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Cells in affine Weyl groups, III

Dedicated to Professor Nagayoshi Iwahori on his sixtieth birthday

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Let $\underline{H}_{r_0}$ be the Hecke algebra (over C) attached by Iwahori and Matsumoto [2] to an affine Weyl group W and to a parameter $r_0=\sqrt{q}\in C^*$.

The simple $\underline{H}_{r_0}$-modules have been recently classified (see [4]), when r_0 is not a root of 1, using methods from equivariant K-theory. Another (conjectural) approach to the same question, using cells in W, was given in [6]. In this paper, we shall use the results of [4] to answer some of the questions raised in [6, 9.10] concerning the relation between simple W-modules and simple modules for $\underline{H}_{r_0}$. Our main tool is the *asymptotic Hecke algebra* $\underline{J}$ defined in [7]; this is a C-algebra J whose structure constants are integers. (See 1.3.)

It turns out that $\underline{J}$ contains all the algebras $\underline{H}_{r_0}$ as subalgebras (see 1.7), in such a way that the simple $\underline{J}$-modules restricted to any $\underline{H}_{r_0}$ form a basis for the Grothendieck group of $\underline{H}_{r_0}$-modules of finite length, at least when r_0 is not a root of 1, or when $r_0=1$. (Theorem 3.4.)

One of the applications of our results is that for a large class of modules over the Hecke algebra there is a canonical direct sum decomposition (indexed by left cells) such that the action of the generators of the Hecke algebra is given in terms of this decomposition by a particularly simple formula (Theorem 3.8).

The results of this paper together with [4] imply the validity of several of the conjectures made in [6, 9.10]; more precisely conjecture A follows (without uniqueness of irreducible quotients), and also conjectures C, F and a variant of conjecture E. (Conjecture B in [*loc. cit*] has been verified in [7]; conjecture D remains open.)

We shall often give references to [6], [7] to results which are proved there for ordinary affine Weyl groups and which we need for extended affine Weyl groups; the results we need for extended affine Weyl groups can be reduced trivially to those for ordinary affine Weyl groups.

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We make the convention that, unless otherwise specified, "module" means "left module".

§ 1. The Hecke algebra and the asymptotic Hecke algebra

1.1. Let (W_1, S_1) be an irreducible affine Weyl group regarded as a Coxeter group; S_1 is the set of simple reflections. There is a unique subgroup Q of W_1 which is abelian of finite index and is maximal with these properties.

Let Ω be the group of all automorphisms of W_1 which leave S_1 stable and whose restriction to Q coincides with the restriction of some inner automorphism of W_1. We form the semidirect product $W=\Omega\cdot W_1$ with W_1 normal. Let $l: W\to N$ be the function extending the usual length function of W_1 and such that $l(\omega w_1)=l(w_1)$, $\omega\in\Omega$, $w_1\in W_1$.

Let X be the centralizer of Q in W. It is a free abelian normal subgroup of W and $X/Q\cong\Omega$. We can find a simple reflection $s_0\in S_1$ such that the set $S'=S_1-\{s_0\}$ generates a finite subgroup W' of W which is complementary to X. Thus, W is a semidirect product $W'\cdot X$ with X normal.

Let w_0 be the longest element of W'.

Let $X^{++}=\{x\in X | l(sx)>l(x) \text{ for all } s\in S'\}$. We have

(a) $l(w_0x)=l(w_0)+l(x)$, $l(x^{-1}w_0)=l(x^{-1})+l(w_0)$ for all $x\in X^{++}$.

(b) Any $x\in X$ can be written as $x=x_1x_2^{-1}$, with $x_1, x_2\in X^{++}$.

(c) For any $y\in W$ we can find $s_1, s_2, \cdots, s_p\in S_1$ such that $y'=ys_1\cdots s_p$ satisfies $l(y')=l(y)+p$ and $l(y's')>l(y')$ for all $s'\in S'$.

1.2. Let r be an indeterminate, and let $\underline{A}=C[r, r^{-1}]$. Let $\underline{H}$ be the Hecke algebra of W over $\underline{A}$, that is the free $\underline{A}$-module with basis $\tilde{T}_w$ $(w\in W)$ and multiplication defined by $(\tilde{T}_s+r^{-1})(\tilde{T}_s-r)=0$ if $s\in S_1$ and $\tilde{T}_w\tilde{T}_{w'}=\tilde{T}_{ww'}$ if $l(ww')=l(w)+l(w')$.

Define a polynomial $P_{y,w}$ $(y=\omega y_1, w=\omega' w_1, \omega, \omega'\in\Omega, y_1, w_1\in W)$ to be P_{y_1,w_1} of [3] if $\omega=\omega'$ and to be 0 if $\omega\neq\omega'$.

For each $w\in W$, the element

$$C_w=\sum_y(-r)^{l(w)-l(y)}P_{y,w}(r^{-2})\tilde{T}_y\in\underline{H}$$

is well defined, see [3].

If $\underline{B}$ is a commutative $\underline{A}$-algebra with 1, we shall write $\underline{H}_B=\underline{H}\otimes_{\underline{A}}\underline{B}$;

we shall denote the images of $\tilde{T}_w \otimes 1, C_w \otimes 1$ in $\underline{H}_B$, again by $\tilde{T}_w, C_w$.

We write $C_x C_y = \sum_z h_{x,y,z} C_z \in \underline{H}$, $h_{x,y,z} \in \underline{A}$.

For each $z \in W$, there is a well defined integer $a(z) \geq 0$ such that

$$r^{a(z)} h_{x,y,z} \in C[r] \quad \text{for all } x, y \in W$$
$$r^{a(z)-1} h_{x,y,z} \notin C[r] \quad \text{for some } x, y \in W$$

see [6, 7.3]. We have $a(z) = a(z^{-1})$ and $a(z) \leq l(w_0)$.

1.3. Let $\gamma_{x,y,z}$ be the constant term of $(-r)^{a(z)} h_{x,y,z^{-1}} \in C[r]$; we have $\gamma_{x,y,z} \in N$. Moreover,

(a) $$\gamma_{x,y,z} \neq 0 \Longrightarrow a(x) = a(y) = a(z).$$

Let $\underline{J}$ be the C vector space with basis $(t_w)_{w \in W}$. This is an associative C-algebra with multiplication $t_x t_y = \sum_{z \in W} \gamma_{x,y,z} t_{z^{-1}}$. (This is a finite sum.) It has a unit element $1 = \sum_{d \in \mathcal{D}} t_d$ where $\mathcal{D}$ is a certain finite subset of W_1 consisting of involutions. (See [7, 2.3].) For any C-algebra B with 1 we denote $\underline{J}_B = \underline{J} \otimes_C B$.

The $\underline{A}$-linear map $\phi : \underline{H} \to \underline{J}_A$ defined by

(b) $$\phi(C_w) = \sum_{\substack{d \in \mathcal{D} \\ z \in W \\ a(d) = a(z)}} h_{w,d,z} t_z$$

is a homomorphism of $\underline{A}$-algebras with 1, [7, 2.4].

(c) If $a(w) = i$, we have

$$\phi((-r)^i C_w) = \sum_{\substack{z \in W \\ a(z) = i}} \pi_{w,z} t_z + \sum_{\substack{z \in W \\ a(z) > i}} \pi'_{w,z} t_z$$

where $\pi'_{w,z} \in \underline{A}$, $\pi_{w,z} \in C[r]$ and the constant term of $\pi_{w,z}$ is 1 for $z = w$ and 0 for $z \neq w$, [7].

Let $\underline{J}^i$ be the C-subspace of $\underline{J}$ spanned by the t_w, $(a(w) = i)$.

(d) $\underline{J}^i$ is a two sided ideal of $\underline{J}$, see (a), and we have clearly $\underline{J} = \bigoplus_i \underline{J}^i$.

If B is a commutative $\underline{A}$-algebra with 1 we shall write $\phi_B : \underline{H}_B \to J \otimes_C B$ for the B-algebra homomorphism defined by $\phi : \underline{H}_B \to J \otimes_C \underline{A}$.

1.4. For each $r_0 \in C^*$ we denote $\underline{H}_{r_0} = \underline{H} \otimes_A C$ where C is regarded as an $\underline{A}$-algebra with r acting as scalar multiplication by r_0.

Let ϕ_{r_0}: $\underline{H}_{r_0} \to \underline{J}$ be the C-algebra homomorphism induced by ϕ: $\underline{H} \to \underline{J}_A$.

Let $\underline{H}_{r_0}^{\geq i}$ be the C-subspace of $\underline{H}_{r_0}$ spanned by all C_w ($w \in W$, $a(w) \geq i$). This is a two-sided ideal of $\underline{H}_{r_0}$. Let $\underline{H}_{r_0}^i = \underline{H}_{r_0}^{\geq i} / \underline{H}_{r_0}^{\geq i+1}$; this is an $\underline{H}_{r_0}$-bimodule. It has as C-basis the images $[C_w]$ of $C_w \in \underline{H}_{r_0}^{\geq i}$, ($a(w)=i$). Hence for $f \in \underline{H}_{r_0}^i$, $h, h' \in \underline{H}_{r_0}$, we have $(hf)h' = h(fh')$.

