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Perturbation of moduli spaces of self-dual connections

Dedicated to Professor Itiro Tamura on his 60th birthday

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§ 0. Introduction.

The possible postive definite unimodular forms over Z which are represented as intersection forms of smooth 4-manifolds are much restricted by the following theorems. Let M be a smooth closed oriented 4-manifold with $H_1(M, R)=0$ and positive definite intersection form $Q(M)$.

THEOREM (Donaldson [D1]). *If there is no homomorphism from* $\pi_1(M)$ *to* $SU(2)$ *except the trivial one, then* $Q(M)$ *can be diagonalized over* Z.

THEOREM (Fintushel-Stern [FS 1]). *If* $H_1(M, Z)$ *contains no 2-torsion, the number* $\mu(\alpha_0)$ *defined by*

$$\mu(\alpha_0)=1/2 \sharp \{\alpha \in H^2(M, Z) \mid \alpha \equiv \alpha_0 \bmod 2, \alpha^2=\alpha_0^2\}$$

is even for any $\alpha_0 \in H^2(M, Z)$ *with* $\alpha_0^2=2$ *or* 3. *Moreover if* $H^2(\pi_1(M), Z_2)=0$, *then* $\mu(\alpha_0)$ *is even for any* $\alpha_0 \in H^2(M, Z)$ *with* $\alpha_0^2=4$.

Fintushel and Stern's theorem implies that for example if a positive definite unimodular quadratic form has a direct factor isomorphic to E_8 type, then it can not be realized by a smooth 4-manifold.

The proofs of above theorems depend on the structure of the moduli spaces of self-dual connections on certain $SU(2)$ or $SO(3)$-bundles respectively.

We shall show in this paper that we can replace the assumption Hom $(\pi_1(M), SU(2))=1$ in Donaldson's theorem by Hom $(\pi_1(M), S^1)=1$.

MAIN THEOREM. *If* $H_1(M, Z)=0$, *then* $Q(M)$ *can be diagonalized over* Z.

The framework of the proof is the same as Donaldson's [D1]. (See also [FU].) We explain it here briefly. Let P_1 be a principal $SU(2)$-

bundle with $c_2(P)=-1$ over M. Fix a metric on M and we call a principal connection A on P_1 is self-dual if the anti-self-dual part $p_-F(A)$ of the curvature $F(A)$ of A vanishes. Here p_- is defined by $p_-=(1-*)/2$ using Hodge's star operator $*$ acting on 2-forms with coefficient in the adjoint bundle $\mathrm{ad}\,P_1=P_1\times_{\mathrm{Ad}} su(2)$. The moduli space $\mathcal{M}$ of self-dual connections is defined by $\mathcal{M}=\{\text{self-dual connections}\}/\mathcal{G}(P_1)$, where $\mathcal{G}(P_1)=\{\text{automorphisms of } P_1 \text{ covering the identity of } M\}$ is the gauge group of P_1.

Roughly speaking, Donaldson showed that if we "perturb" $\mathcal{M}$ slightly, we get a singular 5-manifold $\mathcal{M}^{\sigma_1}$ which gives a cobordism between a certain number of $CP(2)$ and ends of $\mathcal{M}^{\sigma_1}$. The crucial points in his proof seem to be the following.

(1) If $H_1(M, \boldsymbol{Z})=0$, then $\mathcal{M}^{\sigma_1}$-{singular points} is orientable.

(2) If $\mathrm{Hom}\,(\pi_1(M), SU(2))=1$, then $\mathcal{M}$ has only one end diffeomorphic to $M\times(0,1)$.

Since the perturbation in [D1] is such as changing a compact piece of $\mathcal{M}$, the above (2) also holds for $\mathcal{M}^{\sigma_1}$. If $\mathrm{Hom}\,(\pi_1(M), SU(2))$ has a non trivial element, the statement (2) might not be true. But if $\mathrm{Hom}\,(\pi_1(M), S^1)=1$, then we can prove that there is a perturbation such that any other ends than $M\times(0,1)$ in (2) disappear (Theorem 5.4). The point is to consider perturbation over different bundles at the same time. Such perturbations are also used in [D2].

After we had completed the preliminary version of this article we were informed from Donaldson that Main Theorem holds without any assumption on $H_1(M, \boldsymbol{Z})$ [D3]. He considers a similar perturbation of (anti-)self-duality equation, and moreover he shows the orientability of moduli space even in the case when $H_1(M, \boldsymbol{Z})$ is not trivial. Donaldson's perturbation is rather concrete and has an advantage when it is to be extended on different bundles from the original bundle on which the equation is perturbed. Our perturbation is more abstract and could be formulated in a context of Banach manifold, Banach bundle and Fredholm operator. It seems that self-duality equation gives a "Fredholm section" on some Banach bundle and should be understood in a similar context of "Bordism theory of Fredholm maps" as in [K]. With this understanding we believe it is still of some interest to present our method of perturbation in this paper.

We deal with preliminary local lemmas in § 1. In § 2, a weak compact theorem for a certain family of "almost self-dual connections" is proved. We consider "almost self-dual connections" on principal $SU(2)$-

bundles P_0 and P_1 ($c_2(P_k)=-k$) in § 3 and § 4 respectively. The required perturbation is constructed in § 5. In § 6 we apply the perturbation method to some orbifold case and prove the statement announced in Remark 1.5 of [F].

We refer to [D1], [FU] and [L] for general back ground material.

The first version of the paper contained a gap in the proof of Proposition 3.3. In revising I have profited from the conversations with S. Donaldson I had at Berkeley before the gap was pointed out to me by the referee. I am grateful to him for that enlightening conversation and also wish to thank Professor A. Hattori for his advice and encouragement.

Notations.

B_r: the open r-ball in $\boldsymbol{R}^4$ centered at the origin.

$B(x, r)$: the open r-ball centered at x.

$\| \ \|_{L^p_k(U,A)}$: L^p_k-Sobolev norm on open set U defined by using the covariant derivative ∇_A associated with Riemannian metric and connetion A.

l: a fixed integer larger than 5.

$L^p_k(\Omega^i(E))$: L^p_k i-forms with coefficient in E.

p_-: the projection $(\mathrm{id}-*)/2$ on anti-self-dual part.

$\Omega^2_-(E)=p_-\Omega^2(E)$.

$\mathcal{C}(P)$: the set of L^2_l-principal connections on P.

$\mathcal{G}(P)$: the set of L^2_{l+1}-gauge transformations.

$\mathcal{B}(P)=\mathcal{C}(P)/\mathcal{G}(P)$: the moduli space of connections.

$F(A)$: the curvature of connection A.

REMARK. After the preliminary version of this manuscript was written down, I was informed that Fintushel and Stern [FS 2] proved Main Theorem under the condition that $H_1(M, Z/2)=0$. They used a compact moduli space of self-dual connections with a certain property on a principal $SO(3)$-bundle.

§ 1. Local lemmas.

In this section we state some local lemmas. We fix a principal $SU(2)$-bundle $B_1 \times SU(2)$ on the 4-dimensional standard unit disk B_1 and its adjoint bundle $B_1 \times su(2)$.

DEFINITION 1.1. We call two norms $\| \quad \|_1$ and $\| \quad \|_2$ are c-equivalent if they satisfy the following for every f.

$$\frac{1}{c}\|f\|_1 \leqq \|f\|_2 \leqq c\|f\|_1.$$

LEMMA 1.2. *For any $\varepsilon>0$ there exists $m=m(\varepsilon)>0$ such that if a connection A on B_1 and a flat connection A_0 satisfy*

$$\|a\|_{L_1^2(B_1, A_0)} < m \quad (A=A_0+a),$$

then $L_1^2(B_1, A_0)$ and $L_1^2(B_1, A)$ are $(1+\varepsilon)$-equivalent.

PROOF.
$$\begin{aligned} |\|f\|_{L_1^2(B_1, A)} - \|f\|_{L_1^2(B_1, A_0)}| &\leqq \|[a \otimes f]\|_{L^2} \\ &\leqq \|a\|_{L^4(B_1)} \|f\|_{L^4(B_1)} \leqq c\|a\|_{L_1^2(A_0)} \|f\|_{L_1^2(A_0)}. \end{aligned}$$

Therefore if m satisfies $1+cm$, $1/(1-cm) \leqq 1+\varepsilon$, then we have the result.

LEMMA 1.3. *For any $\varepsilon>0$ there exists $m=m(\varepsilon)>0$ such that if a connection A on B_1 and a flat connection A_0 satisfy*

$$\|a\|_{L_2^2(B_1, A_0)} < m \quad (A=A_0+a),$$

then $L_2^2(B_1, A_0)$ and $L_2^2(B_1, A)$ are $(1+\varepsilon)$-equivalent.

