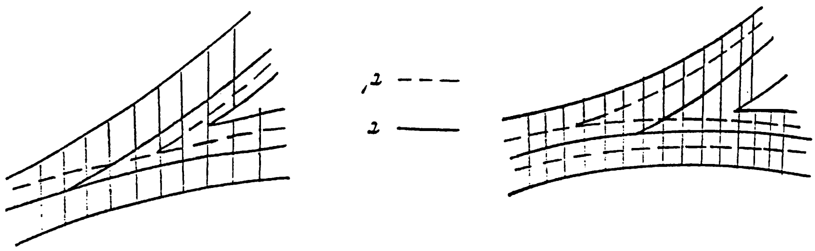
DEFINITION (2.1). τ' is said to be *carried by* τ *w. r. t.* $(N, \mathcal{J})$, if $\tau' \subset \mathring{N}$, $\tau' \pitchfork \mathcal{J}$ and there exists a continuous projection $\tau' \to \tau$ along leaves of $\mathcal{J}$.

See *Figure* 1. Note that this definition differs slightly from the usual one; here one is not allowed to move τ' by isotopy. Let $m(\mathcal{J})$ be the maximal length of the leaves of $\mathcal{J}$.

PROPOSITION (2.2). *Let* $\phi : F \to F$ *be a pseudo Anosov diffeomorphism. Then there exists a sequence of train tracks* τ_n $(n \geqq 1)$ *with adapted neighbourhoods* N_n *and transverse foliations* $\mathcal{J}_n$ *such that*

(1) $\phi(\tau_n)$ *is carried by* τ_n *w. r. t.* $(N_n, \mathcal{J}_n)$
(2) $m(\tau_n) \to 0$ *as* $n \to \infty$.

PROOF. At each singular point p of $\mathcal{F}^u$, the unstable foliation of ϕ, consider neighbourhoods Δ_p and Δ'_p of p which are of the form depicted in *Figure* 2, such that $\Delta'_p \subset \mathring{\Delta}_p$, $\Delta_p \cap \Delta_q \neq \emptyset$ if $p \neq q$ and $\phi(\mathring{\Delta}_p) \supset \Delta'_q$ if $\phi(p) = q$. On $N = F \setminus \bigcup_p \mathring{\Delta}_p$, the stable foliation $\mathcal{F}^s$ of ϕ is a foliation by intervals. The quotient space of N by $\mathcal{F}^s$ is a compact branched 1-manifold τ_0. Further one can find a smooth cross section $i : \tau_0 \to N$ of the natural projection. Then $\tau_1 = i(\tau_0)$ is a train track which satisfies (1), with $N_1 = F \setminus \bigcup_p \mathring{\Delta}'_p$ and



τ' is carried by τ $\qquad$ τ' is not carried by τ

Figure 1

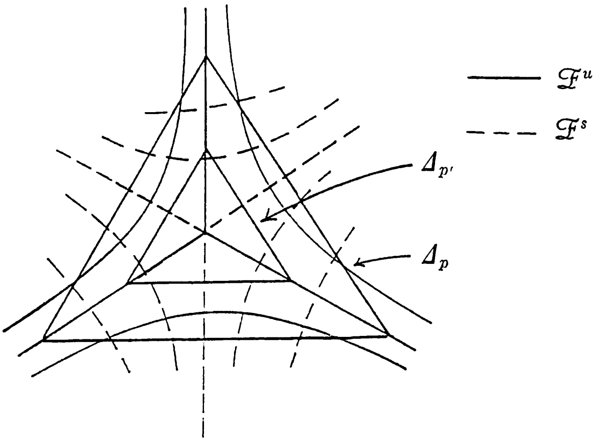


Figure 2

$\mathcal{J}_1=\mathscr{F}^s|_{N_1}$. For $n>1$, define $\tau_n=\phi^n(\tau_1)$, $N_n=\phi^n(N_1)$ and $\mathcal{J}_n=\mathscr{F}^s|_{N_n}$.

q. e. d.

