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On the global existence of real analytic solutions and hyperfunction solutions of linear differential equations

Dedicated to Professor S. Itô on his sixtieth birthday

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§ 0. Introduction.

The purpose of this paper is to show that microlocal analysis enables us to derive global existence theorems for hyperfunction solutions and real analytic solutions of linear differential equations from some closed range property of the relevant differential operator (Theorem 1.0 below), which has recently been verified by Takei and the author in [8].

Let us briefly describe our strategy: First we present a geometric condition which, combined with the above mentioned closed range property of a linear differential operator $P(x, D_x)$, entails its surjectivity when it acts upon the space $\mathcal{A}(K)$ of real analytic functions on a compact subset K of $\boldsymbol{R}^n$. Here, and in what follows, we always assume that the differential operator P in question is with real analytic coefficients and with real simple characteristics. Once we get this surjectivity result, the surjectivity of P acting on the space $\mathcal{A}(\Omega)$ of real analytic functions on a relatively compact open set Ω follows from the surjectivity of P acting on the quotient space $\mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega})$, where $\bar{\Omega}$ denotes the closure of Ω. In order to prove the surjectivity of

$$P: \mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega}) \longrightarrow \mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega})$$

under some geometric conditions on Ω, we make full use of the flabbiness of the sheaf $\mathcal{B}$ of hyperfunctions, that of the sheaf C of microfunctions and the structure theorem for microdifferential equations. See [3] or [12] for these topics. It might be of some interest to observe that the geometric conditions we use resemble the geometric interpretation of (strong) P-convexity, which Hörmander [2], Chap. III and Chap. VIII, gave in studying the solvability of equations in the framework of distributions and C^∞-functions. We also note that the reduction of the problem to the surjectivity of P acting on $\mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega})$ was also crucial in the reasoning of

Kawai [6], which deals with operators with constant coefficients.

Now that we get a global existence theorem for real analytic solutions, it is natural to try to obtain some global existence theorem for hyperfunction solutions by studying the structure of $(\mathcal{B}/\mathcal{A})$-solutions of the equation $Pu=0$. One important thing to be noted here is that, in principle, we do not need any geometric conditions on the boundary of the domain in question to obtain a global existence theorem for hyperfunction solutions, because we are concerned here only with a single equation and because the sheaf of hyperfunctions is flabby. (Cf. Harvey [1] and Komatsu [10].) This comment applies also to $(\mathcal{B}/\mathcal{A})$-solutions. However, in order to prove the global existence of $(\mathcal{B}/\mathcal{A})$-solutions, we first suppose some geometric condition on the domain $U(1)$ in question to the effect that it is exhausted out by a family of open sets which are convex with respect to the bicharacteristic curves of P. (See Proposition 2.0 for the precise statement.) Of course, the global existence theorem for $(\mathcal{B}/\mathcal{A})$-solutions on any open subset Ω of $U(1)$ immediately follows from the result for $U(1)$ because of the flabbiness of the sheaf $\mathcal{B}/\mathcal{A}$. (Theorem 2.1.) The reason why we employ such an indirect way of the reasoning is that the elementary solution (not the parametrix, i.e., with some residual regular functions admitted) so far known is a local one when P is with variable coefficients except for the case where P is strictly hyperbolic (Kawai [5]) ; the geometric conditions assumed in Proposition 2.0 are used to enlarge the domain on which the first cohomology group vanishes, i.e., the domain where the global existence of solutions is guaranteed, starting from a tiny domain where the explicitly constructed local elementary solution solves the problem. Once we obtain a global existence theorem for $(\mathcal{B}/\mathcal{A})$-solutions on Ω and a global existence theorem for real analytic solutions on Ω, then we can easily obtain a global existence theorem for hyperfunction solutions on Ω. In fact, by some cohomological machinery, we can obtain the following long exact sequence in our case:

$$\cdots \longrightarrow H^1(\Omega\,;\mathcal{A}^P) \longrightarrow H^1(\Omega\,;\mathcal{B}^P) \longrightarrow H^1(\Omega\,;(\mathcal{B}/\mathcal{A})^P) \longrightarrow \cdots.$$

Here, and in what follows, $\mathcal{A}^P$ etc. denote the real analytic solution sheaf etc. of P, i.e., $\mathcal{A}^P=\mathrm{Ker}\,(P:\mathcal{A}\rightarrow\mathcal{A})$ etc.

Hence, by combining a result for $(\mathcal{B}/\mathcal{A})$-solutions and that for real analytic solutions, we finally obtain the required result for hyperfunction solutions, i.e., the vanishing of $H^1(\Omega\,;\mathcal{B}^P)$.