We may regard $\underline{H}_{r_0}^i$ as a $\underline{J}$-bimodule with multiplication defined by the rule

$$\text{(a)} \quad \begin{cases} t_x \circ [C_w] = \sum_{\substack{w' \\ a(w')=i}} \gamma_{x,w,w'^{-1}} [C_{w'}] \\ [C_w] \circ t_x = \sum_{\substack{w' \\ a(w')=i}} \gamma_{w,x,w'^{-1}} [C_{w'}], \qquad (w, x \in W,\ a(w)=i); \end{cases}$$

this simply expresses the fact that $\underline{J}^i$ is a two-sided ideal of $\underline{J}$. We have, for all $f \in \underline{H}_{r_0}^i$, $h \in \underline{H}_{r_0}$, $j \in J$:

$$\text{(b)} \qquad hf = \phi_{r_0}(h) \circ f,\ (j \circ f)h = j \circ (fh),\ (hf) \circ j = h(f \circ j).$$

This follows from [7, 2.4 (d)].

1.5. Note that $\underline{H}_1$ is naturally the group algebra of W over C. (The basis $(\tilde{T}_w)$ of $\underline{H}_1$ is the standard basis of the group algebra.) Let Z be the centre of $\underline{H}_1$. It is easy to see that

(a) $\underline{H}_1$ is finitely generated as a Z-module and Z is a finitely generated C-algebra.

1.6. PROPOSITION.

(i) *$\underline{J}$ is a finitely generated module over its centre.*

(ii) *The centre of $\underline{J}$ is a finitely generated C-algebra.*

(iii) *Any simple $\underline{J}$-module is finite dimensional over C.*

PROOF. (i) Let Z be as in 1.5 and let Z' be the centre of $\underline{J}$. We first show that $\phi_1(Z)$ is contained in Z'. It is enough to show that $\phi_1(z)t_x = t_x\phi_1(z)$ for all $z \in Z$, $x \in W$. Assume that $a(x)=i$.

Let $f_i = \sum_{\substack{d \in \mathcal{D} \\ a(d)=i}} [C_d] \in \underline{H}_1^i$. We have

$$\text{(a)} \qquad t_x \circ f_i = f_i \circ t_x = [C_x],$$

see 1.4 (a). Since $z \in Z$, we have

$$\text{(b)} \qquad zf = fz, \quad \text{for all } f \in \underline{H}_1^i.$$

We have

$$(\phi_1(z)t_x)\circ f_i=\phi_1(z)\circ t_x\circ f_i=\phi_1(z)\circ[C_x]=z[C_x]$$
$$(t_x\phi_1(z))\circ f_i=t_x\circ(\phi_1(z)\circ f_i)=t_x\circ(zf_i)=t_x\circ(f_iz)=(t_x\circ f_i)z=[C_x]z=z[C_x]$$

see 1.4 (a), (b). Hence

(c) $$(\phi_1(z)t_x)\circ f_i=(t_x\phi_1(z))\circ f_i.$$

Writing $\phi_1(z)t_x=\sum_{a(x')=i}\alpha_{x'}t_{x'}$, $t_x\phi_1(z)=\sum_{a(x')=i}\beta_{x'}t_{x'}$ $(\alpha_{x'},\beta_{x'}\in C)$, we see from (a), (c) that $\sum_{a(x')=i}\alpha_{x'}[C_{x'}]=\sum_{a(x')=i}\beta_{x'}[C_{x'}]$ hence $\alpha_{x'}=\beta_{x'}$ for all x', $a(x')=i$. Thus $\phi_1(z)t_x=t_x\phi_1(z)$, as required.

It is now enough to show that $\underline{J}$ is finitely generated as a $\phi_1(Z)$-module. Clearly, the left $\underline{J}$-module $\underline{J}$ (left regular representation) is isomorphic to $\bigoplus_i \underline{H}_1^i$, with $\underline{J}$ acting by $j:f\to j\circ f$. Hence it is enough to show that for each i, $\underline{H}_1^i$ is a finitely generated $\phi_1(Z)$-module. (We have $\underline{H}_1^i=0$ for all but finitely many i.)

Since $\underline{H}_1^i$ is a subquotient of $\underline{H}_1$, we see from 1.5 (a) that $\underline{H}_1^i$ is finitely generated as a Z-module; let $\psi_1,\cdots,\psi_N$ be generators. For any $\psi\in\underline{H}_1^i$, there exist $z_1,\cdots,z_N\in Z$ such that $\psi=\sum_{i=1}^N z_i\psi_i$. By 1.4 (b) we have also $\psi=\sum_{i=1}^N\phi_1(z_i)\circ\psi_i$ hence ψ_i are also generators of $\underline{H}_1^i$ as a $\phi_1(Z)$-module. This proves (i).

(ii) Z' contains $\phi_1(Z)$, a finitely generated C-algebra. Moreover, Z' is a $\phi_1(Z)$-submodule of the finitely generated $\phi_1(Z)$-module $\underline{J}$, hence Z' is a finitely generated $\phi_1(Z)$-module. Hence, by 1.5 (a), Z' is a finitely generated C-algebra.

(iii) Let E be a simple $\underline{J}$-module. Since $\underline{J}$ has countable dimension over C, a known argument of Dixmier, see [8], shows that $\mathrm{End}_J(E)=C$. Hence Z' acts on E by scalar multiplications. Using (i), it follows that the image of $\underline{J}$ in $\mathrm{End}_C(E)$ is a finite dimensional C-vector space.

Since E is simple, it follows that $\dim_C E<\infty$.

(d) REMARK. The previous proposition is also true for $\underline{H}_{r_0}$ instead of $\underline{J}$. (Bernstein).

1.7. PROPOSITION. *For any $r_0\in C^*$, the map $\phi_{r_0}:\underline{H}_{r_0}\to\underline{J}$ is injective.*

PROOF. Assume that $h\in\underline{H}_{r_0}$ is a non-zero element in the kernel of ϕ_{r_0}. We express h as a C-linear combination of basis elements $\tilde{T}_w$ and let $y \in W$ be an element such that $\tilde{T}_y$ appears in h with $\neq 0$ coefficient, with $l(y)$ maximum possible. Let $y' = ys_1s_2 \cdots s_p$ be as in 1.1 (c). Let $h' = h\tilde{T}_{s_1}\tilde{T}_{s_2} \cdots \tilde{T}_{s_p}$. Then h' is an element in the kernel of ϕ_{r_0} in which $\tilde{T}_{y'}$ appears with $\neq 0$ coefficient and $l(y') \geq l(y'')$ whenever $\tilde{T}_{y''}$ appears with $\neq 0$ coefficient in h'. We have $C_{w_0} = \sum_{z \in W'} (-r_0)^{l(w_0)-l(z)} \tilde{T}_z$.

We have $a(w_0) = l(w_0)$ and $a(w) \leq l(w_0)$ for all $w \in W$. By 1.4 (b), we have $h'f = 0$ for all $f \in \underline{H}_{r_0}^{a(w_0)}$, hence $h'\underline{H}_{r_0}^{\geq a(w_0)} \subset \underline{H}_{r_0}^{\geq a(w_0)+1} = 0$; in particular, $h'C_{w_0} = 0$. But the coefficient of $\tilde{T}_{y'w_0}$ in $h'C_{w_0}$ is the same as the coefficient of $\tilde{T}_{y'}$ in h', hence is non-zero. Thus, we have $h'C_{w_0} \neq 0$, a contradiction. The proposition is proved.

1.8. Any left $\underline{J}$-module E gives rise, via $\phi_{r_0} : \underline{H}_{r_0} \to \underline{J}$ to a left $\underline{H}_{r_0}$-module E_{r_0}. We denote by $K(\underline{J})$ (resp. $K(\underline{H}_{r_0})$) the Grothendieck group of left $\underline{J}$-modules (resp. $\underline{H}_{r_0}$-modules) of finite length, or equivalently, of finite dimension over $\boldsymbol{C}$. The correspondence $E \to E_{r_0}$ defines a homomorphism $(\phi_{r_0})_* : K(\underline{J}) \to K(\underline{H}_{r_0})$.

1.9. LEMMA. *For any $r_0 \in \boldsymbol{C}^*$, $(\phi_{r_0})_* : K(\underline{J}) \to K(\underline{H}_{r_0})$ is surjective.*

PROOF. Let M be a simple $\underline{H}_{r_0}$-module. We attach to M an integer $a = a_M$ by the following two requirements:

$$C_w M = 0 \quad \text{for all } w \in W,\ a(w) > a$$
$$C_w M \neq 0 \quad \text{for some } w \in W,\ a(w) = a.$$

This is well defined since $a(w)$ is bounded on W.

