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Twisted differential operators and affine Weyl groups

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

By Toshiyuki TANISAKI

§ 0. Introduction.

0.1. In our previous papers [13] and [21] we gave some relation between representations of the Weyl group and $\mathcal{D}$-modules on the flag manifold. The purpose of this paper is to give its generalization using the affine Weyl group and the sheaves of twisted differential operators instead of the Weyl group and the sheaf of (non-twisted) differential operators, respectively.

0.2. We first review results of [13] and [21].

For a non-singular algebraic variety Y over $\boldsymbol{C}$ we denote by $\mathcal{D}_Y$ the sheaf of (algebraic) differential operators.

Let G be a connected reductive algebraic group over $\boldsymbol{C}$ and X its flag variety. We denote by $\mathcal{T}$ the abelian category consisting of coherent $\mathcal{D}_{X\times X}$-modules with G-actions. It is easily seen that each irreducible component of the characteristic variety $\mathrm{Ch}(M)$ of $M\in\mathcal{T}$ coincides with the closure of the conormal bundle $T^*_Y(X\times X)$ of some G-orbit Y on $X\times X$. Let Z be the union of $T^*_Y(X\times X)$ for G-orbits on $X\times X$. It is a closed subvariety of $T^*(X\times X)$ with pure dimension $2\dim X$. Taking into account the multiplicities at the irreducible components we have the characteristic cycle $\mathbf{Ch}(M)\in\bigoplus_Y \boldsymbol{Z}_{\geq 0}[\overline{T^*_Y(X\times X)}]$ of $M\in\mathcal{T}$, where Y runs through G-orbits on $X\times X$. Let $K(\mathcal{T})$ be the Grothendieck group of the abelian category $\mathcal{T}$. By the additivity of $\mathbf{Ch}$ we have a $\boldsymbol{Z}$-linear map:

$$(*)\qquad \mathbf{Ch}: K(\mathcal{T})\longrightarrow H_{2\dim Z}(Z),$$

where $H_{2\dim Z}(Z)=\bigoplus_Y \boldsymbol{Z}[\overline{T^*_Y(X\times X)}]$ is the top Borel-Moore homology group of Z.

Using basic operations of $\mathcal{D}$-modules such as direct images and inverse images, $K(\mathcal{T})$ is endowed with a ring structure and is isomorphic to the group ring $Z[W]$ of the Weyl group W (see Section 3 below). Especially $K(\mathcal{T})$ is a $W\times W$-module (the two-sided regular representation). On the other hand a $W\times W$-module structure on $H_{2\dim Z}(Z)$ is defined in [15] (a version of the Springer representation [20]).

THEOREM ([13]). *The Z-linear map* (*) *is an isomorphism of* $W\times W$-*modules.*

We can generalize this theorem a little more. Let F be a closed subgroup of G (with respect to the Zariski topology) so that the number of F-orbits on X is finite. Let $\mathcal{S}_0^F$ be the abelian category of coherent $\mathcal{D}_X$-modules with F-actions and Λ the union of the conormal bundles of F-orbits on X. We have also a Z-linear map:

$$(**) \qquad \mathbf{Ch}: K(\mathcal{S}_0^F) \longrightarrow H_{2\dim\Lambda}(\Lambda)$$

and W-module structures on $K(\mathcal{S}_0^F)$ and $H_{2\dim\Lambda}(\Lambda)$, as before.

THEOREM ([21]). *The Z-linear map* (**) *is a homomorphism of W-modules.*

0.3. Our aim is to generalize above theorems using the affine Weyl group instead of the Weyl group. Here the affine Weyl group is the semidirect product $W_a = W \ltimes P$ with respect to the action of the Weyl group W on the weight lattice P of (a maximal torus of) G. Hence we hope to construct two W_a-modules corresponding to $K(\mathcal{S}_0^F)$ and $H_{2\dim\Lambda}(\Lambda)$, and to show that there exists a natural homomorphism of W_a-modules from the one (corresponding to $K(\mathcal{S}_0^F)$) to the other.

Let us consider first about the counterpart of $K(\mathcal{S}_0^F)$. For $\lambda\in P$ let $\mathcal{O}(\lambda)$ be the invertible sheaf consisting of the sections of the G-equivariant line bundle on X so that the action of B_x on the fiber at $x\in X$ is given by λ. Here B_x is the Borel subgroup of G corresponding to $x\in X$. Set $\mathcal{D}_X^\lambda = \mathcal{O}(\lambda)\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{O}(-\lambda)$. It is a sheaf of rings which is locally isomorphic to $\mathcal{D}_X$. Let $\mathcal{S}_\lambda^F$ be the abelian category of coherent $\mathcal{D}_X^\lambda$-modules with F-actions. We can define a W_a-module structure on $\bigoplus_{\lambda\in P} K(\mathcal{S}_\lambda^F)$ extending the W-module structure on $K(\mathcal{S}_0^F)$ (see Section 3 below).

Next consider the counterpart of $H_{2\dim\Lambda}(\Lambda)$. For an algebraic variety Y with an action of an algebraic group F we denote by $K^F(Y)$ the equivariant K-homology group (see Section 2.2 below). Kazhdan-Lusztig ([16]) and Ginsburg ([6], [7]) have defined an $(H(W_a), H(W_a))$-bimodule structure on $K^{G\times C^*}(Z)$, where $H(W_a)$ is the Hecke algebra of W_a. It follows immediately from this that there exists a natural action of $W_a\times W_a$ on $K^G(Z)$. Similarly an action of W_a on $K^F(\Lambda)$ is defined.

We have a natural map:

$$(***) \qquad \mathrm{gr}: \bigoplus_{\lambda\in P} K(\mathcal{S}_\lambda^F) \longrightarrow K^F(\Lambda).$$

THEOREM. *The Z-linear map* (***) *is a surjective homomorphism of* W_a*-modules.*

0.4. A motivation of this paper is to shed light to a conjecture of Lusztig ([18]) which claims that there exists a natural one-to-one correspondence between the set of unipotent orbits in G and the set of two-sided cells in W_a. In order to prove the conjecture it seems to be important to generalize the theorems in 0.2 using the Hecke algebra $H(W_a)$ of W_a instead of the Weyl group W and to specify elements C'_w ($w\in W_a$) of $H(W_a)$ ([14]) in this situation. In [22] we gave a generalization of the theorems in 0.2 using the Hecke algebra $H(W)$ of W by the aid of Hodge modules. Since $H(W_a)$ is a twisted tensor product of $H(W)$ and the group ring $Z[P]$ over Z (Bernstein), we should add the translation part P in the picture. The result of this paper seems to show that this might be done using twisted differential operators. But we do not know how to do it at present.

0.5. The result of this paper was reported on the occasion of the symposium "Representations of algebraic groups and Lie algebras, Hopf algebras, and Galois theory" held at Osaka University on January 20–23, 1986 ([23]).

§ 1. $\mathcal{D}$-modules.

1.1. In order to fix notations and make clear what will be used later we review basic facts concerning modules over the sheaf of (twisted) differential operators. We refer the reader to [3], [4], [8] as expositions.

