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Stochastic differential equation corresponding to the spatially homogeneous Boltzmann equation of Maxwellian and non-cutoff type

Dedicated to Professor Seizô Itô on his 60th birthday

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Introduction

It is known that a Markov process can be associated with a certain nonlinear equation of Boltzmann type ([2][3][4][5][7]). In the case of the spatially homogeneous Boltzmann equation of Maxwellian molecules, the associated Markov process was constructed by solving certain stochastic differential equation (abbreviated: SDE) based on a Poisson random measure ([7], see also [5][6]). The purposes of this paper are to simplify the proof of existence of solutions of the SDE of [7] by modifying the form of the SDE and also to give some remarks concerning the uniqueness of solutions.

We consider the Boltzmann equation of Maxwellian molecules:

$$(1) \qquad \frac{\partial u}{\partial t}=\int_{(0,\pi)\times(0,2\pi)\times R^3}(u'u_1'-uu_1)Q(\theta)d\theta\hat{d}\epsilon dx_1,\ t\geq 0,\ x\in \boldsymbol{R}^3,$$

where $u=u(t,x)$, $u_1=u(t,x_1)$, $u'=u(t,x')$, $u_1'=u(t,x_1')$ and $\hat{d}\epsilon=d\epsilon/2\pi$. $Q(\theta)$, $0<\theta<\pi$, is a positive function determined by the intermolecular repulsive force which is inversely proportional to the fifth power of their distance and has the property: $Q(\theta)\sim const.\,\theta^{-3/2}$, $\theta\downarrow 0$; so $\int_0^\pi Q(\theta)d\theta=\infty$ (non-cutoff) but

$$(\mathrm{I}) \qquad \int_0^\pi \theta Q(\theta)d\theta<\infty.$$

However, the special form of $Q(\theta)$ is not important in our methods and hence in this paper we assume that $Q(\theta)$ is an arbitrary nonnegative function satisfying only the condition (I) or even the following weaker one:

(II) $$\int_0^\pi \theta^2 Q(\theta)d\theta<\infty.$$

The two cases (I) and (II) are discussed separately.

A molecule with velocity x collides with a similar test molecule with velocity x_1; the post-collision velocities are denoted by x' and x_1', respectively. If $S(x, x_1)$ denotes the 2-dimensional sphere with center $(x+x_1)/2$ and radius $|x-x_1|/2$, then x' and x_1' are always on $S(x, x_1)$, or more precisely, $S(x', x_1')=S(x, x_1)$. Taking a spherical coordinate system on $S(x, x_1)$ with north pole x, denote by θ (resp. ϵ) the colatitude (resp. longitude) of x'. Then x' and x_1' can be regarded as functions of x, x_1, θ and ϵ. We set

$$a(x, x_1, \theta, \epsilon)=x'-x.$$

A probability measure valued function $u(t)$, $t>0$, is called a weak solution of (1) if

$$\frac{d}{dt}\langle u(t), \varphi\rangle=\langle u(t)\otimes u(t), K\varphi\rangle, \quad \varphi\in C_0^\infty(\boldsymbol{R}^3),$$

where $(K\varphi)(x, x_1)=\int_{(0,\pi)\times(0,2\pi)}\{\varphi(x')-\varphi(x)\}Q(\theta)d\theta\hat{d}\epsilon$ (see Appendix of [7]).

In [7] the following SDE was considered in connection with the Boltzmann equation (1) under the assumption (I):

$$(2)\quad X(t, \omega)=X(0, \omega) + \int_{(0,t]\times(0,\pi)\times(0,2\pi)\times(0,1)} a(X(s-, \omega), Y(s-, \alpha), \theta, \epsilon)N(dsd\theta d\epsilon d\alpha).$$

Here, $N(\cdot)$ is a Poisson random measure on $(0,\infty)\times(0,\pi)\times(0,2\pi)\times(0,1)$ with intensity measure $dsQ(\theta)\hat{d}\epsilon d\alpha$, and the solution process $\{X(t, \omega), t\geq 0\}$ is to be found on a basic probability space $\{\Omega, P\}$ under the condition that the process $\{Y(t, \alpha), t\geq 0\}$, defined on the probability space $\{(0, 1), d\alpha\}$ and describing the motion of a test molecule, is equivalent in law to $\{X(t, \omega), t\geq 0\}$. The relation between the Boltzmann equation (1) and the SDE (2) is that the probability distribution of $X(t)$ is a weak solution of (1) (general theory of SDE's including jump parts goes back to K. Itô [1]).

The modification we are making for the SDE (2) in proving existence theorem is as follows:

(i) $N(\cdot)$ is replaced by a Poisson random measure (again denoted by $N(\cdot)$) on $(0, \infty)\times(0, \pi)\times S^2\times\Omega_1$ with intensity measure $dsQ(\theta)d\theta\hat{d}\sigma dP_1$, where $\{\Omega_1, P_1\}$ is a copy of the basic probability space $\{\Omega, P\}$ and $\hat{d}\sigma$ is the uniform probability distribution on the 2-dimensional unit sphere S^2.

(ii) $Y(s-, \alpha)$ is replaced by $X(s-, \omega_1)$.

(iii) $a(x, x_1, \theta, \epsilon)$ is replaced by $b(x, x_1, \theta, \sigma)$ (the definition is given in § 1).

