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A remark on Maass wave forms attached to real quadratic fields

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

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In this note, we would like to discuss from hyperfunction theoretic point of view on wave forms given by Maass [M.1] attached to real quadratic fields (or more precisely, to zeta functions of such fields with Grössen-characters). We study some kind of theta series with Grössen-characters (1.2) which turn out to be "automorphic" hyperfunctions (see (1.3) and (1.6)). We see that these hyperfunctions naturally define certain sections of line bundles over the real projective line (Sect. 2) and obtain the above wave forms as the Poisson integrals of such sections, (3.5).

1. Let $K=\boldsymbol{Q}(\sqrt{D})$ be the real quadratic field with discriminant $D>0$, $\mathcal{O}$ the ring of integers in K and $\mathfrak{a}$ a non-zero ideal in $\mathcal{O}$ with $N(\mathfrak{a})=A$. We let ε be the fundamental unit with $\varepsilon>1$ and put $c=\pi/\log\varepsilon$. Fix a natural number Q. For $\rho\in\mathfrak{a}$ and $n\in\boldsymbol{Z}$, we define some theta series by

$$(1.1)\qquad \theta_{\pm}(z_{\pm};\rho,\mathfrak{a},n,Q\sqrt{D})=\sum_{\substack{\mu\in\mathcal{O},\ \mu\equiv\rho(\mathfrak{a}Q\sqrt{D})\\ \mu\neq 0,\ \pm N(\mu)>0\\ (\mu)_{Q\sqrt{D}p_{\infty}}}}|\mu|^{2nci}\,e^{2\pi i(N(\mu)/AQD)z_{\pm}}$$

generalizing the one in Hecke's work [H.2]. Here the symbol $(\mu)_{Q\sqrt{D}p_{\infty}}$ indicates that the sum is taken over all representative elements modulo the multiplication by totally positive units $\equiv 1$ $(Q\sqrt{D})$. Hence $\theta_{\pm}$ are defined for $z_{\pm}\in H_{\pm}=\{z_{\pm}\in\boldsymbol{C};\ \pm\mathrm{Im}\,z_{\pm}>0\}$, respectively. Now we can define a hyperfunction $\theta(x;n,\rho,\mathfrak{a},Q\sqrt{D})$ on $\boldsymbol{R}$ which is the main object of this note;

$$(1.2)\qquad \theta(x;n,\rho,\mathfrak{a},Q\sqrt{D})=\theta_{+}(x+i0;n,\rho,\mathfrak{a},Q\sqrt{D})+\theta_{-}(x-i0;n,\rho,\mathfrak{a},Q\sqrt{D}).$$

We remark here that this "theta series" which may be called a theta hyperfunction is analogous to the usual theta series for a positive definite quadratic form with some spherical function. Also we see that this hyperfunction is actually a distribution in view of the growth of the

Fourier coefficients.

In what follows, we shall give transformation formulas for $\theta(x, n, \rho, \mathfrak{a}, Q\sqrt{D})$ under $x \to x+1$ and $x \to -1/x$. The first one is simply given as

$$\theta(x+1; n, \rho, \mathfrak{a}, Q\sqrt{D}) = e^{2\pi i (N(\rho)/AQD)} \theta(x; n, \rho, \mathfrak{a}, Q\sqrt{D}). \tag{1.3}$$

To obtain the second formula, here, we appeal to the corresponding formulas for $\theta_\pm(z_\pm; n, \rho, \mathfrak{a}, Q\sqrt{D})$, the case $n=0$ for which we can find in [H.2]. Let

$$\zeta_k(s; n, \rho, \mathfrak{a}, Q\sqrt{D}) = \sum_{\substack{\mu \equiv \rho \ (\mathfrak{a}Q\sqrt{D}) \\ \mu \neq 0, \ (\mu)_{Q\sqrt{D}p_\infty}}} (\operatorname{sgn} N(\mu))^k \left| \frac{\mu}{\mu'} \right|^{nci} \frac{1}{|N(\mu)|^s} \qquad (k=0, 1)$$

be zeta functions of real quadratic field K with Grössen-characters (see, e. g. [M.1] for the properties we need of these zeta functions). Put

$$\zeta_\pm(s; n, \rho) = \frac{1}{2}\{\zeta_0(s; n, \rho, \mathfrak{a}, Q\sqrt{D}) \pm \zeta_1(s; n, \rho, \mathfrak{a}, Q\sqrt{D})\};$$