For a non-singular algebraic variety X over C we denote the structure sheaf by $\mathcal{O}_X$ and the sheaf of (algebraic) differential operators on X by $\mathcal{D}_X$. For an invertible $\mathcal{O}_X$-module L we set $\mathcal{D}_X^L = L\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X} L^{-1}$, where $L^{-1}=\mathcal{H}om_{\mathcal{O}_X}(L,\mathcal{O}_X)$. It is a sheaf of rings locally isomorphic to $\mathcal{D}_X$. A sheaf of $\mathcal{D}_X^L$-modules which is quasi-coherent as an $\mathcal{O}_X$-module is called a

$\mathcal{D}_X^L$-module. The derived category consisting of bounded complexes of $\mathcal{D}_X^L$-modules is denoted by $D^b(\mathcal{D}_X^L)$. The full subcategory of $D^b(\mathcal{D}_X^L)$ consisting of $M \in D^b(\mathcal{D}_X^L)$ whose cohomology sheaves $\mathcal{H}^i(M)$ $(i \in Z)$ are coherent $\mathcal{D}_X^L$-modules is denoted by $D^b_{coh}(\mathcal{D}_X^L)$.

1.2. Operations.

(a) Shift by invertible sheaves.

Let L_1 and L_2 be invertible $\mathcal{O}_X$-modules. For a $\mathcal{D}_X^{L_1}$-module M we have a $\mathcal{D}_X^{L_1 \otimes L_2}$-module: $L_2 \otimes M := L_2 \underset{\mathcal{O}_X}{\otimes} M$.

(b) Inverse image and direct image ([10], [11]).

Let $f: X \to Y$ be a morphism of non-singular varieties and L an invertible $\mathcal{O}_Y$-module. Set

$$\mathcal{D}^L_{X\to Y} = \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{D}^L_Y, \quad \mathcal{D}^L_{Y\leftarrow X} = f^{-1}(\mathcal{D}^L_Y \underset{\mathcal{O}_Y}{\otimes} \Omega_Y^{-1}) \underset{f^{-1}\mathcal{O}_Y}{\otimes} \Omega_X,$$

where Ω_X and Ω_Y are sheaves of differential forms of the highest degree on X and Y respectively. Then $\mathcal{D}^L_{X\to Y}$ is a $(\mathcal{D}_X^{f^*L}, f^{-1}\mathcal{D}^L_Y)$-bimodule and $\mathcal{D}^L_{Y\leftarrow X}$ is an $(f^{-1}\mathcal{D}^L_Y, \mathcal{D}_X^{f^*L})$-bimodule. The functors:

$$Lf^* : D^b(\mathcal{D}^L_Y) \longrightarrow D^b(\mathcal{D}_X^{f^*L}) \quad \text{and} \quad \int_f : D^b(\mathcal{D}_X^{f^*L}) \longrightarrow D^b(\mathcal{D}^L_Y)$$

are defined by:

$$Lf^*M = \mathcal{D}^L_{X\to Y} \overset{L}{\underset{f^{-1}\mathcal{D}^L_Y}{\otimes}} f^{-1}M \quad \text{and} \quad \int_f M = Rf_*\left(\mathcal{D}^L_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}^L_X}{\otimes}} M\right).$$

Sometimes it is more convenient to use $f^! = Lf^*[\dim X - \dim Y]$. When f is smooth and M is a $\mathcal{D}^L_Y$-module, we have $\mathcal{H}^j(Lf^*M)=0$ for $j \neq 0$ and we write f^*M instead of $\mathcal{H}^0(Lf^*M)$.

(c) Tensor product.

For $M_1 \in D^b(\mathcal{D}_X^{L_1})$ and $M_2 \in D^b(\mathcal{D}_X^{L_2})$ we have:

$$M_1 \overset{L}{\otimes} M_2 := L\Delta^*(M_1 \boxtimes M_2) \in D^b(\mathcal{D}_X^{L_1 \otimes L_2}),$$

where $\Delta : X \to X \times X$ is the diagonal embedding.

1.3. Fundamental properties.

LEMMA 1.1. *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of non-singular varieties.*

(i) *$L(g \circ f)^* = Lf^* \circ Lg^*$ and hence $(g \circ f)^! = f^! \circ g^!$.*

(ii) $\int_g \circ \int_f = \int_{g\circ f}$.

(iii) $Lf^*(M_1 \overset{L}{\otimes} M_2) = Lf^*(M_1) \overset{L}{\otimes} Lf^*(M_2)$. *Hence for an invertible* $\mathcal{O}_Y$*-module* L *we have* $Lf^*(L \otimes (\cdot)) = f^*L \otimes Lf^*(\cdot)$.

PROPOSITION 1.2 ([4], see also [11]). *Let* Y *be a non-singular closed subvariety of a non-singular variety* X. *Set* $U = X - Y$, $i : Y \hookrightarrow X$, $j : U \hookrightarrow X$. *For* $M \in D^b(\mathcal{D}_X^L)$ *we have the following distinguished triangle in* $D^b(\mathcal{D}_X^L)$.

$$\int_i i^! M \longrightarrow M \longrightarrow \int_j j^! M \longrightarrow \int_i i^! M[1].$$

PROPOSITION 1.3 (base change theorem, see [4]). *Consider the following cartesian diagram of non-singular varieties*:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} .$$

For an invertible $\mathcal{O}_Y$*-module* L *two functors* $g^! \circ \int_f$ *and* $\int_{f'} \circ g'^!$ *from* $D^b(\mathcal{D}_X^{f^*L})$ *to* $D^b(\mathcal{D}_{Y'}^{g^*L})$ *are naturally equivalent.*

PROPOSITION 1.4 (projection formula, corollary of 1.3). *Let* $f : X \to Y$ *be a morphism of non-singular varieties. For* $M \in D^b(\mathcal{D}_Y^{L_1})$ *and* $N \in D^b(\mathcal{D}_X^{f^*L_2})$ *we have* $\int_f ((Lf^*M) \overset{L}{\otimes} N) \simeq M \overset{L}{\otimes} \int_f N$ *in* $D^b(\mathcal{D}_Y^{L_1 \otimes L_2})$. *Especially we have* $\int_f (f^*L \otimes (\cdot)) \simeq L \otimes \int_f (\cdot)$ *for an invertible* $\mathcal{O}_Y$*-module* L.

PROPOSITION 1.5 ([10]). $D^b_{coh}(\ )$ *is stable under inverse images relative to smooth morphisms, and direct images relative to proper morphisms.*

1.4. Holonomic modules.

For a non-singular variety X we denote the cotangent bundle by $p_X : T^*X \to X$. Let $\mathcal{D}_{X,i}$ be the sheaf of differential operators of order $\leq i$. Then $\operatorname{Gr} \mathcal{D}_X := \bigoplus_i (\mathcal{D}_{X,i} / \mathcal{D}_{X,i-1})$ is a commutative $\mathcal{O}_X$-algebra isomorphic to $(p_X)_* \mathcal{O}_{T^*X}$. For an invertible $\mathcal{O}_X$-module L $\mathcal{D}_{X,i}^L$ is defined similarly and we have $\operatorname{Gr} \mathcal{D}_X^L = \operatorname{Gr} \mathcal{D}_X = (p_X)_* \mathcal{O}_{T^*X}$.