Let T_ϕ be the mapping torus of ϕ. That is,

$$T_\phi=F\times[0, 1]/(x, 1)\sim(\phi(x), 0).$$

Let us define a smoothly embedded branched surface in T associated to τ_n. First of all, construct a smooth homotopy $\pi^n=(\pi_t^n):F\times[0, 1]\to F$ such that

(1) $\pi_t^n(x)=x$ if $t\leqq 1/3$ or if $x\notin N_n$
(2) $\pi_t^n=\pi_1^n$ if $t\geqq 2/3$
(3) π_t^n is a diffeomorphism of F if $t<2/3$
(4) If $x\in N_n$, $\pi_t^n(x)\in N_n$ and $\pi_t^n(x)$ and x lie on the same leaf of $\mathcal{J}_n$
(5) $\pi_1^n(\phi\tau_n)\subset\tau_n$.

The image L_n of $\phi\tau_n\times[0, 1]$ by π^n is a branched surface in T_ϕ. See *Figure* 3. Its branching locus consists of two parts. One is "horizontal", i.e. the part which lies in $F\times\{2/3\}$ and the other is "vertical", corresponding to (switches of τ_n)$\times[0, 1]$. The intersection of the two is called *rectangular points.*

The vector field $\partial/\partial t$ on $F\times[0, 1]$ gives rise to a vector field X on T_ϕ. Likewise $\pi_*^n(\partial/\partial t)$ defines a vector field Y_n on L_n.

By virtue of the condition (2) of Proposition (2.2), one has;

PROPOSITION (2.3). *The maximal angle between X and Y_n on L_n tends towards zero, as n tends towards the infinity.*

At this point, let us return to our initial situation. We are given a

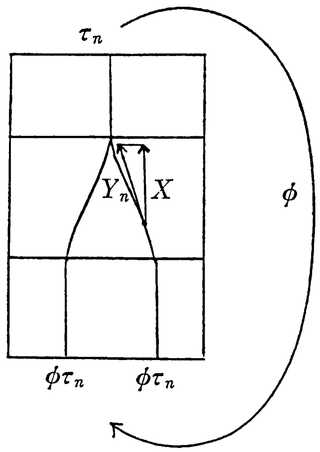


Figure 3

manifold M which satisfies (1.1). $\mathcal{C}$ is the open cone over a top cell e of $\mathcal{S}$. It is a component of $\mathfrak{N}$.

In the rest, fix once and for all an irreducible integral class Ξ in $\mathcal{C}$, as the reference point. Ξ determines the structure of a fibration over S^1, with fiber F, a connected surface and monodromy $\phi : F \to F$, a pseudo Anosov diffeomorphism. Now M is homeomorphic to the mapping torus T_ϕ. Fix once and for all such a homeomorphism, herewith identifying M with T_ϕ. Thus the branched surface L_n, the flow X and the semiflow Y_n are considered to lie in M. Our starting point is the following interesting fact due to Fried [F].

THEOREM (2.4). *Each element* $[\omega]$ *of* $\mathcal{C}$ *can be represented by a nonsingular closed* 1-*form* ω *such that* $\omega(X)>0$ *on the entire manifold* M. *Furthermore suppose* $[\omega]\in\mathcal{C}_Q$. *Let* S *be a compact leaf of the foliation by* ω. *Then the first return map of the flow* X *is a pseudo Anosov diffeomorphism of* S.

A subset K of $\mathcal{C}$ is called *projectively compact* if K is the cone over a compact subset in e.

PROPOSITION (2.5). *For any projectively compact subset* K *of* $\mathcal{C}$, *there exists a number* n *such that each element of* K *is represented by a nonsingular closed* 1-*form* ω *such that* $\omega(X)>0$ *on* M *and* $\omega(Y_n)>0$ *on* L_n.

The proof follows at once from Proposition (2.3) and the fact that a nonsingular closed 1-form ω can be taken locally continuously w. r. t. its class in $\mathcal{C}$.

Now we shall raise some properties of the semiflow Y_n.

PROPOSITION (2.6). *The semiflow Y_n has dense forward trajectories.*

PROOF. It is enough to show for $n=1$. As is well known, the pseudo Anosov diffeomorphism $\phi: F\to F$ has a dense forward orbit. Let $\psi_1: \tau_1\to\tau_1$ be the first return map of the semiflow Y_1. It is easy to show that ψ_1 is in fact connected to ϕ as follows. For each singularity p of $\mathcal{F}^u$, delete an open neighbourhood C_p of p in the (singular) leaf of $\mathcal{F}^u$ passing through p. Let $F'=F\setminus\bigcup C_p$. Then $\phi(F')\subset F'$. Choosing C_p's in appropriate size, one has that $\psi_1: \tau_1\to\tau_1$ is the projected image of $\phi: F'\to F'$, by the canonical projection of F' onto $F'/\mathcal{F}^s=\tau_1$. This shows that ψ_1 has dense orbits.

q. e. d.