Let $\tilde{M} = \underline{H}_{r_0}^a \underset{\underline{H}_{r_0}}{\otimes} M$ where $\underline{H}_{r_0}^a$ is regarded as a right $\underline{H}_{r_0}$-module ($h : f \to fh$) and M as a left $\underline{H}_{r_0}$-module. Then $\tilde{M}$ is an $\underline{H}_{r_0}$-module ($h : (f \otimes m) \to (hf) \otimes m$). We have a natural homomorphism $p : \tilde{M} \to M$ defined by $p(f \otimes m) = \dot{f}m$ where $\dot{f} \in \underline{H}_{r_0}^{\geq a}$ is a representative for $f \in \underline{H}_{r_0}^a$. This map is correctly defined, by the definition of $a = a_M$. It is clearly a homomorphism of left $\underline{H}_{r_0}$-modules. It is non-zero since, otherwise, $\underline{H}_{r_0}^{\geq a} M = 0$, contradicting the definition of a. Since M is simple, it follows that p is surjective. We now show that

(a) $$\underline{H}_{r_0} \cdot \operatorname{Ker}(\tilde{M} \xrightarrow{p} M) = 0.$$

Let $\sum_i f_i \otimes m_i \in \ker p$, ($f_i \in \underline{H}_{r_0}^a$, $m_i \in M$) and let $\dot{f}_i \in \underline{H}_{r_0}^{\geq a}$ be representatives for f_i. Then $\sum_i \dot{f}_i m_i = 0$ in M. Let $f' \in \underline{H}_{r_0}^a$ and let $\dot{f}' \in \underline{H}_{r_0}^{\geq a}$ be a representative for f'. We have

$$\dot{f}'(\sum_i f_i\otimes m_i)=\sum_i(\dot{f}'f_i)\otimes m=\sum_i(f'\dot{f}_i)\otimes m_i=\sum_i(f'\otimes(\dot{f}_i m_i))$$
$$=f'\otimes(\sum_i \dot{f}_i m_i)=0$$

and (a) follows.

Next we show that $\tilde{M}$ is finite dimensional over C. From 1.6 (d) we see that $\underline{H}^a_{r_0}$ is finitely generated as a module over the centre of $\underline{H}_{r_0}$ hence it is generated by, say, N elements as a right $\underline{H}_{r_0}$-module. The definition of $\tilde{M}$ shows then that $\dim_C \tilde{M}\leq N\cdot\dim_C M<\infty$.

Let $\tilde{M}_J$ be the $\underline{J}$-module whose underlying C-vector space is $\tilde{M}$ and $\underline{J}$ acts by $j:(f\otimes m)\to(j\circ f)\otimes m$. (This is well defined by 1.4 (b).) The image of $\tilde{M}_J$ under $(\phi_{r_0})_*$ is the class of the $\underline{H}_{r_0}$-module with underlying C-vector space $\tilde{M}$ and $\underline{H}_{r_0}$-action $h:f\otimes m\to(\phi_{r_0}(h)\circ f)\otimes m=(hf)\otimes m$ (see 1.4 (b)), hence it is just the $\underline{H}_{r_0}$-module $\tilde{M}$ defined earlier. Thus, $\tilde{M}$ is in the image of $(\phi_{r_0})_*$. From (a) we see that in $K(\underline{H}_{r_0})$, $\tilde{M}$ is equal to M plus a sum of simple $\underline{H}_{r_0}$-modules M' satisfying $a_{M'}<a_M$. We may assume by induction that any M' with $a_{M'}<a_M$ is in the image of $(\phi_{r_0})_*$. Since $\tilde{M}$ is in the image of $(\phi_{r_0})_*$, it follows that M is in the image of $(\phi_{r_0})_*$. (To begin the induction we note that if $a_M=0$, we must have $\tilde{M}=M$.) The lemma is proved.

§ 2. Simple $\underline{J}$-modules

2.1. We consider a simply connected reductive algebraic group G over C with a fixed maximal torus $T_0\subset G$ and a fixed Borel subgroup B_0 containing T_0, such that (W', S') is identified with the Weyl group of G with respect to T_0 with simple reflections determined by B_0, X is identified with the group of characters of T_0; the elements of X^{++} correspond to the inverses of the characters by which T_0 acts on the B_0-stable lines of the various simple rational G-modules.

The complex varieties $G, T_0, B_0, \cdots$ will be generally identified with their sets of C-points.

Let K be an algebraic closure of $C(r)$. Any complex variety Z gives rise to an algebraic variety over K with set of K-points Z_K. We shall identify algebraic varieties over K with their sets of K-points. In particular $G_K, (T_0)_K, (B_0)_K, \cdots$ are well defined.

2.2. We now consider a unipotent element $u\in G$; let $f:SL_2(C)\to G$ be a homomorphism of C-algebraic groups such that $f\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=u$. Let

$$M(u)=\{(\gamma,\alpha)\in G\times \boldsymbol{C}^*|\gamma u\gamma^{-1}=u^\alpha\}$$

$$M_f=\left\{(\gamma,\alpha)\in G\times \boldsymbol{C}^*|\gamma f(A)\gamma^{-1}=f\left(\begin{pmatrix}\alpha^{1/2} & 0\\ 0 & \alpha^{-1/2}\end{pmatrix}A\begin{pmatrix}\alpha^{-1/2} & 0\\ 0 & \alpha^{1/2}\end{pmatrix}\right)\right.$$
$$\left.\text{for all } A\in SL_2(\boldsymbol{C})\right\}.$$

Let (s,β) be a semisimple element in $M(u)$. Define

$$M(u,s)=\{(\gamma,\alpha)\in M(u)|\gamma s=s\gamma\}$$
$$M_f(s)=\{(\gamma,\alpha)\in M_f|\gamma s=s\gamma\}.$$

It is known [4] that

(a) M_f is a maximal reductive subgroup of $M(u)$.

This implies that

(b) $M_f(s)$ is a maximal reductive subgroup of $M(u,s)$.

In particular:

(c) $$M_f(s)/M_f^0(s)\cong M(u,s)/M^0(u,s)\overset{\text{def}}{=}\bar{M}(u,s).$$

Let $\mathcal{B}$ be the variety of all Borel subgroup of G with the natural $G\times\boldsymbol{C}^*$ action $(\gamma,\alpha):B\to\gamma B\gamma^{-1}$. Let $\mathcal{B}_u^s$ be the variety of all Borel subgroups of G which contain u and s. Then $M(u,s)$ (a subgroup of $G\times\boldsymbol{C}^*$) leaves $\mathcal{B}_u^s$ stable and induces an action of $\bar{M}(u,s)$ on the étale cohomology of $\mathcal{B}_u^s$. Let $\rho_0\bar{M}(u,s)$ be the set of isomorphism classes of irreducible representations of the finite group $\bar{M}(u,s)$ which appear in the representation of $\bar{M}(u,s)$ on the total étale cohomology of $\mathcal{B}_u^s$.

2.3. Now let u be a unipotent element of G_K and let s be a semisimple element of G_K such that $sus^{-1}=u^\beta$ for some $\beta\in K^*$. We define a K-algebraic group

$$M^{(K)}(u,s)=\{(\gamma,\alpha)\in G_K\times K^*|\gamma u\gamma^{-1}=u^\alpha,\ \gamma s=s\gamma\};$$

let $\bar{M}^{(K)}(u,s)$ be the group of components of $M^{(K)}(u,s)$.

Let $(\mathcal{B}_K)_u^s$ be the variety of Borel subgroups of G_K which contain u and s. As in 2.2, $\bar{M}^{(K)}(u,s)$ acts naturally on the étale cohomology of $(\mathcal{B}_K)_u^s$ and in terms of this action we define $\rho_0\bar{M}^{(K)}(u,s)$ to be the set of isomorphism classes of irreducible representations of the finite group $\bar{M}^{(K)}(u, s)$ which appear in the representation of $\bar{M}^{(K)}(u, s)$ on the total étale cohomology of $(\mathcal{B}_K)_u^s$.

2.4. Given a semisimple conjugacy class C in G_K, we define a new conjugacy class $\hat{C}$ in G_K as follows. Let $s \in C$; then $Z_{G_K}(s)$ has a unique open orbit on the variety of unipotent elements $u' \in G_K$ such that $su's^{-1} = u'^{r^2}$. Choose an element u in this open orbit. We can find a homomorphism of K-algebraic groups $f: SL_2(K) \to G_K$ such that $f\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u$ and $(s, r^2) \in M_f^{(K)}$. Then s commutes with $f\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$. We set $s_1 = sf\begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix}$. Then s and u are in $Z_{G_K}(s_1)$ and we define $\hat{C}$ to be the G_K-conjugacy class of $s_1 u = u s_1$. One can check that $\hat{C}$ is independent of the choices made; it depends only on C.