Let $\{M_i\}_{i\in Z}$ be a good filtration of a coherent $\mathcal{D}_X^L$-module M. Then $\mathrm{Gr}(M):=\bigoplus_i(M_i/M_{i-1})$ is a coherent $\mathrm{Gr}\,\mathcal{D}_X^L(\simeq(p_X)_*\mathcal{O}_{T^*X})$-module and hence

$$\mathrm{gr}(M):=\mathcal{O}_{T^*X}\underset{p_X^{-1}\mathrm{Gr}\,\mathcal{D}_X^L}{\otimes}p_X^{-1}\mathrm{Gr}(M)$$

is a coherent $\mathcal{O}_{T^*X}$-module. We call $\mathrm{Ch}(M)=\mathrm{Supp}(\mathrm{gr}(M))$ the characteristic variety of M. $\mathrm{Ch}(M)$ together with the multiplicities of $\mathrm{gr}(M)$ at the irreducible components of $\mathrm{Ch}(M)$ is called the characteristic cycle and is denoted by $\mathbf{Ch}(M)$. $\mathrm{Ch}(M)$ and $\mathbf{Ch}(M)$ do not depend on the choice of a good filtration.

It is known that $\mathrm{Ch}(M)$ is an involutive subvariety of T^*X (Sato-Kawai-Kashiwara) and hence its irreducible components have dimension $\geqq\dim X$. A coherent $\mathcal{D}_X^L$-module M with $\dim\mathrm{Ch}(M)=\dim X$ is called a holonomic $\mathcal{D}_X^L$-module.

Although we do not give definition here, there is a good class of holonomic $\mathcal{D}_X$-modules, that is, regular holonomic $\mathcal{D}_X$-modules, which has been investigated intensively by Kashiwara-Kawai ([12]) et al. Note that since we are working in the algebraic category, the regularity here includes the regularity at infinity and hence it is not a local property (see [4]). We denote the full subcategory of $D^b_{coh}(\mathcal{D}_X)$ consisting of M with $\mathcal{H}^i(M)$ holonomic (resp. regular holonomic) for any i by $D^b_h(\mathcal{D}_X)$ (resp. $D^b_{rh}(\mathcal{D}_X)$).

PROPOSITION 1.6 (see [4]). *$D^b_h(\ )$ and $D^b_{rh}(\ )$ are stable under inverse images and direct images.*

§ 2. $\mathcal{D}$-modules with group actions and equivariant K-theory.

2.1. $\mathcal{D}$-modules with group actions.

Let G be an algebraic group and X a non-singular variety with G-action. We denote by $m: G\times G\to G$ and $\sigma: G\times X\to X$ the product in G and the action of G on X, respectively.

A coherent $\mathcal{O}_X$-module L is said to have a G-action if an isomorphism $\phi:\sigma^*L\to p_2{}^*L$ of $\mathcal{O}_{G\times X}$-modules satisfying the cocycle condition: $(p_{23}{}^*\phi)\circ((1_G\times\sigma)^*\phi)=(m\times 1_X)^*\phi$ is given. Here $p_2: G\times X\to X$ and $p_{23}: G\times G\times X\to G\times X$ are projections. This formulation is due to Mumford.

Let L be an invertible $\mathcal{O}_X$-module with G-action. A $\mathcal{D}_X^L$-module M is said to have a G-action if an isomorphism $\sigma^*M\to p_2{}^*M$ of $\mathcal{D}_{G\times X}^{\sigma^*L}\ (=\mathcal{D}_{G\times X}^{p_2{}^*L})$-modules satisfying the cocycle condition is given.

We denote the category of coherent $\mathcal{D}_X^L$-modules (resp. $\mathcal{D}_X$-modules) with G-actions by $\mathcal{S}(X, G, L)$ (resp. $\mathcal{S}(X, G)$). When $f: X \to Y$ is a G-equivariant morphism of non-singular G-varieties and L is an invertible $\mathcal{O}_Y$-module with G-action, $\boldsymbol{Z}$-linear maps between Grothendieck groups:

$$\int_f : K(\mathcal{S}(X, G, f^*L)) \longrightarrow K(\mathcal{S}(Y, G, L)),$$
$$f^* : K(\mathcal{S}(Y, G, L)) \longrightarrow K(\mathcal{S}(X, G, f^*L))$$

are defined by $\int_f([M]) = \left[\int_f M\right]$ and $f^*([M]) = [Lf^*M]$. Here $[N]$ stands for $\sum_j (-1)^j [\mathcal{H}^j(N)]$ for a bounded complex N. When L_i $(i=1, 2)$ are invertible $\mathcal{O}_X$-modules with G-actions, we have a bilinear map:

$$K(\mathcal{S}(X, G, L_1)) \times K(\mathcal{S}(X, G, L_2)) \longrightarrow K(\mathcal{S}(X, G, L_1 \otimes L_2))$$
$$([M_1], [M_2]) \longrightarrow [M_1] \otimes [M_2] := [M_1 \overset{L}{\otimes} M_2].$$

LEMMA 2.1. *Let X be a homogeneous space of G.*

(i) *Any object of $\mathcal{S}(X, G)$ is regular holonomic as a $\mathcal{D}_X$-module and is locally free of finite rank as an $\mathcal{O}_X$-module.*

(ii) *The category $\mathcal{S}(X, G)$ is equivalent to the category of finite dimensional representations over $\boldsymbol{C}$ of the component group $G^x/(G^x)^0$ for any point $x \in X$.*

PROOF. (i) Choose a point $x \in X$. We define $f: G \to X$ and $i: G \to G \times X$ by $f(g) = g \cdot x$ and $i(g) = (g, x)$. Set $\mathrm{pt} = \{x\}$, $r: G \to \mathrm{pt}$ and $j: \mathrm{pt} \hookrightarrow X$. For $M \in \mathcal{S}(X, G)$ we have $f^*M = (\mathcal{H}^0 i^*)\sigma^*M \simeq (\mathcal{H}^0 i^*)p_2{}^*M = r^*(\mathcal{H}^0 j^*)M = \mathcal{O}_G \otimes_{\boldsymbol{C}} (\mathcal{H}^0 j^*M)$ and hence f^*M is isomorphic to a product of finite copies of $\mathcal{O}_X$ as a $\mathcal{D}_X$-module. Since f is smooth, M is a regular holonomic $\mathcal{D}_X$-module which is locally free of finite rank as an $\mathcal{O}_X$-module.

(ii) By (i) and the Riemann-Hilbert correspondence the category $\mathcal{S}(X, G)$ is equivalent to the category of local systems on X_{an} (=the associated complex manifold) with G-actions and hence it is equivalent to the category of finite dimensional representations of $G^x/(G^x)^0$.

Using Lemma 2.1, Proposition 1.2, Proposition 1.6 and the induction on the number of G-orbits we have the following:

PROPOSITION 2.2. *Let X be a non-singular G-variety with finitely many G-orbits. We denote by Λ the union of the conormal bundles of the*

G-orbits on X.

