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Examples of non-smoothable actions on the interval

Dedicated to Professor Itiro Tamura on his 60th birthday

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§ 1. Introduction

In this paper we give several examples of the actions on the interval of class C^r of a finitely presented group Γ which are not topologically conjugate to actions of class C^{r+1}, where r is a non-negative integer.

The group Γ is generated by two elements a and b subjected to the relation $[a, [a, b]]=1$. An action on the interval of class C^r of Γ is a homomorphism $\Gamma \longrightarrow D^r$, where D^r denotes $\mathrm{Diff}^r_+([0,1])$, the group of orientation preserving diffeomorphisms of class C^r of the interval $[0,1]$.

For a free abelian group A of finite rank, by using a result of D. Pixton [6], we see that every homomorphism $A \longrightarrow D^0$ is topologically conjugate to a homomorphism $A \longrightarrow D^1$. By a result of N. Kopell [5], there are homomorphisms $A \longrightarrow D^1$ which are not topologically conjugate to homomorphisms $A \longrightarrow D^2$, however, any homomorphism $A \longrightarrow D^2$ is topologically conjugate to a homomorphism $A \longrightarrow D^\infty$. The same phenomenon has been well known as the existence of the Denjoy flows in class C^1.

The question of the smoothability of homomorphisms to the group of diffeomorphisms of the circle or of the interval was posed to the author by T. Natsume. After the author gave examples of homomorphisms $\Gamma \longrightarrow D^0$ which are not conjugate to homomorphisms $\Gamma \longrightarrow D^1$, E. Ghys showed him the preprint [1] of J. Cantwell and L. Conlon, where they gave examples of foliations of class C^r which are not topologically conjugate to foliations of class C^{r+1}. A little later, the author recognized that his construction gives homomorphisms $\Gamma \longrightarrow D^r$ which are not topologically conjugate to $\Gamma \longrightarrow D^{r+1}$. By the suspension of the homomorphisms $\Gamma \longrightarrow D^r$, the author's examples give rise to the examples of foliations of class C^r which are not topologically conjugate to foliations of class C^{r+1}. These examples should have essentially the same roots as the examples in [1], because the criterion for the differentiability in [1] is similar to Lemma 2 in this paper. The construction of [1] is more delicate and they also obtained a nice result on the totally proper foliations. The examples in this paper may be of interest because the construction is simple and the group Γ is solvable.

§ 2. Two lemmas

We give two lemmas which serve as criteria for a homeomorphism of an interval to be of class C^r.

LEMMA 1. *Let f and g be C^1-diffeomorphisms of the interval* $[0, 1]$ *satisfying the following three conditions.*

i) $f(x)>x$ $(x\in(0,1))$.

ii) $g|(0,1)$ *has a unique fixed point c such that* $g(x)<x$ $(x\in(0,c))$ *and* $g(x)>x$ $(x\in(c,1))$.

iii) $gfg^{-1}=f^n$ *for an integer n greater than* 1.

Then there exist an integer j $(0\leq j<n)$ and a point $x\in(f^j(c),f^{j+1}(c))$ such that $(d/dx)(gf^{-j})(x)\geq n$.

PROOF. First note that $(gf^{-j})([f^j(c),f^{j+1}(c)])=[c,f^n(c)]$. Since the union of $[f^j(c),f^{j+1}(c)]$ $(0\leq j<n)$ coincides with $[c,f^n(c)]$, there is an integer j such that the ratio of the length of $[c,f^n(c)]$ and the length of $[f^j(c),f^{j+1}(c)]$ is not smaller than n. Hence the lemma follows from the mean value theorem.

LEMMA 2. *Let $0=x_0<x_1<\cdots$ be a strictly increasing sequence of real numbers such that* $\lim x_i=1$. *Let* $h:[0,1]\longrightarrow[0,1]$ *be a diffeomorphism of class C^r such that* $h(x_i)=x_i$ $(i\geq 0)$. *Then*

$$\sum_{i=0}^{\infty}\Big(\sup_{x\in[x_i,x_{i+1}]}|(d/dx)(h-id)(x)|\Big)^{1/(r-1)}<\infty.$$

PROOF. For any i $(i\geq 0)$ and j $(1\leq j\leq r)$, there is a point $x\in(x_i, x_{i+j})$ such that $(d/dx)^j(h-id)(x)=0$. Then $|(d/dx)(h-id)|$ on $[x_i, x_{i+1}]$ is bounded by $(x_{i+2}-x_i)(x_{i+3}-x_i)\cdots(x_{i+r}-x_i)|h-id|_r$, hence by $(x_{i+r}-x_i)^{r-1}|h-id|_r$, where $|\ \ |_r$ is the C^r norm. The sum of the lemma is bounded by r times $(|h-id|_r)^{1/(r-1)}$.