Thus in the case (I) the modified SDE can be written as

$$X(t, \omega)=X(0, \omega)+\int_{S_t} b(X(s-, \omega), X(s-, \omega_1), \theta, \sigma)N(dsd\theta d\sigma d\omega_1), \tag{3}$$

where $S_t=(0, t]\times(0, \pi)\times S^2\times\Omega_1$ (in the case (II) the modified SDE is given by (3.3) in § 3). Advantage of the modified SDE (3) is that the new coefficient $b(x, x_1, \theta, \sigma)$ is Lipschitz continuous in (x, x_1) as an $L^1(\hat{d}\sigma)$-valued function for each fixed θ (see Lemma 1) and that the process describing the motion of a test molecule is exactly a copy of the solution process $X(t, \omega)$, and in fact, by virtue of these, (3) can be solved easily by using a routine iteration method. The proof of pathwise uniqueness for (3) is also easy.

Most of the discussions on the uniqueness in the law sense are essentially the same as the proof of Theorem 4.1 of [7] but they are somewhat simplified. In formulating the uniqueness in the law sense we further modify the SDE (3) as follows:

(i′) $\{\Omega_1, P_1\}$ is replaced by a probability space $\{\tilde{\Omega}, \tilde{P}\}$ which need not be a copy of $\{\Omega, P\}$.

(ii′) $X(s-, \omega_1)$ is replaced by $\tilde{X}(s, \tilde{\omega})$ which is an arbitrary measurable process defined on $\{\tilde{\Omega}, \tilde{P}\}$ such that it has the same distribution as the solution $X(s, \omega)$ for each s.

The uniqueness in the law sense is proved for this modified SDE so (in the case (I)) the solution process has the same law as the solution process of (3) (and also (2)).

Similar discussions in the case (II) are also given.

§ 1. L^1-Lipschitz continuity of $b(x, x_1, \theta, \sigma)$

Think of $S(x, x_1)$ as a celestial globe with north pole x and let $C(x, x_1, \theta)$ denote the circle on $S(x, x_1)$ with constant colatitude θ. Given

$\sigma \in S^2$, let $s(x, x_1, \sigma) = 2^{-1}|x - x_1|\sigma + 2^{-1}(x + x_1)$, let $M(x, x_1, \sigma)$ denote the meridian on $S(x, x_1)$ passing through $s(x, x_1, \sigma)$ and set

$$b_0(x, x_1, \theta, \sigma) = C(x, x_1, \theta) \cap M(x, x_1, \theta) = \text{a point on } S(x, x_1),$$
$$b(x, x_1, \theta, \sigma) = b_0(x, x_1, \theta, \sigma) - x.$$

Then for fixed $x, x_1 \in \boldsymbol{R}^3$, $x \neq x_1$ and $\theta \in (0, \pi)$, $b_0(x, x_1, \theta, \sigma)$ is uniformly distributed on $C(x, x_1, \theta)$ as a random variable defined on the probability space $\{S^2, \hat{d}\sigma\}$. When $x = x_1$, we set $b(x, x_1, \theta, \sigma) = 0$.

LEMMA 1. *For any* $x, x_1, y, y_1 \in \boldsymbol{R}^3$ *and* $\theta \in (0, \pi)$,

$$\int_{S^2} |b(x, x_1, \theta, \sigma) - b(y, y_1, \theta, \sigma)| \hat{d}\sigma \leq const. \{|x - y| + |x_1 - y_1|\}\theta, \tag{1.1}$$

where const. is independent of x, x_1, y, y_1 *and* θ.

PROOF. First we consider a special case.

(i) Special case: $S(x, x_1) = S(y, y_1) = S^2$.

In this case the integral on the left in (1.1) depends only on θ and the angle ξ $(0 \leq \xi \leq \pi)$ between x and y. Therefore, it is enough to consider the case

$$x = (0, 0, 1), \quad y = (0, \sin\xi, \cos\xi), \quad x_1 = -x, \quad y_1 = -y, \tag{1.2}$$

and prove that the integral on the left in (1.1) is dominated by const. $\theta\xi$. Let A be the rotation in $\boldsymbol{R}^3$ around the x^1-axis by the angle ξ. Then in the case (1.2) we have

$$b(y, y_1, \theta, \sigma) = A^{-1} b(x, x_1, \theta, A\sigma). \tag{1.3}$$

A point $\sigma \in S^2$ is expressed as $\sigma = (r, \sqrt{1-r^2}\cos\varphi, \sqrt{1-r^2}\sin\varphi)$ where $-1 \leq r \leq 1$, $0 \leq \varphi < 2\pi$. We assume $0 \leq r \leq 1$ for simplicity. We notice that $A\sigma = (r, \sqrt{1-r^2}\cos(\varphi + \xi), \sqrt{1-r^2}\sin(\varphi + \xi))$. Next, we define α and $\tilde{\alpha}$, respectively, by

$$\cos\alpha = \frac{r}{\sqrt{r^2 + (1-r^2)\cos^2\varphi}}, \qquad \sin\alpha = \frac{\sqrt{1-r^2}\cos\varphi}{\sqrt{r^2 + (1-r^2)\cos^2\varphi}}$$
$$\cos\tilde{\alpha} = \frac{r}{\sqrt{r^2 + (1-r^2)\cos^2(\varphi+\xi)}}, \quad \sin\tilde{\alpha} = \frac{\sqrt{1-r^2}\cos(\varphi+\xi)}{\sqrt{r^2 + (1-r^2)\cos^2(\varphi+\xi)}}.$$

Then

$$b(x, x_1, \theta, \sigma)=(\sin\theta\cos\alpha, \sin\theta\sin\alpha, \cos\theta-1)$$
$$b(x, x_1, \theta, A\sigma)=(\sin\theta\cos\tilde{\alpha}, \sin\theta\sin\tilde{\alpha}, \cos\theta-1),$$

and hence from (1.3)

$$b(y, y_1, \theta, \sigma)=(\sin\theta\cos\tilde{\alpha}, \sin\theta\sin\tilde{\alpha}\cos\xi+(\cos\theta-1)\sin\xi, \\ -\sin\theta\sin\tilde{\alpha}\sin\xi+(\cos\theta-1)\cos\xi).$$