A similar argument shows the local eventual surjectivity of the first return map $\psi_n: \tau_n\to\tau_n$ of the semiflow Y_n.

PROPOSITION (2.7). *For each interval J in τ_n, there exists $N>0$ such that $\psi_n^N(J)=\tau_n$.*

§ 3. Graded measures on branched surfaces.

In this section, we introduce, after [LO], graded measures on the branched surface L_n. From now on, for simplicity, let us denote L_n, τ_n, Y_n and ψ_n by L, τ, Y and ψ, if there is no fear of confusion.

Suppose ω is a nonsingular closed 1-form on M such that $\omega(X)>0$ on M and that $\omega(Y)>0$ on L.

DEFINITION (3.1). A positive valued function f on L is called a *graded measure* for ω with growth rate λ ($\lambda>1$), if

(1) f is smooth on the interior of each sector of L,

(2) *branching condition*: For each branch point x of L, there exist a neighbourhood N of x and a submersion $\pi: N\to \mathring{D}^2$ such that the function $\bar{f}: \mathring{D}^2\to \boldsymbol{R}_+$, defined by $\bar{f}(y)=\sum_{\pi(x)=y} f(x)$ is smooth,

(3) $df/f=-\log\lambda\cdot\omega$ on the interior of each sector of L.

In (3) above, we denote by ω the restriction of ω to L.

PROPOSITION (3.2). *Suppose ω is a closed 1-form on M such that $\omega(X)>0$ on M and $\omega(Y)>0$ on L. Assume also $[\omega]$ is an integral class. Then there exists a graded measure f for ω such that $df/f=-h(\omega)\cdot\omega$.*

where $h(\omega)$ is the topological entropy of the corresponding monodromy. Further f is unique up to a multiplicative constant.

Here is an outline of the proof. Let S be a leaf of the foliation by ω. The first return map $\bar{\phi}: S \to S$ of the flow X is a pseudo Anosov diffeomorphism, by Theorem (2.4). Thus on $\sigma = L \cap S$ there must exist a weight w corresponding to the unstable foliation, or the unstable measured lamination, of $\bar{\phi}$. The weight w then yields the required graded measure f on L by the differential equation $df/f = -h(\omega)\cdot\omega$. $\bar{\phi}_*(w) = e^{h(\omega)}w$ implies the continuity of f.

Although it is easy to show that σ is the correct one to carry such a weight w, we shall directly carry out (perhaps, well known) construction of w. This will greatly fascilitate the exposition of the next section.

DEFINITION (3.3). A smoothly embedded, boundaryless, compact, branched 1-manifold σ in a closed oriented surface S is called a *pseudo train track* if no connected component of $S-\sigma$ is a nullgon or a monogon.

Thus the difference from train tracks is that bigons and annuli are allowed to exist in the complement. Let ω be the 1-form of Proposition (3.2) and again let S be a leaf of the foliation by ω. One may suppose that S does not pass through the rectangular points of L. Also, it is no loss of generality to suppose that S is transverse to the branching locus of L. For, if not, isotope L. Let $\sigma = S \cap L$ and let $\bar{\phi}: \sigma \to \sigma$ be the first return map of the semiflow Y.

LEMMA (3.4). *For each $n>0$, $\bar{\phi}^n$ has dense forward orbits.*

The proof is analogous to the one for Proposition (2.7). But here one has to consider an appropriate finite covering of M. The details are left to the reader.

LEMMA (3.5). *σ is a pseudo train track of S.*

PROOF. Suppose on the contrary that there exists a nullgon or a monogon, say B, in the complement. Let $F \subset M$ be the surface corresponding to the reference point Ξ in C. By walking along the semiflow Y, one obtains a submersion of ∂B into the train track τ of F. Since F is incompressible, the image loop γ is homotopically trivial in F. A standard argument shows the existence of a nullgon or a monogon in the complement of τ. But τ is a train track. A contradiction. q. e. d.

