J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
34 (1987), 833-851.

The limiting absorption principle for Schrödinger operators with long-range time-periodic potentials

Dedicated to Professor Seizô Itô on his 60th birthday

By Yoshihiro KUWABARA and Kenji YAJIMA

§ 1. Introduction.

Consider the time-periodic Schrödinger equation with period $\omega>0$

$$i\frac{\partial}{\partial t}u(t, x)=(-\Delta+V(t, x))u(t, x)\ , \qquad u(t, \cdot)\in L^{2}(\boldsymbol{R}^{n})\ , \tag{1.1}$$

$$V(t+\omega, x)=V(t, x)\ , \qquad t\in\boldsymbol{R},\ x\in\boldsymbol{R}^{n}\ , \tag{1.2}$$

where $\Delta=\sum_{j=1}^{n}\partial^{2}/\partial x_{j}^{2}$. Under suitable conditions on $V(t, x)$ ([16]), Eq. (1.1) generates a unique unitary propagator $\{U(t, s) : -\infty<t, s<\infty\}$ in $L^{2}(\boldsymbol{R}^{n})$ and the solution $u(t)$ of (1.1) with the initial condition $u(s)=u_{0}\in L^{2}(\boldsymbol{R}^{n})$ is given as $u(t)=U(t, s)u_{0}$. By virtue of (1.2) $U(t, s)$ satisfies the periodicity $U(t+\omega, s+\omega)=U(t, s)$ so that the large time behavior of $U(t, s)u_{0}$ is closely related to the spectral property of the Floquet operator $U(s+\omega, s)$. On the other hand the spectral property of $U(s+\omega, s)$ may be studied by means of an explicit operator $-i\partial/\partial t-\Delta+V(t, x)$: If we let $\mathcal{K}=L^{2}(\boldsymbol{T})\otimes L^{2}(\boldsymbol{R}^{n})$, $\boldsymbol{T}=\boldsymbol{R}/\omega\boldsymbol{Z}$, and K be the natural selfadjoint realization of $-i\partial/\partial t-\Delta+V(t, x)$ on $\mathcal{K}$, then via the unitary operator

$$(\mathcal{U}_{s}f)(t)=U(t, s)f(t) \qquad \text{for } s\leqq t<s+\omega$$

we have

$$e^{-i\omega K}=\mathcal{U}_{s}(1\otimes U(s+\omega, s))\mathcal{U}_{s}^{*}\ .$$

In this paper we study the behavior of the resolvent $R(\zeta)=(K-\zeta)^{-1}$ of K near the real axis and prove the so-called limiting absorption principle for K, the existence of the boundary values $\lim_{\varepsilon\downarrow 0}R(\lambda\pm i\varepsilon)$ for $\lambda\in\boldsymbol{R}$ in certain weighted spaces. As a byproduct, it will be also shown that any of the eigenfunctions of K is smooth and all its derivatives are rapidly decreasing with respect to x. The application of this result for the scattering theory for (1.1) will be the theme of a forthcoming paper.

We assume that the potential $V(t, x)$ satisfies the following assumption.

$D_t=-i\partial/\partial t$ and $D_x=(-i\partial/\partial x_1, -i\partial/\partial x_2, \cdots, -i\partial/\partial x_n)$.

ASSUMPTION (V). $V(t, x)$ can be written as $V(t, x)=V^L(t, x)+V^S(t, x)$ where $V^L(t, x)$ and $V^S(t, x)$ are real-valued functions periodic in t with period $\omega>0$, and satisfy the following properties:

(1) $V^L(t, x)\in C^\infty(\boldsymbol{R}^{n+1})$ and for any multi-indices α and β,

$$|D_t^\alpha D_x^\beta V^L(t, x)|\leqq C_{\alpha\beta}(1+|x|)^{-m(|\beta|)}, \qquad (t, x)\in \boldsymbol{R}^{n+1}, \tag{1.3}$$

where $m(j)$ is the sequence defined by

$$m(j)=\delta \qquad \text{when } j=0,$$

$$m(j)=1+\delta j \qquad \text{when } j\geqq 1,\ 0<\delta\leqq 1.$$

(2) When $n\geqq 3$, $(1+|x|)^{1+\delta}V^S(t, x)\in L^\infty(\boldsymbol{T}, L^p(\boldsymbol{R}^n))+L^\infty(\boldsymbol{T}\times\boldsymbol{R}^n)$ for some $p>n$. When $n=2$, $(1+|x|)^{1+\delta}V^S(t, x)(H_0+1)^{-\gamma/4}$ is a bounded operator in $\mathcal{K}$ for some $0<\gamma<1$, where $H_0=-\Delta$. When $n=1$, $(1+|x|)^{1+\delta}V^S(t, x)\in L^\infty(\boldsymbol{T}\times\boldsymbol{R}^n)$.

We note that if we let K_0 be the unique selfadjoint extention of $-i\partial/\partial t-\Delta|_{C_0^\infty(\boldsymbol{T}\times\boldsymbol{R}^n)}$ on $\mathcal{K}$ and V be the multiplication operator by $V(t, x)$, then V is K_0-bounded with K_0-bound 0 under Assumption (V), and $K=K_0+V$ with domain $\mathcal{D}(K)=\mathcal{D}(K_0)$ is selfadjoint.

For stating our theorems we need some of the notation. The space $B=B(\boldsymbol{R}^n, L^2(\boldsymbol{T}))$ is the Banach space defined by

$$B=\left\{u\in\mathcal{K} : \|u\|_B=\sum_{j=1}^\infty\left(R_j\int_{X_j}\int_0^\omega|u(t, x)|^2dtdx\right)^{1/2}<\infty\right\}$$

where

$$R_0=0,\ R_j=2^{j-1} \qquad \text{when } j>0,\ X_j=\{x\,;\,R_{j-1}<|x|<R_j\}.$$

Its dual space B^* is given as

$$B^*=\left\{u\in L^2_{\mathrm{loc}} : \|u\|_{B^*}=\sup_{j>0}\left(R_j^{-1}\int_{X_j}\int_0^\omega|u(t, x)|^2dtdx\right)^{1/2}<\infty\right\}.$$

$\mathring{B}^*$ is defined by

$$\mathring{B}^*=\left\{u\in L^2_{\mathrm{loc}} : R^{-1}\int_{|x|<R}\int_0^\omega|u(t, x)|^2dtdx\to 0 \text{ as } R\to\infty\right\}.$$

It is well known that $C_0^\infty(\boldsymbol{T}\times\boldsymbol{R}^n)$ is dense in B and that $\mathring{B}^*$ is the closure of $C_0^\infty(\boldsymbol{T}\times\boldsymbol{R}^n)$ in B^*. We note that

$$\|u\|_{B^*}^2 \leqq \sup_{R>1} R^{-1}\int_{|x|<R}\int_0^\omega |u(t, x)|^2 dtdx \leqq 4\|u\|_{B^*}^2 . \tag{1.4}$$

For $s\in \boldsymbol{R}$, $L_s^2 = L_s^2(\boldsymbol{T}\times \boldsymbol{R}^n)$ is the Hilbert space defined by

$$L_s^2 = \left\{u\in L_{\mathrm{loc}}^2 : \|u\|_s^2 = \int_0^\omega\int_{\boldsymbol{R}^n} \langle x\rangle^{2s}|u(t, x)|^2 dtdx < \infty\right\},$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$. For a Banach space X, $\mathcal{L}(X)$ is the Banach space of all bounded linear operators on X. For operators A and B, $[A, B] = AB-BA$. We introduce the following Hörmander's notation. For $0\leqq\delta\leqq 1$ and $l, m\in \boldsymbol{R}$,

$$S(\langle x\rangle^l(\langle \xi\rangle^2+\langle k\rangle)^m, G_\delta)$$
$$=\{p\in C^\infty(\boldsymbol{R}^{n+1}\times \boldsymbol{R}^{n+1}) : \quad \text{for } j\geqq 0, |p|_j^\delta = \max_{|\alpha+\beta+\gamma+\eta|=j}\ \sup_{\tilde{x},\tilde{\xi}\in \boldsymbol{R}^{n+1}}$$
$$\langle x\rangle^{-l+\delta|\beta|}(\langle \xi\rangle^2+\langle k\rangle)^{-m+(|\gamma|+|\eta|)/2}|D_t^\alpha D_x^\beta D_k^\gamma D_\xi^\eta p(t, x, k, \xi)| < \infty\},$$

where $\tilde{x} = (t, x)$ and $\tilde{\xi} = (k, \xi)$. For $p\in S(\langle x\rangle^l(\langle \xi\rangle^2+\langle k\rangle)^m, G_\delta)$, the pseudo-differential operator (Ps. D. Op.) P with symbol $p(\tilde{x}, \tilde{\xi})$ is defined by

$$Pf(\tilde{x}) = p(\tilde{x}, D)f(\tilde{x}) = (2\pi)^{-(n+1)/2}\int_{\boldsymbol{R}^{n+1}} e^{i\tilde{x}\cdot\tilde{\xi}} p(\tilde{x}, \tilde{\xi})\hat{f}(\tilde{\xi})d\tilde{\xi},$$

where $\hat{f}(\tilde{\xi})$ is the Fourier transform of f:

$$\hat{f}(\tilde{\xi}) = (2\pi)^{-(n+1)/2}\int_{\boldsymbol{R}^{n+1}} e^{\ i\tilde{x}\cdot\tilde{\xi}} f(\tilde{x})d\tilde{x} .$$