§ 3. Construction of examples

For the construction of examples, we use the following action of

$A(1)=\left\{\begin{pmatrix} e^\beta & \alpha \\ 0 & 1 \end{pmatrix};\ \alpha, \beta \in \boldsymbol{R}\right\}$ on the interval. First, $A(1)$ acts on the set of the rays of $\boldsymbol{R}^2$ from the origin. This is a C^∞-action of $A(1)$ on S^1 fixing two points. Hence we can take one of the two invariant intervals of S^1 and we get $A(1) \subset \mathrm{Diff}^\infty_+([0,1])$. Secondly, by the conjugation by a homeomorphism of $[0,1]$ used in [7], we have a homomorphism $\mathrm{Diff}^\infty_+([0,1]) \longrightarrow \mathrm{Diff}^\infty_\infty([0,1])$, where $\mathrm{Diff}^\infty_\infty([0,1])$ denotes the group of diffeomorphisms of $[0,1]$ infinitely tangent to the identity at $\{0,1\}$. Thus we obtain an injective homomorphism $A(1) \longrightarrow \mathrm{Diff}^\infty_\infty([0,1])$. Let f^α and g^β be the images of $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} e^\beta & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Then we have $g^\beta f^\alpha g^{-\beta} = f^{e^\beta \alpha}$.

We are going to define the images F and G of the generators a and b of the group Γ.

For a positive integer n, put $g_n = g^{(\log 2)/n}$. Then we have $(g_n)^n f^\alpha (g_n)^{-n} = f^{2\alpha}$. Let r be a non-negative integer. Let $0 = y_0 < y_1 < \cdots$ be a sequence of real numbers such that $\lim y_n = 1$ and $y_n - y_{n-1} = n^{-(1+\varepsilon)}$ for sufficiently large n, where ε is a positive real number such that $\varepsilon < 1/(r-1)$ if $r \geq 2$.

Let G denote the homeomorphism of $[0,1]$ such that $G(y_n) = y_n$ $(n \geq 0)$ and $G|[y_{n-1}, y_n]$ is the conjugate of g_{n^r} by the affine map sending $[0,1]$ onto $[y_{n-1}, y_n]$. We claim that G is of class C^r. It is obvious that G is of class C^r on $[0,1)$. For $r=0$, it is obvious that G is continuous at 1. For $r=1$, there is a positive real number C such that $\left|G|[y_{n-1}, y_n] - id\right|_1 < Cn^{-1}$. For $r \geq 2$, we see that there exists a positive real number C' such that $\left|G|[y_{n-1}, y_n] - id\right|_r < C' n^{(1+\varepsilon)(r-1)}/n^r = C' n^{\varepsilon(r-1)-1}$. Since these quantities tend to zero as n tends to the infinity, G is of class C^r on $[0,1]$.

Let F be a diffeomorphism of $[0,1]$ such that $F(y_n) = y_n$ and $F|[y_{n-1}, y_n]$ is the conjugate of f^{α_n} by the affine map sending $[0,1]$ onto $[y_{n-1}, y_n]$, where $0 < \alpha_n < n^{-r}$. Then F is also of class C^r by a similar argument.

Since F and G satisfy $[F, [F, G]] = id$, we obtain a homomorphism $\Gamma \longrightarrow D^r$. We show that this cannot be topologically conjugate to $\Gamma \longrightarrow D^{r+1}$. If it is the case, we obtain the conjugates of F and G which are of class C^{r+1}. We write them by the same letters. Then we have the sequence $0 = x_0 < x_1 < \cdots$ of the fixed points of both F and G. For $F_n = F|[x_{n-1}, x_n]$ and $G_n = G|[x_{n-1}, x_n]$, we have $(G_n)^{n^r} F_n (G_n)^{-n^r} = (F_n)^2$. By Lemma 1, for $j=0$ or 1, we have a point $x \in [x_{n-1}, x_n]$ such that $(d/dx)((G_n)^{n^r}(F_n)^{-j})(x) \geq 2$. Since F is of class C^1, there is an integer N such that $|dF/dx| < 2^{1/2}$ on $[x_N, 1]$. Thus, for $n > N$, we have a point x' in

$[x_{n-1}, x_n]$ such that $(d/dx)((G_n)^{n^r})(x') \geq 2^{1/2}$. For $r=0$, this implies that G is not of class C^1 at 1. For $r \geq 1$, we see that there is a point x'' in $[x_{n-1}, x_n]$ such that $(d/dx)G_n(x) \geq 2^{1/(2n^r)}$. Since $\sum (2^{1/(2n^r)} - 1)^{1/r}$ diverges in a way similar to $\sum 1/n$, G cannot be of class C^{r+1} by Lemma 2.

§ 4. Remarks

REMARK 1. There are uncountably many variations of our examples which are not topologically conjugate. For example, we can replace several fixed points by intervals of fixed points.

REMARK 2. For the actions on the manifolds of dimensions greater than one, see J. Harrison [3, 4].

REMARK 3. The question of conjugation of class C^1 is treated in [2].

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