PROOF. We can assume $L_1^2(B_1, A)$ and $L_1^2(B_1, A_0)$ are $(1+\varepsilon')$-equivalent for $\varepsilon'<\varepsilon$. Then we have

$$\begin{aligned} \|f\|_{L_2^2(A)} &\leqq \|f\|_{L^2} + (1+\varepsilon')\|\nabla_A f\|_{L_1^2(A_0)} \\ &\leqq \|f\|_{L^2} + (1+\varepsilon')\|\nabla_{A_0} f\|_{L_1^2(A_0)} + \|[a \otimes f]\|_{L_1^2(A_0)}) \\ &\leqq \|f\|_{L^2} + (1+\varepsilon')(\|\nabla_{A_0} f\|_{L_1^2(A_0)} + c\|a\|_{L_2^2(A_0)} \|f\|_{L_2^2(A_0)}) \\ &\leqq (1+\varepsilon')(1+c\|a\|_{L_2^2(A_0)}) \|f\|_{L_2^2(A_0)}. \end{aligned}$$

Similarly we have

$$\|f\|_{L_2^2(A_0)} \leqq (1+\varepsilon')\|f\|_{L_2^2(A)} + c\|a\|_{L_2^2(A_0)} \|f\|_{L_2^2(A_0)}.$$

Therefore if m satisfies

$$1+\varepsilon \geqq (1+\varepsilon')(1+cm) \quad \text{and} \quad 1+\varepsilon \geqq (1+\varepsilon') + cm(1+\varepsilon),$$

then we get the result.

LEMMA 1.4. *For any $m>0$ there is a constant $c=c(m)>1$ such that if a connection A on B_1 and a flat connection A_0 on B_1 satisfy*

$$\|a\|_{L_l^2(B_1, A_0)} < m \quad (A = A_0 + a),$$

then $L_k^2(B_1, A)$ and $L_k^2(B_1, A_0)$ are c-equivalent for $k \leqq l+1$. We can take $c(m)$ so that $c(m) \to 1$ $(m \to 0)$.

PROOF. If $k=0$, the equality holds for $c=1$. Estimating

$$\nabla_A f = \nabla_{A_0} f + [a, f] \quad \text{and} \quad \nabla_{A_0} f = \nabla_A f - [a, f]$$

in $L_k^2(B_1, A)$-norm and in $L_k^2(B_1, A_0)$-norm respectively, and using $L_l^2 \times L_k^2 \to L_k^2$ $(l \geqq 3, k \leqq l)$ we can get the result by induction on k.

The following a priori estimate is known ([FU] Proposition 8.3).

THEOREM (Uhlenbeck). *There are constants $\varepsilon > 0$ and $c > 0$ for any $0 < r < 1$ such that if a self-dual connection A on B_1 satisfies*

$$\|F(A)\|_{L^2(B_1)} < \varepsilon,$$

then for a flat connection A_0 on B_1, we have

$$d_{A_0}^* a = 0, \ {*a}|S^3 = 0 \ (A = A_0 + a)$$

and

$$\|a\|_{L_{l+1}^2(B_r, A_0)} \leqq c \|F(A)\|_{L^2(B_1)}.$$

For non self-dual connections the following estimate holds.

PROPOSITION 1.5. *There are constants $\varepsilon_0 > 0$ and $m_0 > 0$ such that for any $0 < r < 1$, there is a constant $c > 0$ and if a connection A on B_1 satisfy*

$$\|F(A)\|_{L^2(B_1)} < \varepsilon_0$$

and

$$\|p_- F(A)\|_{L_l^2(B_1, A)} < m_0,$$

then for a flat connection A_0 on B_1, we have

$$d_{A_0}^* a = 0, \ {*a}|S^3 = 0 \ (A = A_0 + a)$$

and

$$\|a\|_{L_{l+1}^2(B_r, A_0)} < c(\|F(A)\|_{L^2(B_1)} + \|p_- F(A)\|_{L_l^2(B_1, A)}).$$

PROOF. We use the next theorem of Uhlenbeck [U2].

LEMMA. *There are $\varepsilon > 0$ and $c > 0$ such that when a connection A on B_1 satisfies*

$$\|F(A)\|_{L^2(B_1)}<\varepsilon,$$

then there is a flat connection A_0 *such that, if we write* $A=A_0+a$,

$$d_{A_0}^* a=0,\ *a|S^3=0 \quad \textit{and} \quad \|a\|_{L_1^2(B_1,A_0)}\leqq c\|F(A)\|_{L^2(B_1)}.$$

We write $f\in L_k^p$ if $f=f(A, A_0)$ is estimated as follows for some $c>0$ dependent only on m_0, ε_0 and r.

$$\|f\|_{L_k^p(B_r,A_0)}<c(\|F(A)\|_{L^2(B_1)}+\|p_-F(A)\|_{L_l^2(B_1,A)}).$$

We follow the proof of [FU] Proposition 8.3. Take A_0 as in the above Lemma. Then we have $a\in L_1^2$ and the 2-equivalence of $L_1^2(B_1, A_0)$ and $L_1^2(B_1, A)$ from Lemma 1.2 if ε_0 is small. We also have the following equation for a.

(1) $La+(1/2)(p_-d)^*p_-[a\wedge a]=\sigma,$
$(d=d_{A_0},\ L=dd^*+(p_-d)^*(p_-d),\ \sigma=(p_-d)^*p_-F(A)).$

For fixed $\phi\in C_0^\infty(B_1)$ and $\mu\in \boldsymbol{R}$, we have

(2) $(L+\mu)(\phi a)+(a, d(\phi a))_0=h_1a+(h_2, da)_2+h_3(a, a)_3+\mu\phi a+\phi\sigma,$

and

(3) $(L+\mu)(\phi a)=h_4a+(h_5, da)_5+h_6(a, da)_6+\mu\phi a+\phi\sigma.$

Here $h_i\in C_0^\infty(B_1)$ depends on ϕ and $(\ ,\ )_i$ is a certain bilinear form. When $\|a\|_{L_1^2(B_1,A_0)}$ $(<c\|F(A)\|_{L^2(B_1)})$ is sufficiently small, the next bounded maps are invertible for sufficiently large μ.

(4) $L+\mu+(a, d(\ \ ))_0:\ L_{1,0}^2\longrightarrow L_{-1}^2,$

and

(5) $L+\mu+(a, d(\ \ ))_0:\ L_{2,0}^2\longrightarrow L^2.$

Here $L_{k,0}^p=L_k^p\cap L_{1,0}^p$ and $L_{1,0}^p$ is the L_1^p-closure of C_0^∞.

We can show $a\in L_2^2$ since the R.H.S. of (2) is L^2-bounded and (4) and (5) are invertible. Then $L_2^2(B_r, A_0)$ and $L_2^2(B_r, A)$ are 2-equivalent from Lemma 1.3 if ε_0 and m_0 are small.

So far we can not say the R.H.S of (3) is in L_1^2, but for any small $\delta>0$, it is in $L_1^{2-\delta}$ because $L_2^2\times L_1^2\rightarrow L_1^{2-\delta}$. This implies $a\in L_3^{2-\delta}$. So now we can show the R.H.S. of (3) is in L_1^2 because $L_3^{2-\delta}\times L_2^2\rightarrow L_1^2$. This implies that $a\in L_3^2$. We can use the following processes inductively to show $a\in L_{l+1}^2$.

(6) For $k\geqq 3$, if $a\in L_k^2$ and $\sigma\in L_{k-1}^2$, then $a\in L_{k+1}^2$.

(7) For $3 \leqq k \leqq l$, if $a \in L_k^2$, then $L_{k+1}^2(B_r, A_0)$ and $L_{k+1}^2(B_r, A)$ are 2-equivalent if ε_0 and m_0 are small.

(8) For $3 \leqq k \leqq l$, if $L_k^2(B_r, A_0)$ and $L_k^2(B_r, A)$ are 2-equivalent, then $\sigma \in L_{k-1}^2$.

REMARK 1.6. (1) Definition 1.1, Lemmas 1.2, 1.3 and 1.4 are valid on any compact manifold with boundary.

(2) Proposition 1.5 also holds even if the metric of B_1 is not standard, but is very close to the standard metric in L_{l+1}^2-norm.

§ 2. Global compactness.

In this section we fix a compact oriented Riemannian 4-manifold M and a principal $SU(2)$-bundle P on M.

PROPOSITION 2.1. *Let $\{U_\alpha\}$ $(U_\alpha = B(x_\alpha, r_\alpha))$ be a finite open covering of M and suppose a sequence $\{A_i\}$ of connections satisfies the following condition for some $m>0$. For any A_i and any U_α there is a flat connection $A_{i,\alpha}$ on U_α such that*

$$\|a_{i,\alpha}\|_{L_{l+1}^2(U_\alpha, A_{i,\alpha})} < m \quad (A_i = A_{i,\alpha} + a_{i,\alpha}).$$

Then there are a subsequence $\{A_{i'}\}$, a sequence of gauge transformations $\{g_{i'}\}$ and a connection A_∞ such that $\{g_{i'}(A_{i'})\}$ is L_{l+1}^2-weakly convergent (hence L_l^2-convergent) to A_∞.