The symbol $p(\tilde{x}, \tilde{\xi})$ of P is denoted by $\sigma(P)$. We write the set of all Ps. D. Op.'s with symbols in $S(\langle x\rangle^l(\langle \xi\rangle^2+\langle k\rangle)^m, G_\delta)$ by $\mathrm{Op}\, S(\langle x\rangle^l(\langle \xi\rangle^2+\langle k\rangle)^m, G_\delta)$. When $p(\tilde{x}, \tilde{\xi})$ is periodic in t, P maps periodic functions into periodic functions. Thus P is considered as linear operators on function spaces over $\boldsymbol{T}\times \boldsymbol{R}^n$. $\mathcal{D} = (\langle D_x\rangle^4 + \langle D_t\rangle^2)^{1/2}$ is the Ps. D. Op. with symbol $(\langle \xi\rangle^4+\langle k\rangle^2)^{1/2}$. $H_{\alpha,\gamma}$ is the weighted Sobolev space:

$$H_{\alpha,\gamma} = \{u\in \mathcal{S}' : \|u\|_{\alpha,\gamma} = \|\langle x\rangle^\gamma \mathcal{D}^\alpha u\|_{\mathcal{K}} < \infty\} .$$

For $x\in \boldsymbol{R}$, $[x]$ is the greatest integer not greater than x. $\boldsymbol{C}^\pm = \{\zeta\in \boldsymbol{C} : \mathrm{Im}\,\zeta \gtrless 0\}$, $\boldsymbol{C}_\mu^\pm = \{\zeta\in \boldsymbol{C}^\pm : \mu\leqq \mathrm{Re}\,\zeta\leqq 1-\mu\}$, $0<\mu<1/2$. $\boldsymbol{R}^+ = \{x\in \boldsymbol{R} : x>0\}$.

Now we can state our main theorems.

THEOREM 1.1. *Let $V(t, x)$ satisfy Assumption* (V). *Then*:

(1) *The eigenvalues $\lambda\in \boldsymbol{R}\setminus(2\pi/\omega)\boldsymbol{Z}$ of K are of finite multiplicity, and form a set Λ which is discrete in $\boldsymbol{R}\setminus(2\pi/\omega)\boldsymbol{Z}$.*

(2) *The corresponding eigenfunctions belong to* $H_{1/2,\gamma}$ *for every* $\gamma\in\boldsymbol{R}$.

(3) *For every* $\lambda\in\boldsymbol{R}\backslash(\Lambda\cup(2\pi/\omega)\boldsymbol{Z})$ *and* $f\in B$, $\lim_{C_{\pm}\ni\zeta\to\lambda}\mathcal{D}^{1/2}R(\zeta)f=\mathcal{D}^{1/2}R(\lambda\pm i0)f$ *exists as a weak* limit in* B^*.

(4) *For every* $f\in B$, $\mathcal{D}^{1/2}R(\lambda\pm i0)f$ *is continuous in* λ *in the weak* topology of* B^*.

(5) $u=R(\lambda\pm i0)f$ *is a unique solution of* $(-i\partial/\partial t-\Delta+V(t,x)-\lambda)u=f$ *which satisfies* $\mathcal{D}^{1/2}u\in B^*$ *and* $a(x,D)u\in\mathring{B}^*$ *for every* $a(x,k,\xi)\in S((\langle\xi\rangle^2+\langle k\rangle)^{1/4},G_1)$ *vanishing on* $N_{\pm}(\lambda)=\{(x,k,\xi)\in\boldsymbol{R}^{2n+1};\ x=2l\xi,\ l\gtrless 0,\ \xi^2+2\pi k/\omega=\lambda,\ \text{and}\ k\notin([\omega\lambda/2\pi],\ [(\omega\lambda/2\pi)+1])\}$.

THEOREM 1.2. *Assume that* $V^S(t,x)=0$ *in addition to Assumption* (V). *Then every eigenfunction* φ *of* K *corresponding to the eigenvalue* $\lambda\in\boldsymbol{R}\backslash(2\pi/\omega)\boldsymbol{Z}$ *is smooth and*

$$|D_t^{\alpha}D_x^{\beta}\varphi(t,x)|\leqq C_{\alpha\beta\gamma}\langle x\rangle^{-\gamma}\tag{1.5}$$

or all multi-indices α, β *and for every* $\gamma\in\boldsymbol{R}$.

It is needless to say that the limiting absorption principle (LAP) plays important roles in the scattering theory. Up to middle 1970s this was the most powerful tool for proving the asymptotic completeness (AC). Indeed Agmon [1] and Kuroda [9] (resp. Kitada [8] and Ikebe-Isozaki [5]) have settled the AC problem for short-range (resp. long-range) potentials using the LAP at the crucial steps of their proofs. Though its role for the AC problem became less eminent after the appearence of the time-dependent method (Enss [3]), the LAP is still indispensable for studying properties of, for example, the scattering matrix.

The LAP for Schrödinger operators with long-range potentials was first proved by Ikebe-Saito [6] and independently by Lavine [10]. These results were subsequently extended to general elliptic operators by Agmon [2] (cf. also Hörmander [4]) and to some non-elliptic operators (cf. Tamura [13], Iwashita [7] and Mochizuki-Uchiyama [11]). We should also mention Mourre [12] which elaborated the Lavine's commutator method.

Our operator differs from the ones previously studied in the respect that K_0 is not elliptic and the energy surface $\xi^2+2\pi k/\omega=\lambda$ is not compact, and that the space $\boldsymbol{T}\times\boldsymbol{R}^n$ is compact in t-direction. We shall show, nonetheless, that the standard method for elliptic operators summarized in Hörmander's monograph [4] still applies to our operator with a few modifications. Note that when $V\equiv 0$, our operator $(K_0-\zeta)^{-1}\cong\oplus_{k=-\infty}^{\infty}(-\Delta-2\pi k/\omega-\zeta)^{-1}$ and we can control $(K_0-\zeta)^{-1}$ once we establish estimates for $(-\Delta-\zeta)^{-1}$ which is uniform in $\zeta\in\boldsymbol{C}$. Thus we need keep track of the λ-dependence of various estimates for elliptic operators.

The organization of our paper is as follows: In Section 2 we briefly review some of the properties of Ps. D. Op.'s and prepare several lemmas needed in the next section. In Section 3 we establish a priori estimates under the assumption that $V^S \equiv 0$ and obtain Theorem 1.2 as a corollary. In Section 4 we take a short-range perturbation into consideration and prove Theorem 1.1.

§ 2. Preliminaries.

In this section we review some basic facts about function spaces and Ps. D. Op.'s, and prepare some lemmas which will be needed in what follows.

The function spaces B, B^* and L_s^2 are as in the introduction. We have the following interpolation theorem.

THEOREM 2.1. *Let a linear operator T satisfy that $\|Tf\|_{-s} \leqq A\|f\|_{-s}$ and that $\|Tf\|_s \leqq A\|f\|_s$ for some $s>1/2$. Then there is a constant C_s such that $\|Tf\|_B \leqq C_s A\|f\|_B$.*

PROOF. The proof of Theorem 14.1.4 in Hörmander [4] obviously applies to vector-valued functions and yields the theorem.

The space of symbols $S(\langle x\rangle^l(\langle\xi\rangle^2+\langle k\rangle)^m, G_\delta)$ with the topology defined by the seminorms $|\cdot|_j^\delta$ is a Fréchet space. When $\Lambda \subset \boldsymbol{R}^+$ and $c \in \boldsymbol{R}$, we say $\{a_\lambda\}_{\lambda\in\Lambda}$ is bounded in $S(\lambda^c\langle x\rangle^l(\langle\xi\rangle^2+\langle k\rangle)^m, G_\delta)$ with respect to $\lambda \in \Lambda$ if $\{\lambda^{-c}a_\lambda\}_{\lambda\in\Lambda}$ is bounded in the Fréchet space $S(\langle x\rangle^l(\langle\xi\rangle^2+\langle k\rangle)^m, G_\delta)$.

THEOREM 2.2. (1) *Let $P_j \in \mathrm{Op}\, S(\langle x\rangle^{l_j}(\langle\xi\rangle^2+\langle k\rangle)^{m_j}, G_\delta)$, $j=1, 2$. Then $P_1P_2 \in \mathrm{Op}\, S(\langle x\rangle^{l_1+l_2}(\langle\xi\rangle^2+\langle k\rangle)^{m_1+m_2}, G_\delta)$ and*

$$\sigma(P_1P_2) - \sum_{j<N} (i\langle D_{\tilde y}, D_{\tilde\eta}\rangle)^j \sigma(P_1)(\tilde x, \tilde\eta)\sigma(P_2)(\tilde y, \tilde\xi)|_{\tilde\eta=\tilde\xi, \tilde y=\tilde x}/j!$$
$$\in S(\langle x\rangle^{l_1+l_2-\delta N}(\langle\xi\rangle^2+\langle k\rangle)^{m_1+m_2-N/2}, G_\delta)\,, \qquad N=0, 1, 2, \cdots.$$

(2) *Let $P \in \mathrm{Op}\, S(\langle x\rangle^l(\langle\xi\rangle^2+\langle k\rangle)^m, G_\delta)$. Then P^* belongs to the same class and*

$$\sigma(P^*) - \sum_{j<N} (i\langle D_{\tilde x}, D_{\tilde\xi}\rangle)^j \overline{\sigma(P)(\tilde x, \tilde\xi)}/j! \in S(\langle x\rangle^{l-\delta N}(\langle\xi\rangle^2+\langle k\rangle)^{m-N/2}, G_\delta)\,,$$
$$N=0, 1, 2, \cdots.$$

(3) *Let $p \in S(1, G_\delta)$ and $p(t+\omega, x, k, \xi)=p(t, x, k, \xi)$. Then $p(\tilde x, D) \in \mathcal{L}(\mathcal{K})$ and $\|\cdot\|_{\mathcal{L}(\mathcal{K})}$ is a continuous seminorm in $S(1, G_\delta)$.*

(4) (The sharp Gårding inequality) *Let* $0\leqq p\in S(\langle x\rangle^{\delta}(\langle\xi\rangle^{2}+\langle k\rangle)^{1/2}, G_{\delta})$ *and* $p(t+\omega, x, k, \xi)=p(t, x, k, \xi)$. *Then*

$$\mathrm{Re}(p(\tilde{x}, D)u, u)_{\mathcal{K}}\geqq-C\|u\|_{\mathcal{K}}^{2}, \qquad u\in C_{0}^{\infty}(\boldsymbol{T}\times\boldsymbol{R}^{n}). \tag{2.1}$$

PROOF. See Appendix.