$l(Q\sqrt{D})$ = the logarithm of the fundamental unit (>1) mod $Q\sqrt{D}p_\infty$;

and

$$\delta(n, \rho) = \begin{cases} 1 & \text{when } n=0 \text{ and } \rho \equiv 0 \ (\mathfrak{a}Q\sqrt{D}) \\ 0 & \text{otherwise.} \end{cases}$$

Define the functions $M_\pm(w; n, \rho)$ by

$$M_\pm(w; n, \rho) = \int_{3/2-i\infty}^{3/2+i\infty} \left(\frac{AQD}{2\pi w}\right)^s \frac{\Gamma(s)}{\sin \pi(s-2nci)} \zeta_\pm(s-nci; n, \rho)\, ds.$$

As is pointed out in [H.2], these $M_\pm(w; n, \rho)$ can be continued analytically to $-(3/2)\pi < \arg w < (3/2)\pi$. Henceforth we assume that $-\pi < \arg w < \pi$ for any variable w unless otherwise stated; but we always think that $\arg(we^{i\theta}) = \arg w + \theta$. Now we can write down the desired transformation formulas for $\theta_\pm(z_\pm; n, \rho) = \theta_\pm(z_\pm; n, \rho, \mathfrak{a}, Q\sqrt{D})$ in two ways:

$$\theta_\pm\left(-\frac{1}{z_\pm}; n, \rho\right) = \delta(n, 0) \frac{l(Q\sqrt{D})}{\pi Q\sqrt{D}} \left(\frac{z_\pm}{\pm i}\right) \mp \frac{1}{2}\zeta_1(-nci; n, \rho) + \frac{z_\pm^{1+2nci}}{Q\sqrt{D}} \sum_{\substack{\beta \bmod \mathfrak{a}Q\sqrt{D} \\ \beta \equiv 0 \ (\mathfrak{a})}} e^{2\pi i \operatorname{Tr}(\rho'\beta/AQD)} \theta_\pm(z_\pm; n, \beta) \tag{1.4}$$

$$\mp\frac{\cos\pi(nci)}{2\pi i\,Q\sqrt{D}}(z_\pm e^{(\pi/2)i})^{1+2nci}\sum_{\substack{\beta \bmod \mathfrak{a}Q\sqrt{D}\\ \beta\equiv 0\ (\mathfrak{a})}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}$$
$$\times M_+(z_\pm e^{(\pi/2)i};\,n,\,\beta)$$
$$\pm\frac{\cos\pi(nci)}{2\pi i\,Q\sqrt{D}}(z_\pm e^{-(\pi/2)i})^{1+2nci}\sum_{\substack{\beta \bmod \mathfrak{a}Q\sqrt{D}\\ \beta\equiv 0\ (\mathfrak{a})}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}$$
$$\times M_-(z_\pm e^{-(\pi/2)i};\,n,\,\beta),$$