2.5. Let M be a simple $\underline{H}_K$-module. It is necessarily of finite dimension over K. We attach to M a semisimple class C_M in G_K as follows. We can find a non-zero vector $\xi \in M$ and a homomorphism $\chi: X \to K^*$ such that $\tilde{T}_x \xi = \chi(x)\xi$ for all $x \in X^{++}$. (Note that the operators $\tilde{T}_x: M \to M$, $(x \in X^{++})$, commute with each other.) Then there is a unique element $s \in (T_0)_K$ such that $\chi(x) = x(s)$ for all $x \in X = \mathrm{Hom}(T_0, \boldsymbol{C}^*) = \mathrm{Hom}((T_0)_K, K^*)$. Let C_M be the conjugacy class of s in G_K. It is known that C_M is an invariant of M (it is independent of the choice of ξ).

(a) Let Y be the set of isomorphism classes of simple $\underline{H}_K$-modules M such that the conjugacy class $\hat{C}_M$ in G_K contains some $\boldsymbol{C}$-point of G.

2.6. Let $V: K^* \to \boldsymbol{R}$ be a homomorphism such that $V(r) > 0$. An element g of G_K (or its conjugacy class in G_K) is said to be V-tempered if all eigenvalues of $\mathrm{Ad}(g)$ on $\mathrm{Lie}(G_K)$ (or, equivalently, all eigenvalues of g in all finite dimensional rational G_K-modules) are in the kernel of V.

An H_K-module M of finite dimension over K is said to be V-tempered (resp. V-antitempered) if all the eigenvalues λ of $\tilde{T}_x: M \to M$ satisfy $V(\lambda) \leq 0$ (resp. $V(\lambda) \geq 0$) for all elements $x \in X^{++}$.

One of the main results of [4] is:

(a) A simple H_K-module is V-tempered if and only if $\hat{C}_M$ is a V-tempered conjugacy class in G_K.

(See [4, 7.12, 8.2]; these results are applicable since the field K is (non canonically) isomorphic to C.)

For any commutative $\underline{A}$-algebra B with unit, let $h \to h^*$ be the unique B-algebra automorphism of $\underline{H}_B$ such that

$$*\tilde{T}_s = -\tilde{T}_s^{-1}, \quad (s \in S'), \quad *\tilde{T}_x = \tilde{T}_x^{-1}, \quad (x \in X^{++}).$$

If M is an $\underline{H}_B$-module, then composition with $*$ gives a new $\underline{H}_B$-module $*M$. It is clear that

(b) $*$ interchanges V-tempered and V-antitempered simple $\underline{H}_K$-modules.

2.7. Let $^- : K \to K$ be any field automorphism inducing identity on C and satisfying $\bar{r} = r^{-1}$. Following [3], we extend $^- : K \to K$ to a ring homomorphism $^- : H_K \to H_K$ by $\bar{\tilde{T}}_w = \tilde{T}_{w^{-1}}^{-1}$, $(w \in W)$. For any C-variety Z, the map $^- : K \to K$ induces a natural bijection $^- : Z_K \to Z_K$. In particular, it induces a bijection $^- : G_K \to G_K$ which is the identity on the C-points of G.

2.8. LEMMA. *Let $V_0 : C(r)^* \to Z$ be the homomorphism defined by attaching to a rational function its order of vanishing at $r=0$. Let $\lambda \in K^*$ be an integral element over $C[r, r^{-1}]$ such that for any valuation $V : K^* \to R$ extending V_0, we have $V(\lambda) = V(\bar{\lambda}) = 0$, ($\bar{\lambda}$ as in* 2.7). *Then $\lambda \in C^*$.*

PROOF. We define V (as above) at $0 \in K$ by $V(0) = \infty$. Let $\lambda_1, \cdots, \lambda_n$ be the conjugates of λ under the Galois group of K over $C(r)$, and let ε_i be the i-th elementary symmetric function in $\lambda_1, \cdots, \lambda_n$. If V is as in the lemma we have $V(\lambda_1) = \cdots = V(\lambda_n) = 0$ hence $V(\varepsilon_i) \geq 0$, so that $\varepsilon_i \in C(r)$ has no pole at 0. Similarly, we have $V(\bar{\lambda}_1) = \cdots = V(\bar{\lambda}_n) = 0$ hence $V(\bar{\varepsilon}_i) \geq 0$ hence $\varepsilon_i \in C(r)$ has no pole at ∞. Since λ is integral over $C[r, r^{-1}]$, we see that ε_i is integral over $C[r, r^{-1}]$; this is integrally closed in $C(r)$, hence $\varepsilon_i \in C[r, r^{-1}]$. As ε_i has no poles at 0 and ∞, it is in C. Since $\varepsilon_i \in C$ for all i, we must have $\lambda_j \in C$ for all j and the lemma is proved.

2.9. PROPOSITION. *Let M be a simple $\underline{H}_K$-module. Assume that* (a), (b), (c) *below hold.*

(a) $\mathrm{Tr}_K(\tilde{T}_w, M) \in C[r, r^{-1}]$ *for all $w \in W$.*

(b) *M is V-tempered for any V as in Lemma* 2.8.

(c) *There exists a C-isomorphism $^- : M \to M$ such that $\overline{hm} = \bar{h}\bar{m}$ for all $h \in \underline{H}_K$, $m \in M$.*

Then $M \in Y$. (See 2.5 (a).)

PROOF. From (a), it follows that $\mathrm{Tr}_K(\tilde{T}_w^n, M) \in C[r, r^{-1}]$ for any $n \in Z$.

Hence all eigenvalues of $\tilde{T}_w$ and $\tilde{T}_w^{-1}: M \to M$ are integral over $C[r, r^{-1}]$ for any $w \in W$. Let $\xi \in M$ be a non-zero vector such that for some $s_0 \in (T_0)_K$, we have $\tilde{T}_x \xi = x(s_0)\xi$ for all $x \in X^{++}$. We then have also $\tilde{T}_x^{-1}\xi = x^{-1}(s_0)\xi$ for all $x \in X^{++}$. By the argument above, $x(s_0)$ and $x^{-1}(s_0)$ are integral over $C[r, r^{-1}]$ for all $x \in X^{++}$. Using 1.1 (b), we deduce that $x(s_0)$ is integral over $C[r, r^{-1}]$ for any $x \in X$.

By definition, we have $s_0 \in \mathcal{C}_M$. Starting with an element $s \in \mathcal{C}_M$, we define $u, f: SL_2(K) \to G_K$, s_1 as in 2.4; then $s_1 u \in \hat{\mathcal{C}}_M$. We can find an element $g \in G_K$ and a homomorphism of C-algebraic groups $f': SL_2(C) \to G$ such that $f'(A) = gf(A)g^{-1}$ for all $A \in SL_2(C)$. Hence, replacing s, u, f, s_1 by their conjugate under g, we can assume that f and u are defined over C.

We want to show that the G_K-conjugacy class of $s_1 u$ contains some C-point of G. For this, it is enough to show that for any rational G_K-module $\mathcal{V}$ of finite dimension over K, any eigenvalue $\lambda \in K^*$ of $s_1: \mathcal{V} \to \mathcal{V}$ is actually in C^*. We shall verify that λ satisfies the hypothesis of Lemma 2.8.

Since $x(s_0)$ is integral over $C[r, r^{-1}]$ for all $x \in X$, it follows that the eigenvalues of $s_0: \mathcal{V} \to \mathcal{V}$ are integral over $C[r, r^{-1}]$. Since s is conjugate to s_0 in G_K, the eigenvalues of $s: \mathcal{V} \to \mathcal{V}$ are integral over $C[r, r^{-1}]$. We have $s = s_1 f\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = f\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} s_1$ and the eigenvalues of $f\begin{pmatrix} r & \\ & r^{-1} \end{pmatrix}$ on $\mathcal{V}$ are clearly of form r^j $(j \in Z)$. Then λ is equal to r^{-j} times an eigenvalue of s on $\mathcal{V}$; hence λ is integral over $C[r, r^{-1}]$.

Now fix $V: K^* \to R$ as in Lemma 2.8. Since M is V-tempered we see from 2.6 (a) that $s_1 u$ is V-tempered, hence s_1 is V-tempered, hence $V(\lambda) = 0$.

To check that $V(\bar{\lambda}) = 0$ it will be enough to check that $\bar{\lambda}$ is an eigenvalue of $s_1: \mathcal{V} \to \mathcal{V}$.

For any $x \in X^{++}$, we have $\bar{\tilde{T}}_x = \tilde{T}_{w_0} \tilde{T}^{-1}_{w_0 x^{-1} w_0^{-1}} \tilde{T}^{-1}_{w_0}$. (This is equivalent to $\tilde{T}_{w_0} \tilde{T}_x = \tilde{T}_{w_0 x w_0^{-1}} \tilde{T}_{w_0}$; both expressions are equal to $\tilde{T}_{w_0 x}$, see 1.1 (a).)