(i) *If $M\in\mathcal{S}(X,G)$, then M is regular holonomic and its characteristic variety* $\mathrm{Ch}(M)$ *is a subvariety of Λ.*

(ii) *Set $i_O:O\hookrightarrow X$ for a G-orbit O on X. Then the map $\int_{i_O}:K(\mathcal{S}(O,G))\to K(\mathcal{S}(X,G))$ is injective for any G-orbit O on X and $K(\mathcal{S}(X,G))=\bigoplus_O \mathrm{Image}\left(\int_{i_O}\right)$, where O is running through G-orbits on X.*

REMARK. The proof of Propositions 2.1, 2.2 given above was communicated to the author by Kashiwara. Note also that the regularity in Proposition 2.2 does not hold in general in the analytic category.

2.2. Equivariant K-theory (see [24]).

Let G be an algebraic group and $i:X\to Y$ a G-equivariant closed immersion of G-varieties. We denote the abelian category of coherent $\mathcal{O}_Y$-modules with G-actions supported in X by $C^G(X,Y)$ and its Grothendieck group by $K^G(X,Y)$. Since the exact functor $i_*:C^G(X,X)\to C^G(X,Y)$ induces an isomorphism $i_*:K^G(X,X)\to K^G(X,Y)$, $K^G(X,Y)$ does not depend on the choice of the ambient space Y. When we do not have to specify Y we denote it by $K^G(X)$. We set $R_G=K^G(\mathrm{pt})$, where pt is the algebraic variety consisting of a single point. R_G is a ring and $K^G(X)$ is an R_G-module via the tensor product.

Let $f:Y_1\to Y_2$ be a G-equivariant morphism of G-varieties and X_i $(i=1,2)$ be G-stable closed subvarieties of Y_i. When $f(X_1)\subset X_2$ and $X_1\to X_2$ is proper, we have an R_G-linear map:

$$f_*:K^G(X_1,Y_1)\longrightarrow K^G(X_2,Y_2)$$
$$f_*([M])=[\boldsymbol{R}f_*(M)]=\sum_i(-1)^i[\mathcal{H}^i(\boldsymbol{R}f_*(M))].$$

When $f^{-1}(X_2)\subset X_1$ and Y_2 is non-singular, we have an R_G-linear map:

$$f^*:K^G(X_2,Y_2)\longrightarrow K^G(X_1,Y_1)$$
$$f^*([M])=[\boldsymbol{L}f^*(M)].$$

Let X_i $(i=1,2,3)$ be G-stable closed subvarieties of a non-singular G-variety Y such that $X_1\cap X_2\subset X_3$. Then we have an R_G-bilinear map:

$$K^G(X_1,Y)\times K^G(X_2,Y)\longrightarrow K^G(X_3,Y)$$

$$([M_1],\ [M_2])\longmapsto[M_1]\otimes[M_2]=[M_1\overset{L}{\underset{\mathcal{O}_Y}{\otimes}}M_2].$$

We note the following facts for the future reference.

PROPOSITION 2.3 (projection formula). *Let $f: Y_1\to Y_2$ be a morphism of non-singular varieties and M_i $(i=1, 2)$ be coherent $\mathcal{O}_{Y_i}$-modules. We assume that* $\operatorname{Supp} M_1\to Y_2$ *is proper. Then we have*:

$$\boldsymbol{R}f_*(M_1)\overset{L}{\underset{\mathcal{O}_{Y_2}}{\otimes}}M_2=\boldsymbol{R}f_*\Big(M_1\overset{L}{\underset{\mathcal{O}_{Y_1}}{\otimes}}\boldsymbol{L}f^*(M_2)\Big).$$

PROPOSITION 2.4 (smooth base change theorem). *Let $f: Y_1\to Y_2$ and $g: Y_2'\to Y_2$ be morphisms of algebraic varieties. Set $Y_1'=Y_1\underset{Y_2}{\times}Y_2'$, $g': Y_1'\to Y_1$, $f': Y_1'\to Y_2'$. We assume that Y_2 is non-singular and g is smooth. Let M be a coherent $\mathcal{O}_{Y_1}$-module such that* $\operatorname{Supp} M\to Y_2$ *is proper. Then we have*:

$$\boldsymbol{L}g^*\circ \boldsymbol{R}f_*(M)=\boldsymbol{R}f'_*\circ \boldsymbol{L}g'^*(M).$$

PROPOSITION 2.5 (Thom isomorphism). *For a G-vector bundle $p:E\to X$ on a non-singular G-variety we have*:

$$p^*: K^G(X)\xrightarrow{\sim}K^G(E).$$

LEMMA 2.6. *Let $p: E\to X$ be a G-vector bundle on a non-singular G-variety and E_1 a G-stable subbundle of E. We denote the locally free $\mathcal{O}_X$-module consisting of sections of $(E/E_1)^*$ by L. Set $i: E_1\to E$ and $n=$* $\operatorname{rank}(L)$. *Then we have*:

$$i_*[\mathcal{O}_{E_1}]=\sum_{k=0}^{n}(-1)^k p^*([\overset{k}{\wedge}L])$$

in $K^G(E)$.

2.3. Let X be a non-singular G-variety with finitely many G-orbits. We denote the union of the conormal bundles of G-orbits by $\Lambda_{(X,G)}$. It is a G-stable closed subvariety of T^*X with dimension $\dim X$. Let L be an invertible $\mathcal{O}_X$-module with G-action. For a coherent $\mathcal{D}_X^L$-module M with G-action $\operatorname{gr}(M)$ is a coherent $\mathcal{O}_{T^*X}$-module with G-action supported in $\Lambda_{(X,G)}$. Hence we have a $\boldsymbol{Z}$-linear map:

$$\operatorname{gr}: K(\mathcal{S}(X, G, L))\longrightarrow K^G(\Lambda_{(X,G)}, T^*X).$$

LEMMA 2.7. *Let X be a non-singular variety. For coherent $\mathcal{O}_X$-*

modules L_i $(i=1,2)$ *and a coherent* $\mathcal{D}_X^{L_1}$*-module* M *we have*:

$$\mathrm{gr}(L_2 \otimes M) = p_X{}^* L_2 \underset{\mathcal{O}_{T^*X}}{\otimes} \mathrm{gr}(M).$$

For a morphism $f: X \to Y$ of non-singular varieties we set

$$\Omega_{X/Y} = \Omega_X \underset{\mathcal{O}_X}{\otimes} f^* \Omega_Y^{-1}.$$

LEMMA 2.8 (see [17]). *Let* $f: X \to Y$ *be a* G*-equivariant morphism of non-singular* G*-varieties with finitely many* G*-orbits and* L *an invertible* $\mathcal{O}_Y$*-module with* G*-action. Let*

$$\varpi : T^*Y \underset{Y}{\times} X \longrightarrow T^*Y, \quad \rho : T^*Y \underset{Y}{\times} X \longrightarrow T^*X$$

be the natural morphisms.