PROOF. For $f \in \Gamma(U_\alpha \cap U_\beta, \operatorname{ad} P)$ using Lemma 1.4 twice we have

$$\|f\|_{L_{l+1}^2(U_\alpha \cap U_\beta, A_{i,\alpha})} \leqq c \|f\|_{L_{l+1}^2(U_\alpha \cap U_\beta, A_{i,\beta})}.$$

Apply this for $f = a_{i,\beta}$, and we find that $L_{l+1}^2(U_\alpha \cap U_\beta, A_{i,\alpha})$-norm of $A_{i,\alpha} - A_{i,\beta} = a_{i,\beta} - a_{i,\alpha}$ is bounded. We fix a trivialization $h_{i,\alpha}: B_1 \times SU(2) \to P|U_\alpha$ such that the pullback of $A_{i,\alpha}$ is equal to the trivial flat connection θ on $B_1 \times SU(2)$. The gauge transformation $g_{i,\alpha,\beta} = h_{i,\alpha} h_{i,\beta}^{-1}$ on $U_\alpha \cap U_\beta$ transforms $A_{i,\beta}$ to $A_{i,\alpha}$. Using bootstrapping to

$$d_{A_{i,\alpha}} g_{i,\alpha,\beta} = g_{i,\alpha,\beta}(A_{i,\beta} - A_{i,\alpha}),$$

we can show that $L_{l+2}^2(U_\alpha \cap U_\beta, A_{i,\alpha})$-norm of $g_{i,\alpha,\beta}$ is uniformly bounded. Therefore we can choose a subsequence $\{i'\}$ such that

(1) $\{h_{i',\alpha}^* a_{i',\alpha}\}$ is $L_l^2(B_1)$-convergent for any α.

and

(2) $\{g_{i',\alpha,\beta}\}$ *is* $L^2_{l+1}(U_\alpha \cap U_\beta)$-convergent for any α and β.

Then by the discussion in [U2] § 3, there is a sequence of gauge transformation $\{g_{i'}\}$ such that $\{g_{i'}(A_{i'})\}$ is L^2_{l+1}-weak convergent.

We suppose that the injectivity radius of M is larger than 3 and any ball with radius smaller than 1 is flat enough to satisfy the local lemmas in § 1.

DEFINITION 2.2. We define $\mathcal{K}(m)$ to be the set of $A \in \mathcal{C}$ satisfying $\|p_-F(A)\|_{L^2_l(M,A)} \leqq m$.

PROPOSITION 2.3. *Let* m_0 *be the constant in Proposition* 1.5. *If* $\{A_i\}$ *is a sequence of connections in* $\mathcal{K}(m_0)$ *and* $\{\|F(A_i)\|_{L^\infty}\}$ *is bounded, then there are a subsequence* $\{A_{i'}\}$ *and a sequence of gauge transformations* $\{g_{i'}\}$ *such that* $\{g_{i'}(A_{i'})\}$ *is* L^2_{l+1}*-weak convergent* (*hence* L^2_l*-convergent*).

PROOF. Since $\|F(A_i)\|_{L^\infty}$ is bounded, we can take a finite covering $\{V_\alpha\}$ such that

$$V_\alpha = B(x_\alpha, r_\alpha), \quad 2r_\alpha < 1, \quad M = \cup V_\alpha,$$

and

$$\|F(A_i)\|_{L^2(B(x_\alpha, 2r_\alpha))} < \varepsilon \quad \text{for any } \alpha \text{ and } i.$$

Take a flat connection $A_{i,\alpha}$ on V_α as in Proposition 1.3. Then, if we write $A_{i,\alpha} + a_{i,\alpha}$ for $A_i | V_\alpha$,

$$\|a_{i,\alpha}\|_{L^2_{l+1}(V_\alpha, A_{i,\alpha})}$$

is uniformly bounded, so that we can apply Proposition 2.1 to get the result.

COROLLARY 2.4. $\mathcal{K}(m_0)$ *is a closed subset of* $\mathcal{C}(P)$. *Here the topology of* $\mathcal{C}(P)$ *is defined by* L^2_l*-norm.*

PROOF. When $\{A_i\}$ is L^2_{l+1}-weak convergent to A_∞, $\{p_-F(A_i)\}$ is L^2_l-weak convergent to $p_-F(A_\infty)$, so we have

$$\|p_-F(A_\infty)\|_{L^2_l(M, A_\infty)} \leqq \liminf \|p_-F(A_i)\|_{L^2_l(M, A_i)}.$$

COROLLARY 2.5. $\{A \in \mathcal{K}(m_0) \,|\, \|F(A)\|_{L^\infty} \leqq D\} / \mathcal{G}$ *is compact for any* $D > 0$.

COROLLARY 2.6. *$\mathcal{K}(m_0)/\mathcal{G}$ is locally compact and paracompact. Here the topology of $\mathcal{K}(m_0)/\mathcal{G}$ is defined by the quotient topology.*

For convenience we assume $m_0<2\pi$.

PROPOSITION 2.7. *If $\{A_i\}$ is a sequence of connections in $\mathcal{K}(m_0)$ with $\|F(A_i)\|_{L^\infty}\to+\infty$, then there are a subsequence $\{A_i'\}$ and sequences $\{x_{i'}\}$ $(x_{i'}\in M)$, $\{\lambda_{i'}\}$ $(\lambda_{i'}>0,\ \lambda_{i'}\to 0)$ and $\{e_{i'}\}$ $(e_{i'}: \boldsymbol{R}^4\times SU(2)\to P$ is a bundle map covering $v\mapsto \exp_{x_{i'}}\lambda_{i'}v$, where we identify $TM_{x_i}=\boldsymbol{R}^4$.) which satisfy the followings.*

(1) *$e_{i'}^*A_{i'}$ is $L^2_{l,\mathrm{loc}}$-convergent to a self-dual connection I on $\boldsymbol{R}^4\times SU(2)$.*

(2) *$8\pi^2\leqq\|F(\mathrm{I})\|^2_{L^2(\boldsymbol{R}^4)}\leqq 8\pi^2(-c_2(P))$ and $\|F(I)\|_{L^\infty}=1$.*

PROOF. Fix x_i and λ_i so that $\|F(A_i)\|_{L^\infty}=|F(A_i)|_{x_i}$ and $|e_i^*F(A)|_{\text{at }0}=1$. Since $\{\lambda_i\}$ is convergent to 0, the metrics on $\boldsymbol{R}^4$ induced by $\{e_i\}$ are C^∞_{loc}-convergent to the standard flat metric (after multiplying by constant $1/\lambda_i$). So we assume for simplicity that the induced metrics are flat in some fixed bounded domain in $\boldsymbol{R}^4$. We can apply Proposition 2.4, which is also valid on $\boldsymbol{R}^4$, to $\{e_i^*A_i\}$ so that after changing gauges we have a subsequence $\{i'\}$ such that $\{e_{i'}A_{i'}\}$ is $L^2_{l,\mathrm{loc}}$-convergent to a connection I on $\boldsymbol{R}^4$. Then we have

$$\begin{aligned}\|F(I)\|^2_{L^2(\boldsymbol{R}^4)}&\leqq \liminf \|F(A_{i'})\|^2_{L^2(M)}\\&=8\pi^2(-c_2(P))+2\liminf\|p_-F(A_{i'})\|^2_{L^2(M)}\\&\leqq 8\pi^2(-c_2(P))+2m_0^2.\end{aligned}$$

Especially $\|p_-F(I)\|^2_{L^2(\boldsymbol{R}^4)}$ is bounded. On the other hand the estimate $\|\nabla_{A_{i'}}p_-F(A_{i'})\|_{L^2(B(x_{i'},\lambda_{i'}),A_{i'})}\leqq m_0$ implies

$$\|e_{i'}^*\nabla_{A_{i'}}p_-F(A_{i'})\|_{L^2(B_1,e_{i'}^*A_{i'})}\leqq\lambda_{i'}m_0.$$

Therefore $p_-F(I)$ is parallel on B_1 for I and similarly we can see that $p_-F(I)$ is parallel on $\boldsymbol{R}^4$. Then $\|p_-F(I)\|_{L^2}<+\infty$ implies that $p_-F(I)=0$. When I is self-dual, $\|F(I)\|^2_{L^2}$ is a multiple of $8\pi^2$ from removability of singularity [U1]. It is not zero from $\|F(I)\|_{L^\infty}=1$. Therefore we get the required inequality.

§ 3. The case $c_2(P)=0$.

In this section we assume that the intersection form of M is positive definite and that $H_1(M, Z)=0$. Let P_0 be a principal $SU(2)$-bundle with $c_2(P_0)=0$, i.e., $P_0\simeq M\times SU(2)$.

PROPOSITION 3.1. *$\mathcal{K}(m_0)(P_0)/\mathcal{G}(P_0)$ is a compact subset of $\mathcal{B}(P)$. Here the topology of $\mathcal{K}(m_0)(P_0)$ is defined by L_l^2-norm and the topolgy of $\mathcal{K}(m_0)(P_0)/\mathcal{G}(P_0)$ is the quotient topology.*

PROOF. Assume that $\{A_i\}$ is a sequence in $\mathcal{K}(m_0)(P_0)$. If $\{\|F(A_i)\|_{L^\infty}\}$ is not bounded, Proposition 2.7 (2) contradicts $c_2(P_0)=0$. Then Corollary 2.6 implies the required result.