Let W be the vector space of all the weights of σ. A weight is a function on σ, constant on each edge, satisfying the "switch condition". $\bar{\phi}$ defines a linear map $\bar{\phi}_*: W \to W$ by

$$\bar{\phi}_*(w)(y) = \sum_{\bar{\phi}(x)=y} w(x) .$$

A standard argument of linear algebra shows that there exists a set of edges $e_1, \cdots, e_n$ in σ such that a set of arbitrary values on each e_j yields a unique weight on σ and that any weight is so obtained. Define $w_j \in W$ by the condition $w_j(e_i) = \delta_{ji}$. Then w_j's form a linear basis of W. Let A be the matrix which represents $\bar{\phi}_*$ w.r.t. this basis.

LEMMA (3.6). *A is a nonnegative matrix and A^n is a positive matrix for some $n > 0$.*

The proof is an easy exercise. The key fact is Lemma (3.4). Show in the first place that all the diagonal entries of some power of A are positive.

PROOF OF (3.2). Apply Perron-Frobenius theorem to $\bar{\phi}_*: W \to W$. Let λ be the Perron-Frobenius eigenvalue and let $w \in W$ be the corresponding eigenvector. We shall show that w is in fact a positive weight. Consider first a nonnegative weight w_0 on σ. (For any pseudo train track, there always exists one.) w_0 is represented by a nonnegative column vector w.r.t. the above basis. Therefore, by the positivity of the matrix, Lemma (3.6), $\bar{\phi}_*^n w_0$ tends projectively towards w. This shows that w is nonnegative as a weight. Now Lemma (3.4) implies w is actually positive.

By using the flow line of the semiflow Y, one can construct, extending w, the graded measure f such that $df/f = -\log \lambda \cdot \omega$. Now as is the case with train tracks, clearly the positive weight w on the pseudo train track also gives birth to a measured lamination μ on S. It is not difficult to show that $\bar{\phi}_* \mu = \lambda \mu$, where $\bar{\phi}: S \to S$ is the first return map of the flow X. This shows λ is the distortion of the pseudo Anosov diffeomorphism $\bar{\phi}$. That is, $\log \lambda = h(\omega)$.

The uniqueness of the graded measure follows at once from the uniqueness of the Perron-Frobenius eigenvector. q. e. d.

REMARK (3.7). As will be shown in § 5, there exists a graded measure not only for ω in C_Z but also for any ω in C, provided $\omega(X) > 0$ and $\omega(Y) > 0$. It seems plausible to expect the uniqueness for such general ω. In fact this would simplify some of our argument in § 5. But unfortunately we cannot prove it at present.

§ 4. Local convexity.

The purpose of this section is to show the following proposition.

PROPOSITION (4.1). *Let ω_j be a nonsingular closed 1-form which represents an element of C_Q such that $\omega_j(X)>0$ on M and $\omega_j(Y)>0$ on L $(j=0,1)$. Suppose $[\omega_0]\neq[\omega_1]$. Then there exists $C>1$ such that for any $t\in[1/3, 2/3]$, we have*

$$h^{-1}((1-t)\omega_0+t\omega_1)\geqq C((1-t)h^{-1}(\omega_0)+th^{-1}(\omega_1)).$$

In § 5, Proposition (4.1) is generalized from C_Q to C by studying the dependence of C upon ω_0 and ω_1. The proof of (4.1) occupies this section.

First of all, by Proposition (3.2), let us construct a graded measure f_j for ω_j. Though Proposition (3.2), as it is, only states the existence of graded measures for C_Z, it is a matter of fact that this implies the existence for C_Q. Thus $df_j/f_j=-h(\omega_j)\omega_j$. By the homogeneity of h^{-1}, it suffices to show

(4.2) $\quad h(\eta_t)\leqq C^{-1}$, where $\eta_t=(1-t)h(\omega_0)\omega_0+th(\omega_1)\omega_1$ for

(4.3) $\quad h(\omega_0)/h(\omega_0)+2h(\omega_1)=t'\leqq t\leqq t''=2h(\omega_0)/2h(\omega_0)+h(\omega_1)$.

Also suppose:

(4.4) $\quad \eta_t=p\omega$, where $[\omega]\in C_Z$ is an irreducible integral point and $0<p<1$.

Of course there are dense values of t which satisfy (4.4). Therefore by the continuity of h, it suffices for our purpose to show (4.2) for those values of t which satisfy (4.3) and (4.4). However one has to be careful so that C does not depend upon each value of t.

