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On the statistical solutions of Navier-Stokes equation

Dedicated to Professor Seizô Itô on his sixtieth birthday

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Introduction.

In the theory of statistical hydrodynamics, which aims at a phenomenalistic description of turbulence, one of the principal problems is to determine probability distributions of velocity fields and their time evolution.

This leads to the study of so-called Hopf equation introduced in [6].

But, in order to obtain full informations on the statistical nature of turbulence, it is necessary to know the joint distributions of velocity fields at several different times.

The equations satisfied by them, the necessity of which had been pointed out by E. Hopf himself, are given and discussed by M. Visik and A. Fursikov in [1].

On the other hand, the collection of all joint distributions is essentially the same as a probability measure concentrated on the solutions of Navier-Stokes equation. Such a measure is called a statistical solution of Navier-Stokes equation.

The construction of statistical solutions in bounded domains is given in [1] Chap. 4.

In this work we shall prove the existence of statistical solutions in general domains, improving the method in [1] so as to be applied to arbitrary domains in $\boldsymbol{R}^N$.

§1. Statement of Theorem.

Let Ω be a domain in $\boldsymbol{R}^N$. In the sequel we shall frequently use the space $\mathcal{H}_{s,r}(\Omega)$ (or simply $\mathcal{H}_{s,r}$), which is defined as the completion of

$$\mathcal{V}=\{u\in C_0^\infty(\Omega)^N\,;\ \operatorname{div} u=0\}$$

with respect to the norm

$$\|u\|_{s,r}=\left[\int_\Omega (1+|x|^2)^r \sum_{|\alpha|\leq s} |D^\alpha u|^2 dx\right]^{1/2} \qquad (r\in\boldsymbol{R},\ s\in\boldsymbol{N})\,.$$

We shall write $\mathcal{H}_s$ for $\mathcal{H}_{s,0}$ and $\mathcal{H}$ for $\mathcal{H}_0$. Identifying $\mathcal{H}'$ with $\mathcal{H}$ by means of inner product, we may regard $\mathcal{H}$ as a linear subspace of $\mathcal{H}'_s$. The norm of $\mathcal{H}_s$, $\mathcal{H}$ and $\mathcal{H}'_s$ will be denoted by $\| \|_s$, $\| \|$ and $\| \|_{-s}$ respectively. Throughout this work, we shall assume $s \geqq [(n+1)/2]$.

For $f \in L^2(0, T; \mathcal{H}'_1)$ we consider the following equation (variational form of Navier-Stokes equation: cf. [3] p. 69)

$$\frac{d}{dt}\langle u, v\rangle + \nu a(u, v) + b(u, u, v) = \langle f, v\rangle^{1)} \qquad \forall v \in \mathcal{V} \tag{1}$$

where

$$a(u, v) = \sum_{i,j=1}^{N} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx$$

$$b(u, v, w) = \sum_{j,k=1}^{N} \int_\Omega u_k \frac{\partial v_j}{\partial x_k} w_j dx$$

ν is a positive constant (viscosity).

We call $u(t)$ a (individual) solution of (1), if $u \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{H}_1)$ and satisfies (1).

The main result of this work is the following

THEOREM. *Let μ be a given (Borel) probability measure on $\mathcal{H}$, such that $\int \|u_0\|^2 d\mu(u_0) < \infty^{2)}$. Then, there exists a (Borel) probability measure P on $Z = C(0, T; \mathcal{H}'_s) \cap L^2(0, T; \mathcal{H})$ with the following properties.*

1° *$\gamma_0 P = \mu$, where γ_0 is defined by $\gamma_0 u = u(0)$.*

2° *There exists a Borel set W in Z, consisting of solutions of (1) with full measure: $P(W) = 1$.*

3° *(Energy inequality)*

$$\int \left(\|u\|^2_{L^\infty(0,T;\mathcal{H})} + \|u\|^2_{L^2(0,T;\mathcal{H}_1)} + \left\|\frac{du}{dt}\right\|_{L^2(0,T;\mathcal{H}'_s)} \right) dP(u) \leqq C \int (1 + \|u_0\|^2) d\mu(u_0)$$

(C is a positive constant which is independent of μ).

P is called a statistical solution of Navier-Stokes equation with "initial value" μ.