Therefore

$$\begin{aligned}(1.4)\qquad & b(x, x_1, \theta, \sigma)-b(y, y_1, \theta, \sigma)\\ &=(\sin\theta(\cos\alpha-\cos\tilde{\alpha}), \sin\theta(\sin\alpha-\sin\tilde{\alpha}\cos\xi)+(1-\cos\theta)\sin\xi,\\ &\qquad \sin\theta\sin\tilde{\alpha}\sin\xi-(1-\cos\theta)(1-\cos\xi)),\end{aligned}$$

and hence

$$\begin{aligned}&\int_{S^2}|b(x, x_1, \theta, \sigma)-b(y, y_1, \theta, \sigma)|\hat{d}\sigma\\ &=\frac{1}{2\pi}\int_0^1\int_0^{2\pi}|b(x, x_1, \theta, \sigma)-b(y, y_1, \theta, \sigma)|drd\varphi\leq \text{const.}\,\theta\xi\end{aligned}$$

follows from (1.4) once we prove the following estimates.

$$(1.5)\qquad \int_0^1\int_0^{2\pi}|\cos\alpha-\cos\tilde{\alpha}|drd\varphi\leq\text{const.}\,\xi.$$

$$(1.6)\qquad \int_0^1\int_0^{2\pi}|\sin\alpha-\sin\tilde{\alpha}\cos\xi|drd\varphi\leq\text{const.}\,\xi.$$

The proof of (1.5) is as follows. Setting $f(r, \varphi, \xi)=\cos\alpha-\cos\tilde{\alpha}$ and $f_\xi(r, \varphi, \xi)=\partial f/\partial\xi$, we have

$$\begin{aligned}\int_0^1\int_0^{2\pi}|\cos\alpha-\cos\tilde{\alpha}|drd\varphi&=\int_0^1\int_0^{2\pi}\left|\int_0^\xi f_\xi(r, \varphi, \eta)d\eta\right|drd\varphi\\ &\leq\int_0^\xi d\eta\int_0^1\int_0^{2\pi}|f_\xi(r, \varphi, \eta)|drd\varphi\leq\text{const.}\,\xi,\end{aligned}$$

because

$$\begin{aligned}\int_0^1\int_0^{2\pi}|f_\xi(r, \varphi, \eta)|drd\varphi&=\int_0^1\int_0^{2\pi}r(1-r^2)\{r^2+(1-r^2)\cos^2\varphi\}^{-3/2}|\cos\varphi\sin\varphi|drd\varphi\\ &=4\int_0^1\int_0^1 r(1-r^2)\{r^2+(1-r^2)x^2\}^{-3/2}xdrdx\\ &\leq 4\int_0^1\int_0^1 rx(rx)^{-3/2}drdx<\infty\quad(\text{use } r^2+(1-r^2)x^2\geq rx).\end{aligned}$$

As for (1.6), it is enough to prove

$$\int_0^1\int_0^{2\pi} |\sin\alpha - \sin\tilde{\alpha}|\,drd\varphi \leq \text{const.}\,\xi,$$

and for this it is also enough to prove that

$$\int_0^1\int_0^{2\pi} |g_\xi(r, \varphi, \xi)|\,drd\varphi = \text{indep. of } \xi < \infty, \tag{1.7}$$

where $g(r, \varphi, \xi) = \sin\alpha - \sin\tilde{\alpha}$ and $g_\xi(r, \varphi, \xi) = \partial g/\partial\xi$. But the left hand side of (1.7) is dominated by $I_1 + I_2$, where

$$\begin{aligned}
I_1 &= \int_0^1\int_0^{2\pi} \sqrt{1-r^2}\{r^2 + (1-r^2)\cos^2\varphi\}^{-1/2}|\sin\varphi|\,drd\varphi \\
&\leq 4\int_0^1\int_0^1 \{r^2 + (1-r^2)x^2\}^{-1/2}drdx < \infty \quad (\text{because } r^2 + (1-r^2)x^2 \geq rx). \\
I_2 &= \int_0^1\int_0^{2\pi} (1-r^2)^{3/2}\{r^2 + (1-r^2)\cos^2\varphi\}^{-3/2}\cos^2\varphi|\sin\varphi|\,drd\varphi \\
&\leq 4\int_0^1\int_0^1 \{r^2 + (1-r^2)x^2\}^{-3/2}x^2drdx < \infty.
\end{aligned}$$

(ii) General case: Since

$$b(x, x_1, \theta, \sigma) = b\left(\frac{x-x_1}{2}, -\frac{x-x_1}{2}, \theta, \sigma\right) = \frac{|x-x_1|}{2} b(e_1, -e_1, \theta, \sigma),$$

$$e_1 = \frac{x-x_1}{|x-x_1|}, \quad e_2 = \frac{y-y_1}{|y-y_1|},$$

we have

$$\begin{aligned}
&\int_{S^2} |b(x, x_1, \theta, \sigma) - b(y, y_1, \theta, \sigma)|\,\hat{d}\sigma \\
&\leq \int_{S^2} \frac{|x-x_1|}{2} \cdot |b(e_1, -e_1, \theta, \sigma) - b(e_2, -e_2, \theta, \sigma)|\,\hat{d}\sigma \\
&\quad + \left|\frac{|x-x_1|}{2} - \frac{|y-y_1|}{2}\right| \cdot \int_{S^2} |b(e_2, -e_2, \theta, \sigma)|\,\hat{d}\sigma \\
&\leq \text{const.}\,\frac{|x-x_1|}{2} \cdot |e_1 - e_2|\theta + \left|\frac{|x-x_1|}{2} - \frac{|y-y_1|}{2}\right|\theta \\
&\leq \text{const.}\left\{\left|(x-x_1) - \frac{|x-x_1|}{|y-y_1|}(y-y_1)\right| + \Big||x-x_1| - |y-y_1|\Big|\right\}\theta,
\end{aligned}$$

where we have used the result of case (i). Now (1.1) follows from the following trivial inequalities.