Notice that $\omega(X)>0$ and $\omega(Y)>0$. Let us define a new function f on L by

$$f=f_0^{1-t}f_1^{t}.$$

Clearly $df/f=-p\omega$. But f is not a graded measure on L, because it does no longer satisfy the branching condition, (2) of (3.1). Let S be a leaf of the foliation by ω. As before we suppose that S is transverse to the branching locus of L and that S does not pass through rectangular points. Let $\sigma=S\cap L$. Then $f|_\sigma$, also denoted by f, is a positive function, constant on each edge, which however does not satisfy the switch condition. Let $\bar{\phi}:\sigma\to\sigma$ be the first return map of the semiflow Y.

(4.5) $\quad$ The cardinality of $\bar{\phi}^{-n}(y)\to\infty$, uniformly on $y\in\sigma$, as $n\to\infty$.

This follows at once from the fact that the Perron-Frobenius root of $\bar{\Phi}_*: W \to W$ is >1, where W is the space of the weights on σ.

Now extending the definition of $\bar{\Phi}_*$, let us consider $\bar{\Phi}_*(f)$, a function on σ. $\bar{\Phi}_*^n(f)$ is no longer constant on each edge. However there are only a bounded number of points of discontinuity. Let $|\cdot|$ denote the supremum norm.

(4.6) $\bar{\Phi}_*^n f/|\bar{\Phi}_*^n f|$ has a subsequence which converges unifomly to a Perron-Frobenius eigenvector $w \in W$.

For the proof, using (4.5) and the boundedness of the number of points of discontinuity, show that some subsequence converges to a nonnegative weight. Next apply the diagonal argument. The details are left to the reader.

Let λ be the eigenvalue of w. Then $h(w)=\log \lambda$. So (4.2) reduces to the following (4.7)

$$\log \lambda \leqq C^{-1} p\,. \tag{4.7}$$

The rest of this section is devoted to the proof of (4.7). Recall that f satisfies the following.

$$df/f = -p\omega\,. \tag{4.8}$$

An easy computation shows that (4.8) implies the following.

(4.9) If f should satisfy the branching condition, then we would obtain $\bar{\Phi}_* f(y) = e^p f(y)$, $y \in \sigma$.

But actually f does not satisfy the branching condition. We shall begin by investigating how it fails.

Let z be a point in the horizontal branching locus of L. Suppose a sector D of L approaches z from above and r sectors $D_1, \cdots, D_r$ from below. See *Figure* 4. Let α_i (resp. β_i) be the limit of f_0 (resp. f_1) at z, approximated in D_i. α and β are defined by the approximations in D. Since f_0 and f_1 satisfy the branching condition, we have

$$\alpha = \sum_i \alpha_i\,, \qquad \beta = \sum_i \beta_i\,. \tag{4.10}$$

Note that the corresponding limits for $f_0^{1-t} f_1^t$ are, of course, $\alpha_i^{1-t}\beta_i^t$ etc.. We have;

LEMMA (4.11). $\displaystyle\sum_i \alpha_i^{1-t}\beta_i^t \leqq \alpha^{1-t}\beta^t\,.$

The proof is an induction on r. The case $r=2$ is immediate from the

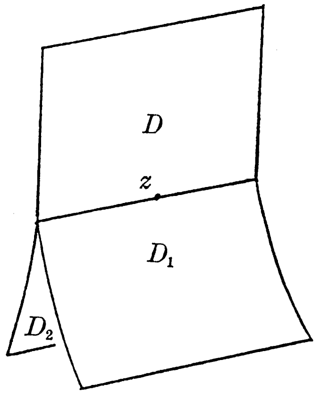


Figure 4

concavity of the function $T(x, y)=x^{1-t}y^t$ on the first quadrant. Note that T is homogeneous of total degree one.

This lemma, combined with (4.9), yields;

$$\bar{\phi}_* f(y) \leqq e^p f(y), \qquad y \in \sigma. \tag{4.12}$$

Suppose, by (4.6), that a sequence $\bar{\phi}_*^{n_i} f / |\bar{\phi}_*^{n_i} f|$ converges to the Perron-Frobenius eigenvector w. Then (4.12) implies

$$e^p w \geqq \bar{\phi}_* w = \lambda w .$$

Hence $\log \lambda \leqq p$. This is the argument employed in [LO] to show Theorem (1.3). But we need more. The first task is to refine (4.11).