Note. 1. Let E_i $(i=1, 2)$ be a Banach space, which is a linear sub-

1) $\langle , \rangle$ denotes the pairing between $\mathcal{H}_1$ and $\mathcal{H}'_1$. Henceforth we shall use the same notation to indicate the pairing between E and E' for any Banach space E.

2) If the domain of integration is not specified, then the integral is taken over the whole space.

space of another Banach space E and for which the canonical injection $E_i \to E$ is continuous.

Then, $E_1 \cap E_2$ can be identified with the closed subspace of $E_1 \oplus E_2$ by the diagonal mapping $u \mapsto (u, u)$.

Thus it may be regarded as a Banach space, which is separable if E_1 and E_2 are both separable.

In particular, Z is a separable Banach space, hence a polish space (i.e. a separable metrizable space which is complete with respect to a metric compatible with its topology).

2. In the above situation, suppose E_1 is separable. We extend $\|u\|_{E_1}$ (the norm of E_1) to E, setting $\|u\|_{E_1} = \infty$ for $u \in E \setminus E_1$. Then $\|u\|_E$ is Borel measurable and hence its restriction to E_2 is also Borel measurable. This is an immediate consequence of the classical theorem of Lusin:

Let f be a continuous injection of a polish space X into a Hausdorff space Y. Then, for every Borel set B in X, $f(B)$ is a Borel set in Y.

If f is not necessarily injective, $f(B)$ is universally measurable, i.e. μ-measurable for every finite Borel measure μ on Y.

(For the proof, see e.g. [4] Chap II.)

In case $E = L^2(0, T; \mathcal{H})$, $E = L^2(0, T; \mathcal{H}_1)$ and $E_2 = Z$, the above result implies the measurability of $\|u\|_{L^2(0,T;\mathcal{H}_1)}$ on Z.

Similarly $\|u\|_{L^\infty(0,T;\mathcal{H})} = \lim_{p\to\infty} \|u\|_{L^p(0,T;\mathcal{H})}$ is also Borel measurable on Z. In § 3 it will be shown that $\|du/dt\|_{L^2(0,T;\mathcal{H}'_s)}$ is P-measurable.

3. Let X, Y and f be as in the theorem of Lusin. For a Borel measure ν on Y, $f^{-1}\nu(A) = \nu(f(A))$ defines a Borel measure on X (pull-back of ν) by virtue of that theorem. Then it is obvious that $f^{-1}f\mu = \mu$, where $f\mu$ (image of μ) is defined by $f\mu(B) = \mu(f^{-1}(B))$.

In particular, μ is uniquely determined by its image. They are often identified and are denoted by the same notation.

Thus, when E_2 is also separable, we may "compare" a measure on E_1 and a measure on E_2.

The proof of Theorem will be given in § 3. Here we shall give some of its consequences.

COROLLARY (Existence of individual solutions). *For every $u_0 \in \mathcal{H}$, there exists a solution of* (1) *with* $u(0) = u_0$ (*more precisely*, $\lim_{t \downarrow 0} u(t) = u_0$ *in* $\mathcal{H}'_s$).

PROOF. Let $W_0 = \mathcal{H} \cap \gamma_0 W$. Then we have

$$\mu(W_0) = \mu(\gamma_0 W) = P(\gamma_0^{-1}\gamma_0 W) \geqq P(W) = 1 .$$

(Note that $\gamma_0 W$ is μ-measurable and hence $\gamma_0^{-1}\gamma_0 W$ is P-measurable.)

From this we can conclude that, for μ-a.e. u_0, there exists a solution $u(t)$ satisfying $u(0)=u_0$. In particular, taking δ_{u_0} (the point measure with support $\{u_0\}$) as μ, we see that, for every $u_0 \in \mathcal{H}$, there exists a solution with initial value u_0.

COROLLARY (uniqueness of a statistical solution). *In case the uniqueness of an individual solution is established, (for example, in case $N=2$), a statistical solution with prescribed initial value is also unique.*

PROOF. See [1] Chap. 4 Theorem 6.

§ 2. Preliminaries.

LEMMA 2.1. *For some constant $C>0$, we have*

$$|b(u, v, w)| \leqq C\|w\|_s(\|u\|\,\|u\|_1\|v\|\,\|v\|_1)^{1/2}.$$

PROOF. See [3] Chap., 6.3. As a consequence, we have

$$b(u, v, w)=\langle B(u, v), w\rangle \qquad (u, v \in \mathcal{H}_1, w \in \mathcal{H}_s)$$

where $B: \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}'_s$ is a continuous bilinear mapping. Similarly, there exists a continuous linear mapping $A: \mathcal{H}_1 \to \mathcal{H}'_1$ such that $\langle Au, v\rangle = a(u, v)$ $(u, v \in \mathcal{H}_1)$.