or

$$(1.5)\quad \theta_\pm\Big(-\frac{1}{z_\pm};\,n,\,\rho\Big)=\delta(n,0)\frac{l(Q\sqrt{D})}{\pi Q\sqrt{D}}\Big(\frac{z_\pm}{\pm i}\Big)\mp\frac{1}{2}\zeta_1(-nci;\,n,\,\rho)$$
$$+\frac{(z_\pm e^{\mp\pi i})^{1+2nci}}{Q\sqrt{D}}\sum_{\substack{\beta \bmod \mathfrak{a}Q\sqrt{D}\\ \beta\equiv 0\ (\mathfrak{a})}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}\,\theta_\pm(z_\pm;\,n,\,\beta)$$
$$\mp\frac{\cos\pi(nci)}{2\pi i\,Q\sqrt{D}}(z_\pm e^{-(\pi/2)i\mp\pi i})^{1+2nci}\sum_{\substack{\beta \bmod \mathfrak{a}Q\sqrt{D}\\ \beta\equiv 0\ (\mathfrak{a})}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}$$
$$\times M_+(z_\pm e^{-(\pi/2)i\mp\pi i};\,n,\,\beta)$$
$$\pm\frac{\cos\pi(nci)}{2\pi i\,Q\sqrt{D}}(z_\pm e^{(\pi/2)i\mp\pi i})^{1+2nci}\sum_{\substack{\beta \bmod \mathfrak{a}Q\sqrt{D}\\ \beta\equiv 0\ (\mathfrak{a})}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}$$
$$\times M_-(z_\pm e^{(\pi/2)i\mp\pi i};\,n,\,\beta).$$

The proof of (1.4) and (1.5) is same as the one in [H. 2]. But, for example, in the 3rd term of (1.5) for θ_+, we can continue analytically

$$(z_+e^{-(3/2)\pi i})^{1+2nci}\,M_+(z_+e^{-(3/2)\pi i};\,n,\,\beta)$$

which we originally regarded as a function on H_+ to $0<\arg z_+<(3/2)\pi$. Hence, if we put $z_+=z_-e^{2\pi i}$ $(-(3/2)\pi<\arg z_-<-\pi/2)$, we get the equality

$$(z_+e^{-(3/2)\pi i})^{1+2nci}\,M_+(z_+e^{-(3/2)\pi i};\,n,\,\beta)=(z_-e^{(\pi/2)i})^{1+2nci}\,M_+(z_-e^{(\pi/2)i};\,n,\,\beta),$$

the right hand side of which appears in the 3rd term of (1.5) for θ_-. So, in "the sum" of θ_+ and θ_-, the 3rd terms in (1.5) cancel each other on the right half-plane. In this way, (1.4) (for $x>0$) and (1.5) (for $x<0$) imply the following transformation formula for the theta hyperfunction,

$$(1.6)\qquad \theta\Big(-\frac{1}{x};\,n,\,\rho,\,\mathfrak{a},\,Q\sqrt{D}\Big)=\frac{|x|^{1+2nci}}{Q\sqrt{D}}\sum_{\substack{\beta \bmod \mathfrak{a}Q\sqrt{D}\\ \beta\equiv 0\ (\mathfrak{a})}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}$$
$$\times\theta(x;\,n,\,\beta,\,\mathfrak{a},\,Q\sqrt{D})\qquad (x\neq 0).$$

It is to be noted that this formula in the case $n=0$ was already found about 20 years ago by Sato and Orihara (unpublished) as a hyperfunction theoretic interpretation of [H.2]. (The present author learned this fact after noticing these formulas independently.) Also we would like to point out that similar results hold for certain Fourier series arising from indefinite quadratic forms, cf. [H.1], [Si]. For example, if we take a binary zero form instead of anisotropic ones as in the above, we get transformation formulas for some Lambert series corresponding to real analytic Eisenstein series (via Poisson integrals); cf. [H.2; p. 170]. Actually we can handle the Fourier series associated with some kind of Dirichlet series [M.1], [M.2] in the same manner, but we stick to "theta series" here.

2. Put $G=SL_2(\boldsymbol{R})$ and $P=$the group of upper triangular matrices in G. For $\lambda\in\boldsymbol{C}$, we denote by L_λ a real analytic line bundle over G/P whose hyperfunction section is given by

$$\mathscr{B}(G/P, L_\lambda)=\left\{\varphi\in\mathscr{B}(G);\ \varphi\left(g\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix}\right)=|a|^{-1-2\lambda}\varphi(g)\quad(\forall g\in G)\right\}.$$

