J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
34 (1987), 299-307.

Removable singularities for Yang-Mills connections in higher dimensions

By Hiraku NAKAJIMA

(Communicated by T. Ochiai)

§ 1. Introduction

We shall study on the removability of point singularities of Yang-Mills connections in higher dimensions. In 4-dimension K. Uhlenbeck [U1] has proved that *point singularities of Yang-Mills connections with curvature in L^2 can be removed by a gauge transformation.* But in higher dimensions, this is false if we only assume that the Yang-Mills connections have curvature in L^2 as shown by examples in [U1]. L. M. Sibner (for the case $n=3$ and $n\geqq 5$) and P. D. Smith (for the case $n=2$) have proved that point singularities in Yang-Mills connections with curvature in $L^{n/2}$ can be removed by a gauge transformation ([S1], [S2], [Sm]).

In this paper we shall strengthen L. M. Sibner's theorems. We shall prove that point singularities in Yang-Mills connections whose curvature has sufficiently small L^2-norm can be removed by a gauge transformation.

(1.1) THEOREM. *Let B be the unit ball $B_1(0)\subset \boldsymbol{R}^n$ $(n\geqq 4)$ with a Riemannian metric g which satisfies*

$$\left|\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}\right| \leqq \Lambda$$

with a constant Λ, where we assume that the coordinates $(x_1, \cdots, x_n)$ are the normal coordinates around 0 with respect to g. Let D be a smooth Yang-Mills connection in a G-vector bundle E over $B-\{0\}$ with respect to the metric g. Then there exists a constant $\varepsilon=\varepsilon(n, \Lambda, G)>0$ such that if D satisfies $\int_B |R^D|^2 dV \leqq \varepsilon$, then for some gauge transformation γ, $\gamma^*(E)$ extends to a smooth G-vector bundle $\tilde{E}$ over the full ball B and $\gamma^*(D)$ extends to a smooth Yang-Mills connection $\tilde{D}$ in $\tilde{E}$.

As a corollary of theorem (1.1), we can prove L. M. Sibner's results in dimension $n \geqq 4$, by using the Hölder inequality and the rescaling argument. But our method of proof is limited to higher dimensions, since we use the monotonicity formula which is available only for dimension $n \geqq 4$.

As a special case, we study the following situation. Let D be a smooth Yang-Mills connection in a G-vector bundle E over S^n $(n \geqq 4)$ with respect to the standard metric, and we define $f: B^{n+1}-\{0\} \to S^n$ by $f(x)=\frac{x}{|x|}$. Then f^*D is a Yang-Mills connection in f^*E over $B-\{0\}$ and has curvature $R^{f^*D}(x)=|x|^{-2}R^D\left(\frac{x}{|x|}\right)$. So f^*D satisfies

$$\int_B |R^{f^*D}|^2 dx = \frac{1}{n-3}\int_{S^n} |R^D|^2 dV.$$

If $\int_{S^n} |R^D|^2 dV \leqq (n-3)\varepsilon(n, 0, G)$, then by theorem (1.1), f^*D extends to a smooth Yang-Mills connection over B, which means $R^D=0$ on S^n. (In fact our proof of theorem (1.1) also shows that this gap theorem holds for a general metric g). Thus we have;

(1.2) COROLLARY. *Let g be a metric on S^n $(n \geqq 4)$. There exists a constant $\varepsilon=\varepsilon(n, g, G)>0$ such that if D is a smooth Yang-Mills connection in a G-vector bundle E over S^n with respect to the metric g, and satisfies*

$$\int_{S^n} |R^D|^2 dV \leqq \varepsilon,$$

then D is flat (i.e. $R^D=0$ on S^n).

To prove theorem (1.1), using a priori estimates obtained in [Na] and the monotonicity formula proved in [Pr], we first show that $|x|^2|R(x)|$ is bounded. Then we can take "the broken Hodge gauges" of E due to K. Uhlenbeck [U1]. In this gauge we can prove that for some $\alpha>0$, $|x|^{2-\alpha}|R(x)|$ is bounded. This implies that $R \in L^p$ for some $p>n/2$, from which our assertion follows by the result of K. Uhlenbeck [U2].

Corresponding results for the case of the harmonic maps have been proved in [Li] (see also [Ta]). Our results are inspired by their results.

We would like to thank Prof. T. Ochiai for his advice, and Prof. J. Kazdan who informed us the reference [S1], [S2].