LEMMA 2.2. *For $r>0$, the canonical injection $\mathcal{H}_{1,r} \to \mathcal{H}$ is compact.*

PROOF. We may restrict ourselves to the case $\Omega = \boldsymbol{R}^N$. Let B a bounded set in $\mathcal{H}_{1,r}$. Then, for a given $\varepsilon>0$, there exists a finite number of elements in $B: u_1, \cdots, u_n$ such that for each $u \in B$, there is a j with

$$\|u-u_j\| < \frac{\varepsilon}{2} \qquad \text{in} \quad L^2(S_R)$$

where S_R is the ball of radius R in $\boldsymbol{R}^N$ with center at origin.

This is possible because the injection $H_1(S_R) \to L^2(S_R)$ is compact.

As we can easily see, $\{u_i\}$ forms a ε-net for B in $\mathcal{H}$ if R is sufficiently large.

This lemma ensures the existence of a complete orthogonal system $\{e_n\}$ in $\mathcal{H}_{s,r}$ which is, at the same time, an orthogonormal base of $\mathcal{H}$.

We denote by E_m the subspace spanned by $\{e_1, \cdots, e_m\}$.

The orthogonal projection π_m onto E_m (in $\mathcal{H}$), which is expressed as

$$\pi_m u = \sum_{i=1}^{m} (u, e_i) e_i$$

can obviously be extended to a continuous linear mapping from $\mathcal{H}'_{s,r}$ to E_m.

Moreover, as is easily seen,

$$\pi_m \longrightarrow I \qquad \text{(identity operator)}$$

strongly in $\mathcal{H}_{s,r}$.

From this, in particular, follows

$$\|\pi_m u\|_{s,r} \leqq \|u\|_{s,r}\,. \tag{2}$$

In E_m we consider the approximate equation

$$\frac{d}{dt}u_m + \pi_m A u_m + \pi_m B(u_m, u_m) = \pi_m f\,. \tag*{$(1)_m$}$$

LEMMA 2.3 (a priori estimate).

1) $\displaystyle \|u_m(t)\|^2 + \nu\int_0^t \|u_m\|_1^2 d\tau \leqq e^{2\nu t}\Big(\|u_m(0)\|^2 + \frac{1}{\nu}\|f\|^2_{L^2(0,T;\mathcal{H}'_1)}\Big)$

2) $\displaystyle \Big\|\frac{d}{dt}u_m\Big\|_{L^2(0,T;\mathcal{H}'_{s,r})} \leqq C(1 + \|u_m\|^2_{L^\infty(0,T;\mathcal{H})} + \|u_m\|^2_{L^2(0,T;\mathcal{H})})$

where C is a positive constant which is independent of m.

PROOF. Since $b(u, v, v) = 0$ for $u \in \mathcal{H}_1$, $v \in \mathcal{H}_s$, we have

$$\frac{d}{dt}\|u_m\|^2 + 2\nu\|\nabla u_m\|^2 = 2\langle f, u_m\rangle \leqq \nu\|u_m\|_1^2 + \frac{1}{\nu}\|f\|_{-1}^2\,,$$

or

$$\frac{d}{dt}\|u_m\|^2 + \nu\|u_m\|_1^2 \leqq 2\nu\|u_m\|^2 + \frac{1}{\nu}\|f\|_{-1}^2\,,$$

from which follows

$$\frac{d}{dt}(e^{-2\nu t}\|u_m\|^2) + \nu e^{-2\nu t}\|u_m\|^2 \leqq \frac{1}{\nu}e^{-2\nu t}\|f\|_{-1}^2\,.$$