If ξ, s_0 are as in the beginning of the proof, we set $\xi' = \tilde{T}^{-1}_{w_0} \bar{\xi}$; using (c), we have

$$\begin{aligned} \tilde{T}_x \xi' &= \tilde{T}_x \tilde{T}^{-1}_{w_0} \bar{\xi} = \overline{\bar{\tilde{T}}_x \tilde{T}_{w_0} \xi} = \overline{\tilde{T}_{w_0} \tilde{T}^{-1}_{w_0 x^{-1} w_0^{-1}} \xi} \\ &= \overline{(w_0 x^{-1} w_0^{-1})(s_0)^{-1} \tilde{T}_{w_0} \xi} = \overline{x(w_0(s_0))}\, \tilde{T}^{-1}_{w_0} \bar{\xi} = x(w_0(\bar{s}_0))\xi' \end{aligned}$$

for all $x \in X^{++}$. It follows that $w_0(\bar{s}_0) \in (T_0)_K$ is in $\mathcal{C}_M$. (See 2.5.) Let $\sigma: G \to G$ be the unique automorphism of C-algebraic groups such that

$\sigma B_0=B_0$, $\sigma T_0=T_0$ and $\sigma(t)=w_0(t^{-1})$ for all $t\in T_0$. Then σ extends to an automorphism $\sigma: G_K\to G_K$ which commutes with $^{-}: G_K\to G_K$. (See 2.7.) We have $\sigma(\bar{s}_0^{-1})\in\hat{\mathcal{C}}_M$. Since s_0, s are G_K-conjugate, we have also $\sigma(\bar{s}^{-1})\in\hat{\mathcal{C}}_M$.

We now apply the definition 2.4 of $\hat{\mathcal{C}}_M$ starting from $\sigma(\bar{s}^{-1})$ instead of s. Since u is in the open $Z_{G_K}(s)$-orbit on the set of unipotents $u'\in G_K$ satisfying $su's^{-1}=u'^{r^2}$, we see that the element $\sigma(\bar{u})=\sigma(u)$ is in the open $Z_{G_K}(\sigma(\bar{s}^{-1}))$-orbit on the set of unipotents $u''\in G_K$ satisfying $\sigma(\bar{s}^{-1})u''\sigma(\bar{s}^{-1})^{-1}=u''^{r^2}$.

Let ${}^{\sigma}f=\sigma\circ f$. Since f is defined over $\mathbf{C}$, $f\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=u$, and $(s, r^2)\in(M_f)_K$, we see that ${}^{\sigma}f$ is defined over $\mathbf{C}$, ${}^{\sigma}f\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\sigma(u)$ and $\sigma(\bar{s}^{-1})\in(M_{{}^{\sigma}f})_K$. We have $\sigma(\bar{s}^{-1})\,{}^{\sigma}f\begin{pmatrix}r^{-1} & 0\\ 0 & r\end{pmatrix}=\sigma\left(\overline{\left(sf\begin{pmatrix}r^{-1} & 0\\ 0 & r\end{pmatrix}\right)^{-1}}\right)=\sigma(\bar{s}_1^{-1})$. Hence the definition 2.4 of $\hat{\mathcal{C}}_M$ starting from $\sigma(\bar{s}^{-1})$ instead of s leads to $\sigma(\bar{s}_1^{-1})\sigma(u)$ instead of s_1u.

Since s_1u, $\sigma(\bar{s}_1^{-1})\sigma(u)$ are in the same G_K-conjugacy class $\hat{\mathcal{C}}_M$, it follows that s_1, $\sigma(\bar{s}_1^{-1})$ are conjugate in G_K. Since any semisimple element $t\in G_K$ is G_K-conjugate to $\sigma(t^{-1})$, it follows that $s_1, \bar{s}_1$ are G_K-conjugate. We can find a $\mathbf{C}$-linear isomorphism $^{-}: \mathcal{CV}\to\mathcal{CV}$ such that $\overline{gy}=\bar{g}\bar{y}$ for all $g\in G_K$, $y\in\mathcal{CV}$, and $\overline{\xi y}=\bar{\xi}\bar{y}$ for all $\xi\in K$, $y\in\mathcal{CV}$. It follows that the eigenvalues of $\bar{s}_1:\mathcal{CV}\to\mathcal{CV}$ are obtained by applying $^{-}: K\to K$ to the eigenvalues of $s_1:\mathcal{CV}\to\mathcal{CV}$. Since λ is an eigenvalue of $s_1:\mathcal{CV}\to\mathcal{CV}$, it follows that $\bar{\lambda}$ is an eigenvalue of $\bar{s}_1:\mathcal{CV}\to\mathcal{CV}$. Since $s_1, \bar{s}_1$ are G_K-conjugate, $\bar{\lambda}$ is also an eigenvalue of $s_1:\mathcal{CV}\to\mathcal{CV}$. Since s_1 is V-tempered we have $V(\bar{\lambda})=0$.

Thus, we have verified that λ satisfies the hypothesis of Lemma 2.8; it follows that $\lambda\in\mathbf{C}^*$. Hence the G_K-conjugacy class of s_1u contains some $\mathbf{C}$-point of G. The proposition is proved.

2.10. Let E be a simple $\underline{J}$-module and let $\underline{B}$ be an $\underline{A}$-algebra with 1. Then $E_B=E\underset{\mathbf{C}}{\otimes}B$ is naturally a $\underline{J}_B$-module and it can be also regarded as an $\underline{H}_B$-module, via $\phi_B:\underline{H}_B\to\underline{J}_B$. Hence, for $B=K$, ${}^*E_K$ (see 2.6) is again an $\underline{H}_K$-module.

2.11. PROPOSITION.

(a) *If E is a simple $\underline{J}$-module, then ${}^*E_K$ is a simple $\underline{H}_K$-module in Y.* (*See* 2.5 (a).)

(b) *If E' is another simple $\underline{J}$-module and E'_K is isomorphic to E_K as an $\underline{H}_K$-module then E' is isomorphic to E as a $\underline{J}$-module.*

PROOF. (a) From 1.3 (d) and the fact that E is simple it follows that there exists $i \in N$ such that $t_w E \neq 0 \Longrightarrow a(w)=i$. By Burnside's theorem, the endomorphisms $t_w: E \to E$, $(a(w)=i)$, span the C-vector space $\mathrm{End}_C(E)$. Hence we can find $w_1, \cdots, w_r \in W$, $r=\dim_C(E)^2$, $a(w_1)=\cdots=a(w_r)=i$, such that $t_{w_1}, \cdots, t_{w_r}: E \to E$ form a C-basis of $\mathrm{End}_C(E)$. From 1.3 (c), we see that $\phi_K((-r)^i C_{w_j}) = \sum_{\substack{z\in W\\ a(z)=i}} \pi_{w_j,z} t_z$ as endomorphisms of $E \underset{C}{\otimes} K$, $(1 \leq j \leq r)$; moreover $\pi_{w_j,z} \in C[r]$ has constant term 1 if $z=w_j$ and 0 if $z \neq w_j$. Expressing $t_z: E \to E$ as C-linear combination of $t_{w_k}: E \to E$, we see that $\phi_K((-r)^i C_{w_j}) = \sum_{1\leq k\leq r} \psi_{j,k} t_{w_k}$, as endomorphisms of E_K $(1 \leq j \leq r)$; moreover $\psi_{j,k} \in C[r]$ has constant term 1 if $j=k$ and 0 if $j \neq k$. In particular, $\det(\psi_{j,k}) \in C[r]$ has constant term 1, so it is $\neq 0$. Thus $\phi_K((-r)^i C_{w_j}): E_K \to E_K$ span $\mathrm{End}_K(E_K)$ as a K-vector space. Using again Burnside's theorem we see that the $\underline{H}_K$-module E_K is simple; hence the $\underline{H}_K$-module $*E_K$ is simple.

From 1.3 (c) we see that $\phi_K((-r)^i C_w) = \sum_{\substack{z\in W\\ a(z)=i}} \pi_{w,z} t_z$ as endomorphisms of $E \underset{C}{\otimes} K$, $(w \in W)$, $\pi_{w,z} \in C[r]$. Hence $\phi_K((-r)^i C_w)$ maps the $C[r]$-submodule $E \underset{C}{\otimes} C[r]$ of E_K into itself, $(w \in W)$. The same holds for $\phi_K((-r)^i \tilde{T}_w)$: $E_K \to E_K$, $(w \in W)$ since $\tilde{T}_w \in H$ is a finite $C[r]$-linear combination of elements $C_{w'}$. It follows that all the eigenvalues in K of $\phi_K((-r)^i \tilde{T}_w)$: $E_K \to E_K$ are integral over $C[r]$. We apply this to w a power x^n $(n \geq 1)$ of an element $x \in X^{++}$. It is known that $l(x^n)=nl(x)$ hence $\tilde{T}_{x^n} = \tilde{T}_x^n$. Thus, all the eigenvalues of $\phi((-r)^i \tilde{T}_x^n): E_K \to E_K$ are integral over $C[r]$. Hence if λ is an eigenvalue of $\phi(\tilde{T}_x): E_K \to E_K$ then $(-r)^i \lambda^n$ is integral over $C[r]$ for $n=1, 2, 3, \cdots$. This implies that λ is integral over $C[r]$. Hence if $V: K^* \to \boldsymbol{R}$ is as in Lemma 2.8, then $V(\lambda) \geq 0$. (Note that $\lambda \neq 0$ since $\tilde{T}_x$ is invertible in $\underline{H}$.) It follows that the $\underline{H}_K$-module E_K is V-antitempered (see 2.6) and therefore the $\underline{H}_K$-module $*E_K$ is V-tempered (see 2.6 (a)).