By virtue of Theorem 2.1 and 2.2, one easily proves the following

THEOREM 2.3. *Let* $P\in\mathrm{Op}\,S(1, G_{\delta})$ *and let* $\sigma(P)(t+\omega, x, k, \xi)=\sigma(P)(t, x, k, \xi)$. *Then P is bounded in B and in* B^{*}.

We shall use the following symbols in Section 3. $\tilde{\rho}(s)\in C_{0}^{\infty}(-1/4, 1/4)$ is a decreasing function of s^{2} such that $0\leqq\tilde{\rho}\leqq 1$ and $\tilde{\rho}(s)=1$ when $|s|\leqq 1/8$. For $0<\varepsilon<1/2$, $\tilde{\chi}_{\varepsilon}(s)\in C_{0}^{\infty}(\varepsilon/4, 1-\varepsilon/4)$ is a function such that $0\leqq\tilde{\chi}_{\varepsilon}\leqq 1$ and $\tilde{\chi}_{\varepsilon}(s)=1$ when $\varepsilon/2\leqq s\leqq 1-\varepsilon/2$. Set $\chi_{\varepsilon}(k, \xi; \lambda)=\tilde{\rho}((\xi^{2}+k-\lambda)/\varepsilon\langle k-\lambda\rangle)$ and $\underline{\chi}_{\varepsilon}(k, \xi; \lambda)=\chi_{\varepsilon}(k, \xi; \lambda)(1-\tilde{\chi}_{\varepsilon}(k))$. The following lemma is easy to prove.

LEMMA 2.4. (1) *For each* $0<\varepsilon<1/2$, χ_{ε} *and* $\underline{\chi}_{\varepsilon}$ *are bounded in* $S(1, G_{\delta})$ *with respect to* λ *when* $\varepsilon\leqq\lambda\leqq 1-\varepsilon$.

(2) *There exists a positive constant c such that* $|\xi|>c$ *in* $\mathrm{supp}\,\underline{\chi}_{\varepsilon}$.

(3) *If* $V_{1}(x)\in C^{\infty}(\boldsymbol{R}^{n})$ *is such that* $\sup_{x\in\boldsymbol{R}^{n}}|V_{1}(x)|<\varepsilon/32$ *and* $|D_{x}^{\alpha}V_{1}(x)|\leqq C_{\alpha}\langle x\rangle^{-m(|\alpha|)}$, *then for* $\zeta\in\{z\in\boldsymbol{C}; \varepsilon\leqq\mathrm{Re}\,z\leqq 1-\varepsilon\}$,

$$e_{\zeta}(x, k, \xi)=(\xi^{2}+k+V_{1}(x)-\zeta)^{-1}(1-\chi_{\varepsilon}(k, \xi; \lambda)), \qquad \lambda=\mathrm{Re}\,\zeta, \tag{2.2}$$

is bounded in $S((\langle\xi\rangle^{2}+\langle k\rangle)^{-1}, G_{\delta})$.

(4) *Set* $\Xi_{l}(k, \xi)=\tilde{\rho}((\langle\xi\rangle^{4}+\langle k\rangle^{2})^{1/4}/l)$ *and* $\zeta_{l}(x)=\tilde{\rho}(|x|/l)$. *Then for every* $c\geqq 0$, $\{\Xi_{l}-1\}_{l\geqq 1}$ *and* $\{\zeta_{l}-1\}_{l\geqq 1}$ *are bounded in* $S(l^{-c}(\langle\xi\rangle^{2}+\langle k\rangle)^{c/2}, G_{1})$ *and* $S(l^{-c}\langle x\rangle^{c}, G_{1})$, *respectively.*

Hörmander [4, pp. 287 and 290 of vol. 4] shows the following

LEMMA 2.5. *There exist functions* $\Psi_{1}(x, y)$ *and* $\Psi_{2}(x, y)$ *which satisfy the following properties*:

(1) $\Psi_{j}\in C^{\infty}(\boldsymbol{R}^{n}\times(\boldsymbol{R}^{n}\backslash\{0\}))$.

(2) $\Psi_{j}(x, \lambda y)=\Psi_{j}(x, y)$, $\lambda\neq 0$.

(3) $\Psi_{j}(x, y)\geqq 0$ *and* $\Psi_{j}'(x, y)\geqq 0$ *for* $\Psi_{j}'(x, y)\equiv\langle y, \partial\Psi_{j}(x, y)/\partial x\rangle$.

(4) $|D_{x}^{\alpha}D_{y}^{\beta}\Psi_{j}(x, y)|\leqq C_{\alpha\beta}(1+|x|)^{-|\alpha|}|y|^{-|\beta|}$.

(5) *For some* $\phi\in C_{0}^{\infty}(\{x; 1/2<|x|<3\})$, $\phi\equiv$ *constant* >0 *on* $1\leqq|x|\leqq 2$,

$$|\phi(x)|^{2}\leqq\Psi_{1}'(x, y)\Psi_{1}(x, y)/|y|. \tag{2.3}$$

(6) *There is a constant* $c>0$ *such that* $\Psi_2(x, y)=0$ *when* $|x|<c$, *and*

$$|(1-\rho_\varepsilon(x, -y))\phi(x)|^2 \leqq \Psi_2'(x, y)\Psi_2(x, y)/|y| . \tag{2.4}$$

Here $\rho_\varepsilon(x, y)=\tilde{\rho}(\varepsilon^{-1}(1-\langle x, y\rangle/|x||y|))$, $0<\varepsilon\leqq 2$.

Using these functions, we define

$$q_{j,R}(x, k, \xi)=\Psi_j(x/R, -2\xi)\underline{\chi}_\varepsilon(k, \xi ; \lambda) , \qquad j=1, 2,$$

$$S_{1,R}(x, k, \xi)=-q_{1,R}\langle 2\xi, \partial q_{1,R}/\partial x\rangle - cR^{-1}(\langle\xi\rangle^4+\langle k\rangle^2)^{1/4}|\phi(x/R)\underline{\chi}_\varepsilon(k, \xi ; \lambda)|^2$$

and

$$S_{2,R}(x, k, \xi)=-q_{2,R}\langle 2\xi, \partial q_{2,R}/\partial x\rangle - cR^{-1}(\langle\xi\rangle^4+\langle k\rangle^2)^{1/4}|\Phi_{R,\varepsilon}(x, k, \xi)|^2 ,$$

where

$$\Phi_{R,\varepsilon}(x, k, \xi)=(1-\rho_\varepsilon(x/R, 2\xi))\phi(x/R)\underline{\chi}_\varepsilon(k, \xi ; \lambda) .$$

In view of (2.3), (2.4) and Lemma 2.4, (2), we assume that $S_{j,R}\geqq 0$ by taking $c>0$ small enough.

LEMMA 2.6. *Set* $Q_{j,R}=q_{j,R}(x, D)$ *and* $S_{j,R}=S_{j,R}(x, D)$. *Then*:

(1) $\{q_{1,R}\}_{R\geqq 1}$ *and* $\{S_{1,R}\}_{R\geqq 1}$ *are bounded in* $S(1, G_1)$ *and in* $S(\langle x\rangle^{-1}(\langle\xi\rangle^2+\langle k\rangle)^{1/2}, G_1)$, *respectively.*

(2) *If* $g\in C^\infty(\boldsymbol{R}^+)$ *is a positive increasing function such that* $|D^\alpha g(s)| \leqq C_\alpha |s|^{\gamma-|\alpha|}$ *for some* $\gamma\geqq 0$. *Then* $\{g(R)q_{2,R}\}_{R\geqq 1}$ *and* $\{g(R)S_{2,R}\}_{R\geqq 1}$ *are bounded in* $S(g(\langle x\rangle), G_1)$ *and* $S(g(\langle x\rangle)(\langle\xi\rangle^2+\langle k\rangle)^{1/2}, G_1)$, *respectively.*

(3) *For any* $\gamma\geqq 0$,

$$\mathrm{Re}(S_{1,R}u, u)\geqq -C\|u\|_{-1} , \qquad R\geqq 1 , \tag{2.5}$$

$$R^{2\gamma}\,\mathrm{Re}(S_{2,R}u, u)\geqq -C\|u\|_{\gamma-1} , \qquad R\geqq 1 . \tag{2.6}$$

(4) *For every* $s\in\boldsymbol{R}$,

$$\|\mathcal{D}^s Q_{j,R}u\|_{B^*}\leqq C\|\mathcal{D}^s\chi_\varepsilon u\|_{B^*} , \qquad \mathcal{D}^s u\in B^*,\ R\geqq 1 , \tag{2.7}$$

$$\|\mathcal{D}^s Q_{j,R}f\|_B\leqq C\|\mathcal{D}^s\chi_\varepsilon f\|_B , \qquad \mathcal{D}^s f\in B,\ R\geqq 1 . \tag{2.8}$$

(5) *For every* $\gamma\geqq 0$ *and* $s\in\boldsymbol{R}$, $\{\sigma([Q_{2,R}, \mathcal{D}^s])\}_{R\geqq 1}$ *is bounded in* $S(R^{-\gamma}\langle x\rangle^{\gamma-1}(\langle\xi\rangle^2+\langle k\rangle)^{s-1/2}, G_1)$.