(i) *When* f *is smooth, we have the following commutative diagram*:

$$\begin{array}{ccc} K(\mathcal{S}(Y, G, L)) & \xrightarrow{\ f^*\ } & K(\mathcal{S}(X, G, f^*L)) \\ {\scriptstyle \mathrm{gr}}\downarrow & & \downarrow{\scriptstyle \mathrm{gr}} \\ K^G(\Lambda_{(Y,G)},\ T^*Y) & \xrightarrow{\ \rho_* \circ \varpi^*\ } & K^G(\Lambda_{(X,G)},\ T^*X). \end{array}$$

(ii) *When* f *is proper, we have the following commutative diagram*:

$$\begin{array}{ccc} K(\mathcal{S}(X, G, f^*L)) & \xrightarrow{\ \int_f\ } & K(\mathcal{S}(Y, G, L)) \\ {\scriptstyle \mathrm{gr}}\downarrow & & \downarrow{\scriptstyle \mathrm{gr}} \\ K^G(\Lambda_{(X,G)},\ T^*X) & \xrightarrow{\ F\ } & K^G(\Lambda_{(Y,G)},\ T^*Y), \end{array}$$

where $F(m) = \varpi_* \circ \rho^*([p_X{}^* \Omega_{X/Y}] \otimes m)$.

§ 3. Affine Weyl groups and twisted differential operators.

In the rest of this paper G is a connected reductive algebraic group over $\boldsymbol{C}$ and X is the flag variety of G.

3.1. The set of G-orbits on $X \times X$ with respect to the diagonal action is parametrized by the Weyl group W. In fact identifying X with G/B for a Borel subgroup B and setting $Y_w = G \cdot (eB, wB)$ for $w \in W$, we have $X\times X=\bigcup_{w\in W} Y_w$ (disjoint union). We denote by $q_j: X\times X\to X$ $(j=1, 2)$ the projection. Set $i_w: Y_w \hookrightarrow X\times X$ and $q_j^w=q_j\circ i_w$.

The affine Weyl group W_a is the semidirect product of the Weyl group W and the weight lattice P. An element $\lambda\in P$ is denoted by t_λ when it is regarded as an element of W_a.

Let F be a closed subgroup of G so that the number of F-orbits on X is finite. Set $\mathcal{S}_\lambda^F=\mathcal{S}(X, F, \mathcal{O}(\lambda))$ for $\lambda\in P$.

PROPOSITION 3.1. *An action of* W_a *on* $\bigoplus_{\lambda\in P} K(\mathcal{S}_\lambda^F)$ *is given by the following. Let* $m\in K(\mathcal{S}_\lambda^F)$.

(a) $t_\mu\cdot m=[\mathcal{O}(-\mu)]\otimes m\in K(\mathcal{S}_{\lambda-\mu}^F)$ $\qquad(\mu\in P)$,

(b) $w\cdot m=(-1)^{l(w)}\int_{q_1^w}(q_2^w)^*m\in K(\mathcal{S}_{w\lambda}^F)$ $\qquad(w\in W)$.

Here $l(w)$ *is the length of* w.

We explain the action of $w\in W$. When M is a $\mathcal{D}_X^\lambda$ $(=\mathcal{D}_X^{\mathcal{O}(\lambda)})$-module, $(q_2^w)^*M$ is a $\mathcal{D}_{Y_w}^{(q_2^w)^*\mathcal{O}(\lambda)}$-module. Since $(q_2^w)^*\mathcal{O}(\lambda)$ is isomorphic to $(q_1^w)^*\mathcal{O}(w\lambda)$, $(q_2^w)^*M$ is a $\mathcal{D}_{Y_w}^{(q_1^w)^*\mathcal{O}(w\lambda)}$-module and hence $\int_{q_1^w}((q_2^w)^*M)$ is a complex of $\mathcal{D}_X^{w\lambda}$ $(=\mathcal{D}_X^{\mathcal{O}(w\lambda)})$-modules.

3.2. Proof of Proposition 3.1.

Set $\mathcal{T}=\mathcal{S}(X\times X, G)$. Let $p_{ij}: X\times X\times X\to X\times X$ be the projection onto the (i, j)-factor.

LEMMA 3.2 (compare with [19]). (i) *A ring structure on* $K(\mathcal{T})$ *is given by:*

$$m_1\cdot m_2=\int_{p_{13}}(p_{12}{}^*m_1\otimes p_{23}{}^*m_2) \qquad (m_1, m_2\in K(\mathcal{T})).$$

(ii) *A* $K(\mathcal{T})$*-module structure on* $K(\mathcal{S}_0^F)$ *is given by*:

$$m\cdot n=\int_{q_1}(m\otimes q_2{}^*n) \quad (m\in K(\mathcal{T}),\ n\in K(\mathcal{S}_0^F)).$$

PROOF. (i) It is sufficient to show the relation:

$$(m_1\cdot m_2)\cdot m_3=m_1\cdot(m_2\cdot m_3) \quad (m_1, m_2, m_3\in K(\mathcal{T})).$$

Let $r_{ijk}: X\times X\times X\times X\to X\times X\times X$ and $r_{ij}: X\times X\times X\times X\to X\times X$ be the obvious projections. Then we have:

$$(m_1\cdot m_2)\cdot m_3$$
$$=\int_{p_{13}}\left(\left(p_{12}{}^*\int_{p_{13}}(p_{12}{}^*m_1\otimes p_{23}{}^*m_2)\right)\otimes p_{23}{}^*m_3\right)$$
$$=\int_{p_{13}}\left(\left(\int_{r_{134}}r_{123}{}^*(p_{12}{}^*m_1\otimes p_{23}{}^*m_2)\right)\otimes p_{23}{}^*m_3\right)\quad\text{(Proposition 1.3)}$$
$$=\int_{p_{13}}\int_{r_{134}}(r_{123}{}^*p_{12}{}^*m_1\otimes r_{123}{}^*p_{23}{}^*m_2\otimes r_{134}{}^*p_{23}{}^*m_3)\quad\text{(Lemma 1.1, Corollary 1.4)}$$
$$=\int_{r_{14}}(r_{12}{}^*m_1\otimes r_{23}{}^*m_2\otimes r_{34}{}^*m_3)\quad\text{(Lemma 1.1)}.$$

By the similar argument we also have:

$$m_1\cdot(m_2\cdot m_3)=\int_{r_{14}}(r_{12}{}^*m_1\otimes r_{23}{}^*m_2\otimes r_{34}{}^*m_3)$$

and (i) is proved. The statement (ii) is also shown in the same manner.

We denote by S the set of simple reflections in W. For $s\in S$ let $\bar{i}_s:\bar{Y}_s\to X\times X$ be the inclusion. Set $M_w=\int_{i_w}\mathcal{O}_{Y_w}$ and $L_s=\int_{\bar{i}_s}\mathcal{O}_{\bar{Y}_s}$ for $w\in W$ and $s\in S$. Note that since i_w and $\bar{i}_s$ are affine morphisms, $\mathcal{H}^j\left(\int_{i_w}\mathcal{O}_{Y_w}\right)=0$ and $\mathcal{H}^j\left(\int_{\bar{i}_s}\mathcal{O}_{\bar{Y}_s}\right)=0$ for $j\neq 0$. By Proposition 2.2 we have $K(\mathcal{T})=\bigoplus_{w\in W}\mathbf{Z}[M_w]$ and by Proposition 1.2 we have $[L_s]=[M_s]-[M_e]$ for $s\in S$.

Let $s\in S$. We denote the generalized flag variety consisting of parabolic subgroups with semisimple rank one corresponding to s by X^s. The natural map from X to X^s is denoted by π_s. Set $\bar{q}_j^s=q_j\circ\bar{i}_s$ for $j=1,2$.