The $\underline{H}_K$-modules E_K, $*E_K$ come by extension of scalars from $\underline{A}$ to K from $\underline{H}$-modules on the underlying $\underline{A}$-modules E_A. Hence $\mathrm{Tr}_K(\tilde{T}_w, E_K)$ and $\mathrm{Tr}_K(\tilde{T}_w, *E_K)$ are in $\underline{A}$ for any $w \in W$.

Let $^-: E \underset{C}{\otimes} K \to E \underset{C}{\otimes} K$ be the map defined by $e \otimes \xi \to e \otimes \bar{\xi}$, $(e \in E, \xi \in K)$, where $^-: K \to K$ is as in 2.7. The elements C_w are known to be fixed by $^-: \underline{H}_K \to \underline{H}_K$. Hence the elements $h_{x,y,z} \in C[r, r^{-1}] \subset K$ (see 1.2) are fixed by $^-: K \to K$. From 1.3 (b) it now follows that

$$\overline{\phi(C_w)(e\otimes\xi)} = \overline{\sum_{\substack{d\in\mathcal{D}\\ z\in W\\ a(d)=a(z)}} (t_z e) \otimes h_{w,d,z}\xi}$$

$$= \sum_{\substack{d \in \mathcal{D} \\ z \in W \\ a(d)=a(z)}} (t_z e) \otimes h_{w,d,z} \bar{\xi} = \phi(C_w) \overline{(e \otimes \xi)} = \phi(\bar{C}_w) \overline{(e \otimes \xi)},$$

$(w \in W, e \in E, \xi \in K)$, hence $\overline{hm} = \bar{h}\bar{m}$ for all $h \in \underline{H}_K$, $m \in E_K$. We then have also $\overline{(^*h)m} = {}^*(\bar{h})\bar{m}$, $(h \in \underline{H}_K, m \in E_K)$, since $\overline{(^*h)} = {}^*(\bar{h})$ for all $h \in \underline{H}_K$.

Thus we have verified that the $\underline{H}_K$-module ${}^*E_K$ satisfies the hypothesis of 2.9. It follows that ${}^*E_K \in Y$ and (a) is proved.

Now let E' be another simple $\underline{J}$-module such that E'_K, E_K are isomorphic as $\underline{H}_K$-modules.

It follows that $E'_{C(r)}$, $E_{C(r)}$ are isomorphic as $\underline{H}_{C(r)}$-modules, hence $E'_{C((r))}$, $E_{C((r))}$ are isomorphic as $\underline{H}_{C((r))}$-modules.

We must show that $E' \cong E$ as $\underline{J}$-modules.

As in the beginning of the proof, we can find $i, i' \in N$ such that $t_w E \neq 0 \Longrightarrow a(w) = i$ and $t_w E' \neq 0 \Longrightarrow a(w) = i'$. We may assume that $i' \leq i$.

Let $\alpha : E_{C((r))} \to E'_{C((r))}$ be an isomorphism of $\underline{H}_{C((r))}$-modules. Replacing if necessary α by $r^k \alpha$, $(k \in Z)$, we may assume that α maps $\mathcal{E} = E \otimes_C C[[r]]$ into $\mathcal{E}' = E' \otimes_C C[[r]]$ but not into $r\mathcal{E}'$.

Hence, if we identify in the natural way $\mathcal{E}/r\mathcal{E} \cong E$, $\mathcal{E}'/r\mathcal{E}' \cong E'$, we see that there exists a C-linear map $\tilde{\alpha} : E \to E'$ such that

(c) $$\alpha(e \otimes \xi) = \tilde{\alpha} e \otimes \xi \pmod{r\mathcal{E}'}$$

for any $e \in E$, $\xi \in C[[r]]$.

Let $w \in W$ be such that $a(w) = i$. Since $i' \leq i$, we see from 1.3 (c) and the definition of i, i' that:

$\phi_{C(r)}((-r)^i C_w)$ maps $\mathcal{E}$ into $\mathcal{E}$ and $\mathcal{E}'$ into $\mathcal{E}'$. Moreover, it commutes with $\alpha : \mathcal{E} \to \mathcal{E}'$. We shall express this commutation using 1.3 (c):

$$\alpha\Big(\sum_{\substack{z \in W \\ a(z)=i}} t_z(e) \otimes \pi_{w,z} \xi\Big) = \Big(\sum_{\substack{z \in W \\ a(z)=i}} \pi_{w,z} t_z\Big) \alpha(e \otimes \xi)$$

for all $e \in E, \xi \in C[[r]]$. Using (c), this can be written as follows

$$\sum_{\substack{z \in W \\ a(z)=i}} \tilde{\alpha}(t_z(e)) \otimes \pi_{w,z} \xi = \sum_{\substack{z \in W \\ a(z)=i}} t_z(\tilde{\alpha}(e)) \otimes \pi_{w,z} \xi + \text{element in } r\mathcal{E}'.$$

Here $\pi_{w,z} \in C[r]$ has constant term 1 if $w = z$ and 0 if $w \neq z$; hence the previous equality implies

$$\tilde{\alpha}(t_w(e)) \otimes \xi = t_w(\tilde{\alpha}(e)) \otimes \xi + \text{element in } r\mathcal{E}'$$

hence

$$\tilde{\alpha}(t_w(e))=t_w(\tilde{\alpha}(e)) \quad \text{for all } e\in E,\ w\in W,\ a(w)=i.$$

Since $t_w: E\to E$, $(a(w)=i)$ span $\mathrm{End}_C(E)$, and $\tilde{\alpha}\neq 0$, we can find $e\in E$, $w\in W$ $(a(w)=i)$ such that $\tilde{\alpha}(t_w(e))\neq 0$. For that e, w we have $t_w(\tilde{\alpha}(e))\neq 0$, hence $t_wE'\neq 0$ hence $i=i'$ by the definition of i'.

From the definition of i, i' and the equality $i=i'$, we see that the equality $\tilde{\alpha}(t_w(e))=t_w(\tilde{\alpha}(e))$, $(e\in E)$, holds also if $a(w)\neq i$. Hence it holds for all $w\in W$. Hence $\tilde{\alpha}$ is a $\underline{J}$-module homomorphism. Since E, E' are simple and $\tilde{\alpha}\neq 0$, we see that $\tilde{\alpha}$ must be an isomorphism. The proposition is proved.

§ 3. The main results

3.1. Fix $r_0\in \boldsymbol{C}^*$. To any simple $\underline{H}_{r_0}$-module M, we can attach a semisimple element $s\in G$ (a $\boldsymbol{C}$-point of G) well defined up to conjugacy, by the procedure of 2.5: we can find a non-zero vector $\xi\in M$ and a homomorphism $\chi: X\to \boldsymbol{C}^*$ such that $\tilde{T}_x\xi=\chi(x)\xi$ for all $x\in X^{++}$; then there is a unique element $s\in T_0$ such that $\chi(x)=x(s)$ for all $x\in X=\mathrm{Hom}(T_0, \boldsymbol{C}^*)$. This is the required semisimple element. We then have a canonical decomposition

$$K(\underline{H}_{r_0})=\bigoplus_s K(\underline{H}_{r_0})_s$$

indexed by a set of representatives for the semisimple classes in G, where $K(\underline{H}_{r_0})_s$ is spanned by the simple $\underline{H}_{r_0}$-modules M such that the semisimple class associated above to M is that of s.

Then $K(\underline{H}_{r_0})_s$ is a free abelian group of finite rank.

If r_0 is not a root of 1, then

(a) $$\mathrm{rank}\ K(\underline{H}_{r_0})_s=\sum_u \sharp(\rho_0\overline{M}(u, s))$$

where u runs over a set of representatives for the $Z(s)$-orbits on the set of unipotent elements $u'\in G$ such that $su's^{-1}=u'^{r_0^2}$ and $\rho_0\overline{M}(u, s)$ is as in 2.2. This follows from [4].

The equality (a) remains true for $r_0=1$, by the Springer correspondence [9] for the irreducible representations of the Weyl group of $Z(s)$.