Next we perturb the map $p_-F: \mathcal{C}(P_0)\to L_{l-1}^2(\Omega_-^2(\operatorname{ad} P_0))$ following Donaldson [D1].

DEFINITION 3.2. We write $\mathcal{P}$ for the set of the maps

$$\sigma: M\times \mathcal{C}(P_0)\to L_l^2(\Omega_-^2(\operatorname{ad} P_0))$$

satisfying the followings.

(1) σ is smooth and $\mathcal{G}(P_0)$-equivariant.

(2) $\|\sigma\|_{\mathcal{P}}=\sup_{x, A}\|\sigma(x, A)\|_{L_l^2(M, A)}<+\infty$.

(3) $\operatorname{supp}\sigma(x, A)\subset M-B(x, 3)$.

PROPOSITION 3.3. *There is $\sigma\in\mathcal{P}$ with arbitrary small $\|\sigma\|_{\mathcal{P}}$ such that for any $x\in M$ and $A\in\mathcal{C}(P_0)$, we have $p_-F(A)+\sigma(x, A)=0$, iff A is globally flat.*

PROOF. *Step* 1. We review the local description of moduli space by using the Kuranishi map ([AHS], [D1], [FU]). When $c_2(P_0)$ is zero, self-dual connections are flat connections. Fix a flat connection A. We write H_A^0, H_A^1 and H_A^2 for the cohomology groups of the following complex.

$$(3)\quad 0\to L_{l+1}^2(\Omega^0(\operatorname{ad} P_0))\xrightarrow{d_A} L_l^2(\Omega^1(\operatorname{ad} P_0))\xrightarrow{p_-d_A} L_{l-1}^2(\Omega_-^2(\operatorname{ad} P_0))\to 0.$$

Set $\Gamma_A=\{g\in\mathcal{G}(P_0)\mid g^*A=A\}$ and fix a Γ_A-invariant complement subspace $H_A^{2\prime}$ of $\operatorname{Im} p_-d_A$ in $L_{l-1}^2(\Omega_-^2(\operatorname{ad} P_0))$. Then by Kuranishi's method we can find a Γ_A-equivariant diffeomorphic map L from a neighbourhood $\mathcal{U}$ of 0 in $\operatorname{Ker}(d_A^*: L_l^2(\Omega^1(\operatorname{ad} P_0))\to L_{l-1}^2(\Omega^0(\operatorname{ad} P_0)))$ to $\mathcal{U}$ itself $(L(0)=0)$, and a smooth map ϕ from a neighbourhood of 0 in H_A^1 to a neighbourhood of 0 in $H_A^{2\prime}(\phi(0)=0)$ which satisfy the followings.

(4) A coordinate of the neighbourhood of $[A]\in\mathcal{B}(P_0)$ is given by

$$\mathcal{U}/\Gamma_A \subset \mathcal{B}(P_0), \quad [a] \mapsto [A+a].$$

(5) The following diagram is commutative.

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{p_-F} & L^2_{l-1}(\Omega^2_-(\mathrm{ad}\, P_0)) \\ L\downarrow & & \| \\ H^1_A \oplus \mathrm{Im}\,(p_-d_A)^* & \xrightarrow[\phi\oplus p_-d_A]{} & H^{2\prime}_A \oplus \mathrm{Im}\, p_-d_A \end{array}$$

Especially the neighbourhood of $[A]$ in $\mathcal{M}(P_0)=(p_-F)^{-1}(0)/\mathcal{G}(P_0)$ is homeomorphic to $\phi^{-1}(0)/\Gamma_A$.

Step 2. A reducible connection, i. e., a connection with $L_A \neq \{\pm 1\}$, is given by an element of $\mathrm{Hom}(\pi_1(M), S^1)$ which has the only element zero since $H_1(M, \boldsymbol{Z})=0$. Therefore reducible connections are globally flat. When A is globally flat, $[A]$ is an isolated point of $\mathcal{M}$ since $H^1_A = H^1(M, su(2))=0$.

Step 3. Let x_0 be a point of M and A be an irreducible flat smooth connection. Since $\Gamma_A=\{\pm 1\}$ acts on $\mathcal{C}(P_0)$ trivially, $\mathcal{U}$ is a neighbourhood of $[A]$ in $\mathcal{B}(P_0)$. Using Unique Continuation Principle for second order elliptic system with scalar principal symbol, we can find $H^{2\prime}_A$ in Step 1 so that for any $\alpha \in H^{2\prime}_A$, α is smooth and the support of α is contained in $B(x_0, 1)$.

Step 4. Fix two points x_0 and x_0' on M such that the distance of x_0 and x_0' is larger than 8. Since $\mathcal{M}=\mathrm{Hom}(\pi_1(M), SU(2))/SU(2)$ is compact and the point of globally flat connection is isolated, we can get from Step 3 a finite number of $\sigma_1, \cdots, \sigma_n \in \mathcal{P}$ with support in $B(x_0, 1)$ and $\sigma_1', \cdots, \sigma'_{n'} \in \mathcal{P}$ with support in $B(x_0', 1)$ such that the Fredholm section

$$(A, t_1, \cdots, t_n, t_1', \cdots, t'_{n'}, t) \to p_-F(A) + t\sum_{i=1}^{n} t_i\sigma_i + (1-t)\sum_{j=1}^{n'} t_j'\sigma_j'$$

of the Banach bundle

$$\mathcal{C} \times_{\mathcal{G}} L^2_l(\Omega^2_-(\mathrm{ad}\, P_0)) \times \boldsymbol{R}^n \times \boldsymbol{R}^{n'} \times \boldsymbol{R} \to \mathcal{B} \times \boldsymbol{R}^n \times \boldsymbol{R}^{n'} \times \boldsymbol{R}$$

is transversal to the zero section on an open neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{V}'\times\mathcal{W}$ of $\mathcal{M}'\times 0\times 0\times[0,1]\subset\mathcal{B}\times\boldsymbol{R}^n\times\boldsymbol{R}^{n'}\times\boldsymbol{R}$, where $\mathcal{M}'=\mathcal{M}-\{$globally flat connection$\}$. Let $\beta: \mathcal{B}\to[0,1]$ be a smooth cut off function satisfying $\beta=1$ on $\mathcal{M}'$ and $\beta=0$ outside $\mathcal{U}$. Then

$$(A, t_1, \cdots, t_n, t_1', \cdots, t_{n'}', t) \to p_-F(A) + \beta(A)\Big(t\sum_{i=1}^{n} t_i\sigma_i + (1-t)\sum_{j=1}^{n'} t_j'\sigma_j'\Big)$$

is transversal to the zero section on $\mathcal{B}\times\mathcal{V}\times\mathcal{V}'\times\mathcal{W}$. Atiyah-Singer index theorem [AHS] says the index of the complex (3) is $+3$, so the index of this Fredholm section is $-3+n+n'+1$ and the zero set is a smooth $n+n'-2$ dimensional submanifold of $\mathcal{B}\times\mathcal{V}\times\mathcal{V}'\times\mathcal{W}$. The projection image of the zero set to $\mathcal{V}\times\mathcal{V}'$ is nowhere dense, so we can take arbitrary small $(t_1, \cdots, t_n, t_1', \cdots, t_{n'}')\in\mathcal{V}\times\mathcal{V}'$ from the complement of the projection image. Let ρ be a smooth function on M such that $0\leqq\rho\leqq 1$ on M, $\rho=0$ on $B(x_0, 4)$ and $\rho=1$ on $B(x_0', 4)$. Now we can define σ as follows.

$$\sigma=\beta(\rho\sum t_i\sigma_i+(1-\rho)\sum t_j'\sigma_j').$$

PROPOSITION 3.4. *There exist m and a disjoint decomposition $\mathcal{K}(m)(P_0)=\mathcal{K}^0\cup\mathcal{K}^1$ which satisfy the followings.*

(1) *$\mathcal{K}^0$ and $\mathcal{K}^1$ are $\mathcal{G}(P_0)$-invariant closed sets.*

(2) *Every globally flat connection is contained in $\mathcal{K}^0$.*

(3) *Every irreducible flat connection is contained in $\mathcal{K}^1$.*

PROOF. In Step 4 of the proof of Proposition 3.3, we find the point of globally flat connections is isolated in $\mathcal{M}$. From the definition we have $\mathcal{M}=\bigcap_m \mathcal{K}(m)(P_0)/\mathcal{G}(P_0)$. So we get the result from the compactness of $\mathcal{K}(m)(P_0)/\mathcal{G}(P_0)$ $(m\leqq m_0)$.