$$\Big| |x-x_1|-|y-y_1| \Big| \leq |x-y|+|x_1-y_1| .$$

$$\left| (x-x_1)-\frac{|x-x_1|}{|y-y_1|}(y-y_1) \right| \leq 2(|x-y|+|x_1-y_1|) .$$

§ 2. Stochastic differential equation—I

In this section we assume that $Q(\theta)$ satisfies the condition (I).

2.1. Existence theorem

We assume that a basic probability space $\{\Omega, \mathcal{F}, P\}$, equipped with a filtration $\{\mathcal{F}_t\}$ of increasing sub-σ-fields of $\mathcal{F}$, satisfies the following conditions.

(2.1) The σ-field $\mathcal{F}_0$ contains all P-negligible sets and is rich enough in the sense that, for any probability distribution μ in $\boldsymbol{R}^3$, there exists an $\mathcal{F}_0$-measurable $\boldsymbol{R}^3$-valued random variable with distribution μ.

(2.2) There exists an $\mathcal{F}_t$-adapted Poisson random measure $N(\cdot)$ on $(0, \infty)\times(0, \pi)\times S^2\times\Omega_1$ with intensity measure $dsQ(\theta)d\theta\hat{d}\sigma dP_1$ where $\{\Omega_1, P_1\}$ is a copy of $\{\Omega, P\}$.

A Poisson random measure $N(\cdot)$ on $(0, \infty)\times(0, \pi)\times S^2\times\Omega_1$ is said to be $\mathcal{F}_t$-adapted if, for each $t\geq 0$, $\mathcal{F}_t^0\subset\mathcal{F}_t$ and $\mathcal{F}_t$ is indepedent of $\mathcal{F}_\infty^t$, where $\mathcal{F}_t^0$ (resp. $\mathcal{F}_\infty^t$) is the smallest σ-field on Ω with respect to which the random variables $N(A)$, $A\in\mathcal{A}_t^0$ (resp. $A\in\mathcal{A}_\infty^t$), are measurable; here $\mathcal{A}_t^0$ (resp. $\mathcal{A}_\infty^t$) denotes the class of measurable subsets of $S_t=(0, t]\times(0, \pi)\times S^2\times\Omega_1$ (resp. $S_\infty^t=(t, \infty)\times(0, \pi)\times S^2\times\Omega_1$).

REMARK 1. The conditions (2.1) and (2.2) are not severe restrictions on $\{\Omega, \mathcal{F}, P\}$; in fact, it is easy to see that even the unit interval (0, 1) with the Lebesgue measure satisfies these conditions.

The SDE we are going to discuss is the following (=(3)):

$$X(t, \omega)=X(0, \omega)+\int_{S_t} b(X(s-, \omega), X(s-, \omega_1), \theta, \sigma)N(dsd\theta d\sigma d\omega_1). \tag{2.3}$$

By a solution of (2.3) we mean an $\mathcal{F}_t$-adapted process $X(t, \omega)$, $t\geq 0$, which is right continuous and has left limits for almost all ω. $X(t, \omega)$, $t\geq 0$, is said to be integrable if $\int_0^T E|X(t,\omega)|dt<\infty,\ 0<{}^\forall T<\infty.$

THEOREM 1. *Let the condition* (I) *be satisfied and let* $X(0,\omega)$ *be a given* $\mathcal{F}_0$*-measurable random variable with* $E|X(0,\omega)|<\infty$. *Then there exists a unique integrable solution of* (2.3).

PROOF. We set $X_0(t,\omega)=X(0,\omega)$, $t\geq 0$, and define $X_n(t,\omega)$, $n\geq 1$, successively by

$$X_n(t,\omega)=X(0,\omega)+\int_{S_t} b(X_{n-1}(s-,\omega), X_{n-1}(s-,\omega_1),\theta,\sigma)N(dsd\theta d\sigma d\omega_1). \tag{2.4}$$

The stochastic integral is well-defined for each n by virtue of the estimate $|b(x,x_1,\theta,\sigma)|\leq|x-x_1|\theta/2$. By Lemma 1 we have

$$\begin{aligned}
&E\Big\{\sup_{0\leq s\leq t}|X_{n+1}(s)-X_n(s)|\Big\}\\
&\leq E\Big\{\int_{S_t}|b(X_n(s-,\omega),X_n(s-,\omega_1),\theta,\sigma)\\
&\qquad\qquad -b(X_{n-1}(s-,\omega),X_{n-1}(s-,\omega_1),\theta,\sigma)|N(dsd\theta d\sigma d\omega_1)\Big\}\\
&\leq \text{const.}\, E\Big[\int_{S_t}\{|X_n(s-,\omega)-X_{n-1}(s-,\omega)|\\
&\qquad\qquad +|X_n(s-,\omega_1)-X_{n-1}(s-,\omega_1)|\}\theta dsQ(\theta)d\theta P_1(d\omega_1)\Big]\\
&\leq \text{const.}\int_0^t E|X_n(s,\omega)-X_{n-1}(s,\omega)|ds,
\end{aligned}$$