3.2. We now attach to each simple $\underline{H}_K$-module $M\in Y$ (see 2.5 (a)) a semisimple element $s\in G$ (defined up to conjugacy). The definition of s is given in terms of a fixed $r_0\in \boldsymbol{C}^*$. Let s_1 be a semisimple element of

G, u a unipotent element of $Z(s_1)$ (both defined over C) such that $s_1u\in\hat{C}_M$. Choose a homomorphism of C-algebraic groups $f: SL_2(C)\to Z(s_1)$ such that $f\begin{pmatrix}1&1\\0&1\end{pmatrix}=u$ and define $s=s_1f\begin{pmatrix}r_0&0\\0&r_0^{-1}\end{pmatrix}$. The conjugacy class of s in G depends only on M. The fibres of the map $M\to s$ will be denoted Y_s. Thus, we have a partition $Y=\coprod_s Y_s$ indexed by a set of representatives for the semisimple conjugacy classes in G.

From the results of [4] applied to $\underline{H}_K$ we see that

(a) $$\sharp(Y_s)=\sum_f \rho_0\bar{M}^{(K)}\left(f\begin{pmatrix}1&1\\0&1\end{pmatrix}, sf_K\begin{pmatrix}rr_0^{-1}&0\\&r^{-1}r_0\end{pmatrix}\right)$$

where f runs over a set of representatives for the $Z(s)$-conjugacy classes of homomorphisms of C-algebraic groups $f: SL_2(C)\to G$ such that $(s,r_0)\in M_f$ and $f_K: SL_2(K)\to G_K$ is the natural extension of f. (See 2.2, 2.3.)

For any such f, we denote

$$u=f\begin{pmatrix}1&1\\0&1\end{pmatrix},\ s_1=sf\begin{pmatrix}r_0^{-1}&0\\0&r_0\end{pmatrix};\ \text{then } M^{(K)}\left(u, s_1f_K\begin{pmatrix}r&0\\0&r^{-1}\end{pmatrix}\right)$$
$$=\left\{(\gamma,\alpha)\in G_K\times K^*|\gamma u\gamma^{-1}=u^\alpha, \gamma s_1=s_1\gamma, \gamma f_K\begin{pmatrix}r&0\\0&r^{-1}\end{pmatrix}=f_K\begin{pmatrix}r&0\\0&r^{-1}\end{pmatrix}\gamma\right\}$$
$$=\left\{(\gamma,\alpha)\in G_K\times K^*|\gamma u\gamma^{-1}=u^\alpha, \gamma s=s\gamma, \gamma f_K\begin{pmatrix}r&0\\0&r^{-1}\end{pmatrix}=f_K\begin{pmatrix}r&0\\0&r^{-1}\end{pmatrix}\gamma\right\}=M_f(s)_K,$$

(see 2.2). Hence $\bar{M}^{(K)}\left(u, s_1f_K\begin{pmatrix}r&0\\0&r^{-1}\end{pmatrix}\right)$ may be identified with the group of components of $M_f(s)$, hence with $\bar{M}(u,s)$, (see 2.2 (c)). Moreover the variety $(\mathcal{B}_K)_u^s$ is obtained by extension of scalars from $\mathcal{B}_u^s$; hence these two varieties (one over K, one over C) have the same étale cohomology; it follows that $\rho_0\bar{M}^{(K)}\left(u, s_1f_K\begin{pmatrix}r&0\\0&r^{-1}\end{pmatrix}\right)\cong\rho_0\bar{M}(u,s)$. Hence

$$\sharp(Y_s)=\sum_f \rho_0\bar{M}\left(f\begin{pmatrix}1&1\\0&1\end{pmatrix}, s\right)$$

where f is as in (a). It follows that

(b) $$\sharp(Y_s)=\text{right hand side of 3.1 (a).}$$

3.3. Let Y' be the set of isomorphism classes of $\underline{H}_{C(r)}$-modules M' with the properties (a), (b) below.

(a) $M'\underset{C(r)}{\otimes}K$ is a simple $\underline{H}_K$-module in Y (see 2.5 (a)).

(b) There exists an $\underline{H}$-submodule M_1' of M such that M_1' is of finite A-rank equal to $\dim_{C(r)}M'$.

Let Y'' be the set of isomorphism classes of simple $\underline{J}$-modules. We have natural maps

$$Y'' \xrightarrow{\alpha} Y' \xrightarrow{\beta} Y$$

defined as follows. If E is a simple $\underline{J}$-module, then $E_{C(r)} = E \underset{C}{\otimes} C(r)$ may be regarded as an $\underline{H}_{C(r)}$-module, via $\phi_{C(r)} : \underline{H}_{C(r)} \to \underline{J}_{C(r)}$; by 2.11, we have $^*E_{C(r)} \in Y'$ and we define $\alpha E = {}^*E_{C(r)}$. If M' is an $\underline{H}_{C(r)}$-module in Y', then $M' \underset{C(r)}{\otimes} K$ is in Y and we define $\beta(M') = M' \underset{C(r)}{\otimes} K$.

Let $Z[Y'']$, $Z[Y']$, $Z[Y]$ be the free abelian groups with bases Y'', Y', Y respectively. For fixed $r_0 \in C^*$, we define homomorphisms

$$\begin{array}{ccccc} Z[Y''] & \xrightarrow{\bar{\alpha}} & Z[Y'] & \xrightarrow{\bar{\beta}} & Z[Y] \\ & \searrow^{\varepsilon} & \downarrow{\scriptstyle\delta} & & \\ & & K(\underline{H}_{r_0}) & & \end{array}$$

as follows: $\bar{\alpha}, \bar{\beta}$ are induced by α, β. If E is a simple $\underline{J}$-module we can regard E as an $\underline{H}_{r_0}$-module E_{r_0} via $\phi_{r_0} : \underline{H}_{r_0} \to \underline{J}$ and we define $\varepsilon(E) = {}^*E_{r_0}$. If $M' \in Y'$, we choose M_1' as in (b), and we let $\delta(M')$ be the class of the $\underline{H}_{r_0}$-module $M_1' \underset{A}{\otimes} C$ where C is regarded as an $\underline{A}$-module with r acting as multiplication by r_0. (A standard argument shows that $\delta(M')$ is well defined, i.e., independent of the choice of M_1'.) It is clear that $\varepsilon = \delta \circ \bar{\alpha}$.

3.4. THEOREM. *Assume that r_0 is not a root of 1, or that $r_0 = 1$. Then the maps $\alpha, \beta, \bar{\alpha}, \bar{\beta}, \varepsilon, \delta$ (in 3.3) are isomorphisms.*

PROOF. From 1.9 we see that ε is surjective. Since $\varepsilon = \delta \circ \bar{\alpha}$, it follows that δ is surjective.

The map β is clearly injective. Let $Y_s' = \beta^{-1}(Y_s) \cap Y'$ (s semisimple class in G), where Y_s is as in 3.2. Then we have a direct sum decomposition $Z[Y'] = \bigoplus_s Z[Y_s']$. From the definitions it follows easily that δ (resp. $\bar{\beta}$) defines by restriction a homomorphism $\delta_s : Z[Y_s'] \to K(\underline{H}_{r_0})_s$, (see 3.1) resp. $\bar{\beta}_s : Z[Y_s'] \to Z[Y_s]$. Moreover, each δ_s is surjective, since δ is surjective. Also $\bar{\beta}_s$ is clearly injective. In the diagram

$$Z[Y_s] \xleftarrow{\bar{\beta}_s} Z[Y'_s] \xrightarrow{\delta_s} K(\underline{H}_{r_0}).$$

with $\bar{\beta}_s$ injective, δ_s surjective, the free abelian groups $Z[Y_s]$, $K(\underline{H}_{r_0})_s$ have the same rank. (See 3.1 (a), 3.2 (b).) It follows that δ_s is an isomorphism and $\bar{\beta}_s$ has image of finite index. Since $\bar{\beta}_s$ is induced by an imbedding $Y'_s \to Y_s$, it follows that $Y'_s \to Y_s$ is bijective. Hence $\beta: Y' \to Y$ is bijective. Since δ_s is an isomorphism for all s, we see that δ is an isomorphism.

From 2.11 (b) it follows that $\beta \circ \alpha$ is injective, hence α is injective, hence $\bar{\alpha}$ is injective. Since $\varepsilon = \delta \circ \bar{\alpha}$ with ε surjective, $\bar{\alpha}$ injective and δ bijective, it follows that ε and $\bar{\alpha}$ are both bijective. It also follows that α is bijective. The theorem is proved.

3.5. Let r_0 be as in 3.4. For any simple $\underline{J}$-module E we define $a = a_E \in N$ by the requirement that $\underline{J}^a E \neq 0$ (see 1.3 (d)). For any simple $\underline{H}_{r_0}$-module M we define $a = a_M$ as in the proof of 1.9; we define a $\underline{J}$-module $\tilde{M}_J$ and an H_{r_0}-module $\tilde{M}$, both with underlying C-vector space $\underline{H}^a_{r_0} \otimes_{H_{r_0}} M$ as in the proof of 1.9.