Essentially the same results as those presented here have been announced in Kawai [7]. There the semi-global existence theorem claimed in Kiro [9] is used instead of the result in [8]. However, the reasoning given in Kiro [9] is erroneous, and hence we have avoided the use of the results in [9] in this paper. Accordingly conditions (1.1) and (1.2) in Kawai [7] should be replaced by conditions (1.0), (1.1) and (1.2) in § 1 of this paper.

§ 1. Global existence theorems for real analytic solutions.

Let us first recall the following result in Kawai and Takei [8].

THEOREM 1.0. *Let K be a compact subset of $\boldsymbol{R}^n$ and let $P(x, D_x)$ be a linear differential operator with (not necessarily real-valued) real analytic coefficients defined on an open neighborhood U of K. Suppose that the principal symbol $p_m(x, \xi)$ of $P(x, D_x)$ has a form $q(x, \xi)^l$ for a positive integer l, where $q(x, \xi)$ is a real analytic function in $(x\xi)$ that is a homogeneous polynomial of ξ of degree r $(=m/l)$. Suppose further that there exists a real-valued real analytic function $\phi(x)$ defined on U which satisfies the following two conditions*:

(1.0) $$K=\{x\in U;\ \phi(x)\leqq 0\}\,.$$

(1.1) *There exist strictly positive numbers A_0 and C for which the inequality* (UC) *below holds for every (x, y) in $U\times\boldsymbol{R}^n$ that satisfies $A\phi(x)+A^2\Big(\sum_{j=1}^{n} y_j^2\Big)=1$ and $q\Big(x+\sqrt{-1}y, \frac{1}{2}\operatorname{grad}\phi(x)-\sqrt{-1}Ay\Big)=0$ with $A>A_0$*:

$$\text{(UC)}\quad A\Bigg(\sum_{1\leq j,k\leq n}\frac{1}{2}\frac{\partial^2\phi(x)}{\partial x_j\partial x_k}q^{(j)}\Big(x+\sqrt{-1}y, \frac{1}{2}\operatorname{grad}\phi(x)-\sqrt{-1}Ay\Big)\times \overline{\times q^{(k)}\Big(x+\sqrt{-1}y, \frac{1}{2}\operatorname{grad}\phi(x)-\sqrt{-1}Ay\Big)}$$
$$+\operatorname{Re}\Big(\sum_{1\leq j\leq n}q_{(j)}\Big(x+\sqrt{-1}y, \frac{1}{2}\operatorname{grad}\phi(x)-\sqrt{-1}Ay\Big)\times \overline{\times q^{(j)}\Big(x+\sqrt{-1}y, \frac{1}{2}\operatorname{grad}\phi(x)-\sqrt{-1}Ay\Big)}\Big)\Bigg)$$
$$-\sum_{j=1}^{n}\Big|q_{(j)}\Big(x+\sqrt{-1}y, \frac{1}{2}\operatorname{grad}\phi(x)-\sqrt{-1}Ay\Big)\Big|^2$$
$$\geqq CA(1+A|y|)^{2(r-1)}\,.$$

Here, and in what follows, $q^{(j)}(z, \zeta)$ and $q_{(j)}(z, \zeta)$ respectively denote $\partial q(z, \zeta)/\partial\zeta_j$ and $\partial q(z, \zeta)/\partial z_j$.

Then $P\mathcal{A}(K)$, the range of P acting on the space $\mathcal{A}(K)$ of real analytic functions on K, is a closed subspace of $\mathcal{A}(K)$ (with respect to the natural topology of $\mathcal{A}(K)$).

Since it is essential in our subsequent reasoning to obtain the surjectivity of $P: \mathcal{A}(K) \to \mathcal{A}(K)$, we first prepare the following Proposition 1.1.

Before stating the proposition, we fix some notations which are used there and will be frequently used in this article: For two subsets A and B of a set C, $A \backslash B$ denotes $\{x \in A ; x \notin B\}$. For an open subset U of $\boldsymbol{R}^n$, $\mathring{T}^* U$ denotes the real cotangent bundle T^*U of U deleted its zero-section, i.e., $\mathring{T}^* U = T^*U \backslash T^*_U U$, and $\mathring{\pi}$ denotes the canonical projection from $\mathring{T}^* U$ to U.