This space $\mathscr{B}(G/P, L_\lambda)$ is a G-module under left translations $\varphi(x)\mapsto\varphi(g^{-1}x)$, which is known to be the (hyperfunction-valued) spherical principal series representation of G with parameter λ. Since $G/P\simeq\boldsymbol{P}^1(\boldsymbol{R})$, the real projective line, we can identify $\mathscr{B}(G/P, L_\lambda)$ with

$$\mathscr{B}(\lambda)=\{F=(f_1, f_2)\in\mathscr{B}(\boldsymbol{R})\times\mathscr{B}(\boldsymbol{R});\ f_1(-1/x)=|x|^{1+2\lambda}f_2(x)\quad(x\neq 0)\}$$

by

$$\varphi\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)=\begin{cases} |c|^{-1-2\lambda}f_1\left(\dfrac{a}{c}\right) & (c\neq 0) \\ |a|^{-1-2\lambda}f_2\left(-\dfrac{c}{a}\right) & (a\neq 0). \end{cases}$$

Under this identification, the corresponding G-action on $\mathscr{B}(\lambda)$ is given by

$$g^{-1}\cdot F=(f_1^g, f_2^g)$$

where

$$f_1^g(x)=\begin{cases} |cx+d|^{-1-2\lambda}f_1\left(\dfrac{ax+b}{cx+d}\right) & (cx+d\neq 0) \\ |ax+b|^{-1-2\lambda}f_2\left(-\dfrac{cx+d}{ax+b}\right) & (ax+b\neq 0) \end{cases}$$

and

$$f_2^g(x)=\begin{cases} |c-dx|^{-1-2\lambda} f_1\left(\dfrac{a-bx}{c-dx}\right) & (c-dx\neq 0) \\ |a-bx|^{-1-2\lambda} f_2\left(-\dfrac{c-dx}{a-bx}\right) & (a-bx\neq 0) \end{cases}$$

for $F=(f_1, f_2)$ and $g=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Define an element $\Theta(n, \rho)$ of $\mathcal{B}(\boldsymbol{R})\times\mathcal{B}(\boldsymbol{R})$ by

$$\Theta(n, \rho)=\left(\theta(x, n, \rho), \frac{1}{Q\sqrt{D}} \sum_{\substack{\beta \bmod \mathfrak{a}Q\sqrt{D} \\ \beta\equiv 0\ (\mathfrak{a})}} e^{2\pi i \operatorname{Tr}(\rho'\beta/AQD)} \theta(x; n, \beta)\right). \tag{2.1}$$

By the transformation formulas in the previous section, we see that $\Theta(n, \rho)\in\mathcal{B}(nci)$ and is "automorphic". To be more precise, we have

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}\cdot\Theta(n, \rho) = \frac{1}{Q\sqrt{D}} \sum_{\substack{\beta \bmod \mathfrak{a}Q\sqrt{D} \\ \beta\equiv 0\ (\mathfrak{a})}} e^{2\pi i \operatorname{Tr}(\rho'\beta/AQD)} \Theta(n, \beta) \tag{2.2}$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}\cdot\Theta(n, \rho) = e^{2\pi i(N(\rho)/AQD)}\Theta(n, \rho) \tag{2.3}$$

from (1.3), (1.6) (cf. [H.1]).

3. Let $\mathcal{A}(\lambda)$ be the eigenspace of the Laplacian $\Delta=-y^2\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)$ on the upper half-plane H_+ ($z_+=x+iy$, $y>0$) with the eigenvalue $1/4-\lambda^2$. (Note that $\mathcal{A}(\lambda)=\mathcal{A}(-\lambda)$.) This is a G-module under linear fractional transformations. We define the Poisson integral $P_\lambda\varphi$ of $\varphi\in\mathcal{B}(G/P, L_\lambda)$ by