(6) *For each* $f\in B$,

$$\|Q_{2,R}f\|_B \longrightarrow 0 \qquad \text{as } R\to\infty . \tag{2.9}$$

(7) *Let* $\Phi'_{R,\varepsilon}(x, k, \xi)=(1-\rho_\varepsilon(x/R, -2\xi))\phi(x/R)\underline{\chi}_\varepsilon(k, \xi ; \lambda)$. *For every* $\gamma\geqq 0$,

$s\in\boldsymbol{R}$ *and* $\sigma>0$,

(2.10)
$$R^{-1}\|R^{\gamma}(1+\sigma R)^{-\gamma}\phi(x/R)\chi_{\varepsilon}(D\,;\,\lambda)\mathcal{D}^{s}u\|^{2}$$
$$\leqq C\{R^{-1}\|R^{\gamma}(1+\sigma R)^{-\gamma}\Phi_{R,\varepsilon}\mathcal{D}^{s}u\|^{2}$$
$$+R^{-1}\|R^{\gamma}(1+\sigma R)^{-\gamma}\Phi'_{R,\varepsilon}\mathcal{D}^{s}u\|^{2}+\|\mathcal{D}^{s}u\|_{\gamma-1}^{2}\},$$

where C *is a constant independent of* σ *and* $R\geqq 1$.

PROOF. (1) is obvious. (2) follows from Lemma 2.5. (3) is an immediate consequence of the sharp Gårding inequality (2.1). Set $Q'_{1,R}=\Psi_{1}(x/R,\,-2D_{x})\chi_{2\varepsilon}(D\,;\,\lambda)(1-\tilde{\chi}_{\varepsilon}(D_{t}))$. Then $\mathcal{D}^{s}Q_{1,R}=\mathcal{D}^{s}Q'_{1,R}\mathcal{D}^{-s}\mathcal{D}^{s}\chi_{\varepsilon}$ and $\sigma(\mathcal{D}^{s}Q'_{1,R}\mathcal{D}^{-s})$ is bounded in $S(1,G_{1})$ when $R\geqq 1$. This implies (2.7). (2.8) may be proved similarly. For $\gamma\geqq 0$, $q_{2,R}$ is bounded in $S(R^{-\gamma}\langle x\rangle^{\gamma},G_{1})$ and (5) obviously holds. In view of Lemma 2.5, (6), (2.9) holds for $f\in C_{0}^{\infty}(\boldsymbol{T}\times\boldsymbol{R}^{n})$, which is dense in B, and $\|Q_{2,R}\|_{\mathcal{L}(B)}$ is uniformly bounded for $R\geqq 1$. Hence (2.9) holds for every $f\in B$. Since $0\leqq\rho\leqq 1$ and $\operatorname{supp}\tilde{\rho}\subset(-1/4,1/4)$, we have $\tilde{\rho}(\varepsilon^{-1}(1+t))+\tilde{\rho}(\varepsilon^{-1}(1-t))\leqq 1$ for $0<\varepsilon\leqq 2$, hence

$$\phi(x)\leqq(1-\rho_{\varepsilon}(x,\,-y))\phi(x)+(1-\rho_{\varepsilon}(x,y))\phi(x)\,,\qquad 0<\varepsilon\leqq 2\,.$$

Therefore (2.10) follows from (2.1).

When ε is fixed, the subscript ε will be often suppressed in the following expressions. Some of variables in symbols may also be omitted when no confusion is feared.

§3. A priori estimates.

In this section we assume that Assumption (V) is satisfied and that $V^{S}\equiv 0$. We may set $\omega=2\pi$ without loss of generality. For the later analysis, it is convenient to introduce a gauge transformation: Set

$$V_{0}=(2\pi)^{-1}\int_{0}^{2\pi}V(t,x)dt\,,\qquad F(t,x)=\int_{0}^{t}\{V(s,x)-V_{0}(x)\}ds\,,$$
$$A(t,x)=\operatorname{grad}_{x}F(t,x)\,,$$

and define

$$(\mathcal{V}u)(t,x)=\exp(iF(t,x))u(t,x)\,,\qquad u\in\mathcal{K}\,,$$

and $\tilde{K}=\mathcal{V}K\mathcal{V}^{*}$. It is clear that $\mathcal{V}$ is unitary and that V_{0} and F satisfy

(3.1)
$$|D_{x}^{\alpha}V_{0}(x)|\leqq C_{\alpha}\langle x\rangle^{-m(|\alpha|)}\,,$$

(3.2) $$|D_t^\alpha D_x^\beta F(t,x)| \leqq C_{\alpha\beta}\langle x\rangle^{-m(|\beta|)}.$$

It is also clear that $C_0^\infty(\boldsymbol{T}\times\boldsymbol{R}^n)$ is a core of $\tilde{K}$ and that

$$\begin{aligned}\tilde{K} &= -i\partial/\partial t + (-i\partial/\partial x - A(t,x))^2 + V_0(x)\\ &= -i\partial/\partial t - \Delta + \tilde{V}(\tilde{x}, D_x),\end{aligned}$$

where $\tilde{V}(\tilde{x}, D_x) = -2A(t,x)\cdot D_x + i\,\mathrm{div}_x A(t,x) + A(t,x)^2 + V_0(x)$. Note that the long-range term in $\tilde{K}$ is made time-independent at the cost of introducing the first order short-range perturbation.

In this section we shall prove a priori estimates for solutions u of

(3.3) $$(-i\partial/\partial t - \Delta + \tilde{V}(\tilde{x}, D_x) - \zeta)u = f, \qquad \zeta\in\boldsymbol{C}.$$

It is sufficient to consider the case $0<\mathrm{Re}\,\zeta<1$ since $e^{-ikt}(\tilde{K}-\zeta)e^{ikt} = \tilde{K}+k-\zeta$, $k\in\boldsymbol{Z}$. We arbitrarily take and fix $0<\mu<1/2$, and consider only those ζ which satisfy $\mu\leqq\mathrm{Re}\,\zeta\leqq 1-\mu$. In the followings we denote $\mathrm{Re}\,\zeta=\lambda$ whenever ζ and λ appear in the same statement or formula, and we consider only the boundary value of $(\tilde{K}-\zeta)^{-1}$ from the upper half plane $\boldsymbol{C}^+$. The limit of $(\tilde{K}-\zeta)^{-1}$ from the other half plane can be treated similarly.

We first estimate on the off-shell. χ_ε is as in Section 2.

PROPOSITION 3.1. *Let $0<\varepsilon\leqq\mu$. Suppose $u\in H_{l,\gamma-1-\delta}$ satisfies* (3.3) *with $f\in H_{m,\gamma}$. Then $(1-\chi_\varepsilon(D;\lambda))u\in H_{m+1,\gamma}$ and*

(3.4) $$\|(1-\chi_\varepsilon(D;\lambda))u\|_{m+1,\gamma} \leqq C(\|f\|_{m,\gamma} + \|u\|_{l,\gamma-1-\delta}),$$

where C is independent of u, f and λ.

PROOF. Let us split as

$$\begin{cases} V_0(x) = V_1(x) + V_2(x), \\ \sup|V_1(x)| < \varepsilon/64, \qquad V_2\in C_0^\infty(\boldsymbol{R}^n), \end{cases}$$

and define $e_\zeta(x,k,\xi)$ as (2.2) with ε replaced by $\varepsilon/2$. Then

$$e_\zeta(x,D)(-i\partial/\partial t - \Delta + \tilde{V}(\tilde{x}, D_x) - \zeta) = 1 - \chi_{\varepsilon/2}(D) - r_\zeta(\tilde{x}, D),$$

where r_ζ is bounded in $S(\langle x\rangle^{-1-\delta}(\langle\xi\rangle^2+\langle k\rangle)^{-1/2}, G_0)$ when $\mu\leqq\lambda\leqq 1-\mu$. Letting E_ζ be the asymptotic sum of the symbols of $(1-\chi_\varepsilon(D))r_\zeta(\tilde{x},D)^j e_\zeta(x,D)$, $j=0,1,2,\cdots$, we obtain

$$E_\zeta(\tilde{x},D)(-i\partial/\partial t - \Delta + \tilde{V}(\tilde{x}, D_x) - \zeta) = 1 - \chi_\varepsilon(D) + R_\zeta(\tilde{x}, D),$$

where for every $N\in\boldsymbol{R}$,

$$|D_{\tilde{x}}^{\alpha}D^{\beta}_{\tilde{\xi}}R_{\zeta}(\tilde{x}, \tilde{\xi})| \leqq C_{\alpha\beta N}\langle x\rangle^{-1-\delta}(\langle\xi\rangle^{2}+\langle k\rangle)^{-N}.$$

Hence we have (3.4).

We shall proceed to estimate $\chi_{\varepsilon}(D;\lambda)u$ and prove the following

THEOREM 3.2. *Let* $f\in B$. *Then* $u=(\tilde{K}-\zeta)^{-1}f=\tilde{R}(\zeta)f$ *satisfies* $\mathcal{D}^{1/2}u\in B^{*}$ *and*

$$\|\mathcal{D}^{1/2}u\|_{B^{*}}\leqq C(\|f\|_{B}+\|u\|_{-(1+\delta)/2}), \qquad \zeta\in \boldsymbol{C}_{\mu}^{+}, \tag{3.5}$$

where $C>0$ *is independent of* ζ *and* f.