LEMMA (4.13). *For any $\varepsilon > 0$, there exists $a > 1$ such that*

$$a \sum_i \alpha_i^{1-t} \beta_i^t \leqq \alpha^{1-t} \beta^t ,$$

provided $\varepsilon < t < 1-\varepsilon$, $\varepsilon < \alpha/\beta < \varepsilon^{-1}$ and $\sphericalangle(\alpha_i)(\beta_i) > \varepsilon$.

The proof is a routine calculation and is left to the reader.

LEMMA (4.14). *There is a point z in the horizontal branching locus, such that (α_i) is not a constant multiple of (β_i) at z.*

PROOF. Recall that the reference point $\Xi \in C_Z$ yields a pseudo Anosov diffeomorphism $\phi: F \to F$ and that a homeomorphism between T_ϕ and M is fixed. The horizontal branching locus of L is contained in $F \times \{2/3\}$. By certain abuse, let us denote $F \times \{2/3\}$ by F and $L \cap F \times \{2/3\}$ by τ.

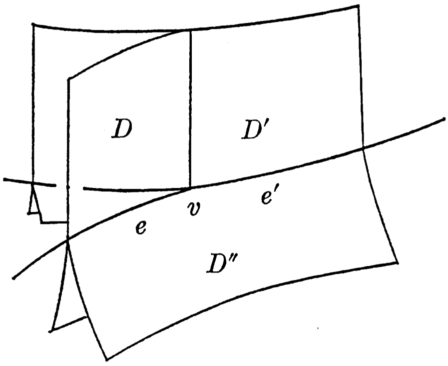


Figure 5

Let us embark upon the proof. Suppose the contrary. Consider the function u_0 and u_1 on τ, defined by $u_i(x)=\lim_{y\to x} f_i(y)$, where y approaches x from above. Let v be a switch of τ. Suppose that an edge e of τ terminates at v and an edge e' initiates at v. See *Figure* 5. Let D (resp. D') be the sector of L above e (resp. e') and let D'' be a sector below e and e'. For $x\in e\cup e'$, let $\tilde{u}_i(x)=\lim_{y\to x} f_i(y)$, where y approaches x in D''. Also let $c_i=\lim_{x\to v} u_i(x)$, where x approaches v in e. Define also c_i' by the limit in e'. Now our assumption (for contradiction) is that $\tilde{u}_0/u_0\equiv\tilde{u}_1/u_1$. This clearly implies $c_0/c_0'=c_1/c_1'$. In summary, we have that at any switch of τ, the ratio of the limits of u_i along two edges are the same for $i=0$ and 1.

Here we have used the presumption that switches are not mapped to switches by the first return map. Clearly we can assume this in the construction of the train track in § 2.

Now f_0 and f_1 are the graded measures for ω_0 and ω_1 and $[\omega_0]\neq[\omega_1]$. Suppose for a while that $[\omega_1-\omega_0]$ is not a constant multiple of Ξ. Then there exists $\gamma\in\pi_1(F)$ such that $\int_\gamma\omega_0\neq\int_\gamma\omega_1$. By the construction of the train track in § 2, all the components of the complement of τ are contractible. This shows the existence of a loop ρ in τ, starting and ending at some switch, such that $\int_\rho\omega_0\neq\int_\rho\omega_1$. Consider the functions u_0 and u_1 along ρ. Let us study the ratio of the initial and terminal values of u_i. We have already observed that the ratio must be identical for u_0 and u_1. Now if you walk around ρ, there are two contributions to this ratio, one from points of discontinuity, i.e., switches, and the other from the continuous parts, i.e., interiors of edges. The first one is identical for u_0 and u_1, while the latter one is not, since $\int_\rho\omega_0\neq\int_\rho\omega_1$. A contradiction.

Now consider the remaining case where $[\omega_1-\omega_0]$ is a constant multiple of Ξ. But this case can easily be ruled out, by considering a closed orbit of Y. Note that Y always has a closed orbit. The details are left to the reader. q. e. d.