If necessary we replace m_0 and assume Proposition 3.4 holds for $m=m_0$. From now on we write $\mathcal{K}(P_0)$ for $\mathcal{K}(m_0)(P_0)$ and we fix $\sigma_0\in\mathcal{P}$ satisfying the condition of Proposition 3.3 and $\|\sigma_0\|_{\mathcal{P}}<m_0/2$. Let $\alpha>0$ be the minimum of $\|p_-F(A)+\sigma_0(x, A)\|_{L^2}$ on the compact set $M\times\mathcal{K}^1/\mathcal{G}(P_0)$.

LEMMA 3.5. *For any $\delta>0$, there is $\rho=\rho(\delta)$ such that when $A_1, A_2\in\mathcal{K}(P_0)$, $x\in M$ and they satisfy*

$$\|A_1-A_2\|_{L_l^2(M-B(x,\rho), A_1)}<\delta/2,$$

then for a gauge transformation g,

$$\|A_1-g(A_2)\|_{L_l^2(M, A_1)}<\delta \text{ and } g|M-B(x, 2\rho)=\mathrm{id}.$$

PROOF. Suppose that $\{A_{1,i}\}$ and $\{A_{2,i}\}$ are sequences of connections in $\mathcal{K}(P_0)$ and satisfy that for some $x_i\in M$ and $\rho_i\to 0$,

$$\|A_{1,i}-A_{2,i}\|_{L_l^2(M-B(x_i,\rho_i), A_{1,i})}<\delta/2.$$

If we take a subsequence and change gauges, we can suppose that $\{A_{1,i}\}$ is L_l^2-convergent to $A_{1,\infty}$ from Proposition 3.1 and $\{x_i\}$ is convergent to x_∞. Let $\{U_\alpha\}$ be finite open covering which has the property as in Proposition 2.1 for every connection of relatively compact set $\mathcal{K}(P_0)/\mathcal{G}(P_0)$. Then $L^2_{l+1}(U_\alpha, A_{1,i,\alpha})$-norm is uniformly equivalent to $L^2_{l+1}(U_\alpha, A_{1,i})$-norm from Lemma 1.4. So similarly

$$L^2_{l+1}(U_\alpha - B(x_i, \rho_i), A_{1,i,\alpha}) \simeq L^2_{l+1}(U_\alpha - B(x_i, \rho_i), A_{2,i})$$
$$\simeq L^2_{l+1}(U_\alpha - B(x_i, \rho_i), A_{2,i,\alpha})$$

($\simeq$ means uniform equivalence).

From the proof of Proposition 2.1, this implies that if we take a subsequence, $\{A_{2,i}|M-\{x_\infty\}\}$ is $L^2_{l,\mathrm{loc}}$-convergent to some connection $A_{2,\infty}$ on $M-\{x_\infty\}$ without changing gauges and $\|A_{2,\infty}-A_{1,\infty}\|_{L^2_l(M-\{x_\infty\}, A_{1,\infty})}$ is bounded, i.e., $A_{2,\infty}\in \mathcal{C}$. On the other hand if we take a subsequence we can find $\{g_i\}$ from Proposition 3.1 so that $\{g_i(A_{2,i})\}$ is convergent to some connection $A'_{2,\infty}$. There is a gauge transformation g_∞ on $M-\{x_\infty\}$ such that $g_\infty(A_{2,\infty})=A'_{2,\infty}$. Since $A_{2,\infty}$ and $A'_{2,\infty}$ are elements of $\mathcal{C}$, g_∞ is an element of $\mathcal{G}$ ([FU] Proposition A. 5). Therefore $\{g_\infty^{-1}g_i(A_{2,i})\}$ is L^2_l-convergent to $A_{2,\infty}$ and

$$\lim_{i\to\infty} \|A_{1,i}-g_\infty^{-1}g_i(A_{2,i})\|_{L^2_l(M, A_{1,i})} = \|A_{1,\infty}-A_{2,\infty}\|_{L^2_l(M, A_{1,\infty})}$$
$$=\lim_{r\downarrow 0}\lim_{i\to\infty} \|A_{1,i}-A_{2,i}\|_{L^2_l(M-B(x_\infty, r), A_{1,i})} \leqq \delta/2.$$

Therefore for sufficiently large i, we have

$$\|A_{1,i}-g(A_{2,i})\|_{L^2_l(M,A_{1,i})}<\delta \quad (g=g_\infty^{-1}g_i).$$

Since $A_{1,i}-A_{2,i}$ is L^2_l-estimated on $M-B(x,\rho_i)$, we can assume that g_i and g_∞ are identity over $M-B(x, 2\rho_i)$.

We suppose that the function $\rho(\delta)$ of δ is monotonely decreasing and $\rho(\delta)\to 0$ $(\delta\to 0)$. The following lemma is easily shown.

LEMMA 3.6. *We can take positive numbers δ_1 and $\rho_1=\rho(\delta_1)$ small enough to satisfy the followings.*

(1) *If $A_1, A_2\in \mathcal{K}(P_0)$, $\|A_1-A_2\|_{L^2_l(M, A_1)}<\delta_1$ and distance$(x_1, x_2)<\rho_1$, then $\|\sigma_0(x_1, A_1)-\sigma_0(x_2, A_2)\|<\alpha/3$.*

(2) *If $A_0\in \mathcal{K}^0$ and $A_1\in \mathcal{K}^1$, then $\|A_0-A_1\|_{L^2_l(M, A_0)}>\delta_1$.*

§ 4. The Case $c_2(P)=-1$.

In this section we assume that M is a closed oriented Riemannian 4-manifold with positive definite intersection form and P_k is a principal

$SU(2)$-bundle with $c_2(P_k)=-k$.

DEFINITION 4.1. For $s=0, 1$, we set

$\mathcal{U}^s(\delta, \rho)=\{A\in\mathcal{C}(P_1)|$ There are $x\in M$, $A'\in\mathcal{K}^s(P_0)$, $c>0$ and a bundle map $h: P_0|M-B(x,\rho)\to P_1|M-B(x,\rho)$ such that $L_l^2(M-B(x,\rho_1/2), A')$ and $L_l^2(M-B(x,\rho_1/2), h^*A)$ are $(3-c)$-equivalent and $\|h^*A-A'\|_{L_l^2(M-B(x,\rho), A')}<\delta\}$.

LEMMA 4.2. *There exists δ_2 such that all the connections in $\mathcal{U}^1(\delta_2, \rho(\delta_2))$ are irreducible and*

$$\mathcal{U}^0(\delta_2, \rho(\delta_2))\cap\mathcal{U}^1(\delta_2, \rho(\delta_2))=\emptyset.$$

PROOF. Since the moduli space of reducible connections is compact and the intersection of all $\mathcal{U}^1(\delta, \rho(\delta))(\delta>0)$ is empty, the first half of the statement follows. Let $\{\delta_i\}$ be a sequence convergent to 0, $A_i\in\mathcal{U}^0(\delta_i, \rho(\delta_i))\cap\mathcal{U}^1(\delta_i, \rho(\delta_i))$ and $A'_{i,0}\in\mathcal{K}^0$ and $A'_{i,1}\in\mathcal{K}^1$ as in Definition 4.1. Taking a subsequence and changing gauges, we can suppose $\{A'_{i,0}\}$ is L_l^2-convergent to $A'_\infty\in\mathcal{K}^0$ and $x_i\to x_\infty$ from Proposition 3.1. We have

$$\|A'_{i,0}-(h_{i,0}^{-1}h_{i,1})A'_{i,1}\|_{L_l^2(M-B(x_\infty, \rho_1, A'_{i,0})}<\delta_1/2$$

for sufficiently large i. From Lemma 3.4, there is $g_i\in\mathcal{G}(P_0)$ such that

$$\|A'_{i,0}-g_i(A'_{i,1})\|_{L_l^2(M, A'_0)}<\delta_1.$$

This contradicts Definition 3.6 (2).

DEFINITION 4.3. We fix D_0 so that

$$D_0>\max\{\|F(A')\|_{L^\infty}|A'\in\mathcal{K}(P_0)\}.$$

For $D>D_0$, we define $\mathcal{K}(P_1, D)$ by

$\mathcal{K}(P_1, D)=\{A\in\mathcal{K}(P_1)|\|F(A)\|_{L^\infty}\geqq D$ and A is self-dual on 2-neighbourhood of $\{x\in M|\,|F(A)|_x>D_0\}.\}$

and $\mathcal{U}^s$ $(s=0, 1)$ by

$\mathcal{U}^s=\{A\in\mathcal{U}^s(\delta_2, \rho(\delta_2))\mid$ There is $x_0\in M$ such that $\{x\mid|F(A)|\geqq D_0\}\subset B(x_0, 1/2)\}$.