§ 2. Notation

We shall describe the notation. Let M be an n-dimensional Riemannian manifold. Let E be a G-vector bundle over M where G is a compact Lie group. We write $\mathrm{Ad}\,E$ for the adjoint bundle with fiber $\mathfrak{g}$, the Lie algebra of G, and $\mathrm{Aut}\,E$ for the automorphism bundle with fiber G. We put a fiber metric on $\mathrm{Ad}\,E$ by some Ad_G-invariant metric on $\mathfrak{g}$.

We write $R=R^D$ for the curvature form of a connection D. We define the Yang-Mills action by

$$\mathcal{YM}(D)=\frac{1}{2}\int_M |R|^2 dV,$$

where dV is the volume element of M. A critical point of the Yang-Mills action is called a Yang-Mills connection. The above action is also defined for $L_1^2\cap L^4$-connections. So we say $L_1^2\cap L^4$-connection D is a weak Yang-Mills connection if D is a critical point of the above action.

It is well-known that a connection D is a Yang-Mills connection if and only if

$$D^*R=0,$$

where D^* is the adjoint operator of D.

A gauge transformation γ is a section of $\mathrm{Aut}\,E$ which acts on connections as follows;

$$\gamma^*(D)=\gamma\cdot D\cdot\gamma^{-1}.$$

Then we have $R^{\gamma^*(D)}=\gamma\cdot R^D\cdot\gamma^{-1}$. The space of Yang-Mills connections is invariant under the action of the gauge transformations.

Later we shall show that the bundle E in theorem (1.1) is trivial $E=(B-\{0\})\times \boldsymbol{R}^N$ (fact (3.5)). In this situation we take the following notation. Let d be the flat connection of $E=(B-\{0\})\times \boldsymbol{R}^N$. Then a connection D in E is given by

$$D=d+A,$$

where A is a $\mathrm{Ad}\,E=(B-\{0\})\times\mathfrak{g}$ valued one form. The curvature form R of D is given by

$$R=dA+\frac{1}{2}[A, A].$$

We denote by δ the adjoint operator of d. The Yang-Mills equations are

$$D^*R=\delta R+{}^*[A, {}^*R]=0.$$

We shall use the radial coordinates $x=(r, \phi)=\left(|x|, \frac{x}{|x|}\right)$. The one form $A=(A^r, A^\phi)$ splits into radial and spherical parts. The two form $R=(R^{r\phi}, R^{\phi\phi})$ also splits into two pieces since $R^{rr}=0$. We denote the flat connection of $E|_{\partial B_r}=\partial B_r\times \boldsymbol{R}^N$ by d^ϕ, and its adjoint operator by δ^ϕ.

§ 3. Proof of the Main Theorem

(3.1) LEMMA (Monotonicity Formula). *If D is a Yang-Mills connection in a bundle E over $B-\{0\}\subset \boldsymbol{R}^n$ $(n\geqq 4)$ for which $\int_B |R|^2 dV<\infty$, then we have for $0<\rho_1\leqq\rho_2\leqq 1$*

$$\exp(C_1\rho_2)\rho_2^{4-n}\int_{B_{\rho_2}(0)}|R|^2dV-\exp(C_1\rho_1)\rho_1^{4-n}\int_{B_{\rho_1}(0)}|R|^2dV$$
$$\geqq 4\int_{B_{\rho_2}(0)-B_{\rho_1}(0)}\exp(C_1r)r^{4-n}|R^{r\phi}|^2dV,$$

where $r=|x|$ and $C_1=C_1(n, \Lambda, G)$.

PROOF. For the case that D is a weakly Yang-Mills connection in the full ball B and stationary under the reparametrizations of B, the above inequality is proved by P. Price [Pr].

We shall show that if the variational vector field X satisfies $|X(x)|\leqq C|x|$ for some constant C, then the first variation vanishes;

$$\text{(3.2)} \qquad \int_B \{|R|^2 \operatorname{div} X-4(R(\nabla_{e_i}X, e_j),\ R(e_i, e_j))\}dV=0.$$

Then we can follow the proof of [Pr], and we get the assertion.

Take a cut-off function $f_\tau\in C^\infty(B)$ so that

i) $f_\tau(x)=0$ if $|x|<\tau$ and $f_\tau(x)=1$ if $|x|>2\tau$,
ii) $|Df_\tau(x)|<4/\tau$.