PROOF. We first regularize u. Take $\Xi_{l}(k,\xi)$ as in Lemma 2.4, (4) and put $u_{l}=\Xi_{l}(D)u$ and $f_{l}=\Xi_{l}(D)f$. Then $f_{l}\in B$ and u_{l} satisfies

$$(\tilde{K}-\zeta)u_{l}=f_{l}+[\tilde{V},\Xi_{l}]u\equiv\tilde{f}_{l}\in B.$$

Let $Q_{1,R}$, $S_{1,R}$, χ_{ε} and ϕ be as in Lemma 2.6 and take ε as $0<\varepsilon\leqq\mu$. We write $Q_{1,R}$ and $S_{1,R}$ as Q_{R} and S_{R} respectively. Since $\tilde{K}$ is selfadjoint and $Q_{R}u_{l}\in\mathcal{D}(\tilde{K})$,

$$\operatorname{Im}(Q_{R}(\tilde{K}-\zeta)u_{l},Q_{R}u_{l})=\operatorname{Im}([Q_{R},\tilde{K}]u_{l},Q_{R}u_{l})-\operatorname{Im}\zeta\|Q_{R}u_{l}\|^{2}. \tag{3.6}$$

Since $\zeta\in\boldsymbol{C}^{+}$, it follows from (3.6) that

$$\operatorname{Re}((Q_{R}^{*}[\tilde{K},Q_{R}]/i)u_{l},u_{l})\leqq-\operatorname{Im}(Q_{R}(\tilde{K}-\zeta)u_{l},Q_{R}u_{l}). \tag{3.7}$$

By Theorem 2.2, $\sigma(Q_{R}^{*}[\tilde{K},Q_{R}]/i)-S_{R}-cR^{-1}\sigma(\underline{\chi}(D)\mathcal{D}^{1/4}\phi(x/R)\mathcal{D}^{1/4}\underline{\chi}(D))$ is bounded in $S(\langle x\rangle^{-1-\delta},G_{0})$, where c is the same constant which appeared in the definition of $S_{1,R}$. By Lemma 2.6, (3), (4) and (3.7),

$$\begin{aligned} &R^{-1}\|\phi(x/R)\mathcal{D}^{1/4}\chi(D)u_{l}\|^{2} \\ &\leqq C(\|\mathcal{D}^{-1/4}\chi(D)\tilde{f}_{l}\|_{B}\|\mathcal{D}^{1/4}\chi(D)u_{l}\|_{B^{*}}+\|u_{l}\|_{-(1+\delta)/2}^{2}). \end{aligned} \tag{3.8}$$

We take the supremum with respect to $R>1$ in (3.8) and obtain

$$\|\mathcal{D}^{1/4}\chi(D)u_{l}\|_{B^{*}}^{2}\leqq C(\|\mathcal{D}^{-1/4}\chi(D)\tilde{f}_{l}\|_{B}\|\mathcal{D}^{1/4}\chi(D)u_{l}\|_{B^{*}}+\|u_{l}\|_{-(1+\delta)/2}^{2}).$$

Hence

$$\|\mathcal{D}^{1/4}\chi(D)u_{l}\|_{B^{*}}\leqq C(\|\mathcal{D}^{-1/4}\chi(D)f_{l}\|_{B}+\|\mathcal{D}^{-1/4}\chi(D)[\tilde{V},\Xi_{l}]u\|_{B}+\|u_{l}\|_{-(1+\delta)/2}).$$

By Lemma 2.4, (4), $\|\mathcal{D}^{-1/4}\chi(D)[\tilde{V},\Xi_{l}]u\|_{B}\to 0$ as $l\to\infty$. Therefore $\mathcal{D}^{1/4}\chi(D)u\in B^{*}$ and

(3.9) $$\|\mathcal{D}^{1/4}\chi(D)u\|_{B^*}\leqq C(\|\mathcal{D}^{-1/4}\chi(D)f\|_B+\|u\|_{-(1+\delta)/2}).$$

Piecing together (3.4) and (3.9) we have $\mathcal{D}^{1/4}u\in B^*$ and

(3.10) $$\|\mathcal{D}^{1/4}u\|_{B^*}\leqq C(\|\mathcal{D}^{-1/4}f\|_B+\|u\|_{-(1+\delta)/2}).$$

Now $\mathcal{D}^{1/4}u_l$ satisfies

$$(\tilde{K}-\zeta)\mathcal{D}^{1/4}u_l=\mathcal{D}^{1/4}\tilde{f}_l+[\tilde{V},\mathcal{D}^{1/4}]u_l.$$

Applying (3.10) to this equation, we have

(3.11) $$\|\mathcal{D}^{1/2}u_l\|_{B^*}\leqq C(\|f_l\|_B+\|[\tilde{V},\Xi_l]u\|_B + \|\mathcal{D}^{-1/4}[\tilde{V},\mathcal{D}^{1/4}]u_l\|_B+\|\mathcal{D}^{1/4}u_l\|_{-(1+\delta)/2}).$$

Since $\mathcal{D}^{1/4}u\in B^*$, $\|[\tilde{V},\Xi_l]u\|_B\to 0$ as $l\to\infty$. We also have

$$\|\mathcal{D}^{-1/4}[\tilde{V},\mathcal{D}^{1/4}]u_l\|_B\leqq C\|u\|_{-(1+\delta)/2}$$

as $\sigma(\mathcal{D}^{-1/4}[\tilde{V},\mathcal{D}^{1/4}]\Xi_l)$ is bounded in $S(\langle x\rangle^{-1-\delta},G_0)$. Letting $l\to\infty$ in (3.11), we have $\mathcal{D}^{1/2}u\in B^*$ and

(3.12) $$\|\mathcal{D}^{1/2}u\|_{B^*}\leqq C(\|f\|_B+\|\mathcal{D}^{1/4}u\|_{-(1+\delta)/2}).$$

Hence we obtain (3.5) by (3.10) and (3.12).

We next show that the limit function of $\tilde{R}(\zeta)f$ as $\zeta\to\lambda+i0$ satisfies the radiation condition.

THEOREM 3.3. *Let $C^+\ni\zeta_j\to\lambda\in[\mu,1-\mu]$, and $(\tilde{K}-\zeta_j)u_j=f_j\in B$. Assume that $\|f_j-f\|_B\to 0$ and that $\mathcal{D}^{2s}u_j\to\mathcal{D}^{2s}u$ in the weak* topology of B^* for some $0\leqq s\leqq 1/4$. Let $a(x,k,\xi)\in S((\langle\xi\rangle^2+\langle k\rangle)^{2s},G_1)$ satisfy $a(x,k,\xi)=0$ on $N_+(\lambda)=\{(x,k,\xi)\in \boldsymbol{R}^{2n+1};\ x=2l\xi,\ l>0,\ \xi^2+k=\lambda$ and $k\notin(0,1)\}$. Then $a(x,D)u\in\mathring{B}^*$.*

Let χ_ε, $\underline{\chi}_\varepsilon$ and Ξ_l be as in Lemma 2.4, and $Q_{2,R}$, $\Phi_{R,\varepsilon}$ and ϕ be as in Lemma 2.6. We start with

LEMMA 3.4. *Let u satisfy the assumption of Theorem 3.3. Then*

(3.13) $$R^{-1}\|\Phi_{R,\varepsilon}(x,D)\mathcal{D}^{s+1/4}u\|^2\longrightarrow 0 \qquad \text{as } R\to\infty.$$

PROOF. We write $Q_{2,R}$ as Q_R and put $u_{j,l}=\Xi_l(D)u_j$ and $f_{j,l}=\Xi_l(D)f_j$. In view of Lemma 2.6, (2), we modify the estimate leading to (3.8) by trading the factor $R^{2\gamma}$ for the increase of the weight in the estimate by γ, and obtain for $0<\gamma<\delta/2$,

$$R^{-1}\|\Phi_{R,\varepsilon}\mathcal{D}^{s+1/4}u_{j,l}\|^2 \leqq C\{-\mathrm{Im}(Q_R(\tilde{K}-\zeta_j)\mathcal{D}^s u_{j,l}, Q_R\mathcal{D}^s u_{j,l})+R^{-2\gamma}\|\mathcal{D}^s u_{j,l}\|^2_{\gamma-(1+\delta)/2}\}. \tag{3.14}$$

By Lemma 2.4, (4), $(Q_R\mathcal{D}^s[\tilde{V}, \Xi_l]u_j, Q_R\mathcal{D}^s u_{j,l})\to 0$ as $l\to\infty$. Taking the limit $l\to\infty$ in (3.14), we obtain

$$\begin{aligned} R^{-1}\|\Phi_{R,\varepsilon}\mathcal{D}^{s+1/4}u_j\|^2 &\leqq C\{-\mathrm{Im}(Q_R f_j, Q_R\mathcal{D}^{2s}u_j)-\mathrm{Im}(Q_R f_j, [\mathcal{D}^s, Q_R]\mathcal{D}^s u_j) \\ &\quad -\mathrm{Im}([Q_R, \mathcal{D}^s]f_j, Q_R\mathcal{D}^s u_j)-\mathrm{Im}(Q_R[\tilde{V}, \mathcal{D}^s]u_j, Q_R\mathcal{D}^s u_j) \\ &\quad +R^{-2\gamma}\|\mathcal{D}^s u_j\|^2_{\gamma-(1+\delta)/2}\}. \end{aligned} \tag{3.15}$$

Since the right-hand side of (3.15) is bounded, $\Phi_{R,\varepsilon}\mathcal{D}^{s+1/4}u_j\to\Phi_{R,\varepsilon}\mathcal{D}^{s+1/4}u$ weakly in $\mathcal{K}$. It follows that (3.15) remains valid for the limit u with f_j replaced by f. Then (3.13) follows from Lemma 2.6, (5) and (2.9).