$$(P_\lambda\varphi)(g)=\int_K \varphi(gk)dk$$

where $K=SO(2)$, a maximal compact subgroup of G and dk is the normalized Haar measure on K. Since $P_\lambda\varphi$ is right K-invariant, by identifying G/K with H_+ $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\mapsto\dfrac{ai+b}{ci+d}\right)$, we see easily that $P_\lambda\varphi\in\mathcal{A}(\lambda)$. Hence we can regard P_λ as a G-equivariant map from $\mathcal{B}(G/P, L_\lambda)$ to $\mathcal{A}(\lambda)$. (At least one of P_λ or $P_{-\lambda}$ is known to be surjective, see [KKMOOT], [O].) The corresponding Poisson integral $\bar{P}_\lambda$: $\mathcal{B}(\lambda)\to\mathcal{A}(\lambda)$ under the identification in Sect. 2 for an element $F=(f_1, f_2)$ is roughly written in the form

$$(\bar{P}_\lambda F)(z_+)=\frac{1}{\pi}\int_\alpha^\beta f_1(x+y\cdot\cot\theta)\,|y^{-1/2}\sin\theta|^{-1-2\lambda}\,d\theta$$
$$+\frac{1}{\pi}\int_\beta^{\alpha+\pi} f_2(-(x+y\cdot\cot\theta)^{-1})|xy^{-1/2}\sin\theta+y^{1/2}\cos\theta|^{-1-2\lambda}\,d\theta$$

where α and β are chosen to satisfy $0<\alpha<\theta_0<\beta<\pi$ for θ_0 with $x+y\cdot\cot\theta_0=0$. A strict definition of the integral is, by changing the variable θ to $u=x+y\cdot\cot\theta$ and $v=-(x+y\cdot\cot\theta)^{-1}$, given as follows:

$$(3.1)\qquad (\bar{P}_\lambda F)(z_+)=\frac{1}{\pi}\int_{-1/a_+}^{-1/(-a)_+} f_{1,+}(u)\left(y+\frac{(u-x)^2}{y}\right)^{-1/2+\lambda}du$$
$$+\frac{1}{\pi}\int_{-1/a_-}^{-1/(-a)_-} f_{1,-}(u)\left(y+\frac{(u-x)^2}{y}\right)^{-1/2+\lambda}du$$
$$+\frac{1}{\pi}\int_{(-a)_+}^{a_+} f_{2,+}(v)\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+\lambda}dv$$
$$+\frac{1}{\pi}\int_{(-a)_-}^{a_-} f_{2,-}(v)\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+\lambda}dv$$
$$+\frac{1}{\pi}\int_{(-a)_-}^{(-a)_+}\left\{f_{2,+}(v)\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+\lambda}\right.$$
$$\left.-f_{1,+}\left(-\frac{1}{v}\right)\left(y+\frac{(vx+1)^2}{yv^2}\right)^{-1/2+\lambda}\frac{1}{v^2}\right\}dv$$
$$+\frac{1}{\pi}\int_{a_-}^{a_+}\left\{f_{1,+}\left(-\frac{1}{v}\right)\left(y+\frac{(vx+1)^2}{yv^2}\right)^{-1/2+\lambda}\frac{1}{v^2}\right.$$
$$\left.-f_{2,+}(v)\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+\lambda}\right\}dv.$$

Here $a>0$, $f_i(x)=f_{i,+}(x+i0)+f_{i,-}(x-i0)$ $(i=1,2)$ and we put $c_\pm=c\pm\varepsilon i$ with sufficiently small $\varepsilon>0$ for any $c\in\boldsymbol{R}$; all paths in the integrals are taken to be horizontal or vertical to the real line $\boldsymbol{R}$; and all branches for complex powers in the integrands take real values on $\boldsymbol{R}$.