Since D is a smooth Yang-Mills connection over $B-\{0\}$, we get from the first variational formula

$$\int_B\{|R|^2 \operatorname{div}(f_\tau X)-4(R(\nabla_{e_i}(f_\tau X), e_j), R(e_i, e_j))\}dV=0.$$

Since f_τ has support in $B_{2\tau}-B_\tau$, $|Df_\tau\cdot X(x)|$ is bounded by

$$\frac{4}{\tau}\cdot C|x|\leqq 8C.$$

We have

$$|\operatorname{div}(f_\tau X)|=|Xf_\tau+f_\tau \operatorname{div} X|\leqq \text{constant independent of } \tau.$$

Similarly we have

$$|\nabla_{e_i}(f_\tau X)|=|e_i f_\tau\cdot X+f_\tau\nabla_{e_i}X|\leqq \text{constant independent of } \tau.$$

So letting $\tau\to 0$, we have got (3.2). Q.E.D.

(3.3) FACT ([Na]). There exist constants $\sigma=\sigma(n, \Lambda, G)$ and $C_2=C_2(n, \Lambda, G)$ $(n\geqq 4)$ such that if D is a Yang-Mills connection over $B_r(x)$ with $r^{4-n}\int_{B_r(x)}|R|^2 dV\leqq\sigma$, then

$$\sup_{B_{r/4}(x)}|R|^2\leqq C_2 r^{-n}\int_{B_r(x)}|R|^2 dV.$$

(3.4) LEMMA. *There exist constants $\varepsilon_1=\varepsilon_1(n, \Lambda, G)$ and $C_3=C_3(n, \Lambda, G)$ $(n\geqq 4)$ such that if D is a Yang-Mills connection in a bundle E over $B-\{0\}$ with $\int_B|R|^2 dV\leqq\varepsilon_1$, then*

$$|x|^4|R(x)|^2\leqq C_3\int_B|R|^2 dV \qquad \textit{for all } x\in B_{1/2}-\{0\}.$$

PROOF. We have the estimate for $x\in B_{1/2}-\{0\}$

$$|x|^{4-n}\int_{B_{|x|}(x)}|R|^2 dV\leqq|x|^{4-n}\int_{B_{2|x|}(0)}|R|^2 dV\leqq C\int_B|R|^2 dV\leqq C\varepsilon_1.$$

In the second inequality we have used (3.1). Thus if we choose ε_1 sufficiently small, then we can apply (3.3) in the ball $B_{|x|}(x)$ to get

$$|R(x)|^2\leqq C_2|x|^{-n}\int_{B_{|x|}(x)}|R|^2 dV\leqq C\,C_2|x|^{-4}\int_B|R|^2 dV. \qquad \text{Q.E.D.}$$

Now let $U_l=B_{2^{-l}}-B_{2^{-l-1}}$, $S_l=\partial B_{2^{-l}}$ for $l\geqq 1$. The next lemma shows the existence of broken Hodge gauges over $B_{1/2}-\{0\}=\bigcup_l U_l$, which is proved by K. Uhlenbeck [U1].

(3.5) FACT (Broken Hodge gauges [U1]). There exists $\gamma_0=\gamma_0(n, \Lambda, G)$ such that if D is a smooth connection in $B_{1/2}-\{0\}$, and the growth of the curvature satisfies $|x|^4|R(x)|^2\leqq\gamma\leqq\gamma_0$, then there exist gauges for $E|_{U_l}$ which are continuously consistent across S_l, and in which $D=d+A$, $A|_{U_l}=A_l$ and $R|_{U_l}=R_l$ have the following properties for all $l\geqq 1$;

(3.6) $\delta A_l=0$ in U_l,

(3.7) $A_l^{\phi}|_{S_l}=A_{l+1}^{\phi}|_{S_l}$,

(3.8) $\delta^{\phi}A_l^{\phi}=0$ on S_l and S_{l+1},

(3.9) $\int_{S_l} A_l^r d\sigma=\int_{S_{l+1}} A_l^r d\sigma=0$,

(3.10) $|A_l|\leqq C_4 2^{-l}\sup_{U_l}|R_l|\leqq C_4 2^l\sqrt{\gamma}$, $C_4=C_4(n, \Lambda, G)$,

(3.11) $(\lambda_1-C_5\gamma)\int_{U_l}|A_l|^2 dV\leqq 2^{-2l}\int_{U_l}|R_l|^2 dV$,
$\lambda_1=\lambda_1(n, \Lambda, G)$, $C_5=C_5(n, \Lambda, G)$,

(3.12) $(\lambda_2-C_6\gamma)\int_{S_1}|A_1^{\phi}|^2 d\sigma\leqq\int_{S_1}|R_1^{\phi\phi}|^2 d\sigma$, $\lambda_2=\lambda_2(n, \Lambda, G)$, $C_6=C_6(n, \Lambda, G)$.