PROOF OF THEOREM 3.3. We may assume without loss of generality that $a(x,k,\xi)=0$ for small x. Since $u\in B^*$ satisfies $(-i\partial/\partial t-\Delta+\tilde{V}(\tilde{x}, D_x)-\lambda)u=f$ with $f\in B$ and $\lambda\in[\mu, 1-\mu]$, it follows from Proposition 3.1 that $a(x,D)(1-\chi_\varepsilon(D))u\in\mathcal{K}\subset\mathring{B}^*$. To study $a\chi_\varepsilon u$, we split a into two parts. Take ρ_ε and ϕ as in Lemma 2.5, and $\phi_1\in C_0^\infty(1,2)$ such that $0\leqq\phi_1\leqq 1$ and that $\phi_1(z)=1$ when $4/3\leqq z\leqq 5/3$. Set

$$a_\varepsilon^{(1)}(x,k,\xi)=a(x,k,\xi)(1-\rho_{2\varepsilon}(x,2\xi))\underline{\chi}_\varepsilon(k,\xi\,;\lambda),$$

$$a_\varepsilon^{(2)}(x,k,\xi)=a(x,k,\xi)\rho_{2\varepsilon}(x,2\xi)\underline{\chi}_\varepsilon(k,\xi\,;\lambda),$$

$$b(x,k,\xi)=a_\varepsilon^{(1)}(x,k,\xi)/\phi(1).$$

Then $b\in S((\langle\xi\rangle^2+\langle k\rangle)^{2s}, G_1)$ and

$$\Gamma_R(x,D)\equiv\phi_1(|x|/R)a_\varepsilon^{(1)}(x,D)\underline{\chi}_\varepsilon(D)-\phi_1(|x|/R)b(x,D)\Phi_{R,\varepsilon}(x,D)$$

is bounded in $S(R^{-1}(\langle\xi\rangle^2+\langle k\rangle)^{2s-1/2}, G_1)$, since $\Phi_{R,\varepsilon}(x,k,\xi)=\phi(1)\underline{\chi}_\varepsilon(k,\xi\,;\lambda)$ if $R\leqq|x|\leqq 2R$ and $\rho_\varepsilon(x,2\xi)=0$. It therefore follows from Lemma 3.4 that $a_\varepsilon^{(1)}(x,D)\underline{\chi}_\varepsilon(D)u\in\mathring{B}^*$. Hence $a_\varepsilon^{(1)}(x,D)u\in\mathring{B}^*$ by Proposition 3.1.

We next consider $a_\varepsilon^{(2)}(x,D)u$. Using Taylor's expansion at $N_+(\lambda)$, we see

$$\sup_{(x,k,\xi)\in R^{2n+1}}|a_\varepsilon^{(2)}(x,k,\xi)(\langle\xi\rangle^4+\langle k\rangle^2)^{-s}|\leqq C\varepsilon^{1/2}, \quad 0<\varepsilon<\min(1/4,\mu).$$

It follows from Theorem 18.1.15 in Hörmander [4] and Theorem 2.1 that

$$\limsup_{R\to\infty} R^{-1}\int_{R<|x|<2R}\int_0^{2\pi}|a_\varepsilon^{(2)}(x, D)u|^2 dtdx \leqq C\varepsilon\|\mathcal{D}^{2s}u\|_{B^*}^2,$$

which can be made as small as we please. Hence $a(x, D)\chi_\varepsilon(D)u\in \mathring{B}^*$ which completes the proof.

Once Theorem 3.3 is proved, Hörmander's proof of Theorem 30.2.7 in [4] directly applies and we obtain

THEOREM 3.5. *Let* u *satisfy* $\mathcal{D}^{1/4}u\in B^*$ *and*

$$(3.16)\qquad (-i\partial/\partial t-\Delta+\tilde{V}(\tilde{x}, D_x)-\lambda)u=f\in B, \qquad \lambda\in[\mu, 1-\mu].$$

Suppose that $a(x, D)u\in\mathring{B}^*$ *for any* $a(x, k, \xi)\in S((\langle\xi\rangle^2+\langle k\rangle)^{1/4}, G_1)$ *with* $a=0$ *on* $N_+(\lambda)$. *Then it follows that*

$$(3.17)\qquad \lim_{R\to\infty} R^{-1}\int_{|x|<R}\int_0^{2\pi}|b(x, D)u|^2 dtdx=2\,\mathrm{Im}(u, f)$$

if $b\in S((\langle\xi\rangle^2+\langle k\rangle)^{1/4}, G_1)$ *satisfies*

$$(3.18)\qquad |b(x, k, \xi)|^2=\langle x/|x|, 2\xi\rangle$$

when $(x, k, \xi)\in N_+(\lambda)$ *and* $|x|$ *is large.*

Noting that $\langle x/|x|, 2\xi\rangle=2|\xi|>0$ on $N_+(\lambda)$, we immediately obtain the following

COROLLARY 3.6. *If the hypotheses of Theorem* 3.5 *are satisfied and in addition* $\mathrm{Im}(u, f)=0$, *then* $\mathcal{D}^{1/4}u\in\mathring{B}^*$.

If $f=0$ in Theorem 3.3, it follows from Corollary 3.6 that $u\in\mathring{B}^*$. We next show that such u is smooth and that all its derivatives rapidly decrease in x.

THEOREM 3.7. *Let* $u\in\mathring{B}^*$ *satisfy* (3.16). *If* $\langle x\rangle^\gamma\mathcal{D}^s f\in B$ *for some* $\gamma\geqq 0$ *and* $s\geqq 0$, *then* $\langle x\rangle^\gamma\mathcal{D}^{s+1/2}u\in B^*$ *and*

$$(3.19)\qquad \|\langle x\rangle^\gamma\mathcal{D}^{s+1/2}u\|_{B^*}\leqq C_{\gamma,s}(\|\langle x\rangle^\gamma\mathcal{D}^s f\|_B+\|u\|_{B^*}).$$

REMARK. The following proof is a slight modification of Hörmander's proof of Theorem 30.2.9 in [4]. We include the proof for completeness.

PROOF. It suffices to show that $\langle x\rangle^\gamma\mathcal{D}^{s+1/4}u\in B^*$ under the conditions that $\langle x\rangle^{\gamma}\mathcal{D}^{s-1/4}f\in B$ and $\langle x\rangle^{\gamma'}\mathcal{D}^{s}u\in B^{*}$ for some γ' with $0\leqq\gamma'\leqq\gamma<\gamma'+\delta/2$ and $s\geqq 0$. Let χ_{ε}, Ξ_{l} and ζ_{l} be as in Lemma 2.4. We define $Q_{2,R}, \Phi_{R,\varepsilon}, \Phi'_{R,\varepsilon}$ and ϕ as in Lemma 2.6 and write $Q_{2,R}$ as Q_{R}. By Proposition 3.1

$$\|\langle x\rangle^{\gamma}\mathcal{D}^{s+1/4}(1-\chi(D))u\|_{B^{*}}\leqq C(\|\langle x\rangle^{\gamma}\mathcal{D}^{s-1/4}f\|_{B}+\|\langle x\rangle^{\gamma-1/2}u\|_{B^{*}})\,. \tag{3.20}$$

Next we estimate $\chi(D)u$. Put $u_{l}=\Xi_{l}(D)u$ and $f_{l}=\Xi_{l}(D)f$, and apply (3.14) to $\zeta_{l'}\mathcal{D}^{s}u_{l}$. Then we have

$$\begin{aligned}(3.21)\qquad & R^{-1}\|\Phi_{R}\mathcal{D}^{1/4}\zeta_{l'}\mathcal{D}^{s}u_{l}\|^{2}\\ &\leqq C\{-\mathrm{Im}(Q_{R}\zeta_{l'}\mathcal{D}^{s}f_{l}, Q_{R}\zeta_{l'}\mathcal{D}^{s}u_{l})-\mathrm{Im}(Q_{R}[\tilde{K},\zeta_{l'}]\mathcal{D}^{s}u_{l}, Q_{R}\zeta_{l'}\mathcal{D}^{s}u_{l})\\ &\quad -\mathrm{Im}(Q_{R}\zeta_{l'}[\tilde{V},\mathcal{D}^{s}\Xi_{l}]u, Q_{R}\zeta_{l'}\mathcal{D}^{s}u_{l})+R^{-2\gamma}\|\zeta_{l'}\mathcal{D}^{s}u_{l}\|^{2}_{\gamma-(1+\delta)/2}\}\,.\end{aligned}$$

By Cauchy-Schwarz' inequality

$$|\mathrm{Im}(Q_{R}[\tilde{K},\zeta_{l'}]\mathcal{D}^{s}u_{l}, Q_{R}\zeta_{l'}\mathcal{D}^{s}u_{l})|\leqq C\Big(l'^{-1}\int_{l'/8\leq|x|\leq l'/4}\int_{0}^{2\pi}|\mathcal{D}^{s}u_{l}|^{2}dtdx\Big)^{1/2}\,. \tag{3.22}$$