PROPOSITION 1.1. *Suppose that a linear differential operator $P(x, D_x)$ and a compact subset K of $\boldsymbol{R}^n$ satisfy the same conditions as in Theorem* 1.0. *Suppose*

(1.2) *$l=1$ and $q(x, \xi)$ $(=p_m(x, \xi))$ is real and with simple characteristics on $\mathring{T}^* U$, i.e., $q(x, \xi)$ is real for (x, ξ) in $\mathring{T}^* U$ and* $\mathrm{grad}_\xi q(x, \xi)$ *never vanishes on $\{(x, \xi) \in \mathring{T}^* U ; q(x, \xi) = 0\}$.*

Suppose further that the pair (P, K) satisfies the following condition:

(1.3) *For any (x, ξ) in $\mathring{T}^* U$ satisfying $p_m(x, \xi) = 0$ with x in K, we can find a bicharacteristic curve b of p_m and a point (y, η) in $\mathring{T}^*(U \backslash K)$ so that b passes through both (x, ξ) and (y, η).*

Then

$$P \mathcal{A}(K) = \mathcal{A}(K)$$

holds.

REMARK 1.2. All the results in this section hold without the assumption $l=1$, and Proposition 1.1 holds under less restrictive assumption than the reality of $q(x, \xi)$. However, in order to make the presentation simpler, we have assumed the condition (1.2).

PROOF OF PROPOSITION 1.1. Let us first recall that the natural topology of $\mathcal{A}(K)$ makes it a DFS-space and that the dual space of $\mathcal{A}(K)$ is the space $\mathcal{B}_K$ of hyperfunctions supported by K. (See Komatsu [10] and Schapira [13], for example.) Since $P\mathcal{A}(K)$ is closed by Theorem 1.0, it suffices to show that ${}^tP$, the adjoint operator of P, is injective as a map from $\mathcal{B}_K$ to $\mathcal{B}_K$.

Now, as ${}^tP$ is also with real simple characteristics, the singularities of the equation ${}^tP\mu = 0$ propagate along bicharacteristic strips. (Kawai [4], Theorem 3.3′ and Sato-Kawai-Kashiwara [11], Chap. III, Theorem 2.1.7.) Suppose now that μ is an element in $\mathcal{B}_K$. Then μ is zero on $U \backslash K$ by its definition, and hence it is analytic there. Hence it follows from the condition (1.3) combined with the above mentioned result on the propagation of singularities that S. S. μ, the singularity spectrum of μ, is void, that is, μ is real analytic on U. Since K is compact, we may assume without loss of generality that U has the form $\bigcup_{j=1}^{N} U_j$, where U_j is a connected component of U and $U_j \cap K \neq \emptyset$. As μ vanishes outside K, the analyticity of μ entails that μ vanishes on each U_j, and hence on U itself. This means that ${}^tP: \mathcal{B}_K \to \mathcal{B}_K$ is injective. Q. E. D.

Once we obtain Proposition 1.1, we can prove a global existence theorem on a relatively compact open set Ω by making full use of microlocal analysis. Our result is the following

THEOREM 1.3. *Let P be a linear differential operator defined on an open subset U of $\boldsymbol{R}^n$, and let $\phi(x)$ be a real analytic function defined on U. Let Ω and K respectively denote $\{x \in U; \phi(x) < 0\}$ and $\{x \in U; \phi(x) \leqq 0\}$. Assume for simplicity that both U and Ω are connected. Suppose that K is compact and that $\phi(x)$ satisfies the condition* (1.1). *Suppose that the pair (P, K) satisfies the condition posed upon it in Proposition* 1.1. *Suppose further that the pair (P, Ω) satisfies conditions* (1.4) *and* (1.5) *below. Then*

$$P\mathcal{A}(\Omega) = \mathcal{A}(\Omega)$$

holds.

(1.4) *For each (x, ξ) in $N \underset{\text{def}}{=} \{(x, \xi) \in \mathring{T}^*U;\ x \in \partial\Omega,\ p_m(x, \xi) = 0\}$, the connected component of bicharacteristic curve $b_{(x,\xi)}$ of p_m passing through (x, ξ) intersects Ω in an open interval.*

(1.5) *For each (x, ξ) in $C \underset{\text{def}}{=} \{(x, \xi) \in N;\ \langle \operatorname{grad}_x \phi(x), \operatorname{grad}_\xi p_m(x, \xi)\rangle = 0\}$, the connected component of the bicharacteristic curve $b_{(x,\xi)}$ of p_m passing through (x, ξ) does not intersect Ω.*

REMARK 1.4. In order to be precise, let us note that the definition of the bicharacteristic curve used above depends upon U: If the domain of definition $\tilde{U}$ of $P(x, D_x)$ is actually larger than U and if it happens that some bicharacteristic curve b goes into $\tilde{U} \setminus U$ and then comes back into U, what we are concerned with here is only the behavior of b before it goes into $\tilde{U} \setminus U$.