From this fact we can extend the vector bundle E over $B-\{0\}$ to the full ball B through the above trivialization.

(3.13) LEMMA. *There exist constants* $\varepsilon_2=\varepsilon_2(n, \Lambda, G)$, $C_7=C_7(n, \Lambda, G)$, *and* $\alpha=\alpha(n, \Lambda, G)$ $(n\geqq 4)$ *such that if* D *is a Yang-Mills connection in a bundle* E *over* $B-\{0\}$ *with* $\int_B|R|^2 dV\leqq\varepsilon_2$, *then for some* $\alpha>0$

$$|x|^{4-\alpha}|R(x)|^2\leqq C_7\int_B|R|^2 dV$$

holds for all $x\in B_{1/2}-\{0\}$.

PROOF. Owing to (3.4) if we choose ε_2 so that $C_3\varepsilon_2\leqq\gamma_0$ and $\varepsilon_2\leqq\varepsilon_1$, we can apply (3.5).

By integration by parts we first obtain

$$\int_{U_l}\left(R_l, R_l+\frac{1}{2}[A_l, A_l]\right)dV=\int_{U_l}(R_l, DA_l)dV=\int_{S_l}-\int_{S_{l+1}}(A_l^{\phi}, R_l^{r\phi})\,d\sigma. \tag{3.14}$$

Here we have used $D^*R_l=0$ since D is a Yang-Mills connection.

From (3.10) and $|x|^4|R(x)|^2 \leqq C_8\varepsilon_2$, we can estimate the inner boundary terms as

$$\left|\int_{S_{l+1}}(A_l^\phi, R_l^{r\phi})d\sigma\right| \leqq C_7 2^l(2^l)^2(2^{-l})^{n-1}=C_7(2^{-l})^{n-4}.$$

So if $n \geqq 5$ this terms vanishes as $l \to \infty$. For $n=4$ this term also vanishes since (3.10) holds and $|x|^4|R(x)|^2 \leqq C_3\int_{B_{2|x|}(0)}|R|^2 dV$.

Thus summing up (3.14) over $l \geqq 1$, we get

$$\int_{B_{1/2}}\left(R, R+\frac{1}{2}[A, A]\right)dV=\int_{S_1}(A_1^\phi, R_1^{r\phi})d\sigma.$$

The other boundary terms cancel since (3.7) holds and the curvature R is continuous across S_l.

Using (3.11), we can estimate the error terms

$$\left|\int_{U_l}\left(R_l, \frac{1}{2}[A_l, A_l]\right)dV\right| \leqq C_8\varepsilon_2 2^{2l}\int_{U_l}|A_l|^2 dV \leqq C_8\varepsilon_2(\lambda_1-C_9\varepsilon_2)^{-1}\int_{U_l}|R_l|^2 dV.$$

If we choose $\lambda_1-C_9\varepsilon_2 \geqq \lambda_1/2$, we have

$$\int_{B_{1/2}}|R|^2 dV \leqq C_{10}\varepsilon_2\int_{B_{1/2}}|R|^2 dV+K\int_{S_1}|A_1^\phi|^2 d\sigma+\frac{1}{K}\int_{S_1}|R^{r\phi}|^2 d\sigma.$$

Here $C_{10}=C_8\cdot 2/\lambda_1$ and K is a constant which we shall fix later. The second term can be estimated from (3.12) as

$$\int_{S_1}|A_1^\phi|^2 d\sigma \leqq (\lambda_2-C_{11}\varepsilon_2)^{-1}\int_{S_1}|R^{\phi\phi}|^2 d\sigma \leqq C_{12}\int_{S_1}|R|^2 d\sigma.$$

Here we have chosen $\lambda_2-C_{11}\varepsilon_2 \geqq \lambda_2/2$.