Integrating this, we obtain

$$\|u_m(t)\|^2 + \nu\int_0^t e^{2\nu(t-\tau)}\|u_m(\tau)\|_1^2 d\tau \leqq e^{2\nu t}\Big(\|u_m(0)\|^2 + \frac{1}{\nu}\int_0^t e^{-2\nu\tau}\|f\|_{-1}^2 d\tau\Big)\,,$$

which proves 1). On the other hand, we have, by $(1)_m$

$$\begin{aligned}\Big|\Big(\frac{d}{dt}u_m, v\Big)\Big| &\leqq \nu|\langle Au_m, \pi_m v\rangle| + |\langle B(u_m, u_m), \pi_m v\rangle| + |\langle f, \pi_m v\rangle| \\ &\leqq (\nu\|u_m\|_1\|\pi_m v\|_1 + \|\pi_m v\|_s\|u_m\|\,\|u_m\|_1 + \|f\|_{-1}\|\pi_m v\|) \\ &\leqq (\nu\|u_m\|_1 + \|u_m\|\,\|u_m\|_1 + \|f\|_{-1})\|\pi_m v\|_{s,r}\,.\end{aligned}$$

Hence, in view of (2), we have

$$\left\|\frac{d}{dt}u_m\right\|_{\mathcal{H}'_{s,r}} \leqq \nu\|u_m\|_1 + \|u_m\|\,\|u_m\|_1 + \|f\|_{-1},$$

which gives

$$\left\|\frac{d}{dt}u_m\right\|_{L^2(0,T;\mathcal{H}'_{s,r})} \leqq \|u_m\|_{L^2(0,T;\mathcal{H}_1)} + \|u_m\|_{L^\infty(0,T;\mathcal{H})}\cdot\|u_m\|_{L^2(0,T;\mathcal{H})} + \|f\|_{L^2(0,T;\mathcal{H}')}.$$

Thus we have 2) for some constant $C=C_{f,\nu}>0$.

A priori estimate 1) ensures the existence of the unique solution of $(1)_m$ $u_m\in C(0,T;E_m)$ with $u_m(0)=u_0$.

By the continuous dependence of a solution on its initial value, the mapping $S_m: E_m\to C(0,T;E_m)$, defined by $(S_m u_0)(t)=u_m(t)$, is continuous.

Setting $\mu_m=\pi_m\mu$, $P_m=S_m\mu_m$, we have

LEMMA 2.4.

1) $\int(\|u\|^2_{L^\infty(0,T;\mathcal{H})}+\|u\|^2_{L^2(0,T;\mathcal{H}_1)})dP_m(u)\leqq C\int(1+\|u_0\|^2)d\mu$

2) $\int\left\|\frac{du}{dt}\right\|_{L^2(0,T;\mathcal{H}'_{s,r})}dP_m(u)\leqq C\int(1+\|u_0\|^2)d\mu(u_0)$

where C is a positive constant which is independent of m and μ.

PROOF. From Lemma 2.3, 1) follows

$$\operatorname*{Max}_{0\leq t\leq T}\|S_m u_0\|^2+\|S_m u_0\|^2_{L^2(0,T;\mathcal{H}_1)}\leqq C(1+\|u_0\|^2),$$

for some constant $C=C_{\nu,T,f}>0$.

Integrating both sides with respect to μ_m, we obtain 1).

Since P_m is concentrated on the space of solutions of $(1)_m$, we have

$$\frac{du}{dt}=F(u)+g(t) \qquad P_m\text{-a.e.}$$

where $F: C(0,T;E_m)\to C(0,T;E_m)$ is continuous and $g\in L^2(0,T;E_m)$.

Hence, there is a continuous function on $C(0,T;E_m)$ which is equal to $\|du/dt\|_{L^2(0,T;\mathcal{H}'_{s,r})}$ P_m-a.e.. That is to say, the latter is P_m-measurable.

Now, 2) is an immediate consequence of 1) and Lemma 2.3, 2).

Let

$$L_r=\left\{u\in L^\infty(0,\,T\,;\,\mathcal{H})\cap L^2(0,\,T\,;\,\mathcal{H}_1)\,;\,\frac{du}{dt}\in L^2(0,\,T\,;\,\mathcal{H}'_{s,r})\right\}$$
$$Z_r=C(0,\,T\,;\,\mathcal{H}'_{s,r})\cap L^2(0,\,T\,;\,\mathcal{H}_{0,-r})$$

(Here du/dt means the derivative of u in the sense of vector-valued distribution).

As was already mentioned in § 1, L_r and Z_r are Banach spaces with norm

$$\|u\|_{L_r}=\|u\|_{L^\infty(0,T;\mathcal{H})}+\|u\|_{L^2(0,T;\mathcal{H}_1)}+\left\|\frac{du}{dt}\right\|_{L^2(0,T;\mathcal{H}'_{s,r})}$$
$$\|u\|_{Z_r}=\|u\|_{C(0,T;\mathcal{H}'_{s,r})}+\|u\|_{L^2(0,T;\mathcal{H}_{0,-r})},$$

respectively.