and hence

$$\sum_{n=0}^{\infty}E\Big\{\sup_{0\leq s\leq t}|X_{n+1}(s)-X_n(s)|\Big\}\leq\sum_{n=0}^{\infty}\frac{c(c't)^n}{n!}<\infty$$

with some constants c and c'. Therefore

$$X(t,\omega)=\lim_{n\to\infty}X_n(t,\omega)$$

exists as a uniform convergence on each finite t-interval (a.s.); clearly $X(t,\omega)$ is an integrable solution of (2.3). To prove the uniqueness, let $X(t,\omega)$ and $Y(t,\omega)$ be any integrable solutions of (2.3). Then we have $E|X(t)-Y(t)|\leq\text{const.}\int_0^t E|X(s)-Y(s)|ds$ and hence $X(t)=Y(t)$, $t\geq 0$, a.s.

2.2. Uniqueness in the law sense

The uniqueness in Theorem 1 asserts that there is only one solution of (2.3) so far as the basic probability space, the initial value and the Poisson random measure are fixed. Different choices of the basic probability space etc. yield different solutions, but we can prove that their probability laws in the path space are the same provided that their initial distributions are the same. We prove this uniqueness in the law sense for a slightly modified SDE (Theorem 2).

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space with a filtration $\{\mathcal{F}_t\}$ satisfying (2.1) and (2.2) as before. But now we replace $\{\Omega_1, P_1\}$ by $\{\tilde{\Omega}, \tilde{P}\}$ which need not be a copy of $\{\Omega, P\}$. A process $Y(t, \tilde{\omega})$ defined on $\{\tilde{\Omega}, \tilde{P}\}$ is said to be integrable if it is jointly measurable and if

$$\int_0^T \int_{\tilde{\Omega}} |Y(t, \tilde{\omega})| dt d\tilde{P} < \infty, \quad 0 < {}^\forall T < \infty .$$

Let W denote the space of $\boldsymbol{R}^3$-valued right continuous paths with left limits.

Given an integrable process $Y(t, \tilde{\omega})$ defined on $\{\tilde{\Omega}, \tilde{P}\}$, we consider the SDE

$$X(t, \omega) = X(0, \omega) + \int_{S_t} b(X(s-, \omega), Y(s, \tilde{\omega}), \theta, \sigma) N(ds d\theta d\sigma d\tilde{\omega}) \tag{2.5}$$

where $S_t = (0, t] \times (0, \pi) \times S^2 \times \tilde{\Omega}$.

PROPOSITION 1. *Let the condition* (I) *be satisfied and let* $X(0, \omega)$ *be a given* $\mathcal{F}_0$*-measurable random variable with* $E|X(0, \omega)| < \infty$. *Then for any given integrable process* $Y(t, \tilde{\omega})$ *there exists an integrable solution of* (2.5). *Also the law uniqueness holds in the following sense: The probability measure on* W *induced by a solution of* (2.5) *is uniquely determined by* u_0 *and* $\tilde{u}(t)$, $t \geq 0$, *where* u_0 *is the probability distribution of* $X(0, \omega)$ *and* $\tilde{u}(t)$ *is that of* $Y(\cdot, \tilde{\omega})$ *at time* t.

PROOF. The existence of a solution is proved by a routine iteration method as in the proof of Theorem 1. The law uniqueness is proved as follows. First we choose a sequence $\{h_n(t), t \geq 0\}$ of step functions such that

(2.6) each $h_n(t)$ is expressed as

$$h_n(t) = \begin{cases} 0 & \text{for } t = 0 \\ t_{nk} & \text{for } t_{nk} < t \leq t_{nk+1} \ (k = 0, 1, \cdots), \end{cases}$$

where $\{t_{nk}\}$ satisfies
$0=t_{n0}<t_{n1}<\cdots,\ \lim_{k\to\infty} t_{nk}=\infty,\ \lim_{n\to\infty}\sup_k(t_{nk+1}-t_{nk})=0$;

$$\lim_{n\to\infty}\int_0^t\int_{\tilde{\Omega}}|Y(s,\tilde{\omega})-Y(h_n(s),\tilde{\omega})|dsd\tilde{P}=0, \quad 0<{}^\forall t<\infty. \tag{2.7}$$

Let $X(t)$ be the solution of (2.5) and let $X_n(t)$ be the solution of

$$X_n(t)=X(0)+\int_{S_t} b(X_n(h_n(s)),\ Y(h_n(s),\tilde{\omega}),\theta,\sigma)N(dsd\theta d\sigma d\tilde{\omega}). \tag{2.8}$$

Then $X_n(t)$ is obtained as follows:

$$X_n(t)=X_n(t_{nk})+\int_{S_t-S_{t_{nk}}} b(X_n(t_{nk}),\ Y(t_{nk},\tilde{\omega}),\theta,\sigma)dN, \qquad t_{nk}<t\leq t_{nk+1} \quad (k\geq 0). \tag{2.9}$$