PROOF OF THEOREM 1.3. Let f be a real analytic function on Ω. Since the sheaf $\mathcal{B}$ of hyperfunctions is flabby (Sato [11]), there exists a hyperfunction $\tilde{f}$ which satisfies the following two conditons:

(1.6) $$\operatorname{supp}\tilde{f}\subset K$$

(1.7) $$f=\tilde{f} \quad \text{holds on } \Omega .$$

We now try to find a hyperfunction v and a real analytic function g which are defined on a neighborhood of K and satisfy

(1.8) $$P(x, D_x)v=\tilde{f}+g$$

and

(1.9) $$v|_{\Omega}\in\mathcal{A}(\Omega) .$$

Once such v and g can be found, then, solving

$$P(x, D_x)w=g$$

in $\mathcal{A}(K)$ by using Proposition 1.1, we can find a real analytic function u defined on Ω which solves $Pu=\tilde{f}$ there. Since $\tilde{f}=f$ holds on Ω, u is the required real analytic solution. Thus the problem is reduced to the problem of finding v and g which satisfy (1.8) and (1.9), with g being analytic. Putting this differently, we claim that we can reduce the problem to verifying the surjectivity of

$$P:\mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega}) \longrightarrow \mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega}) .$$

Now, as we shall show below, the latter problem is, essentially speaking, of microlocal character. Let us first denote by S the subset of $\mathring{T}^*U$ given by

$$\{(x, \xi)\in \mathring{T}^*U ;\ (x, \sqrt{-1}\xi)\in \text{S. S. }\tilde{f}\} .$$

Now, as the bicharacteristic strip $b_{(x,\xi)}$ of p_m passing through a point (x, ξ) in $\mathring{T}^*U$ is a real one-dimensional curve, we can define its positive (resp., negative) part $b^+_{(x,\xi)}$ (resp., negative part $b^-_{(x,\xi)}$) by the non-negativity (resp., non-positivity) of the parameter t used in defining the curve as the solution of the following equations:

(1.10) $$\begin{cases} \dfrac{dx(t)}{dt}=\operatorname{grad}_{\xi} p_m(x(t), \xi(t)) \\ \dfrac{d\xi(t)}{dt}=-\operatorname{grad}_x p_m(x(t), \xi(t)) \\ x(0)=x \\ \xi(0)=\xi . \end{cases}$$

Let us now define subsets $N_{\pm}$ of N as follows:

(1.11) $$N_{\pm}=\{(x, \xi)\in N:\ b^{\pm}_{(x,\xi)} \text{ does not intersect } \Omega\}.$$

Then assumptions (1.4) and (1.5) entail

(1.12) $$N_{+}\cup N_{-}=N.$$

In fact, if (x, ξ) is in C, then it belongs to $N_{+}\cap N_{-}$ by (1.5), and, if (x, ξ) is in $N\backslash C$, then $b_{(x,\xi)}$ and $\partial\Omega$ intersect transversally and hence (1.4) guarantees that the point belongs to either N_{+} or N_{-}. (Actually the assumption (1.4) suffices to show (1.12).) Furthermore the assumption (1.5) guarantees that both N_{+} and N_{-} are closed. To see this, let $(x(n), \xi(n))$ be a sequence of points in N_{+} converging to $(x(\infty), \xi(\infty))$. If $(x(\infty), \xi(\infty))$ belongs to $N\backslash C$, then the transversality of $b_{(x(\infty),\xi(\infty))}$ and $\partial\Omega$ and the continuous dependence of $b_{(x,\xi)}$ upon (x, ξ) guarantees that $(x(\infty), \xi(\infty))$ belong to N_{+}. If $(x(\infty), \xi(\infty))$ belongs to C, then it trivially follows from the definition and the assumption (1.5) that the point belongs to N_{+}. The reasoning for the closedness of N_{-} is exactly the same.