The following is shown in the same manner as Proposition 3.2.

LEMMA 3.3. *Let* $n\in K(\mathcal{S}_0^F)$.

(i) $[M_w]\cdot n=\int_{q_1^w}(q_2^w)^*n \qquad (w\in W)$.

(ii) $[L_s]\cdot n=\int_{\bar{q}_1^s}(\bar{q}_2^s)^*n=\pi_s{}^*\int_{\pi_s}n \qquad (s\in S)$.

LEMMA 3.4. *$K(\mathcal{T})$ is isomorphic to $\mathbf{Z}[W]$ as a ring. The isomorphism is given by $[M_w]\leftrightarrow(-1)^{l(w)}w$ for $w\in W$.*

PROOF. We have only to show the following:

(a) $[M_w]\cdot[M_y]=[M_{wy}] \qquad (w,y\in W,\ l(w)+l(y)=l(wy))$,

(b) $[L_s]\cdot[L_s]=-2[L_s] \qquad (s\in S)$.

The proof of (a) is similar to that of Proposition 3.2. (b) is also easily verified if we note that π_s is a $\boldsymbol{P}^1$-bundle and $\int_\pi(\mathcal{O}_{P^1})=\boldsymbol{R}\Gamma(\boldsymbol{P}^1, \boldsymbol{C}_{P^1})$ [1] for $\pi:\boldsymbol{P}^1\to\text{pt}$.

PROOF OF PROPOSITION 3.1. Define endomorphisms T_μ $(\mu\in P)$ and R_w $(w\in W)$ of the $\boldsymbol{Z}$-module $\bigoplus_{\lambda\in P}K(\mathcal{S}_\lambda^F)$ by:

$$T_\mu(n)=[\mathcal{O}(\mu)]\otimes n\in K(\mathcal{S}_{\lambda+\mu}^F),$$

$$R_w(n)=\int_{q_1^w}(q_2^w)^*n\in K(\mathcal{S}_{w\lambda}^F),$$

for $n\in K(\mathcal{S}_\lambda^F)$. We have only to show the following:

(1) $T_0=R_e=\text{identity}$,

(2) $T_\mu T_\nu=T_{\mu+\nu}$,

(3) $R_w R_y=R_{wy}$,

(4) $R_w T_\mu=T_{w\mu}R_w$.

(1) and (2) are trivial. (4) is proved using Lemma 1.1 and Proposition 1.4. (3) is proved using (4), Lemma 3.3 (i), Lemma 3.3 (ii), Lemma 3.4 if we note that $T_\lambda: K(\mathcal{S}_0^F)\to K(\mathcal{S}_\lambda^F)$ is an isomorphism for each $\lambda\in P$.

COROLLARY 3.5. *As a W_a-module $\bigoplus_{\lambda\in P}K(\mathcal{S}_\lambda^F)$ is isomorphic to the induced module $\boldsymbol{Z}[W_a]\otimes_{\boldsymbol{Z}[W]}K(\mathcal{S}_0^F)$.*

§ 4. On the structure of $K^F(\Lambda)$.

4.1. Let F be a closed subgroup of G with finitely many F-orbits on X and set $\Lambda=\Lambda_{(X,F)}$. We consider $K^F(\Lambda)$ in this section. Ginsburg used the (algebraic) K-theory which we have discussed in Section 2.2, while Kazhdan-Lusztig used the topological K-theory. Let us see that the both coincide for our variety Λ. We denote the (topological) equivariant K-homology group of an F-variety Y defined in [16] by $K_{top}^F(Y)$. It is a module over the complexified representation ring $R_F^C:=\boldsymbol{C}\otimes_{\boldsymbol{Z}}R_F$ of F, in particular a $\boldsymbol{C}$-vector space.

For an F-stable locally closed subvariety V of X we denote the union of the conormal bundles of the F-orbits contained in V by Λ_V.

Step 1. As in [1] we can construct a natural transformation $K^F(\ )\to K_{top}^F(\ )$. We claim that $K^F(\Lambda_V)$ is a free $\boldsymbol{Z}$-module and the map

$C\bigotimes_{Z} K^F(\Lambda_V) \to K^F_{top}(\Lambda_V)$ is an isomorphism for any F-stable locally closed subvariety V of X.

Step 2. Let C be an F-orbit on X. By Proposition 2.5 we have $K^F(C) \simeq K^F(\Lambda_C)$. For $x \in C$ we have $C \simeq F/F^x$ and hence $K^F(C) \simeq R_{F^x}$. On the other hand the same arguments hold for the topological K-groups and we have: $K^F_{top}(\Lambda_C) \simeq R^C_{F^x}$. Therefore $K^F(\Lambda_C)$ is a free Z-module and $C\bigotimes_{Z} K^F(\Lambda_C) \simeq K^F_{top}(\Lambda_C)$ for each F-orbit C.

Step 3. Choose an F-orbit $C \subset V$ which is open in V and set $V_1 = V - C$. Then Λ_{V_1} is closed in Λ_V and Λ_C $(= \Lambda_V - \Lambda_{V_1})$ is open in Λ_V. Hence we have an exact sequence:

$$K^F(\Lambda_{V_1}) \longrightarrow K^F(\Lambda_V) \longrightarrow K^F(\Lambda_C) \longrightarrow 0.$$

On the other hand in the topological K-theory $K^F_{top,1}(\ )$ is defined and we have the following exact sequence:

$$\begin{array}{ccccc} K^F_{top}(\Lambda_{V_1}) & \longrightarrow & K^F_{top}(\Lambda_V) & \longrightarrow & K^F_{top}(\Lambda_C) \\ \uparrow & & & & \downarrow \\ K^F_{top,1}(\Lambda_C) & \longleftarrow & K^F_{top,1}(\Lambda_V) & \longleftarrow & K^F_{top,1}(\Lambda_{V_1}). \end{array}$$

Since $K^F_{top,1}(\Lambda_C) = K^F_{top,1}(C) = K^F_{top,1}(F/F^x) = 0$, we have $K^F_{top,1}(\Lambda_V) = 0$ by induction. Hence we have an exact sequence:

$$0 \longrightarrow K^F_{top}(\Lambda_{V_1}) \longrightarrow K^F_{top}(\Lambda_V) \longrightarrow K^F_{top}(\Lambda_C) \longrightarrow 0.$$

Step 4. By the above consideration we have:

$$\begin{array}{ccccccccc} & & K^F(\Lambda_{V_1}) & \longrightarrow & K^F(\Lambda_V) & \longrightarrow & K^F(\Lambda_C) & \longrightarrow & 0 \quad \text{(exact)} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K^F_{top}(\Lambda_{V_1}) & \longrightarrow & K^F_{top}(\Lambda_V) & \longrightarrow & K^F_{top}(\Lambda_C) & \longrightarrow & 0 \quad \text{(exact).} \end{array}$$

Hence our claim is a consequence of the five lemma and the induction.