Making use of the estimate

$$|b(x,x_1,\theta,\sigma)|\leq|x-x_1|\theta/2, \tag{2.10}$$

and also (2.7), we can easily prove that

$$E|X_n(s)|\leq \text{const.}, \quad 0\leq s\leq t, \tag{2.11}$$

$$\sup\{E|X_n(t_1)-X_n(t_2)|: 0\leq t_1,\ t_2\leq t,\ |t_1-t_2|\leq\varepsilon,\ n\geq 1\}\to 0 \quad \text{as } \varepsilon\downarrow 0, \tag{2.12}$$

where const. may depend on t but not on n. Then, making use of Lemma 1 and then (2.7), (2.12), we have

$$\begin{aligned} E|X_n(t)-X(t)| &\leq \text{const.}\int_0^t E|X_n(h_n(s))-X(s)|ds \\ &\quad +\text{const.}\int_0^t\int_{\tilde{\Omega}}|Y(h_n(s))-Y(s)|dsd\tilde{P} \\ &\leq \text{const.}\int_0^t E|X_n(s)-X(s)|ds+o(1), \end{aligned}$$

and hence by Gronwall's inequality

$$\lim_{n\to\infty} E|X_n(t)-X(t)|=0. \tag{2.13}$$

On the other hand, by (2.9) we have for any $\xi\in \boldsymbol{R}^3$ and $t_{nk}<t\leq t_{nk+1}$

$$E\left[\exp\left\{\sqrt{-1}\xi\cdot X_n(t)\right\}\middle|\mathcal{F}_{t_{nk}}\right]$$

$$= \exp\left[\sqrt{-1}\xi\cdot x+(t-t_{nk})\int_{(0,\pi)\times S^2\times\tilde{\Omega}}\left\{e^{\sqrt{-1}\xi\cdot b(x,Y(t_{nk},\tilde{\omega}),\theta,\sigma)}-1\right\}Q(\theta)d\theta\hat{d}\sigma d\tilde{P}\right]$$

$$= \exp\left[\sqrt{-1}\xi\cdot x+(t-t_{nk})\int_{(0,\pi)\times S^2\times R^3}\left\{e^{\sqrt{-1}\xi\cdot b(x,y,\theta,\sigma)}-1\right\}Q(\theta)d\theta\hat{d}\sigma\tilde{u}(t_{nk},dy)\right]$$

where we put $x=X_n(t_{nk})$. This conditional expectation formula implies that the probability measure on W induced by the process $X_n(t)$ is uniquely determined by u_0 and $\tilde{u}(t)$, $t\geq 0$. Therefore, by (2.13) the probability measure on W induced by $X(t)$ is also uniquely determined by u_0 and $\tilde{u}(t)$, $t\geq 0$. This completes the proof of the proposition.

Now we consider the following SDE for which we are going to prove the law uniqueness:

$$(2.14a)\qquad X(t)=X(0)+\int_{S_t}b(X(s-),\tilde{X}(s,\tilde{\omega}),\theta,\sigma)N(dsd\theta d\sigma d\tilde{\omega}).$$

Here, an $\mathcal{F}_t$-adapted integrable solution $X(t)$ is found under the condition that

(2.14b) $\tilde{X}(t,\tilde{\omega})$ is a measurable process defined on the probability space $\{\tilde{\Omega},\tilde{P}\}$ such that $\tilde{X}(t,\tilde{\omega})$ has the same distribution as $X(t)$ for each t.

REMARK 2. When $\{\tilde{\Omega},\tilde{P}\}=\{\Omega_1,P_1\}$, a solution of (2.3) is also a solution of (2.14).

THEOREM 2. *Let the condition* (I) *be satisfied and let* $X(0,\omega)$ *be any* R^3*-valued and* $\mathcal{F}_0$*-measurable random variable with* $E|X(0,\omega)|<\infty$. *Then the probability measure on* W *induced by any integrable solution of* (2.14) *is uniquely determined by the probability distribution* u_0 *of the initial value* $X(0,\omega)$.

PROOF. Let $\hat{\Omega}=[0,1]$, $\hat{\mathcal{F}}=$the σ-field of Borel subsets of $[0,1]$, $\hat{P}(A)=$ the Lebesgue measure of A $(\in\hat{\mathcal{F}})$ and let $\{\Omega_1,P_1\}$ be a copy of $\{\hat{\Omega},\hat{P}\}$. As in 2.1 we construct, on the probability space $\{\hat{\Omega},\hat{P}\}$, a u_0-distributed random variable $\hat{X}$ and a Poisson random measure $\hat{N}(\cdot)$ on $(0,\infty)\times(0,\pi)\times S^2\times\Omega_1$ with intensity measure $dtQ(\theta)d\theta\hat{d}\sigma dP_1$ so that $\hat{X}$ and $\hat{N}(\cdot)$ are independent. We then consider the SDE of the type (2.3)

$$(2.15)\qquad \hat{X}(t)=\hat{X}+\int_{\hat{S}_t}b(\hat{X}(s-),\hat{X}(s-,\omega_1),\theta,\sigma)d\hat{N},$$

where $\hat{S}_t=(0, t]\times(0, \pi)\times S^2\times\Omega_1$. We are going to prove that for any solution $X(t)$ of (2.14) there exists a solution of (2.15) which (as a process) is equivalent in law to $X(t)$. Once this has been proved, the law uniqueness of solutions of (2.14) follows immediately from the pathwise uniqueness of solutions of (2.15).