LEMMA 2.5. *The canonical injection $L_r\rightarrow Z_r$ is compact.*

Proof proceeds along the same line as in [1] Chap. 4, § 4. We have only to observe that the injections $\mathcal{H}_1\rightarrow\mathcal{H}_{0,-r}$ and $\mathcal{H}\rightarrow\mathcal{H}'_{s,r}$ are compact. (Strictly speaking, the proof in [1] contains a defect. But it is easy to get rid of it).

Let K_R be the closure of $\{u\in L_r\,;\ \|u\|_{L_r}\leqq R\}$ in Z_r, which is compact by virtue of the preceeding lemma. On the other hand, Lemma 2.4 shows

$$P_m(Z_r\backslash K_r)\leqq\frac{1}{R}\int\|u\|_{L_r}dP_m(u)\leqq\frac{C}{R}$$

(C is a constant which is independent of m).

Therefore $\{P_m\}$ contains a subsequence converging weakly to a measure P on Z_r, according to the theorem of Prokhorov:

Let X be a completely regular topological space, whose compact subsets are all metrizable. Suppose that a family of probability Radon measures M is tight, that is, for any $\varepsilon>0$, there exists a compact set $K=K_\varepsilon$, such that

$$P(X\backslash K)<\varepsilon \qquad \textit{for every } P\in M.$$

Then, any sequence in M contains a subsequence which converges weakly on X (For the proof, see e.g. [5] § 5, Theorem 2).

For simplicity, we suppose P_m itself converges to P.

3. Proof of Theorem.

In this paragraph it will be shown that the measure P constructed in

§2 is in fact a statistical solution.

LEMMA 3.1. *Let E be a real separable reflexive Banach space and $\{\mu_n\}$ be a sequence of (Borel) probability measures on E, satisfying for some $\gamma>0$*

$$\int \|x\|^{\gamma} d\mu_n(x) \leqq C \qquad (n\geqq 1).$$

Then, there exists a Borel measure μ on E, such that

$$\int \|x\|^{\gamma} d\mu(x) \leqq C \tag{3}$$

and, for a suitable subsequence $\{\mu_{n'}\}$, we have

$$\hat{\mu}_{n'} \longrightarrow \hat{\mu} \qquad \text{on } E'$$

where $\hat{\mu}(f)=\int e^{i\langle f,x\rangle} d\mu(x)$ —— *characteristic functional of* μ.

PROOF. Since a closed ball in E is weakly compact, $\{\mu_n\}$, regarded as a family of measures on E_σ (E_σ is the space E endowed with the weak topology), is tight by the assumption.

Moreover, separability of E ensures the metrizability of weakly compact sets. In fact, they are homeomorphic to subspaces of I^N (I is a closed interval in R). We also note that since E is a polish space, μ_n is a Radon measure on E, hence a fortiori on E_σ.

Applying Prokhorov's theorem, we are able to extract a subsequence $\mu_{n'}$ which converges weakly to some μ on E_σ.

According to Lusin's theorem, we have $\boldsymbol{B}(E)=\boldsymbol{B}(E_\sigma)$ for a separable Banach space (we denote by $\boldsymbol{B}(X)$ the σ-algebra of all Borel sets in a topological space X).

Thus μ is certainly a Borel measure on E. Because of the weak continuity of $e^{i\langle f,\cdot\rangle}$ ($f\in E'$), we see that $\hat{\mu}_{n'}\to\hat{\mu}$ on E'.

It remains to verify (3).

We take a countable dense subset $\{f_n\}$ in the unit ball in E' (the reflexivity and the separability of E imply the separability of E').

Let us denote by F_n the linear subspace spanned by $\{f_1,\cdots,f_n\}$. Then, as is easily seen, $\|x\|_{(n)}=\sup_{\substack{\|f\|\leqq 1\\ f\in F_n}}|\langle f,x\rangle|$ is weakly continuous and $\|x\|_{(n)}\uparrow\|x\|$ as n tends to infinity. Therefore, setting $\varphi_N(x)=\|x\|_{(N)}^{\gamma}\wedge N$, we have

$$\int \varphi_N(x)d\mu(x)=\lim_{n\to\infty}\int \varphi_N(x)d\mu_n(x)=\varliminf_{n\to\infty}\int\|x\|^{\gamma}d\mu_n(x)\leqq C.$$

Finally the theorem of Beppo Levi shows

$$\int \|x\|^{\gamma} d\mu(x) = \lim_{N\to\infty} \int \varphi_N(x) d\mu(x) \leqq C.$$

Example. Let E be a real separable (infinite dimensional) Hilbert space and $\{e_n\}$ be an orthonormal base of E. Then $\mu_n = \delta_{e_n}$ $(n \geqq 1)$ satisfies (3) for any $\gamma > 0$. In this case we have $\hat{\mu}_n \to \hat{\delta}$ on E.