By the above arguments we have the following:

LEMMA 4.1. (i) *For any F-stable locally closed subvariety V of X $K^F(\Lambda_V)$ is a free Z-module and $C\bigotimes_{Z} K^F(\Lambda_V) \simeq K^F_{top}(\Lambda_V)$.*

(ii) *If V_1, V_2, V_3 are F-stable locally closed subvarieties of X so that $V_1 \subset V_2$, $V_3 = V_2 - V_1$ and V_1 is closed in V_2, then we have an exact sequence:*

$$0 \longrightarrow K^F(\Lambda_{V_1}) \longrightarrow K^F(\Lambda_{V_2}) \longrightarrow K^F(\Lambda_{V_3}) \longrightarrow 0.$$

(iii) *If C is an F-orbit on X, then we have $K^F(\Lambda_C) \simeq R_{F^x}$ for $x \in C$.*

LEMMA 4.2. *The map* gr: $\bigoplus_{\lambda \in P} K(\mathcal{S}_\lambda^F) \to K^F(\Lambda)$ *is surjective.*

PROOF. For $\lambda \in P$ and an F-stable closed subvariety V of X we denote the category of coherent $\mathcal{D}_X^\lambda$-modules with F-actions supported in V by $\mathcal{S}_{\lambda, V}^F$. We shall show that the map gr: $\bigoplus_{\lambda \in P} K(\mathcal{S}_{\lambda, V}^F) \to K^F(\Lambda_V)$ is surjective for any F-stable closed subvariety V of X. By induction the problem is reduced to the case of a single F-orbit C. Fix $x \in C$ and let B be the Borel subgroup of G corresponding to x. Since $R_B = \boldsymbol{Z}[P]$, it is sufficient to show $R_{F^x} = r^*(R_{F^x/(F^x)^0}) \cdot i^*(R_B)$, where $r: F^x \to F^x/(F^x)^0$ and $i: F^x \to B$ are natural group homomorphisms. This follows from elementary facts concerning representations of solvable algebraic groups.

Set $Z = \Lambda_{(X \times X, G)}$. As a corollary to the proof we have the following:

COROLLARY 4.3. (i) *The map* gr: $K(\mathcal{T}) \to K^G(Z)$ *is injective.*

(ii) *Set* $K = \mathrm{Image}(K(\mathcal{T}) \to K^G(Z))$. *We have*:

$$K^G(Z) = \bigoplus_{\lambda \in P} (p_X \times p_X)^*[\mathcal{O}(\lambda) \boxtimes \mathcal{O}_X] \otimes K = \bigoplus_{\lambda \in P} (p_X \times p_X)^*[\mathcal{O}_X \boxtimes \mathcal{O}(\lambda)] \otimes K.$$

§ 5. Affine Weyl groups and equivariant K-theory.

5.1. Let $\mathfrak{g}$ and $\mathfrak{f}$ be the Lie algebras of G and F, respectively. For $x \in X$ we denote the Lie algebra of the unipotent radical of the Borel subgroup of G corresponding to x by $\mathfrak{n}_x$. T^*X is naturally identified with $\{(x, A) \in X \times \mathfrak{g} \mid A \in \mathfrak{n}_x\}$ via the Killing form. Under this identification we have:

$$\Lambda := \Lambda_{(X, F)} = \{(x, A) \in X \times \mathfrak{g} \mid A \in \mathfrak{n}_x \cap \mathfrak{f}^\perp\},$$

with $\mathfrak{f}^\perp = \{A \in \mathfrak{g} \mid (A, \mathfrak{f}) = 0\}$, where $(\,,\,)$ is the Killing form. Especially we have:

$$Z := \Lambda_{(X \times X, G)} = \{(x, y, A, -A) \in X \times X \times \mathfrak{g} \times \mathfrak{g} \mid A \in \mathfrak{n}_x \cap \mathfrak{n}_y\}.$$

Let a be the G-equivariant automorphism of $T^*X \times T^*X$ given by: $a(x, y, A, A') = (x, y, A, -A')$. We set $Z^a = a(Z)$.

5.2. Let $H(W_a)$ be the Hecke algebra of W_a ([9], [16]). It is an algebra over the Laurent polynomial ring $\boldsymbol{Z}[q, q^{-1}]$. For $z \in \boldsymbol{C}^*$ set $H_z =$

$H(W_a)\underset{Z[q,q^{-1}]}{\otimes}C$, where the ring homomorphism $Z[q,q^{-1}]\to C$ is given by: $q\to z$. We have $H_1\simeq C[W_a]$.

Let K be a non-archimedian local field whose residue field consists of q_0 elements. Let G^d be the group of K-rational points of a reductive algebraic group scheme which is dual to G and I an Iwahori subgroup of G^d. It is known that the set of the finite dimensional irreducible representations of the C-algebra H_{q_0} is in one-to-one correspondence with the set of the irreducible admissible representations of G^d which contain non-zero I-fixed vectors (Matsumoto, Borel).

Recently Kazhdan-Lusztig ([16]) and Ginsburg ([6], [7]) have defined an $(H(W_a), H(W_a))$-bimodule structure on $K^{G\times C^*}(Z^a)$ and shown that it is isomorphic to the two-sided regular representation of $H(W_a)$. Using this they have given a classification of the irreducible representation of H_{q_0} (Conjecture of Deligne-Langlands-Lusztig).

It follows immediately from their result that a (W_a, W_a)-bimodule structure is defined on $K^G(Z^a)$ and it is isomorphic to the two-sided regular representation of $Z[W_a]$. More generally a W_a-module structure is defined on $K^F(\Lambda)$.

Our aim is to understand the above facts from the view point of the $\mathcal{D}$-module theory.

5.3. Following Ginsburg ([6], [7]) we define a ring structure on $K^G(Z^a)=K^G(Z^a, T^*X\times T^*X)$ as follows. Let $p_{ij}: T^*X\times T^*X\times T^*X\to T^*X\times T^*X$ be the projection onto the (i,j)-factor. Note that the restriction of p_{13} to $p_{12}^{-1}(Z^a)\cap p_{23}^{-1}(Z^a)$ is projective and its image coincides with Z^a. We define a product on $K^G(Z^a)$ by:

$$m_1\cdot m_2=p_{13*}(p_{12}{}^*m_1\otimes p_{23}{}^*m_2\otimes p_2{}^*p_X{}^*[\Omega_X]).$$

It follows easily from Propositions 2.3 and 2.4 that this product satisfies the associativity.