Now, let us recall the fact the sheaf $\mathcal{C}$ is constant under the dilation map: $(x, \xi)\rightarrow(x, \alpha\xi)$ $(\alpha>0)$ and that, if we regard it as a sheaf on the cotangential sphere bundle using this property, then it is a flabby sheaf. ([12], Chap. III, Corollary 2.1.5; see also [3], Chap. III, Theorem 3.7.1.) For simplicity, we refer to this fact by a slightly loose expression "the flabbiness of $\mathcal{C}$". (As a matter of fact, [3] and [12] call the sheaf considered on the sphere bundle the sheaf of microfunctions and denote it by $\mathcal{C}$.) Using the flabbiness of $\mathcal{C}$ in the above sense, we can decompose $\mathrm{sp}(\tilde{f})$, the microfunction determined by $\tilde{f}$ into the sum of microfunctions $\tilde{f}_{+}$ and $\tilde{f}_{-}$ so that

(1.13) $$\mathrm{supp}\,\tilde{f}_{\pm}\subset\sqrt{-1}S, \qquad \mathrm{supp}\,\tilde{f}_{\pm}\cap\sqrt{-1}N\subset\sqrt{-1}N_{\pm}.$$

Here, and in what follows, $\sqrt{-1}N_{+}$ etc. denote $\{(x, \sqrt{-1}\xi)\in\sqrt{-1}\mathring{T}^{*}U;\ (x, \xi)\in N_{+}\}$ etc.

Now the general structure theorem for microdifferential equations ([12], Chap. II, Theorem 5.1.2; see also [3], Chap. IV, Theorem 4.3.1) guarantees that we can find a finite family $\{\omega_{j}\}$ of open subsets of $\sqrt{-1}\mathring{T}^{*}U$ which satisfy the following two conditions:

(1.14) $$\bigcup_{j}\omega_{j}\supset\sqrt{-1}\mathring{\pi}^{-1}(\partial\Omega)$$

(1.15) A real quantized contact transformation brings the equation $P(x, D_{x})u=0$ to the equation $\partial u/\partial y_{1}=0$ for a canonical coordinate system $(y, \sqrt{-1}\eta)$ on ω_{j}, if $\omega_{j}\cap\sqrt{-1}N\neq\emptyset$.

It then follows from (1.15) that there uniquely exist microfunction $u_{j,\pm}$ on

ω_j which satisfy

(1.16) $$P(x, D_x)u_{j,\pm}=\tilde{f}_\pm|_{\omega_j}$$

(1.17) $$\operatorname{supp} u_{j,\pm}\subset\bigcup_{(x,\xi)\in N_\pm\cap\omega_j}(\sqrt{-1}\,b^\pm_{(x,\xi)}\cap\omega_j)\cup\operatorname{supp}\tilde{f}_\pm .$$

(In the formula (1.17), the sign in the left hand side and that in the right hand side should be taken to be the same, and if $N_+\cap\omega_j$ (resp., $N_-\cap\omega_j$) is void then

$$\bigcup_{(x,\xi)\in N_+\cap\omega_j}(\sqrt{-1}\,b^+_{(x,\xi)}\cap\omega_j)\ \text{(resp.,}\ \bigcup_{(x,\xi)\in N_-\cap\omega_j}(\sqrt{-1}\,b^-_{(x,\xi)}\cap\omega_j))$$

should be understood to be a void set. Note that, if $N_\pm\cap\omega_j$ is void, then $P(x, D_x)$ is a sheaf isomorphism there.) By the uniqueness of $u_{j,\pm}$ mentioned above, we obtain

(1.18) $$u_{j,\pm}|_{\omega_j\cap\omega_k}=u_{k,\pm}|_{\omega_j\cap\omega_k},$$

if $\omega_j\cap\omega_k\neq\emptyset$. Hence we find microfunctions $u_\pm$ on $\bigcup_j\omega_j$ by patching $u_{j,\pm}$ together. It is then clear that $P(x, D_x)u_\pm=\tilde{f}_\pm$ holds on $\bigcup_j\omega_j$ as equalities for microfunctions. It is also clear from (1.17) that

(1.19) $$\operatorname{supp} u_\pm\cap\mathring{\pi}^{-1}\Omega=\emptyset .$$

Hence we may regard $u_\pm$ as microfunctions of $(\bigcup_j\omega_j)\cup(\mathring{\pi}^{-1}U)$ by extending them as zero over $\mathring{\pi}^{-1}\Omega$. Since it follows from (1.14) that

$$(\bigcup_j\omega_j)\cup(\mathring{\pi}^{-1}U)\supset\mathring{\pi}^{-1}(\bar{\Omega}),$$

we thus find that

(1.20) $$P(x, D_x)(u_++u_-)=\operatorname{sp}\tilde{f}$$

holds as an equality for microfunctions on a neighborhood of $\mathring{\pi}^{-1}(\bar{\Omega})$. In view of the cohomological triviality of the sheaf $\mathcal{A}$ of real analytic functions and the fundamental exact sequence in microfunction theory, i.e.,

(1.21) $$0\longrightarrow\mathcal{A}\longrightarrow\mathcal{B}\longrightarrow\mathring{\pi}_*\mathcal{C}\longrightarrow 0,$$

we infer from (1.19) and (1.20) the existence of a hyperfunction v and a real analytic function g which are both defined on some open neighborhood of $\bar{\Omega}$ and which satisfy

(1.22) $$P(x, D_x)v=\tilde{f}+g$$

and

(1.23) $$v|_{\Omega} \in \mathcal{A}(\Omega).$$

Thus we have found the required functions v and g. Q. E. D.