Since $\mathcal{D}^{s}u_{l}\in\mathring{B}^{*}$, the right-hand side of (3.22) vanishes as $l'\to\infty$. Taking the limit $l'\to\infty$ in (3.21), we therefore have

$$\begin{aligned}R^{-1}\|\Phi_{R}\mathcal{D}^{s+1/4}u_{l}\|^{2}\leqq C\{&-\mathrm{Im}(Q_{R}\mathcal{D}^{s}f_{l}, Q_{R}\mathcal{D}^{s}u_{l})\\ &-\mathrm{Im}(Q_{R}[\tilde{V},\mathcal{D}^{s}\Xi_{l}]u, Q_{R}\mathcal{D}^{s}u_{l})+R^{-2\gamma}\|\mathcal{D}^{s}u_{l}\|^{2}_{\gamma-(1+\delta)/2}\}\,.\end{aligned}$$

Since $\sigma(Q_{R}[\tilde{V},\mathcal{D}^{s}\Xi_{l}]\mathcal{D}^{-s})$ is bounded in $S(R^{-2\gamma}\langle x\rangle^{2\gamma-1-\delta}, G_{0})$ by Lemma 2.6, (2), we have

$$\begin{aligned}(3.23)\qquad & R^{-1}\|\Phi_{R}\mathcal{D}^{s+1/4}u_{l}\|^{2}\\ &\leqq C\{\|Q_{R}\mathcal{D}^{s-1/4}f_{l}\|_{B}\|Q_{R}\mathcal{D}^{s+1/4}u_{l}\|_{B^{*}}+\|Q_{R}\mathcal{D}^{s-1/4}f_{l}\|_{B}\|[\mathcal{D}^{1/4}, Q_{R}]\mathcal{D}^{s}u_{l}\|_{B^{*}}\\ &\quad +\|\langle x\rangle[Q_{R},\mathcal{D}^{1/4}]\mathcal{D}^{s-1/4}f_{l}\|_{B}\|\langle x\rangle^{-1}Q_{R}\mathcal{D}^{s}u_{l}\|_{B^{*}}+R^{-2\gamma}\|\mathcal{D}^{s}u_{l}\|^{2}_{\gamma-(1+\delta)/2}\}.\end{aligned}$$

Let $\sigma>0$ and multiply (3.23) by the increasing function $R^{2\gamma}(1+\sigma R)^{-2\gamma}$. Applying Lemma 2.6, (2) with $g(z)=z^{2\gamma}(1+\sigma z)^{-2\gamma}$ and (2.7), we have

$$\begin{aligned}(3.24)\qquad & R^{-1}\|R^{\gamma}(1+\sigma R)^{-\gamma}\Phi_{R}\mathcal{D}^{s+1/4}u_{l}\|^{2}\\ &\leqq C\{\|\langle x\rangle^{\gamma}\mathcal{D}^{s-1/4}f_{l}\|_{B}\|\langle x\rangle^{\gamma}(1+\sigma\langle x\rangle)^{-\gamma}\chi(D)\mathcal{D}^{s+1/4}u_{l}\|_{B^{*}}\\ &\quad +\|\langle x\rangle^{\gamma}\mathcal{D}^{s-1/4}f_{l}\|_{B}\|\langle x\rangle^{\gamma-1}\mathcal{D}^{s}u_{l}\|_{B^{*}}+\|\mathcal{D}^{s}u_{l}\|^{2}_{\gamma-(1+\delta)/2}\}\,.\end{aligned}$$

If we apply (3.24) with $\tilde{K}$ and λ replaced by $-\tilde{K}$ and $-\lambda$ respectively, we obtain (3.24) with Φ_{R} replaced by Φ'_{R}. Hence it follows from (2.10) that

$$R^{-1}\|R^{\gamma}(1+\sigma R)^{-\gamma}\phi(x/R)\chi(D)\mathcal{D}^{s+1/4}u_l\|^2$$
$$\leq C\{\|\langle x\rangle^{\gamma}\mathcal{D}^{s-1/4}f_l\|_B\|\langle x\rangle^{\gamma}(1+\sigma\langle x\rangle)^{-\gamma}\chi(D)\mathcal{D}^{s+1/4}u_l\|_{B^*}$$
$$+\|\langle x\rangle^{\gamma}\mathcal{D}^{s-1/4}f_l\|_B^2+\|\mathcal{D}^s u_l\|_{\gamma-(1+\delta)/2}^2\}\,.$$

Note that $R\sim\langle x\rangle$ in $\operatorname{supp}\phi(x/R)$ and $\|\langle x\rangle^{\gamma}(1+\sigma\langle x\rangle)^{-\gamma}\chi(D)\mathcal{D}^{s+1/4}u_l\|_{B^*}$ is finite for $\sigma>0$. So taking the supremum over $R>1$, we obtain

$$\|\langle x\rangle^{\gamma}(1+\sigma\langle x\rangle)^{-\gamma}\chi(D)\mathcal{D}^{s+1/4}u_l\|_{B^*}\leq C(\|\langle x\rangle^{\gamma}\mathcal{D}^{s-1/4}f_l\|_B+\|\mathcal{D}^s u_l\|_{\gamma-(1+\delta)/2})\,.$$

Letting $l\to\infty$ and $\sigma\to 0$, we conclude that $\langle x\rangle^{\gamma}\mathcal{D}^{s+1/4}\chi(D)u\in B^*$ and that

$$\|\langle x\rangle^{\gamma}\mathcal{D}^{s+1/4}\chi(D)u\|_{B^*}\leq C(\|\langle x\rangle^{\gamma}\mathcal{D}^{s-1/4}f\|_B+\|\mathcal{D}^s u\|_{\gamma-(1+\delta)/2})\,. \tag{3.25}$$

Piecing together (3.20) and (3.25), we have $\langle x\rangle^{\gamma}\mathcal{D}^{s+1/4}u\in B^*$ and

$$\|\langle x\rangle^{\gamma}\mathcal{D}^{s+1/4}u\|_{B^*}\leq C_{\gamma,\gamma',s}(\|\langle x\rangle^{\gamma}\mathcal{D}^{s-1/4}f\|_B+\|\langle x\rangle^{\gamma'}\mathcal{D}^s u\|_{B^*})\,.$$

This completes the proof.

Now we have Theorem 1.2 as a corollary of Theorem 3.7.

PROOF OF THEOREM 1.2. Let φ be an eigenfunction of K with eigenvalue $\lambda\in\boldsymbol{R}\backslash\boldsymbol{Z}$. We may assume that $\lambda\in[\mu,1-\mu]$. Then $\tilde{\varphi}=\mathcal{V}\varphi$ is an eigenfunction of $\tilde{K}$ corresponding to the same eigenvalue. If we apply Theorem 3.7 to $\tilde{\varphi}$ with $f=0$, we have (1.5) for $\tilde{\varphi}$. Hence φ also satisfy (1.5) by (3.2).

We shall close this section by adding a few remarks. In view of (3.2), $\exp(\pm iF(t,x))\in S(1,G_0)$. It follows that $\mathcal{V}$ and $\mathcal{V}^*$ are bounded in B and in B^*. Moreover $\mathcal{V}$ and $\mathcal{V}^*$ maps $\mathring{B}^*$ to $\mathring{B}^*$. Since

$$\sigma([a(x,D),\exp(\pm iF(t,x))])\in S(\langle x\rangle^{-\delta}(\langle\xi\rangle^2+\langle k\rangle)^{m-1/2},G_0)$$

for any $a\in S((\langle\xi\rangle^2+\langle k\rangle)^m,G_1)$, the radiation condition in Theorem 3.3 is invariant under the gauge transformations $\mathcal{V}$ and $\mathcal{V}^*$. Hence Theorem 3.2, 3.3, 3.5, 3.7 and corollary 3.6 hold for K.

§ 4. Completion of the proof of Theorem 1.1.

In this section we take short-range perturbations into account and complete the proof of Theorem 1.1. We assume Assumption (V).

Since $\langle x\rangle^{(1+\delta)/2}V^S$ is K_0-bounded with K_0-bound 0 under Assumption (V) (see Yajima [14] and Yajima-Kitada [15]), $K=K_0+V$ with domain $\mathcal{D}(K)=\mathcal{D}(K_0)$ is selfadjoint. We denote $K_1=K_0+V^L$ and regard the equation $(K-\zeta)u=f$ as

$$(K_1-\zeta)u=f-V^S u\,. \tag{4.1}$$

By the Sobolev embedding theorem we have

$$\|V^S u\|_B \leqq \varepsilon \|\mathcal{D}^{1/2}u\|_{-(1+\delta)/2}+C_\varepsilon \|u\|_{-(1+\delta)/2}\,, \qquad u\in \mathcal{D}(\mathcal{D}^{1/2})\,, \tag{4.2}$$

$$\|\langle x\rangle^{\delta/2} V^S u\|_B \leqq \varepsilon \|\mathcal{D}^{1/2}u\|_{B^*}+C_\varepsilon \|u\|_{B^*}\,, \qquad u\in \mathcal{D}(\mathcal{D}^{1/2})\,, \tag{4.3}$$

for any $\varepsilon>0$. Hence applying the theorems in Section 3 to (4.1) we obtain the following theorems.

THEOREM 4.1. *Let $f\in B$ and $\zeta\in \boldsymbol{C}^+_\mu$. Then $u=(K-\zeta)^{-1}f=R(\zeta)f$ satisfies $\mathcal{D}^{1/2}u\in B^*$ and*

$$\|\mathcal{D}^{1/2}u\|_{B^*}\leqq C(\|f\|_B+\|u\|_{-(1+\delta)/2})\,, \tag{4.4}$$

where $C>0$ is independent of ζ and f.