Let us continue the proof of Proposition (4.1). Given an arc γ, $\int_\gamma dt$ is called the *vertical length* of γ, where t is the coordinate of I, in the identification $M=F\times I/\sim$. For any point $x\in\sigma$, let l_x be the vertical length of the trajectory γ_x of Y beginning at x, ending at $\bar{\phi}x$. We have $\int_{\gamma_x}\omega=1$. This yields the following inequality.

$$|\omega|_{\max}^{-1}\leqq l_x\leqq|\omega|_{\min}^{-1},$$

where $|\omega|_{\max}=\sup\omega(Y)$ and $|\omega|_{\min}=\inf\omega(Y)$. Recall (4.2) and (4.4). Clearly there exists a constant $b>1$ such that

$$b\geqq|\eta_t|_{\max}\geqq|\eta_t|_{\min}\geqq b^{-1}.$$

The constant b depends continuously upon ω_0 and ω_1. We thus have;

$$b^{-1}p\leqq l_x\leqq bp.$$

Let m be the smallest positive integer such that $1\leqq mb^{-1}p$. (Thus, $mb^{-1}p\leqq 2$.) This yields;

$$1\leqq ml_x\leqq 2b^2 \qquad \text{for any} \quad x\in\sigma. \tag{4.15}$$

Thus *the trajectory of the semiflow Y, starting at x and ending at $\bar{\phi}^m x$ meets the horizontal train track τ at least once and not more than $2b^2$ times.*

Now let $z\in\tau$ be the point which is guaranteed by Lemma (4.14). Let J' be a small interval in τ, centered at z and let $\varepsilon>0$ be smaller than the angle formed by the column vectors (α_i) and (β_i) at any point in J'. We may assume that J' does not intersect S. Let $J\subset\sigma$ be the first reaching image of J' by the semiflow Y. Then by Lemmas (4.11) and (4.13), we have;

$$\bar{\phi}_*^m f(x)\leqq e^{mp}f(x), \qquad \forall x\in\sigma \tag{4.16}$$

$$\bar{\phi}_*^m f(x)\leqq a^{-1}e^{mp}f(x), \qquad \forall x\in J, \tag{4.17}$$

where $a>1$ is the constant obtained by (4.13). Here f is understood to be a function on σ. We presume that the above ε is small enough so that $\varepsilon\leqq t'<t''\leqq 1-\varepsilon$ holds for t' and t'' in (4.3) and that $\varepsilon<\alpha/\beta<\varepsilon^{-1}$ holds all over J'. An inspection shows that *a is a constant which depends only* *upon ω_i and f_i. The dependence is continuous in the compact open topology.*

Consider the first return map $\phi:\tau\to\tau$ of Y. By Proposition (2.7), there exists $N>0$ such that $\phi^N(J')=\tau$. For $\bar{\phi}:\sigma\to\sigma$, this says $\bar{\phi}^{Nm}(J)=\sigma$. On the other hand, the trajectory of Y starting at $y\in\sigma$ crosses τ at most $2b^2$ times before it reaches $\bar{\phi}^m(y)$. Hence the cardinality of $\bar{\phi}^{-Nm}(y)$ is at most $d=c^{2b^2N}$, where $c=\sup\{\phi^{-1}(z)\mid z\in\tau\}$. At this point, note that N depends upon J', which, in turn, depends upon the difference of f_0 and f_1. Therefore *d depends only upon ω_i and f_i; the dependence is continuous.*

Now we have

$$\#(\bar{\phi}^{-Nm}(y)\cap J)\geqq 1 \quad\text{and}\quad \#(\bar{\phi}^{-Nm}(y))\leqq d\,, \qquad \forall y\in\sigma\,.$$

This yields

$$k\geqq Nm \Longrightarrow \#(\bar{\phi}^{-k}(y)\cap J)/\#(\bar{\phi}^{-k}(y))\geqq d^{-1} \qquad (\forall y\in\sigma)\,. \tag{4.18}$$

Therefore,

$$\begin{aligned}(\bar{\phi}_*^{k+m}f)(y)&=\sum_{\substack{\bar{\phi}^k(x)=y\\ x\in J}}(\bar{\phi}^m f)(x)+\sum_{\substack{\bar{\phi}^k(x)=y\\ x\notin J}}(\bar{\phi}^m f)(x)\\ &\leqq a^{-1}e^{mp}\sum_{\substack{\bar{\phi}^k(x)=y\\ x\in J}}f(x)+e^{mp}\sum_{\substack{\bar{\phi}^k(x)=y\\ x\notin J}}f(x)\\ &\leqq s^{-1}e^{mp}(\bar{\phi}_*^k f)(y)\,,\end{aligned}$$

where $s>1$ is the following constant.