§ 2. A global existence theorem for hyperfunction solutions.

As we explained in the introduction, we begin our discussion on the global existence of hyperfunction solutions by first studying $(\mathcal{B}/\mathcal{A})$-solutions.

Our first result on the solvability of $(\mathcal{B}/\mathcal{A})$-solutions is the following

PROPOSITION 2.0. *Let U be a connected open subset of $\boldsymbol{R}^n$ and let $P(x, D_x)$ be a linear differential operator defined on U. Suppose that P satisfies the condition* (1.2). *Let $\varphi(x)$ be a real-valued real analytic function defined on U, and let $U(t)$ denote the subset of U given by*

(2.0) $$\{x \in U\,;\, \varphi(x) < t\},$$

where t ranges over all strictly positive real numbers. Suppose the following four conditions on $\varphi(x)$:

(2.1) $\mathrm{grad}_x \varphi(x)$ *never vanishes if* $\varphi(x) = t$ (>0),

(2.2) *For each $t > 0$, $U(t)$ is connected,*

(2.3) $\bigcap_{t>0} U(t)$ *consists of one point,*

(2.4) *For each $t > 0$, $\{x \in U\,;\, \varphi(x) \leqq t\}$ is compact,*

(2.5) *For each $t > 0$, the domain $U(t)$ satisfies conditions* (1.4) *and* (1.5) *with $\psi(x)$ being replaced by $\varphi(x) - t$.*

Under these conditions on P and φ, we find the following:

(2.6) *For any $t > 0$,*

$$P : (\mathcal{B}/\mathcal{A})(U(t)) \longrightarrow (\mathcal{B}/\mathcal{A})(U(t))$$

is surjective.

PROOF. As in the introduction, let us denote by $(\mathcal{B}/\mathcal{A})^P$ the $(\mathcal{B}/\mathcal{A})$-solution sheaf, i.e., $\mathrm{Ker}(P : \mathcal{B}/\mathcal{A} \to \mathcal{B}/\mathcal{A})$. Then it suffices for us to show that

$$H^1(U(t)\,;\,(\mathcal{B}/\mathcal{A})^P) = 0$$

holds for every $t > 0$. We shall prove this by verifying

(2.7) $$\sup\{t\,;H^1(U(s)\,;(\mathcal{B}/\mathcal{A})^P)=0 \text{ for } 0<s<t\}=\infty\,.$$

Since $\bigcap_{t>0} U(t)$ consists of one point by the assumption (2.3), the existence of local elementary solutions for operators with real simple characteristics (Kawai [4]) guarantees the surjectivity of $P:(\mathcal{B}/\mathcal{A})(U(t))\to(\mathcal{B}/\mathcal{A})(U(t))$ for $0<t\ll 1$, i.e., the vanishing of $H^1(U(t)\,;(\mathcal{B}/\mathcal{A})^P)$ for $0<t\ll 1$. Hence we can resort to the reduction to the absurdity by assuming that the supremum in (2.7) were a finite number $c>0$. Then it follows from the definition of c that

(2.8) $$H^1(U(c-\varepsilon)\,;(\mathcal{B}/\mathcal{A})^P)=0 \text{ for any } \varepsilon \text{ such that } 0<\varepsilon<c.$$

Let us now try to verify

(2.9) $$H^1_{U(c+\varepsilon')\setminus U(c-\varepsilon)}(U(c+\varepsilon')\,;(\mathcal{B}/\mathcal{A})^P)=0 \text{ for each non-negative } \varepsilon' \ (\ll 1).$$

If this is proved, then in view of the long exact sequence for the relative cohomology groups (cf. e.g. [3], Chap. I), we can conclude from (2.8) and (2.9) that $H^1(U(c+\varepsilon')\,;(\mathcal{B}/\mathcal{A})^P)$ vanishes. This contradicts the definition of c, completing the proof.