PROPOSITION 4.4. *When $\{A_i\}$ is a sequence of connection in $\mathcal{K}(P_1, D_0)$ such that $\{\|F(A_i)\|_{L^\infty}\}$ is unbounded, there is a subsequence $\{A_{i'}\}$ such that* *for some $x_{i'} \in M$ and $\lambda_{i'} \downarrow 0$ the followings hold.*

(1) *There is a bundle map $e_i : \boldsymbol{R}^4 \times SU(2) \to P_1$ covering $v \in \boldsymbol{R}^4 \mapsto \exp_{x_i} \lambda_i v \in M$ such that $\{e_{i'}^*(A_{i'})\}$ is $L^2_{l,\mathrm{loc}}$-convergent to the standard instanton I_{std} on $\boldsymbol{R}^4$. I_{std} is defined by*

$$I_{\mathrm{std}} = \theta + \mathrm{Im} \frac{x d\bar{x}}{1+|x|^2}, \quad x \in \boldsymbol{H} \simeq \boldsymbol{R}^4.$$

(We identify $\boldsymbol{R}^4$ with the quaternions $\boldsymbol{H}$. θ denotes the trivial connection.)

(2) *$\{x_{i'}\}$ is convergent to some point x_∞ and there is a bundle map*

$$h_{i'} : P_0 | M - \{x_\infty\} \to P_1 | M - \{x_\infty\}$$

such that $\{h_{i'}^* A_{i'}\}$ is $L^2_{l,\mathrm{loc}}$-convergent to some connection A_∞ in $\mathcal{K}(P_0)$ on $P_0 | M - \{x_\infty\}$. A_∞ is self-dual on $B(x_\infty, 1)$.

(3) *For sufficiently large i', $|F(A_{i'})|(x) \in C^0(M)$ takes the maximum value $\|F(A_{i'})\|_{L^\infty}$ at a unique point.*

PROOF. In the situation of Proposition 2.7, we have $\|F(I)\|_{L^2} = 8\pi^2$ and $\|F(I)\|_{L^\infty} = 1$. By the classification of instantons on S^4 [AHS], this implies that I is gauge equivalent to I_{std}. We may assume that $\{x_{i'}\}$ is convergent to x_∞. We claim that $\{\|F(A_{i'})\|_{L^\infty(M - B(x_\infty, \rho_1))}\}$ is bounded for every $\rho_1 > 0$. If not, we get a contradiction as follows. Since Proposition 2.7 can be applied to $M - B(x_\infty, \rho_1)$, there is an instanton I' on $\boldsymbol{R}^4$ with $\|F(I')\|^2_{L^2} \geqq 8\pi^2$ and we have

$$\begin{aligned} (4) \quad 16\pi^2 &\leqq \|F(I)\|^2_{L^2} + \|F(I')\|^2_{L^2} \leqq \liminf \|F(A_{i'})\|^2_{L^2} \\ &= 8\pi^2 + 2 \liminf \|p_- F(A_{i'})\|^2_{L^2} \leqq 8\pi^2 + 2 m_0^2 < 16\pi^2. \end{aligned}$$

This is a contradiction. Therefore we can get $\{h_{i'}\}$ by applying Propositsion 2.3 on $M - B(x_\infty, \rho_1)$ and diagonal argument with $\rho_1 \downarrow 0$. Then A_∞ satisfies

$$\|F(A_\infty)\|_{L^2} \leqq \liminf \|F(A_i)\|^2_{L^2} - 8\pi^2 < 8\pi^2.$$

Since A_∞ is self-dual on $B(x_\infty, 1) - \{x_\infty\}$, using the removability of singularity of Uhlenbeck [U1], we can suppose A_∞ can be extended on M. Then above estimate implies that we can suppose $A_\infty \in \mathcal{C}(P_0)$. Moreover we can see $A_\infty \in \mathcal{K}(P_0)$ in a similar way to the proof of Corollary 2.4.

When $\{e_{i'}^*(A_{i'})\}$ is $L^2_{l+1,\mathrm{loc}}$-weakly convergent to I_{std}, $\{e_{i'}^* F(A_{i'})\}$ is C^2_{loc}-convergent to $F(I_{\mathrm{std}})$. Since $|F(I_{\mathrm{std}})|$ have a unique maximal point at the origin and the Hessian $H(|F(I_{\mathrm{std}})|)$ at the origin is definite, $H(|e_{i'}^* F(A_{i'})|)$ have a unique maximal point near the origin for large i'. If $|F(A_{i'})|$

has another maximal point $x'_{i'}$ for large $A_{i'}$, then $x_{i'}$ and $x'_{i'}$ are apart enough from each other to obtain a contradiction quite similar to (4).

DEFINITION 4.5. From Proposition 4.4 (3), there is a constant $D_0'>0$ such that for any $A\in\mathcal{K}(P_1, D_0')$, we can define $x(A)\in M$ by $\|F(A)\|_{L^\infty}=|F(A)|(x(A))$.

COROLLARY 4.6. *For any* $\delta>0$ *and* $\rho>0$, *there is a constant* $D=D(\delta,\rho)\geqq D_0'$ *which satisfies the following. When* A *is a connection in* $\mathcal{K}(P_1, D)$, *we can find* $A'\in\mathcal{K}(P_0)$ *and a bundle map*

$$h: P_0|M-B(x(A),\rho)\xrightarrow{\cong}P_1|M-B(x(A),\rho)$$

which satisfy

$$(1)\qquad \operatorname{supp} p_-F(A')\subset M-B(x(A),1)$$

and

$$(2)\qquad \|h^*A-A'\|_{L_l^2(M-B(x(A),\rho),A')}<\delta.$$

PROPOSITION 4.7. *There exists* $\delta_3<\min(\delta_1/36,\delta_2)$ *such that if we set* $\rho_3=\rho(\delta_3)$ *and* $D_3=D(\delta_3,\rho_3)$, *then* $\rho_3<\rho_1/2$ *and the following holds. When* A *is a connection in* $\mathcal{K}(P_1, D_3)$, *we can find* $A'\in\mathcal{K}(P_0)$ *and* $h: P_0|M-B(x(A),\rho_3)\to P_1|M-B(x(A),\rho_3)$ *which satisfy*

(1) $\operatorname{supp} p_-F(A')\subset M-B(x(A),1)$,

(2) $\|h^*A-A'\|_{L_l^2(M-B(x(A),\rho_3))}<\delta_3$,

(3) $L_l^2(M-B(x(A),\rho_1/2),A')$ *and* $L_l^2(M-B(x(A),\rho_1/2),h^*A)$ *are* 2-*equivalent*,

and

$$(4)\qquad \|h^*p_-F(A)-p_-F(A')\|_{L^2}<\alpha/3.$$

PROOF. Let $\{\delta_i\}$ be a sequence convergent to 0, $D_i=D(\delta_i,\rho(\delta_i))$ and $A_i\in\mathcal{K}(P_1,D_i)$. From Corollary 4.6 there are $A_i'\in\mathcal{K}(P_0)$ and h_i which satisfy

$$\operatorname{supp} p_-F(A_i')\subset M-B(x(A_i),1)$$

and

$$\|h_i^*A_i-A_i'\|_{L_l^2(M-B(x(A_i),\rho(\delta_i)),A_i')}<\delta_i.$$

Taking subsequence and changing gauges, we can suppose that $\{A_i'\}$ is L_l^2-convergent to $A_\infty'\in\mathcal{K}(P_0)$ and $x_i\to x_\infty$ from Proposition 3.1. Then $L_l^2(M-B(x_\infty,\rho_1/3),A_i')$ and $L_l^2(M-B(x_\infty,\rho_1/3,A_\infty')$ are uniformly equivalent, so $\{h^*A_i|M-B(x_\infty, \rho_1/3)\}$ is L_l^2-convergent to $A'_\infty|M-B(x_\infty, \rho_1/3)$, which implies that (3) and (4) hold for sufficiently large i.

LEMMA 4.8. *There exists $D_4 \geqq D_3$ such that if $A_1, A_2 \in \mathcal{K}(P_1, D_4)$ satisfy $\|A_1-A_2\|_{L_l^2(M, A_1)}<\delta_1/8$, then the distance $d(x(A_1), x(A_2))$ satisfies $d(x(A_1), x(A_2))<\rho_1/2$.*

PROOF. If $A_{1,i}$ and $A_{2,i} \in \mathcal{K}(P_1)$ satisfy

$$\|A_{1,i}-A_{2,i}\|_{L_l^2(M, A_{1,i})}<\delta_1/8,$$
$$\|F(A_{1,i})\|_{L^\infty}, \|F(A_{2,i})\|_{L^\infty} \to +\infty,$$

and

$$d(x(A_{1,i}), x(A_{2,i}))>\rho_1/2,$$

then there is a subsequence such that $x(A_{1,i'}) \to x_{1,\infty}$, $x(A_{2,i'}) \to x_{2,\infty}$ for some $x_{1,\infty}$ and $x_{2,\infty}$ with $d(x_{1,\infty}, x_{2,\infty}) \geqq \rho_1/2$. We use the notation e_i for $A_{i'}=A_{1,i'}$ as in the proof of Proposition 4.4. If we take a subsequence and change gauges, then $\{e_{i'}A_{2,i'}\}$ is $L_{l+1,\mathrm{loc}}^2$-weakly convergent to a self-dual connection as in the proof of Proposition 4.4. The assumption $x_{1,\infty} \neq x_{2,\infty}$ implies the limit is a flat connection, so we get $\|e_{i'}{}^*F(A_{2,i'})\|_{L^2} \to 0$. On the other hand,

$$\begin{aligned}&\limsup \|e_i{}^*(F(A_{1,i})-F(A_{2,i}))\|_{L^2}\\ &\leqq \limsup \|F(A_{1,i})-F(A_{2,i})\|_{L^2} \leqq \limsup (\|A_{1,i}-A_{2,i}\|_{L_l^2(A_{1,i})}+\|A_{1,i}-A_{2,i}\|_{L^2}^2)\\ &\leqq \delta_1/8(\delta_1/8+1).\end{aligned}$$

Since δ_1 is small so that $\delta_1/8(\delta_1/8+1)$ is smaller than $8\pi^2$, this is a contradiction.