Now we shall calculate $\bar{P}_\lambda\Theta(n,\rho)$, which is an element of $\mathcal{A}(nci)$. First, from the formulas (1.4) and (1.5) in Sect. 1, we have

$$(3.2)\qquad \frac{1}{Q\sqrt{D}}\sum_{\substack{\beta \bmod aQ\sqrt{D}\\ \beta\equiv 0\ (n)}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}\theta_\pm(z_\pm;n,\beta)$$
$$=\delta(n,\rho)\frac{l(Q\sqrt{D})}{\pi}\left(\frac{\pm i}{z_\pm}\right)+\left(-\frac{1}{z_\pm}\right)^{1+2nci}\theta_\pm\left(-\frac{1}{z_\pm};n,\rho\right)\mp Z(n,\rho)$$

$$\mp\frac{\cos\pi(nci)}{2\pi i}\left(\left(-\frac{1}{z_\pm}\right)e^{(\pi/2)i}\right)^{1+2nci}M_+\left(\left(-\frac{1}{z_\pm}\right)e^{(\pi/2)i};n,\rho\right)$$

$$\pm\frac{\cos\pi(nci)}{2\pi i}\left(\left(-\frac{1}{z_\pm}\right)e^{-(\pi/2)i}\right)^{1+2nci}M_-\left(\left(-\frac{1}{z_\pm}\right)e^{-(\pi/2)i};n,\rho\right)$$

and

(3.3)
$$\frac{1}{Q\sqrt{D}}\sum_{\substack{\beta \bmod aQ\sqrt{D}\\ \beta\equiv 0\ (a)}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}\theta_\pm(z_\pm;n,\beta)$$

$$=\delta(n,\rho)\frac{l(Q\sqrt{D})}{\pi}\left(\frac{\pm i}{z_\pm}\right)+\left(\left(-\frac{1}{z_\pm}\right)e^{\mp\pi i}\right)^{1+2nci}\theta_\pm\left(-\frac{1}{z_\pm};n,\rho\right)\mp Z(n,\rho)$$

$$\mp\frac{\cos\pi(nci)}{2\pi i}\left(\left(-\frac{1}{z_\pm}\right)e^{-(\pi/2)i\mp\pi i}\right)^{1+2nci}M_+\left(\left(-\frac{1}{z_\pm}\right)e^{-(\pi/2)i\mp\pi i};n,\rho\right)$$

$$\pm\frac{\cos\pi(nci)}{2\pi i}\left(\left(-\frac{1}{z_\pm}\right)e^{(\pi/2)i\mp\pi i}\right)^{1+2nci}M_-\left(\left(-\frac{1}{z_\pm}\right)e^{(\pi/2)i\mp\pi i};n,\rho\right),$$

where

$$Z(n,\rho)=\frac{1}{2Q\sqrt{D}}\sum_{\substack{\beta \bmod aQ\sqrt{D}\\ \beta\equiv 0\ (a)}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}\zeta_1(-nci;n,\beta).$$

(We used the facts $\theta_\pm(z_\pm;n,-\rho)=\theta_\pm(z_\pm;n,\rho)$, $M_\pm(w;n,-\rho)=M_\pm(w;n,\rho)$.) Substitute

(2.1)
$$\Theta(n,\rho)=\left(\theta(x,n,\rho),\ \frac{1}{Q\sqrt{D}}\sum_{\substack{\beta \bmod aQ\sqrt{D}\\ \beta\equiv 0\ (a)}} e^{2\pi i\,\mathrm{Tr}(\rho'\beta/AQD)}\theta(x;n,\beta)\right)$$

in the integrals (3.1). Then we get, by using (3.2) and (3.3),

(3.4) $(\bar{P}_\lambda\Theta(n,\rho))(z_+)$

$$=\frac{1}{\pi}\int_{-1/a_+}^{-1/(-a)_+}\theta_+(u;n,\rho)\left(y+\frac{(u-x)^2}{y}\right)^{-1/2+nci}du$$

$$+\frac{1}{\pi}\int_{-1/a_-}^{-1/(-a)_-}\theta_-(u;n,\rho)\left(y+\frac{(u-x)^2}{y}\right)^{-1/2+nci}du$$

$$+\frac{1}{\pi}\int_{(-a)_+}^{0_+}\theta_+(-1/v;n,\rho)(-1/v)^{1+2nci}\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+nci}dv$$

$$+\frac{1}{\pi}\int_{0_+}^{a_+}\theta_+(-1/v;n,\rho)((-1/v)e^{-\pi i})^{1+2nci}\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+nci}dv$$