Therefore what remains to be proved is (2.9). Let us begin the proof of (2.9) by noting that the sheaf $\mathcal{B}/\mathcal{A}$ is flabby. (The flabbiness of $\mathcal{B}/\mathcal{A}$ is a trivial consequence of that of $\mathcal{C}$, for $\mathcal{B}/\mathcal{A}\cong\mathring{\pi}_*\mathcal{C}$. One can also prove the flabbiness of $\mathcal{B}/\mathcal{A}$ by the flabbiness of $\mathcal{B}$ and the cohomological triviality of $\mathcal{A}$.) Hence relative cohomology groups with $(\mathcal{B}/\mathcal{A})^P$ being their coefficients can be calculated by the flabby resolution of $(\mathcal{B}/\mathcal{A})^P$, i.e.,

$$0 \longrightarrow (\mathcal{B}/\mathcal{A})^P \longrightarrow (\mathcal{B}/\mathcal{A}) \xrightarrow{\ P\ } (\mathcal{B}/\mathcal{A}) \longrightarrow 0\,.$$

Therefore (2.9) follows from the surjectivity of

(2.10) $$P(x,D_x):H^0_{Z(c,\varepsilon,\varepsilon')}(U(c+\varepsilon')\,;\mathcal{B}/\mathcal{A}) \longrightarrow H^0_{Z(c,\varepsilon,\varepsilon')}(U(c+\varepsilon')\,;\mathcal{B}/\mathcal{A})\,,$$

where $Z(c,\varepsilon,\varepsilon')$ is, by definition, $U(c+\varepsilon')\setminus U(c-\varepsilon)$.

To prove the surjectivity of P in (2.10), we employ a method similar to that used in the proof of Theorem 1.3. Let $W_+(c,\varepsilon,\varepsilon')$ (resp., $W_-(c,\varepsilon,\varepsilon')$) denote the following subsets of $\mathring{T}^*U$:

(2.11) $$\bigcup_{c-\varepsilon\leq t<c+\varepsilon'}\{(x,\xi)\in\mathring{T}^*U\,;x\in\partial U(t) \text{ (i.e., } \varphi(x)=t),\ p_m(x,\xi)=0 \text{ and the positive (resp., negative) part } b^+_{(x,\xi)} \text{ (resp., } b^-_{(x,\xi)}) \text{ of the bicharacteristic curve } b_{(x,\xi)} \text{ emanating from } (x,\xi) \text{ does not intersect } U(t)\}.$$

Then, by the same reasoning used in the proof of Theorem 1.3, we find

(2.12) $W_+(c, \varepsilon, \varepsilon') \cup W_-(c, \varepsilon, \varepsilon') = \{(x, \xi) \in \mathring{T}^* U ; x \in Z(c, \varepsilon, \varepsilon') \text{ and } p_m(x, \xi) = 0\}$

and

(2.13) $W_+(c, \varepsilon, \varepsilon')$ and $W_-(c, \varepsilon, \varepsilon')$ are both closed in $\mathring{T}^* U(c+\varepsilon')$.

Since ε and ε' can be fixed arbitrarily small, and since $\partial U(c)$ is compact by the assumption (2.4), we can find a finite family $\{\omega_j\}$ of open subsets of $\mathring{T}^* U$ which satisfy the conditions (2.14) and (2.15) below. Here we have again used the structure theorem for microdifferential equations. ([12], Chap. II.)

(2.14) $\bigcup_j \omega_j \supset \mathring{T}^*(U(c+\tilde{\varepsilon}') \setminus U(c-\tilde{\varepsilon}))$, where $\varepsilon < \tilde{\varepsilon} \ll 1$ and $\varepsilon' < \tilde{\varepsilon}' \ll 1$,

(2.15) A real quantized contact transformation brings the equation $P(x, D_x)u = 0$ to the equation $\partial u / \partial y_1 = 0$ for a canonical coordinate system $(y, \sqrt{-1}\eta)$ on $\sqrt{-1}\omega_j$, if $\omega_j \cap \{(x, \xi) \in \mathring{T}^* U ; p_m(x, \xi) = 0\} \neq \emptyset$.

Now, let us note that, if a point (x, ξ) belongs to $W_+(c, \varepsilon, \varepsilon')$ (resp., $W_-(c, \varepsilon, \varepsilon')$), then $b^+_{(x,\xi)}$ (resp., $b^-_{(x,\xi)}$) does not intersect $U(c-\varepsilon)$, for $\{U(t)\}$ is an increasing family of open sets. Hence, by exactly the same reasoning as in the proof of Theorem 1.3, we can find, for any given μ in $H^0_{Z(c,\varepsilon,\varepsilon')}(U(c+\varepsilon') ; \mathcal{B}/\mathcal{A})$, ν_+ and ν_- in $H^0_{Z(c,\varepsilon,\varepsilon')}(U(c+\varepsilon') ; \mathcal{B}/\mathcal{A})$ so that they satisfy

$$P(x, D_x)(\nu_+ + \nu_-) = \mu .$$

This implies the surjectivity of the map in (2.10), completing the proof of the proposition.