PROPOSITION 4.9. *Let A_1 and A_2 be connections in $\mathcal{K}(P_1, D_4)$. Take A_1', h_1, A_2' and h_2 as in Proposition 4.7 and assume $L_l^2(M, A_1)$ and $L_l^2(M, A_2)$ are 2-equivalent and*

$$\|A_1-A_2\|_{L_l^2(M, A_1)}<\delta_1/8.$$

Then there is $g \in \mathcal{G}(P_0)$ such that

$$\|A_1'-g(A_2')\|_{L_l^2(M, A_1')}<\delta_1$$

and

$$g|M-B(x(A_1), 2\rho_1)=h_1^{-1}h_2|M-B(x(A_1), 2\rho_1).$$

PROOF. Since $d(x(A_1), x(A_2))<\rho_1/2$, we have on $M-B(x(A_1), \rho_1)$

$$L_l^2(h_1^{-1*}A_i')\sim L_l^2(A_1)\sim L_l^2(A_2)\sim L_l^2(h_2^{-1*}A_2'),$$

where $\sim$ means 2-equivalence. Then we get

$$\begin{aligned}\|h_1^{-1*}A_1'-h_2^{-1*}A_2'\|_{L_l^2(M-B(x(A_1),\rho_1),\,h_1^{-1*}A_1')} &\leqq \|h_1^{-1*}A_1'-A_1\|_{L_l^2(M-B(x(A_1),\rho_1),\,h_1^{-1*}A_1')}\\ &\quad +2\|A_1-A_2\|_{L_l^2(M,\,A_1)}+8\|A_2-h_2^{-1*}A_2'\|_{L_l^2(M-B(x(A_2),\rho_1/2),\,h_2^{-1*}A_2')}\\ &\leqq 9\delta_3+2\delta_1/8<\delta_1/2.\end{aligned}$$

Therefore we can apply Lemma 3.5 to $h_1^{-1}h_2$ to get the result.

COROLLARY 4.10. *Under the same assumption as in Proposition* 4.9, *we have*

$$\|h_1^{-1*}\sigma_0(x(A_1),A_1')-h_2^{-1*}\sigma_0(x(A_2),A_2')\|_{L^2}<\alpha/3.$$

PROOF. This is a consequence of the definition of δ_1, the gauge equivariance of σ_0 and the fact that

$$\operatorname{supp}\sigma_0(x(A_1),A_1'),\ \operatorname{supp}\sigma_0(x(A_2),A_2')\subset M-B(x(A_1),2\rho_1).$$

The following lemma is an easy consequence of Corollary 4.6 and the definition of D_0 and $\mathcal{U}^s$.

LEMMA 4.11. *If D_4 is sufficiently large, then we have*

$$\mathcal{K}(P_1,D_4)\subset\mathcal{U}^0\cup\mathcal{U}^1.$$

§ 5. Perturbation of self-dual equation.

We use the notations as in § 3 and § 4. Moreover we assume that the intersection form of M is positive definite and $H_1(M,Z)=0$.

We set

$$\mathcal{K}^s(P_1,D_4)=\mathcal{K}(P_1,D_4)\cap\mathcal{U}^s.$$

For each $A\in\mathcal{K}^1(P_1,D_4)$ we take sufficiently small $\delta(A)<\delta_1/8$ such that

$$\mathcal{U}(A)=\{A+a\in\mathcal{U}^1\mid d_A^*a=0,\ \|a\|_{L_l^2(A)}<\delta(A)\text{ and }L_l^2(A)\text{ and }\\ L_l^2(A+a)\text{ are }(2-c)\text{-equivalent for some }c>0.\}$$

gives a local slice of $\mathcal{C}(P_1)\to\mathcal{B}(P_1)$ at A. Take a locally finite covering $\{[\mathcal{U}(A_\alpha)]\}$ of $\mathcal{K}^1(P_1,D_4)/\mathcal{G}(P_1)$, which is locally compact and paracompact from Corollary 2.7, and a partition of unity $\{\beta_\alpha\}$ of $\{[\mathcal{U}(A_\alpha)]\}$. We define a cut off function $\beta_\infty:\mathcal{B}(P_1)\to[0,1]$ so that $\beta_\infty=1$ on $[\mathcal{K}^1(P_1,D_4)]$ and

$\operatorname{supp}\beta_\infty$ is contained in $\bigcup_\alpha[\mathcal{U}(A_\alpha)]$. We fix $A'_\alpha\in\mathcal{K}^1$ and h_α as in Proposition 4.7.

DEFINITION 5.1. We define $\sigma_1(A)=0$ for $A\notin\mathcal{U}^1$. For $A\in\mathcal{U}^1$ we define $\sigma_1(A)$ as follows. When $[A]\in[\mathcal{U}(A_\alpha)]$ we find a unique g_α up to ±1 such that $g_\alpha(A)\in\mathcal{U}(A_\alpha)$. Let $\tilde{g}_\alpha\in\mathcal{G}(P_0)$ be an extension of $h_\alpha^{-1}g_\alpha h_\alpha\in\mathcal{G}(P_0|M-B(x(A_\alpha),\rho_1/2))$, which exists because g_α is defined globally on M. Since σ_0 is $\mathcal{G}(P_0)$-equivariant and the support of $\sigma_0(x(A_\alpha),\tilde{g}_\alpha^{-1}(A'_\alpha))$ is contained in $M-B(x(A_\alpha),\rho_1/2)$, we can define

$$\sigma_1(A)=\beta_\infty\sum_\alpha\beta_\alpha(A)h_\alpha^{-1*}\sigma_0(x(A_\alpha),\tilde{g}_\alpha^{-1}(A'_\alpha)),$$

which is independent of the choice of $\tilde{g}_\alpha$.

LEMMA 5.2. $\qquad \|\sigma_1(A)\|_{L_l^2(M,A)}\leqq m_0.$

PROOF.
$$\begin{aligned}\|\sigma_1(A)\|_{L_l^2(M,A)}&\leqq2\sum\beta_\alpha\|h_\alpha^{-1*}\sigma_0(x(A_\alpha),\tilde{g}_\alpha^{-1}(A_\alpha))\|_{L_l^2(M,\tilde{g}_\alpha^{-1}(A_\alpha))}\\&\leqq4\sum\beta_\alpha\|\sigma_0(x(A_\alpha),\tilde{g}_\alpha^{-1}(A'_\alpha))\|_{L_l^2(M,\tilde{g}_\alpha^{-1}(A'_\alpha))}\\&\leqq4\sum\beta_\alpha m_0/4=m_0.\end{aligned}$$

LEMMA 5.3. *$p_-F+\sigma_1$ does not vanish on $\mathcal{K}^1(P_1,D_4)$.*

PROOF. Take A' and h for $A\in\mathcal{K}^1(P_1,D_4)$ as in Proposition 4.7. Then A' is in $\mathcal{K}^1$ and we have

$$\begin{aligned}&\|p_-F(A)+\sigma_1(A)\|_{L^2(M)}\\&\geqq\|p_-F(A')+\sigma_0(x(A),A')\|_{L^2(M)}-\|p_-F(A)-h^{-1*}p_-F(A')\|_{L^2(M)}\\&\quad-\sum\beta_\alpha\|h_\alpha^{-1*}\sigma_0(x(A_\alpha),\tilde{g}_\alpha^{-1}(A'_i))-h^{-1*}\sigma_0(x(A),A')\|_{L^2(M)}\\&>\alpha-\alpha/2-\sum\beta_\alpha\alpha/2=0.\end{aligned}$$

In the above estimate we used Proposition 4.7 (4), Corollary 4.8 and Corollary 4.10.

Main Theorem can be deduced from the following theorem.

THEOREM 5.4. *$\mathcal{G}(P_1)$-equivariant smooth map*

$$\sigma_1:\mathcal{C}\to L_l^2(\Omega_-^2(\operatorname{ad}P_1))$$

satisfies that if $\{A_i\}$ is a sequence of zeros of

$$p_-F+\sigma_1:\mathcal{C}\to L_{l-1}^2(\Omega_-^2(\operatorname{ad}P_1)),$$

then there exists a subsequence $\{A_i'\}$ *and a sequence* $\{g_i'\}\subset\mathcal{G}(P_1)$ *such that either of the following holds.*

(1) $\{g_i'(A_i')\}$ *is* L_l^2*-convergent to a zero of* $p\_F+\sigma_1$.