We choose $C_{10}\varepsilon_2 \leqq 1/2$, and using the dilation $y=\frac{1}{r}x$, we apply the above inequality over $B_r=B_r(0)$ to get

$$\text{(3.15)}\quad r^{4-n}\int_{B_r}|R|^2 dV \leqq C_{13}Kr^{5-n}\int_{\partial B_r}|R|^2 d\sigma+C_{13}K^{-1}r^{5-n}\int_{\partial B_r}|R^{r\phi}|^2 d\sigma.$$

We set $F(r)=\exp(C_1 r)r^{4-n}\int_{B_r}|R|^2 dV$. Multiplying $\exp(C_1 r)$ in (3.15) and integrating from $\rho/2$ to ρ, we get

$$\int_{\rho/2}^{\rho} F(r)dr \leqq C_{14}K\exp(C_1\rho)\rho^{5-n}\int_{B_\rho}|R|^2dV$$
$$+C_{14}K^{-1}\int_{\rho/2}^{\rho}\exp(C_1r)r^{5-n}\int_{\partial B_r}|R^{r\phi}|^2d\sigma\, dr.$$

Since $F(r)$ is non-decreasing from (3.1), the left-hand side can be bounded by $\frac{\rho}{2}F\left(\frac{\rho}{2}\right)$ from below. We can also estimate the second term of the right-hand side by $C_{14}K^{-1}\rho\left(F(\rho)-F\left(\frac{\rho}{2}\right)\right)$ from above by (3.1).

Thus we get

$$\left(\frac{1}{2}+C_{14}K^{-1}\right)F\left(\frac{\rho}{2}\right)\leqq(C_{14}K+C_{14}K^{-1})F(\rho).$$

Taking K small so that $\frac{1}{2}>C_{14}K$, we have

$$\mu F\left(\frac{\rho}{2}\right)\leqq F(\rho) \quad \text{for some } \mu=\mu(n, \Lambda, G)>1.$$

By iteration we get

$$F(2^{-l})\leqq\mu^{-l}F\left(\frac{1}{2}\right)=(2^{-l})^{\log_2\mu}F\left(\frac{1}{2}\right) \quad \text{for all } l.$$

Since F is non-decreasing, we finally get

$$F(\rho)\leqq C_{15}\rho^{\beta}\int_B|R|^2dV \quad \text{where } \beta=\log_2\mu.$$

Combining this inequality with

$$|R(x)|^2\leqq C_2|x|^{-n}\int_{B_{2|x|}}|R|^2dV=C_{16}|x|^{-4}F(2|x|),$$

we get the assertion. Q.E.D.

(3.16) LEMMA. *Let D be as in* lemma (3.13). *Then the curvature $R\in L^p$ for some $p>n/2$ and is a weak solution of the Yang-Mills equations in the full ball B.*

The proof is elementary, so we omit it.

Now the main theorem follows from the following theorem of

K. Uhlenbeck [U2].

(3.17) FACT. Let D be a weak Yang-Mills connection in B with $R \in L^p$ for some $p>n/2$. Then there exists L_2^p gauge transformation $\gamma \in L_2^p(B, G)$ such that $\gamma^*(D)$ is smooth.

Since $L_2^p \subset C^0$ for $p>n/2$, the gauge transformation γ does not change the bundle E over $B-\{0\}$. Then $D=d+A$ and $\gamma^*(D)=d+A'$ satisfy the following relation;

$$A=-d\gamma\,\gamma^{-1}+\gamma\, A'\gamma^{-1}.$$

Since A and A' are smooth in $B-\{0\}$, we can conclude that γ is smooth in $B-\{0\}$.

References

[Li] Liao, G., A regularity theorem for harmonic maps with small energy, J. Differential Geom. **22** (1985), 233-241.

[Na] Nakajima, H., Compactness of the moduli space of Yang-Mills connections in higher dimensions, preprint.

[Pr] Price, P., A monotonicity formula for Yang-Mills Fields, Manuscriptia Math. **43** (1983), 131-166.

[S1] Sibner, L. M., Removable singularities of Yang-Mills fields in $\boldsymbol{R}^3$, Compositio Math. **53** (1984), 91-104.

[S2] Sibner, L. M., The isolated point singularity problem for coupled Yang-Mills equations in higher dimensions, Math. Ann. **271** (1985), 125-131.

[Sm] Smith, P. D., in preparation.

[Ta] Takakuwa, S., On removable singularities of stationary harmonic maps, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **32** (1985), 373-395.

[U1] Uhlenbeck, K., Removable singularities for Yang-Mills fields, Comm. Math. Phys. **83** (1982), 11-30.

[U2] Uhlenbeck, K., Connections with L^p-bounds on curvature, Comm. Math. Phys. **83** (1982), 31-42.

(Received October 29, 1986)

Department of Mathematics
University of Tokyo
Hongo, Bunkyo-ku, Tokyo
113 Japan