On the probability space $\{\Omega_1, P_1\}$ we can find an $\boldsymbol{R}^3$-valued right continuous process $\hat{X}_0(t, \omega_1)$ having left limits which is equivalent in law to a solution $X(t)$ of (2.14). Given such a process $\hat{X}_0(t, \omega_1)$, we consider the SDE

$$\hat{X}(t)=\hat{X}+\int_{\hat{S}_t} b(\hat{X}(s-), \hat{X}_0(s-, \omega_1), \theta, \sigma)d\hat{N}. \tag{2.16}$$

Since the both (test) processes $\tilde{X}(t, \tilde{\omega})$ and $\hat{X}_0(t-, \omega_1)$ in (2.14) and (2.16) have the same marginal distribution at each time t, Proposition 1 implies that the unique solution $\hat{X}_1(t)$ of (2.16) is equivalent in law to a solution process $X(t)$ of (2.14). Next we construct $\hat{X}_n(t)$ for $n\geq 2$ by $\hat{X}_n(t)=$the solution of (2.16) with $\hat{X}_0(s-, \omega_1)$ replaced by $\hat{X}_{n-1}(s-, \omega_1)$. Then as in the proof of Theorem 1 we can prove that $\hat{X}_n(t)$ converges to a solution $\hat{X}(t)$ of (2.15) as $n\to\infty$. Since each process $\hat{X}_n(t)$ is equivalent in law to $X(t)$, so is $\hat{X}(t)$. This completes the proof of Theorem 2.

§ 3. Stochastic differential equation—II

In this section we assume

$$\int_0^\pi \theta Q(\theta)d\theta=\infty, \qquad \int_0^\pi \theta^2 Q(\theta)d\theta<\infty. \tag{3.1}$$

Let $\{\Omega, P\}$, $\{\Omega_1, P_1\}$ and $N(\cdot)$ be the same as in 2.1 and set $M(A)=N(A)-\lambda(A)$ for a measurable subset A of $(0, \infty)\times(0, \pi)\times S^2\times\Omega_1$ with $\lambda(A)=\int_A dtQ(\theta)d\theta\hat{d}\sigma dP_1<\infty$. Then the stochastic integral on the right of (2.3) can be written as

$$\int_{S_t} b(X(s-, \omega), X(s-, \omega_1), \theta, \sigma)dM+\int_{S_t} b(X(s-, \omega), X(s-, \omega_1), \theta, \sigma)d\lambda.$$

The first integral in the above will make sense under the condition (3.1) while the second integral equals

$$-c\int_0^t \{X(s, \omega)-\overline{X(s, \omega)}\}ds$$

where $\overline{X(s, \omega)}=E\{X(s, \omega)\}$ and $c=\int_0^\pi 2^{-1}(1-\cos\theta)Q(\theta)d\theta$. So we are led to the following SDE:

$$(3.2) \qquad X(t, \omega)=X(0, \omega)+\int_{S_t} b(X(s-, \omega), X(s-, \omega_1), \theta, \sigma)dM \\ -c\int_0^t \{X(s, \omega)-\overline{X(s, \omega)}\}ds.$$

If $\int_{S^2}|b(x, x_1, \theta, \sigma)-b(y, y_1, \theta, \sigma)|^2\hat{d}\sigma$ were dominated by a constant multiple of $\{|x-y|^2+|x_1-y_1|^2\}\theta^2$, we could solve (3.2) easily. But this is not likely to be true. So we further modify the SDE (3.2) so that it can be solved in an easier way. First we introduce the predictable σ-field $\mathcal{P}$ on $[0, \infty)\times\Omega\times\Omega_1$; it is defined as the smallest σ-field on $[0, \infty)\times\Omega\times\Omega_1$ with respect to which all functions $a(t, \omega, \omega_1)$ satisfying the following conditions (i) and (ii) are measurable.

(i) For each fixed $t\geq 0$, $a(t, \omega, \omega_1)$ is $\mathcal{F}_t\otimes\hat{\mathcal{F}}$-measurable where $\{\Omega_1, \hat{\mathcal{F}}, P_1\}$ is a copy of $\{\Omega, \mathcal{F}, P\}$.
(ii) For fixed ω and ω_1, $a(t, \omega, \omega_1)$ is left continuous in t.

Let $\mathcal{R}$ denote the class of predictable processes (i.e., $\mathcal{P}$-measurable functions on $[0, \infty)\times\Omega\times\Omega_1$) with values in the space $0(3)$ of orthogonal matrices of degree 3. Then our modified SDE can be written as

$$(3.3) \quad X(t, \omega)=X(0, \omega)+\int_{S_t} b(X(s-, \omega), X(s-, \omega_1), \theta, R(s, \omega, \omega_1)\sigma)dM \\ -c\int_0^t \{X(s, \omega)-\overline{X(s, \omega)}\}ds.$$

By a solution of (3.3) we mean an $\mathcal{F}_t$-adapted process $X(t, \omega)$, $t\geq 0$, which is right continuous in t, has left limits for almost all ω and satisfies (3.3) with some $R=R(t, \omega, \omega_1)\in\mathcal{R}$. $X(t, \omega)$ is said to be square integrable if $\int_0^T E\{|X(t, \omega)|^2\}dt<\infty$, $0<{}^\forall T<\infty$.

REMARK 3. If we set

$$\tilde{N}(A)=\int_{S_\infty} \mathbf{1}_A(s, \theta, R(s, \omega, \omega_1)\sigma)N(dsd\theta d\sigma d\omega_1),$$
$$\tilde{M}(A)=\tilde{N}(A)-\lambda(A),$$

then $\tilde{N}(\cdot)$ is also an $\mathcal{F}_t$-adapted Poisson random measure with the same intensity measure λ and (3.3) becomes (3.2) with M replaced by $\tilde{M}$. In this sense (3.2) and (3.3) may be regarded as equivalent. The corresponding martingale problems are the same.

For $\sigma, \sigma' \in S^2$ we denote by $R(\sigma, \sigma')$ the rotation (orthogonal matrix) in $\boldsymbol{R}^3$ which sends σ to σ' along the geodesic connecting σ with σ', and set $R(x, x_1, y, y_1) = R((x-x_1)|x-x_1|^{-1}, (y-y_1)|y-y_1|^{-1})$ (for $x \neq x_1, y \neq y_1$), $=$the identity matrix (otherwise). Then we have the following lemma (see Lemma 3.1 of [7]).