(2) *Every* A_i' *is self-dual and there exists* $x_\infty\in M$ *such that* $\{g_i'(A_i')|M-\{x_\infty\}\}$ *is* $L_{l,\mathrm{loc}}^2(M-\{x_\infty\})$*-convergent to a globally flat connection on* $M-\{x_\infty\}$.

PROOF. Lemma 5.2 and Lemma 5.3 imply $\{A_i\}\subset\mathcal{K}(P_1)-\mathcal{K}^1(P_1,D_4)$. If $\{\|F(A_i)\|_{L^\infty}\}$ is bounded, then we can claim (1) from Corollary 2.6. If not, then we can assume $\{A_i'\}\subset\mathcal{K}^0(P_1,D_4)$ and can claim (2) from Proposition 4.7 and the definition of $\mathcal{K}^0(P_1,\mathrm{D}_4)$.

Now if necessary we once again perturb σ_1 as in [D1], then we can see the perturbed moduli space

$$\mathcal{M}^{\sigma_1}=\{A\in\mathcal{C}(P_1)\,|\,p\_F(A)+\sigma_1(A)=0\}/\mathcal{G}(P_1)$$

satisfies the condition required to show Main Theorem following the discussion in [D1].

REMARK 5.5. As mentioned in Introduction, the orientability of $\mathcal{M}^{\sigma_1}$ is shown in the same way as the case that M is simply connected. In fact it suffices to show that $\mathcal{G}(P_1)/\mathrm{Center}(SU(2))$ is connected ([D1], [FU]). The primary obstruction for any $g\in\mathcal{G}(P_1)$ to be homotopic to the identity map is an element of $H^3(M,\pi_3(SU(2)))\simeq H_1(M,\boldsymbol{Z})$, which vanishes in our case. The secondary obstruction is an element of $H^4(M,\pi_4(SU(2)))\simeq\boldsymbol{Z}/2$, so we have a surjective map $\boldsymbol{Z}/2\to\pi_0(\mathcal{G}(P_1))$. It is easily shown by an explicit construction that the image of $\boldsymbol{Z}/2$ is the same as the image of $\mathrm{Center}(SU(2))$ in $\pi_0(\mathcal{G}(P_1))$ ([FU]), which implies $\pi_0(\mathcal{G}(P_1)/\mathrm{Center}(SU(2)))=0$.

§ 6. Orbifold case.

We can use the perturbation method in the previous sections to prove some properties of orbifolds. Let X be a closed oriented 4-dimensional orbifold with finite singular point $x_0, x_1, \cdots, x_n$. The neighbourhood of x_i is of the form $G_i\backslash\tilde{U}_i$ where G_i is a finite group and $\tilde{U}_i$ is a G_i-invariant neighbourhood of 0 in a 4-dimensional G_i-vector space V_i such that G_i acts on freely on $V_i-\{0\}$.

We assume the following.

(1) X has positive definite intersection form and $H_1(X,\boldsymbol{Z})=0$.

(2) $\sharp G_0 = \max\{\sharp G_i | 0 \leqq i \leqq n\}$.

(3) There is an orientation preserving $\boldsymbol{R}$-linear isomorphism $\alpha_0 : V_0 \to \boldsymbol{H}$ and a homomorphism $\varphi_0 : G_0 \to Sp(1)$ such that α_0 is G_0-equivariant via φ_0 and left multiplication of $Sp(1)$ on $\boldsymbol{H}$.

(4) There is not a homomorphism $\psi : G_0 \to \{\pm 1\} \subset Sp(1)$ such that the homomophism $\varphi_0\psi : G_0 \to Sp(1)$ is conjugate to $\varphi_0 : G_0 \to Sp(1)$ and moreover $\varphi_0\psi$ is a restriction of a homomorphism $\pi_1(X - \{x_0\}) \to Sp(1)$. (It suffices that G_0 is not isomorphic to $\boldsymbol{Z}/4$ or to any binary dihedral groups.)

(5) G_0 is not cyclic, or G_0 is cyclic and the cardinality

$$\sharp\,(\{e \in H^2(X - \{x_0\}, \boldsymbol{Z}) \,|\, \iota(e)^2[X] = \frac{1}{\sharp G_0} \text{ and } e|G_0\backslash \tilde{U}_0 - \{0\} = \pm e_0\}/\pm 1)$$

is even. Here ι denotes the map

$$H^2(X - \{x_0\}, \boldsymbol{Z}) \to H^2(X - \{x_0\}, \boldsymbol{Q}) = H^2(X, \boldsymbol{Q}),$$

and e_0 denotes the Euler class of

$$G_0\backslash(\tilde{U}_0 - \{0\} \times S^1) \to G_0\backslash \tilde{U}_0 - \{0\}.$$

S^1 is the maximal torus of $Sp(1)$ containing $\varphi_0(G_0)$.

THEOREM 6.1. *On the conditions* (1)$\sim$(5), *there exists a singular point* x_1 $(\neq x_0)$ *of* X *which satisfies the following.*

(a) $\sharp G_1 = \max\{\sharp G_i\}$ $(= \sharp G_0)$.

(b) *There is an orientation reversing* $\boldsymbol{R}$*-linear isomorphism* $\alpha_1 : V_1 \to \boldsymbol{H}$ *and a homomorphism* $\varphi_1 : G_1 \to Sp(1)$ *such that* α_1 *is* G_1*-equivariant via* φ_1.

(c) $\varphi_0 : G_0 \to Sp(1)$ and $\varphi_1 : G_1 \to Sp(1)$ *are conjugate to restrictions of a homomorphism* $f : \pi_1(X - \{x_0, x_1\}) \to Sp(1)$.

(d) *When* G_0 *is not cyclic, then we can take* x_1 *so that* G_1 *is not cyclic.*

(e) *When* G_0 *is cyclic and any singular point* x_i *satisfying* (a), (b) *and* (c) *has cyclic* G_i, *then we can take* x_1 *so that* f *is a homomorphism into* S^1 $(\subset Sp(1))$.

PROOF. When X is simply connected, (a), (b) and (c) are shown in [F] Theorem 1.3. We can extend the argument there to weaken the hypothesis $\pi_1(X) = 1$ to $H_1(X, \boldsymbol{Z}) = 0$. In fact, from the perturbation method developed in the previous sections, we only have to show that the index of the following complex is positive.

$$0 \to \Omega^0_{\mathrm{orb}}(\mathrm{ad}\, P_0) \xrightarrow{d_0} \Omega^1_{\mathrm{orb}}(\mathrm{ad}\, P_0) \xrightarrow{p-d_0} \Omega^2_{-\mathrm{orb}}(\mathrm{ad}\, P_0) \to 0.$$

Here $\mathrm{ad}\, P_0$ is the product pseudo-bundle $X \times sp(1)$, Ω^*_{orb} is the set of pseudo-forms on X and d_0 is $d \otimes \mathrm{id}$. The index of the above complex is equal to 1.

In the proof of Main Theorem we need that the corresponding index is larger than 1 in order to define $\sigma_0 \in \mathcal{P}$ in Proposition 3.3 step 4, but here we can adopt a new definition of $\mathcal{P}$ as follows.

DEFINITION 6.2. We write $\mathcal{P}$ for the set of the maps

$$\sigma : \mathcal{C}(P_0) \longrightarrow L^2_l(\Omega^2_{-\mathrm{orb}}(\mathrm{ad}\, P_0))$$

satisfying the followings.

(1) σ is smooth and $\mathcal{G}(P_0)$-equivariant.

(2) $\|\sigma\|_{\mathcal{P}} = \sup \|\sigma(A)\|_{L^2_l(M,A)} < +\infty$.

(3) $\mathrm{supp}\, \sigma(A) \subset M - B(x_0, 3)$.

In this case we do not need to take M as a parameter space as in Definition 3.2. Then the proof of a proposition corresponding to Proposition 3.3 is easier and it suffices to require the index of the above complex is positive.

Next we show (d) and (e). We define a pseudo-bundle P_0' and a flat pseudo-connection on P_0' from $f : \pi_1(X - \{x_0, x_1\}) \to Sp(1)$ in (c). The index of the corresponding complex

$$0 \to \Omega^0_{\mathrm{orb}}(\mathrm{ad}\, P_0') \xrightarrow{d_{A_0}} \Omega^1_{\mathrm{orb}}(\mathrm{ad}\, P_0') \xrightarrow{p-d_{A_0}} \Omega^2_{-\mathrm{orb}}(\mathrm{ad}\, P_0') \to 0$$

is immediately calculated by an excision argument as in [F] Lemma 5.3. The result is that the index equals $\dim sp(1)^{G_1}$, where G_1 acts on $sp(1)$ via φ_1 and adjoint action. Therefore when G_1 is cyclic and then $\dim sp(1)^{G_1}$ is positive, we can assume that A_0 is a reducible connection by the perturbation method, and in particular G_0 must be cyclic. This implies (d) and (e).

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