PROOF. By Theorem 3.2 and the remark at the end of Section 3, $\mathcal{D}^{1/2}u\in B^*$ and

$$\|\mathcal{D}^{1/2}u\|_{B^*}\leqq C(\|f\|_B+\|V^S u\|_B+\|u\|_{-(1+\delta)/2})\,. \tag{4.5}$$

Together with (4.2), this completes the proof.

THEOREM 4.2. *Let $\boldsymbol{C}^+\ni\zeta_j\to\lambda\in[\mu, 1-\mu]$, and $u_j=R(\zeta_j)f_j$ with $f_j\in B$. Assume that $\|f_j-f\|_B\to 0$ and that $\mathcal{D}^{1/2}u_j\to\mathcal{D}^{1/2}u$ in the weak* topology of B^* when $j\to\infty$. Then $a(x, D)u\in \mathring{B}^*$ for any $a(x, k, \xi)\in S((\langle\xi\rangle^2+\langle k\rangle)^{1/2}, G_1)$ with $a=0$ on $N_+(\lambda)$.*

PROOF. It follows from (4.2) that $\|V^S u_j-V^S u\|_B\to 0$. Hence applying Theorem 3.3 to (4.1), we obtain the theorem.

THEOREM 4.3. *Let u satisfy $\mathcal{D}^{1/2}u\in B^*$ and*

$$(-i\partial/\partial t-\Delta+V(x)-\lambda)u=f\in B\,, \qquad \lambda\in[\mu, 1-\mu]\,. \tag{4.6}$$

Suppose that $a(x, D)u\in \mathring{B}^*$ for any $a(x, k, \xi)\in S((\langle\xi\rangle^2+\langle k\rangle)^{1/4}, G_1)$ with $a=0$ on $N_+(\lambda)$. Then it follows that

$$\lim_{R\to\infty} R^{-1}\int_{|x|<R}\int_0^{2\pi}|b(x,D)u|^2dtdx=2\,\mathrm{Im}(u,f) \tag{4.7}$$

if $b\in S((\langle\xi\rangle^2+\langle k\rangle)^{1/4}, G_1)$ *satisfies*

$$|b(x,k,\xi)|^2=\langle x/|x|, 2\xi\rangle \tag{4.8}$$

when $(x,k,\xi)\in N_+(\lambda)$ *and* $|x|$ *is large.*

PROOF. Note that $(V^S u, u)=(u, V^S u)$ and hence $\mathrm{Im}(u, f-V^S u)=\mathrm{Im}(u,f)$ in the right hand side of (3.17). Thus we immediately obtain the theorem from Theorem 3.5.

COROLLARY 4.4. *If the hypotheses of Theorem* 4.3 *are fulfilled and in addition* $\mathrm{Im}(u,f)=0$, *then* $\mathcal{D}^{1/4}u\in \mathring{B}^*$.

In view of (4.3), we obtain the following theorem from Theorem 3.7.

THEOREM 4.5. *Let* $u\in\mathring{B}^*$ *satisfy* $\mathcal{D}^{1/2}u\in B^*$ *and* (4.6). *If* $\langle x\rangle^\gamma f\in B$ *for some* $\gamma\geqq 0$, *it follows that* $\langle x\rangle^\gamma\mathcal{D}^{1/2}u\in B^*$ *and*

$$\|\langle x\rangle^\gamma\mathcal{D}^{1/2}u\|_{B^*}\leqq C_\gamma(\|\langle x\rangle^\gamma f\|_B+\|u\|_{B^*}) . \tag{4.9}$$

Now we can prove Theorem 1.1 by the compactness argument. Since the argument is routine, we omit the details (cf. Hörmander [4, Theorem 30.2.10]).

Appendix. Proof of Theorem 2.2.

Statements (1) and (2) are well known ([4]). We prove (3). We may assume $\omega=2\pi$ without loss of generality. Denote by $\mathcal{F}_t$ (resp. $\tilde{\mathcal{F}}_t$) the Fourier transform with respect to t (resp. its restriction to the periodic functions) :

$$(\mathcal{F}_t f)(k)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-ikt}f(t,x)dt , \qquad k\in \boldsymbol{R}^1 ,$$

$$\left(\text{resp. } (\tilde{\mathcal{F}}_t f)(m)=(2\pi)^{-1/2}\int_0^{2\pi}e^{-imt}f(t,x)dt , \qquad m\in\boldsymbol{Z}\right).$$

Define two unitary operators $\mathcal{W} : L^2(\boldsymbol{R}_t\times\boldsymbol{R}_x^n)\to L^2([0,1],\mathcal{H})$, $\mathcal{H}=l^2(\boldsymbol{Z}, L^2(\boldsymbol{R}_x^n))$ and $\tilde{\mathcal{W}} : \mathcal{K}\to\mathcal{H}$ by

$$(\mathcal{W}f)(k,m,x)=(\mathcal{F}_t f)(k+m,x) , \qquad (k,m,x)\in[0,1]\times\boldsymbol{Z}\times\boldsymbol{R}^n ,$$

$$(\tilde{\mathcal{W}}f)(m, x)=(\tilde{\mathcal{F}}_t f)(m, x)\,, \qquad (m, x)\in \boldsymbol{Z}\times \boldsymbol{R}^n\,.$$

Set $p_j(x, k, \xi)=(2\pi)^{-1/2}(\tilde{\mathcal{F}}_t p(\cdot, x, k, \xi))(j)$ and

$$(P(k)u)(m)=\sum_{j=-\infty}^{\infty} p_j(x, k+m-j, D_x)u(m-j)\,, \qquad u=u(j, x)\in \mathcal{H}\,.$$

Then, by the time-periodicity of $p(t, x, k, \xi)$, $\mathcal{W}$ decomposes $p(\tilde{x}, D)$ into the multicative operators:

$$(\mathcal{W}p(\tilde{x}, D)\mathcal{W}^*f)(k)=P(k)f(k)\,, \qquad f\in L^2([0, 1], \mathcal{H})\,,$$

$$\tilde{\mathcal{W}}P(0)\tilde{\mathcal{W}}^*=p(\tilde{x}, D)|_{\mathcal{K}}\,,$$

and $(P(k)u, v)$ is continuous in k for every $u, v\in \mathcal{H}$. Since $\operatorname{ess.sup}_k \|P(k)\|_{\mathcal{L}(\mathcal{H})} = \|p(\tilde{x}, D)\|_{\mathcal{L}(L^2(\boldsymbol{R}^{n+1}))}<\infty$ by the Calderón-Vaillancourt theorem ([4]), it follows that $\|P(0)\|_{\mathcal{L}(\mathcal{H})}\leqq \|p(\tilde{x}, D)\|_{\mathcal{L}(L^2(\boldsymbol{R}^{n+1}))}<\infty$. This implies (3). Statement (4) follows from (3) by the standard argument (cf. Hörmander [4]).

References

[1] Agmon, S., Spectral properties of Schrödinger operators and scattering theory, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), 151-218.

[2] Agmon, S., Some new results in spectral and scattering theory of differential operators on $\boldsymbol{R}^n$, Seminaire Goulaouic-Schwartz, Ecole Polytechnique, Palaiseau, 1978.

[3] Enss, V., Asymptotic completeness for quantum mechanical potential scattering, I, Short range potentials, Comm. Math. Phys. 61 (1978), 285-291.

[4] Hörmander, L., The Analysis of Linear Partial Differential Operators, Springer-Verlag, New York, Vols. 1 & 2, 1983; Vols. 3 & 4, 1985.

[5] Ikebe, T. and H. Isozaki, Completeness of modified wave operators for long-range potentials, Publ. Res. Inst. Math. Sci. 15 (1979), 679-718.

[6] Ikebe, T. and Y. Saito, Limiting absorption method and absolute continuity for the Schrödinger operator, J. Math. Kyoto Univ. 12 (1972), 513-542.

[7] Iwashita, H., Spectral theory for symmetric systems in an exterior domain, Preprint, Tsukuba Univ., Ibaragi, 1986.

[8] Kitada, H., Scattering theory for Schrödinger operators with long-range potentials, I, Abstract theory, J. Math. Soc. Japan 29 (1977), 665-691. II, Spectral and scattering theory, J. Math. Soc. Japan 30 (1978), 603-632.

[9] Kuroda, S. T., Scattering theory for differential operators, I, Operator theory, J. Math. Soc. Japan 25 (1973), 75-104. II, Self-adjoint elliptic operators, J. Math. Soc. Japan 25 (1973), 222-234.

[10] Lavine, R., Absolute continuity of positive spectrum for Schrödinger operators with long-range potentials, J. Funct. Anal. 12 (1973), 30-54.

[11] Mochizuki, K. and J. Uchiyama, Radiation conditions and "oscillating" long-range potentials, I, the principle of limiting absorption, J. Math. Kyoto Univ. 18 (1978), 377-408.

[12] Mourre, E., Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys. **78** (1981), 391–408.
[13] Tamura, H., The principle of limiting absorption for uniformly propagative systems with perturbations of long-range class, Nagoya Math. J. 82 (1981), 141–174.
[14] Yajima, K., Scattering theory for Schrödinger equations with potentials periodic in time, J. Math. Soc. Japan **29** (1977), 729–743.
[15] Yajima, K. and H. Kitada, Bound states and scattering states for time periodic Hamiltonians, Ann. Inst. H. Poincaré **39** (1983), 145–157.
[16] Yajima, K., Existence of solutions for Schrödinger evolution equations, to appear in Comm. Math. Phys.

(Received April 14, 1987)

Department of Pure and Applied Sciences
University of Tokyo
3-8-1, Komaba, Meguro-ku, Tokyo
153 Japan