As we emphasized in the introduction, the following Theorem 2.2 is an immediate consequence of Proposition 2.1 and the flabbiness of the sheaf $\mathcal{B}/\mathcal{A}$.

THEOREM 2.1. *Let U, P and φ be those given in Proposition* 2.0 *and suppose that they satisfy the conditions given in the proposition. Let t be an arbitrarily fixed strictly positive number. Then, for any open subset Ω of $\{x \in U ; \varphi(x) < t\}$, we find*

(2.16) $$P(x, D_x) : (\mathcal{B}/\mathcal{A})(\Omega) \longrightarrow (\mathcal{B}/\mathcal{A})(\Omega) \quad \textit{is surjective.}$$

Combining Theorem 1.3 and Proposition 2.0, we finally obtain the following global existence theorem for hyperfunction solutions.

THEOREM 2.2. *Let U, P and φ be the same as those given in Proposi-*

tion 2.1. *Set $\psi(x)=\varphi(x)-1$ and define a compact set K by $\{x\in U; \psi(x)\leq 0\}$. Suppose that the pair (P, K) satisfies the conditions posed upon it in Proposition* 1.1. *Then, for any open subset Ω of $U(1)=\{x\in U; \psi(x)<0\}$, we find*

$$P(x, D_x): \mathcal{B}(\Omega) \longrightarrow \mathcal{B}(\Omega) \text{ is surjective.} \tag{2.17}$$

PROOF. Let us first note that the simple characteristicness assumption on P entails as its trivial consequence that, for each x in U, $p_m(x, \xi)$ is not identically zero as a polynomial of ξ. Hence the Cauchy-Kovalevsky theorem guarantees that

$$P(x, D_x): \mathcal{A}_x \longrightarrow \mathcal{A}_x$$

is surjective for every x in U. Here $\mathcal{A}_x$ denotes the stalk at x of the sheaf $\mathcal{A}$. Therefore we find

$$\mathcal{B}^P/\mathcal{A}^P \xrightarrow{\sim} (\mathcal{B}/\mathcal{A})^P.$$

In other words, we obtain the following short sequence:

$$0 \longrightarrow \mathcal{A}^P \longrightarrow \mathcal{B}^P \longrightarrow (\mathcal{B}/\mathcal{A})^P \longrightarrow 0. \tag{2.18}$$

Then the following long exact sequence (2.19) of cohomology groups is derived from (2.18):

$$\begin{aligned} 0 &\longrightarrow H^0(U(1); \mathcal{A}^P) \longrightarrow H^0(U(1); \mathcal{B}^P) \longrightarrow H^0(U(1); (\mathcal{B}/\mathcal{A})^P) \\ &\longrightarrow H^1(U(1); \mathcal{A}^P) \longrightarrow H^1(U(1); \mathcal{B}^P) \longrightarrow H^1(U(1); (\mathcal{B}/\mathcal{A})^P) \\ &\longrightarrow \cdots. \end{aligned} \tag{2.19}$$

On the other hand, it follows from Theorem 1.3 and Proposition 2.0 that both $H^1(U(1); \mathcal{A}^P)$ and $H^1(U(1); (\mathcal{B}/\mathcal{A})^P)$ vanishes. Hence we see from (2.19) that $H^1(U(1); \mathcal{B}^P)$ also vanishes. Putting this differently, we assert that

$$P(x, D_x): \mathcal{B}(U(1)) \longrightarrow \mathcal{B}(U(1)) \text{ is surjective.} \tag{2.20}$$

Now, for any hyperfunction f on $\Omega\subset U(1)$, we use the flabbiness of the sheaf $\mathcal{B}$ of hyperfunctions to find a hyperfunction $\tilde{f}$ on $U(1)$ that coincides with f on Ω. Then, by (2.20), we can find a hyperfunction $\tilde{u}$ satisfying $P(x, D_x)\tilde{u}=\tilde{f}$ on $U(1)$. Hence, by restricting u to Ω, we finally obtain the required hyperfunction u satisfying $P(x, D_x)u=f$ on Ω. Q. E. D.

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