But any subsequence of $\{\mu_n\}$ does not converges weakly on E, because for a bounded continuous function $\varphi(t)$ on R with $\varphi(0) \neq \varphi(1)$, we have

$$\int \varphi(\|x\|) d\mu_n(x) = \varphi(1) \quad \text{and} \quad \int \varphi(\|x\|) d\delta(x) = \varphi(0)\,.$$

The preceeding lemma, with Lemma 2.4, shows that there is a Borel measure P on $E = L^2(0, T; \mathcal{H}_1)$ satisfying

$$\int \|u\|^2_{L^2(0,T;\mathcal{H}_1)} dP'(u) \leqq C \int (1 + \|u_0\|^2) d\mu(u_0) \tag{4}$$

(C is as in Lemma 2.4) and $\hat{P}_{n'} \to \hat{P}'$ for some subsequence $\{P_{n'}\}$.

Since $P_n \to P$ weakly on Z, we must have $P = P'$ (c.f. § 1, Note 3)).

Similarly we have

$$\int \|u\|^2_{L^p(0,T;\mathcal{H})} dP(u) \leqq C \int (1 + \|u_0\|^2) d\mu(u_0)\,, \tag{5}$$

from which follows (by Fatou's lemma)

$$\int \|u\|^2_{L^\infty(0,T;\mathcal{H})} dP(u) \leqq C \int (1 + \|u_0\|^2) d\mu(u_0)\,.$$

LEMMA 3.2. *Let φ_n $(n \geqq 1)$ be a uniformly equicontinuous function on a Banach space E.*

If φ_n converges on a dense subset of E, φ_n converges everywhere to a uniformly continuous function.

Proof is easy.

Let

$$\Phi_t(u, v) = \langle u(t) - u(0), v\rangle + \int_0^t \langle Au, v\rangle d\tau + \int_0^t \langle B(u, u), v\rangle d\tau$$
$$- \int_0^t \langle f, v\rangle d\tau \qquad (t \in [0, T]) \tag{6}$$

and let

$$\varphi_n(v) = \int e^{i\Phi_t(u,v)} dP_n(u) \qquad (n \geqq 1)\,.$$

As in the proof of a priori estimate (Lemma 2.3), we have

$$|\Phi_t(u, v)| \leqq C(\|u(t)\| + \|u(0)\| + \|u\|^2_{L^2(0,T;\mathcal{H}_1)} + \|u\|^2_{L^\infty(0,T;\mathcal{H})} + 1)\|v\|_{s,r}.$$

Therefore Lemma 2.4 proves

$$|\varphi_n(v) - \varphi_n(v_1)| \leqq \int |\Phi_t(u, v-v_1)| dP_n(u) \leqq C\|v-v_1\|_{s,r}$$

(C is independent of n).

We shall next show that

$$\lim_{n\to\infty} \int e^{i\Phi_t(u,v)} dP_n(u) = \int e^{i\Phi_t(u,v)} dP(u) \qquad \text{for} \quad v \in \mathcal{V}. \tag{7}$$

It suffices to verify that, for $t \in [0, T]$ and $v \in \mathcal{V}$, $\Phi_t(u, v)$ is continuous on Z_r.

The continuity of $\langle u(t) - u(0), v\rangle$ is clear. As for the second and the third term in the right side of (6), it is seen from the inequalities:

$$\left|\int_0^t \langle Au, v\rangle d\tau\right| \leqq \nu\sqrt{T}\|u\|_{L^2(0,T;\mathcal{H}_{0,-r})}\|v\|_{2,r}$$

$$\left|\int_0^t \langle B(u, u), v\rangle d\tau\right| \leqq \|u\|^2_{L^2(0,T;\mathcal{H}_{0,-r})} \sup (1+|x|^2)^r |\nabla u|.$$

REMARK. In the same way, we see that $\Phi_t(u, v)$ is continuous in $t \in [0, T]$ for fixed $u \in Z_r$, $v \in \mathcal{V}$ and continuous in $v \in \mathcal{V}$ for fixed $u \in Z_r$, $t \in [0, T]$.