$$s^{-1}=\frac{a^{-1}d^{-1}q+(1-d^{-1})r}{d^{-1}q+(1-d^{-1})r}\,,$$

where $q=\min_\sigma f$ and $r=\max_\sigma f$. The first inequality follows from (4.16) and (4.17) and the latter from (4.18). Taking a subsequence $\{k_i\}$ and passing to the limit, this gives:

$$\lambda^m w=\bar{\phi}_*^m w\leqq s^{-1}e^{mp}w\,,$$

where w is the Perron-Frobenius eigenvector. Therefore,

$$\log\lambda\leqq p-m^{-1}\log s\,.$$

Now by the inequality just before (4.15), we have $m^{-1}\geqq p/2b$. This yields

$$\log\lambda\leqq p(1-(2b)^{-1}\log s)\,.$$

Letting $C^{-1}=(1-(2b)^{-1}\log s)$, we have obtained (4.7). Thus the proof of Proposition (4.1) is now complete. Note that *the constant C depends continuously upon ω_i and f_i.*

§ 5. End of the proof of Theorem (1.4).

The purpose of this section is to prove the following proposition, which clearly implies Theorem (1.4).

PROPOSITION (5.1). *Let a_0 and a_1 be distinct classes in $\mathcal{C}$. Then there exists $C>1$ such that for any $t\in[1/3, 2/3]$, we have*

$$h^{-1}((1-t)a_0+ta_1)\geqq C^{1/2}((1-t)h^{-1}(a_0)+th^{-1}(a_1)).$$

By Proposition (2.5), there exists a closed 1-form ω_i representing a_i such that $\omega_i(X)>0$ on M and $\omega_i(Y_n)>0$ on L_n, for some n $(i=0, 1)$. Again denote L_n and Y_n by L and Y. As is well known, there is a sequence $\omega_i^{(k)}$ of closed 1-forms such that $\omega_i^{(k)}\to\omega_i$ and $[\omega_i^{(k)}]\in\mathcal{C}_Q$. We may assume that $\omega_i^{(k)}(X)>0$ on M and $\omega_i^{(k)}(Y)>0$ on L. By Proposition (4.1), we have;

$$h^{-1}((1-t)\omega_0^{(k)}+t\omega_1^{(k)})\geqq C_k((1-t)h^{-1}(\omega_0^{(k)})+th^{-1}(\omega_1^{(k)}))$$

for some $C_k>1$. The constant C_k above depends not only upon $\omega_i^{(k)}$ but also upon $f_i^{(k)}$, the graded measure for $\omega_i^{(k)}$. The dependence however is continuous. Since $f_i^{(k)}$ can be altered by constant multiples, one may assume that $\|f_i^{(k)}\|_\infty=1$. Also we have;

$$df_i^{(k)}=-h(\omega_i^{(k)})f_i^{(k)}\omega_i^{(k)} \quad \text{and} \quad h(\omega_i^{(k)})\omega_i^{(k)} \longrightarrow h(\omega_i)\omega_i\,.$$

This shows that $\{f_i^{(k)}\}_k$ is equicontinuous. Passing to a subsequence, if necessary, one may suppose that $f_i^{(k)}$ converges to some f_i, in the compact open topology. Clearly f_i satisfies all the conditions for a graded measure for ω_i, except that it may not be strictly positive. But, since f_i satisfies the differential equation $df_i=-h(\omega_i)f_i\omega_i$, if f_i should vanish at some point $x\in L$, f_i would be identically zero on the sector D containing x. Then by the branching condition, f_i also vanishes on the sectors which are adjoined to D from below. By Proposition (2.7), one would finally have $f_i\equiv 0$ on L. This contradiction shows that f_i is strictly positive.

Now notice that Lemma (4.14) also works for f_i of this section, although f_i is a graded measure for a 1-form, not necessarily representing a rational class. Thus by carrying out various estimates of the last section, one can define a constant $C>1$, associated to the present ω_i and f_i. Then clearly one has $C_k>C^{1/2}$ for k large. By the continuity of h^{-1}, this completes the proof of Proposition (5.1).

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