3.6. COROLLARY. *In the setup of* 3.5, *the* $\underline{J}$-*module* $\tilde{M}_J$ *is simple, for any simple* $\underline{H}_{r_0}$-*module* M; *we have* $a_M = a_{\tilde{M}_J}$. *The* $\underline{H}_{r_0}$-*module* $\tilde{M}$ *has* M *as a quotient and all its other simple constituents* M' *satisfy* $a_{M'} > a_M$. *The rule* $M \to \tilde{M}_J$ *defines a* 1-1 *correspondence between the set* Z *of simple* $\underline{H}_{r_0}$-*modules* (*up to isomorphism*) *and the set* Y'' *of simple* $\underline{J}$-*modules* (*up to isomorphism*).

PROOF. Let Y''_a (resp. Z_a) be the set of simple modules $E \in Y''$ (resp. $M \in Z$) such that $a_E = a$ (resp. $a_M = a$). For any $\underline{J}$-module E, we denote by $(\phi_{r_0})_* E$ the $\underline{H}_{r_0}$-module obtained from E, via $\phi_{r_0}: \underline{H}_{r_0} \to \underline{J}$.

(a) If $E \in Y''_a$, then all simple constituents of $(\phi_{r_0})_* E$ are in $\coprod_{a' \leq a} Z_{a'}$.

Indeed, we have $(\phi_{r_0})_* E = E$ as C-vector spaces. Assume that $C_w E \neq 0$ ($C_w \in \bar{H}_{r_0}$, $a(w) > a$). Then $\sum_{\substack{z \in W \\ d \in \mathscr{D} \\ a(z) = a(d) = a}} h_{w,d,z}|_{r=r_0} t_z E \neq 0$. It follows that there exist $y, z \in W$, $a(z)=a$ such that $h_{w,y,z} \neq 0$. It is known that $h_{w,y,z} \neq 0$ implies $a(z) \geq a(w)$. We get $a > a$, contradiction. This proves (a).

(b) If $M \in Z_a$, then all simple constituents of $\tilde{M}_J$ are in Y''_a.

Indeed, from the definition of $\tilde{M}_J$, it is clear that $t_w \tilde{M}_J \neq 0 \Longrightarrow a(w)=a$, and (b) follows.

Let $\varepsilon_1 : Z[Y''] \to Z[Z]$ be the homomorphism of group rings defined by associating to $E \in Y''$ the sum of simple constituents of $(\phi_{r_0})_* E$, with multiplicities. From 3.4, it follows that

(c) ε_1 is an isomorphism.

Note that ε_1 is compatible with the filtrations $\bigoplus_{\substack{a' \\ a' \leq a}} Z[Y''_{a'}]$, $\bigoplus_{\substack{a' \\ a' \leq a}} Z[Z_{a'}]$ of $Z[Y'']$, $Z[Z]$ (see (a)) hence it induces an isomorphism on the associated graded groups: $Z[Y''_a] \xrightarrow{\sim} Z[Z_a]$ for all a. In particular, it follows that

(d) If $E \in Y''_a$, then some simple constituent of $(\phi_{r_0})_* E$ is in Z_a.

Assume that for some $M \in Z_a$, $\tilde{M}_J$ is not simple. From (b) and (d) it then follows that $(\phi_{r_0})_* \tilde{M}_J$ has at least two simple constituents in Z_a. But, as shown in the proof of 1.9, $(\phi_{r_0})_* \tilde{M}_J = \tilde{M}$ has exactly one simple constituent in Z_a. Thus, $\tilde{M}_J$ is simple for all $M \in Z$. The second assertion of the corollary is proved in 1.9. It implies that the map $Z \to Y''$ defined by $M \to \tilde{M}_J$ is injective. It also implies that the restriction of ε_1 to $Z[Y''_0]$, (where Y''_0 is the image of $Z \to Y''$) is an isomorphism $Z[Y''_0] \xrightarrow{\approx} Z[Z]$. From (c) it now follows that $Y''_0 = Y''$ hence $Z \to Y''$ is bijective. This completes the proof.

3.7. Let M be a simple H_K-module such that ${}^*M \in Y$ (see 2.5 (a)). We define a canonical direct sum decomposition $M = \bigoplus_{d \in \mathscr{D}} M_d$ into K-linear subspaces, such that the action of the elements $C_s \in H_K$, $(s \in S_1)$, on M is given in terms of this decomposition by a particularly simple formula.

By 3.4 we can find a simple $\underline{J}$-module E such that $M \cong E \otimes_C K$ where $E \otimes_C K$ is regarded as a $\underline{H}_K$-module via $\phi_K : \underline{H}_K \to J_K$. We have a direct sum decomposition $E = \bigoplus_{d \in \mathscr{D}} E_d$, $E_d = t_d E$. We define $M_d = E_d \otimes_C K$; then $M = \bigoplus_{d \in \mathscr{D}} M_d$.

For any $y, w \in W$, let $\mu(y, w) \in N$ be defined by $\mu(y, w)$=coefficient of $r^{l(w)-l(y)-1}$ in $P_{y,w}(r^2)$, if $y < w$, $\mu(y, w) = \mu(w, y)$, if $w < y$, $\mu(y, w) = 0$, otherwise. (See [3].) For $y, w \in W$, we shall write $y \underset{L}{\sim} w$ whenever y, w are in the same left cell of W.

Let $\delta', \delta \in \mathcal{D}$ be such that, for some $s \in S_1$, we have $s\delta > \delta$, $s\delta' < \delta'$. We define a K-linear map $f_{\delta',\delta}: M_\delta \to M_{\delta'}$ by

$$f_{\delta',\delta}(m) = \sum_{\substack{z \in W \\ z \underset{L}{\sim} \delta \\ z^{-1} \underset{L}{\sim} \delta'}} \mu(z, \delta) t_z m.$$

(Note that for z in the sum we have $t_z m = t_{\delta'} t_z m$, since $z^{-1} \underset{L}{\sim} \delta'$, hence $t_z m \in M_{\delta'}$; by our assumption on δ', δ the sum is finite. It is independent of s.)

3.8. THEOREM. *In the setup of* 3.7, *for any* $s \in S_1$, *the action of* $C_s \in \underline{H}_K$ *on* M *is given by*

$$C_s m = \begin{cases} -(r+r^{-1})m, & \text{if } m \in M_\delta,\ s\delta < \delta \\ \sum_{\substack{\delta' \in \mathcal{D} \\ s\delta' < \delta'}} f_{\delta',\delta}(m), & \text{if } m \in M_\delta,\ s\delta > \delta. \end{cases}$$

PROOF. We have for any $m \in M_\delta$,

$$C_s m = \sum_{\substack{d \in \mathcal{D} \\ z \in W \\ z \underset{L}{\sim} d}} h_{s,d,z} t_z m.$$

For any z in the sum we have $t_z m = t_z t_\delta m$; this is zero unless $z \underset{L}{\sim} \delta$, hence in our sum we can restrict ourselves to those z such that $d \underset{L}{\sim} z \underset{L}{\sim} \delta$; but it is known that two elements of $\mathcal{D}$ are in the same left cell only if they are equal, hence:

$$C_s m = \sum_{\substack{z \in W \\ z \underset{L}{\sim} \delta}} h_{s,\delta,z} t_z m.$$

If $s\delta < \delta$, we have $C_s C_\delta = -(r+r^{-1}) C_\delta$ hence $h_{s,\delta,z} = \begin{cases} -(r+r^{-1}) & \text{if } z = \delta \\ 0 & \text{if } z \neq \delta \end{cases}$ and $C_s m = -(r+r^{-1}) t_\delta m = -(r+r^{-1}) m$. Assume now that $s\delta > \delta$. Then $h_{s,\delta,z} = \begin{cases} \mu(z, \delta) & \text{if } sz < z \\ 0 & \text{if } sz > z \end{cases}$. (See [3].) Hence

$$C_s m = \sum_{\substack{z \in W \\ sz < z \\ z \underset{L}{\sim} \delta}} \mu(z, \delta) t_z m.$$

For any z in the sum there is a unique $\delta' \in \mathcal{D}$ such that $z^{-1} \underset{L}{\sim} \delta'$; we have necessarily $s\delta'<\delta'$. Conversely, if for some z, δ', we have $z^{-1}\underset{L}{\sim}\delta', s\delta'<\delta'$, then $sz<z$. Hence

$$C_s m = \sum_{\substack{\delta'\in\mathcal{D}\\ s\delta'<\delta'}} \sum_{\substack{z\in W\\ z\underset{L}{\sim}\delta\\ z^{-1}\underset{L}{\sim}\delta'}} \mu(z,\delta)t_z m = \sum_{\substack{\delta'\in\mathcal{D}\\ s\delta'<\delta'}} f_{\delta',\delta}(m)$$

and the theorem is proved.

3.9. REMARKS. 1. We can interpret 3.8, as saying that the $\underline{H}_K$-module M admits a W-graph in the sense of [3]. An analogous result for (finite) Weyl groups is proved in [1], using [5].

2. There is a two sided cell $\underline{c}$ of W (depending on M) such that $M_\delta\neq 0 \Longrightarrow \delta\in\underline{c}$.

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