Thus Lemma 3.2 proves that $\lim_{n\to\infty} \int e^{i\Phi_t(u,v)} dP_n(u)$ exists and continuous on $\mathcal{H}_{s,r}$.

On the other hand, in view of the approximate equation $(1)_m$,

$$\Phi_t(u, v) = 0 \qquad P_m\text{-a.e. if } v \in E_n \text{ and } m \geqq n.$$

Hence we have

$$\lim_{n\to\infty} \int e^{i\Phi_t(u,v)} dP_n(u) = 1 \qquad \text{for} \quad v \in \bigcup_{n=1}^\infty E_n.$$

Since $\bigcup_{n=1}^\infty E_n$ is dense in $\mathcal{H}_{s,r}$, this relation holds also for $v \in \mathcal{H}_{s,r}$.

Therefore from (7) follows

$$\int e^{i\Phi_t(u,v)} dP(u) = 1 \qquad (v \in \mathcal{V}),$$

which implies that, for every $v \in \mathcal{V}$ and $t \in [0, T]$, the distribution of

$\Phi_t(u, v)$, regarded as a random variable on Z_r, is equal to δ (Dirac measure). That is to say,

$$\Phi_t(u, v)=0\,, \qquad P\text{-a.e.} \tag{8}$$

Now, we take countable sets $\{t_k\}$ and $\{v_j\}$, which are dense in $[0, T]$ and $\mathcal{V}$, respectively.

Then, from (4), (5), (8) and § 1 Note 2, we see that

$$W=\bigcap_{j,k=1}^{\infty}\{u\in Z_r\,;\,\Phi_{t_k}(u, v_j)=0\}\cap L^2(0, T\,;\mathcal{H}_1)\cap L^\infty(0, T\,;\mathcal{H})$$

is a Borel set in Z_r and $P(W)=1$.

In view of the remark made above, for $u\in W$ we have

$$\Phi_t(u, v)=0 \qquad (t\in[0, T], v\in\mathcal{V})\,.$$

That is to say, u is a solution of (1). Consequently, as in the proof of Lemma 2.3, 2), we have, for $u\in W$,

$$\left|\left\langle\frac{du}{dt}, v\right\rangle\right|\leqq\|v\|_s(\nu\|u\|_1+\|u\|\,\|u\|_1+\|f\|_{-1})\,,$$

from which follows

$$\left\|\frac{du}{dt}\right\|_{L^2(0,T;\mathcal{H}_s')}\leqq C(1+\|u\|^2_{L^\infty(0,T;\mathcal{H})}+\|u\|^2_{L^2(0,T;\mathcal{H}_1)})\,. \tag{9}$$

This, in particular, shows that $W\in\boldsymbol{B}(Z)$, hence P may be thought to be a probability measure on Z.

Let $W_1=\{u\in L^2(0, T\,;\mathcal{H}_1)\,;\, du/dt\in L^2(0, T\,;\mathcal{H}_s')\}$.

Then $W\subset W_1\subset Z$ and, since W_1 is polish, $\|du/dt\|_{L^2(0,T;\mathcal{H}_s')}$ $(=\infty$ for $u\notin W_1)$ is Borel measurable on Z. Therefore, being equal to a Borel measurable function P-a.e., $\|du/dt\|_{L^2(0,T;\mathcal{H}_s')}$ $(=\infty$ if $du/dt\notin L^2(0, T\,;\mathcal{H}_s'))$ is P-measurable.

Now, energy inequality (Theorem. 3°) is a direct consequence of (4), (5) and (9).

To accomplish the proof of Theorem, it remains to show $\gamma_0 P=\mu$.

Let φ be a bounded continuous function $\mathcal{H}_s'$. Then we have

$$\int\varphi(u_0)d(\gamma_0P)(u_0)=\int\varphi(u(0))dP(u)=\lim_{n\to\infty}\int\varphi(u(0))dP_n(u)$$

$$=\lim_{n\to\infty}\int\varphi(\pi_0u_0)d\mu(u_0)=\int\varphi(u_0)d\mu(u_0)\,.$$

Thus all assertions in Theorem have been proved.

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