$$+\frac{1}{\pi}\int_{(-a)_-}^{0_-}\theta_-(-1/v;n,\rho)(-1/v)^{1+2nci}\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+nci}dv$$

$$+\frac{1}{\pi}\int_{0_-}^{a_-}\theta_-(-1/v;n,\rho)((-1/v)e^{\pi i})^{1+2nci}\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+nci}dv$$

$$+\frac{1}{\pi}\int_{\gamma_L}\Big\{-\delta(n,\rho)\frac{l(Q\sqrt{D})}{\pi i}(1/v)-Z(n,\rho)$$

$$-\frac{\cos\pi(nci)}{2\pi i}((-1/v)e^{(\pi/2)i})^{1+2nci}M_+((-1/v)e^{(\pi/2)i};n,\rho)$$

$$+\frac{\cos\pi(nci)}{2\pi i}((-1/v)e^{-(\pi/2)i})^{1+2nci}M_-((-1/v)e^{-(\pi/2)i};n,\rho)\Big\}$$

$$\times\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+nci}dv$$

$$+\frac{1}{\pi}\int_{\gamma_R}\Big\{-\delta(n,\rho)\frac{l(Q\sqrt{D})}{\pi i}(1/v)-Z(n,\rho)$$

$$-\frac{\cos\pi(nci)}{2\pi i}((-1/v)e^{-(3\pi/2)i})^{1+2nci}M_+((-1/v)e^{-(3\pi/2)i};n,\rho)$$

$$+\frac{\cos\pi(nci)}{2\pi i}((-1/v)e^{-(\pi/2)i})^{1+2nci}M_-((-1/v)e^{-(\pi/2)i};n,\rho)\Big\}$$

$$\times\left(yv^2+\frac{(vx+1)^2}{y}\right)^{-1/2+nci}dv.$$

Here $0_\pm=\pm\varepsilon i$; γ_L (resp. γ_R) is the left (resp. right) half of the path (encircling the origin clockwisely) consisting of the edges of the rectangle with vertices $a_\pm, (-a)_\pm$; and we assumed that $\pi/2\leqq\arg(-1/v)\leqq(3/2)\pi$ only in the 8th term of the integrals. Note that $M_\pm(w;n,\rho)$ tend to zero uniformly for $\arg w$ with $-(3/2)\pi+\delta<\arg w<(3/2)\pi-\delta$ $(\delta>0)$ as $w\to\infty$, and also that $\theta_\pm(z_\pm;n,\rho)$ tend to zero uniformly for $\mathrm{Re}(z_\pm)$ as $\mathrm{Im}(z_\pm)\to\pm\infty$. Let us put $a=t/(t^2+\varepsilon_0^2)$, $\varepsilon=\varepsilon_0/(t^2+\varepsilon_0^2)$ for some $\varepsilon_0>0$, and take the limit $t\to+\infty$ of the integrals. Then we have

(3.5)
$$(\bar{P}_\lambda\Theta(n,\rho))(z_+)$$
$$=\delta(n,\rho)\frac{2l(Q\sqrt{D})}{\pi}y^{1/2}+\frac{2(AQD)^{nci}}{\pi^{1/2+nci}\Gamma(1/2-nci)}$$
$$\times\sum_{\substack{\mu\equiv\rho\ (\mathfrak{a}Q\sqrt{D})\\ \mu\neq0,\ (\mu)_{Q\sqrt{D}\mathfrak{p}_\infty}}} y^{1/2}\left|\frac{\mu}{\mu'}\right|^{nci}K_{nci}\left(2\pi\frac{|N(\mu)|}{AQD}y\right)\cdot e^{2\pi i(N(\mu)/AQD)x}$$

by using Basset's formula [W; VI. 6.16] for K_{nci}, the modified Bessel function with the parameter nci. This is the Maass wave form in [M.1; (29)]. Finally we remark that the growth condition [M.1; (12)] on this wave form is deduced from that $\Theta(n, \rho)$ is a distribution section of L_{nci}. Actually these two conditions are equivalent, see [O].

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