Let $q_i: T^*X\times T^*X\to T^*X$ be the projection. Similarly an action of the ring $K^G(Z^a)(=K^G(Z^a, T^*X\times T^*X))$ on $K^F(\Lambda)(=K^F(\Lambda, T^*X))$ is defined by:

$$m\cdot n=q_{1*}(m\otimes q_2{}^*n\otimes q_2{}^*p_X{}^*[\Omega_X]).$$

For $s\in S$ let Z_s^a be the closure of $\{(x, y, A, A)\in Z^a \mid (x, y)\in Y_s\}$. It is seen that $p_s: Z_s^a\to\overline{Y}_s$ is a G-vector bundle. Let $j: T^*X\to T^*X\times T^*X$ be the diagonal embedding and $j_s: Z_s^a\to T^*X\times T^*X$ the natural inclusion. For $\lambda\in P$ and $s\in S$ we define elements $e(\lambda)$ and a_s of $K^G(Z^a, T^*X\times T^*X)$

by:

$$e(\lambda)=j_*p_X^*([\mathcal{O}(-\lambda)\otimes\Omega_X^{-1}]),\quad a_s=j_{s*}p_s^*([\Omega_{\bar{Y}_s/X\times X}]).$$

THEOREM. (i) *$K^G(Z^a)$ is isomorphic to $Z[W_a]$ as a ring. The isomorphism is given by*:

$$e(\lambda)\longleftrightarrow t_\lambda\ (\lambda\in P),\quad a_s\longleftrightarrow-(s+1)\ (s\in S).$$

(ii) *The map $a^*\circ\mathrm{gr}:K(\mathcal{I})\to K^G(Z^a)$ is a ring homomorphism. If we identify $K(\mathcal{I})$ and $K^G(Z^a)$ with $Z[W]$ and $Z[W_a]$ respectively, then $a^*\circ\mathrm{gr}$ coincides with the natural inclusion.*

(iii) *The map $\mathrm{gr}:\bigoplus_{\lambda\in P}K(\mathcal{S}_\lambda^F)\to K^F(\Lambda)$ is a surjective homomorphism of W_a-modules.*

PROOF. Set $u_i^s=q_i\circ j_s$ $(i=1,2,\ s\in S)$. The following relations are proved using Propositions 2.3 and 2.4.

(a) $e(\lambda)\cdot m=((p_X\times p_X)^*([\mathcal{O}(-\lambda)\boxtimes\mathcal{O}_X]))\otimes m\quad(m\in K^G(Z^a),\ \lambda\in P)$,

(a′) $m\cdot e(\lambda)=((p_X\times p_X)^*([\mathcal{O}_X\boxtimes\mathcal{O}(-\lambda)]))\otimes m\quad(m\in K^G(Z^a),\ \lambda\in P)$,

(b) $a_s\cdot m=(u_1^s\times 1)_*(u_2^s\times 1)^*(m\otimes(p_X\times p_X)^*([\Omega_{X/X^s}\boxtimes\mathcal{O}_X]))$
$(m\in K^G(Z^a),\ s\in S)$,

(b′) $m\cdot a_s=(1\times u_2^s)_*(1\times u_1^s)^*(m\otimes(p_X\times p_X)^*([\mathcal{O}_X\boxtimes\Omega_{X/X^s}]))$
$(m\in K^G(Z^a),\ s\in S)$.

By Propositions 2.3, 2.4 and Lemma 2.8 we have:

(c) $(a^*\circ\mathrm{gr})([L_e])=e(0)$,

(d) $(a^*\circ\mathrm{gr})([L_s])=a_s\quad(s\in S)$,

(e) $(a^*\circ\mathrm{gr})([L_s]\cdot m)=a_s\cdot((a^*\circ\mathrm{gr})m)\quad(s\in S,\ m\in K(\mathcal{I}))$,

(e′) $(a^*\circ\mathrm{gr})(m\cdot[L_s])=((a^*\circ\mathrm{gr})m)\cdot a_s\quad(s\in S,\ m\in K(\mathcal{I}))$.

Next we show the following:

(f) $e(\lambda)\cdot a_s-a_s\cdot e(s\lambda)=e(s\lambda)-e(\lambda)\qquad(s\in S,\ \lambda\in P)$.

Set $u:X=Y_e\hookrightarrow\bar{Y}_s$, $v:Y_s\hookrightarrow\bar{Y}_s$, $\bar{i}_s:\bar{Y}_s\hookrightarrow X\times X$. By (a), (a′) we have $e(\lambda)\cdot a_s-a_s\cdot e(s\lambda)=j_{s*}p_s^*(m)$ with

$$m=[\Omega_{\bar{Y}_s/X\times X}]\otimes\bar{i}_s^*([\mathcal{O}(-\lambda)\boxtimes\mathcal{O}_X]-[\mathcal{O}_X\boxtimes\mathcal{O}(-s\lambda)])\in K^G(\bar{Y}_s).$$

Consider the following exact sequence:

$$K^G(X)\xrightarrow{u_*}K^G(\bar{Y}_s)\xrightarrow{v^*}K^G(Y_s)\longrightarrow 0.$$

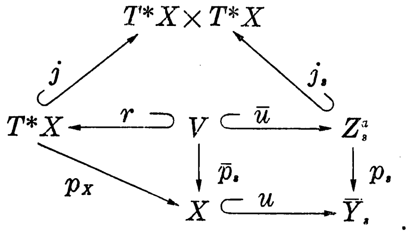
Since $(\mathcal{O}(-\lambda)\boxtimes\mathcal{O}_X)\mid Y_s\simeq(\mathcal{O}_X\boxtimes\mathcal{O}(-s\lambda))\mid Y_s$, we have $v^*(m)=0$. Hence there exists some $m_1 \in K^G(X)$ such that $m = u_*(m_1)$. Since $u^*: K^G(\overline{Y}_s) \to K^G(X)$ is surjective, there exists some $m_2 \in K^G(\overline{Y}_s)$ such that $u^*(m_2) = m_1$. Then we have:

$$\begin{aligned} u^*(m) &= u^* u_* u^*(m_2) = u^*(m_2 \otimes u_*([\mathcal{O}_X])) \\ &= u^*(m_2) \otimes u^* u_*([\mathcal{O}_X]) = m_1 \otimes ([\mathcal{O}_X] - [\Omega^{-1}_{X/\overline{Y}_s}]). \end{aligned}$$

Therefore we have:

$$m_1 \otimes ([\mathcal{O}_X] - [\Omega^{-1}_{X/\overline{Y}_s}]) = u^*([\Omega_{\overline{Y}_s/X \times X}]) \otimes ([\mathcal{O}(-\lambda)] - [\mathcal{O}(-s\lambda)]).$$

Set $V = Z^a_s \cap (p_X \times p_X)^{-1}(Y_e)$ and consider the following commutative diagram.



We have:

$$\begin{aligned} j_{s*} p_s^*(m) &= j_{s*} p_s^* u_*(m_1) = j_{s*} \bar{u}_* \bar{p}_s^*(m_1) = j_* r_* r^* p_X^*(m_1) \\ &= j_*(p_X^*(m_1) \otimes r_*([\mathcal{O}_V])). \end{aligned}$$

By Lemma 2.6 we have $r_*([\mathcal{O}_V]) = p_X^*([\mathcal{O}_X] - [\Omega_{X/\overline{Y}_s}])$. Hence

$$\begin{aligned} j_{s*} p_s^*(m) &= j_* p_X^*(m_1 \otimes ([\mathcal{O}_X] - [\Omega_{X/\overline{Y}_s}])) \\ &= j_* p_X^*(([\mathcal{O}(-s\lambda)] - [\mathcal{O}(-\lambda)]) \otimes [\Omega_{X/\overline{Y}_s}] \otimes u^*[\Omega_{\overline{Y}_s/X \times X}]) \\ &= j_* p_X^*(([\mathcal{O}(-s\lambda)] - [\mathcal{O}(-\lambda)]) \otimes [\Omega_X^{-1}]), \end{aligned}$$

and (f) is proved.

(i) and (ii) are easy consequences of (a)∼(f) and Corollary 4.3. (iii) is proved similarly.

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Mathematical Institute
Tohoku University
Sendai
980 Japan