LEMMA 2. *For any* $x, x_1, y, y_1 \in \boldsymbol{R}^3$

$$|b(x, x_1, \theta, \sigma) - b(y, y_1, \theta, R(x, x_1, y, y_1)\sigma)| \leq \text{const.}\{|x-y| + |x_1 - y_1|\}\theta.$$

THEOREM 3. *Let the condition* (II) *be satisfied and let* $X(0, \omega)$ *be a given* $\mathcal{F}_0$*-measurable random variable with* $E\{|X(0, \omega)|^2\} < \infty$. *Then there exists a square integrable solution of* (3.3). *Moreover, the law uniqueness holds for* (3.3) *in the sense that the probability measure on* W *induced by any square integrable solution of* (3.3) *is uniquely determined by the probability distribution* u_0 *of* $X(0, \omega)$.

PROOF. Define $X_n(t, \omega)$, $n \geq 0$, by

$$\begin{aligned} X_0(t, \omega) &= X(0, \omega), \\ X_n(t, \omega) &= X(0, \omega) + \int_{S_t} b(X_{n-1}(s-, \omega), X_{n-1}(s-, \omega_1), \theta, R_{n-1}\sigma)dM \\ &\qquad - c\int_0^t \{X_{n-1}(s, \omega) - \overline{X_{n-1}(s, \omega)}\}ds, \qquad n \geq 1, \end{aligned}$$

where $R_n = R_n(s, \omega, \omega_1) = \prod_{k=1}^{n} R(X_{k-1}(s-, \omega), X_{k-1}(s-, \omega_1), X_k(s-, \omega), X_k(s-, \omega_1))$. Then, making use of Lemma 2 and the convergence of the second integral of (3.1) we have

$$\begin{aligned} &E\Big\{\sup_{0\leq s\leq t} |X_{n+1}(s) - X_n(s)|^2\Big\} \\ &\leq 8E\Big\{\int_{S_t} |b(X_n(s-, \omega), X_n(s-, \omega_1), \theta, R_n\sigma) - b(X_{n-1}(s-, \omega), X_{n-1}(s-, \omega_1), \\ &\qquad\qquad \theta, R_{n-1}\sigma)|^2 d\lambda\Big\} \\ &\quad + 2c^2 t\int_0^t E\{|X_n(s, \omega) - \overline{X_n(s, \omega)} - X_{n-1}(s, \omega) + \overline{X_{n-1}(s, \omega)}|^2\}ds \\ &\leq \text{const.}\ (1+t)\int_0^t E\{|X_n(s) - X_{n-1}(s)|^2\}ds, \end{aligned}$$

and hence by a routine argument we can prove that

$$(3.4) \qquad \sum_{n=1}^{\infty} \sup_{0\leq s\leq t} |X_n(s, \omega)-X_{n-1}(s, \omega)|$$

is convergent for all $t\geq 0$ with probability 1. We denote by $\tilde{\Omega}$ the set of ω for which (3.4) is convergent for all $t\geq 0$ and also by $\tilde{\Omega}_1$ $(\subset\Omega_1)$ the copy of $\tilde{\Omega}$. We set

$$X(t, \omega)=\begin{cases}\lim_{n\to\infty} X(t, \omega), & \omega\in\tilde{\Omega}\\ 0 \quad , & \text{otherwise,}\end{cases}$$

$$\Gamma=\{(t, \omega, \omega_1)\in[0, \infty)\times\tilde{\Omega}\times\tilde{\Omega}_1 : X(t-, \omega)\neq X(t-, \omega_1)\}.$$

Then $\Gamma\in\mathcal{P}$. We first claim that $R_n(t, \omega, \omega_1)$ is convergent as $n\to\infty$ for each fixed $(t, \omega, \omega_1)\in\Gamma$. Using the notation $\|A\|=\sup\{|Ax| : |x|=1\}$ for a matrix A, we have

$$\begin{aligned}&\|R_n(t, \omega, \omega_1)-R_{n-1}(t, \omega, \omega_1)\|\\ &=\|\{R(X_{n-1}(t-, \omega), X_{n-1}(t-, \omega_1), X_n(t-, \omega), X_n(t-, \omega_1))-I\}R_{n-1}(t, \omega, \omega_1)\|\\ &\leq\sqrt{3}\,\|R(X_{n-1}(t-, \omega), X_{n-1}(t-, \omega_1), X_n(t-, \omega), X_n(t-, \omega_1))-I\|\\ &\leq\sqrt{3}\left|\frac{X_{n-1}(t-, \omega)-X_{n-1}(t-, \omega_1)}{|X_{n-1}(t-, \omega)-X_{n-1}(t-, \omega_1)|}-\frac{X_n(t-, \omega)-X_n(t-, \omega_1)}{|X_n(t-, \omega)-X_n(t-, \omega_1)|}\right|\\ &\leq\frac{2\sqrt{3}\{|X_{n-1}(t-, \omega)-X_n(t-, \omega)+X_{n-1}(t-, \omega_1)-X_n(t-, \omega_1)|\}}{|X_n(t-, \omega)-X_n(t-, \omega_1)|}.\end{aligned}$$

If $(t, \omega, \omega_1)\in\Gamma$, then $|X_n(t-, \omega)-X_n(t-, \omega_1)|\geq\varepsilon$ for some $\varepsilon>0$, and for all sufficiently large n (say, for $n\geq n_0$) and hence

$$\begin{aligned}&\sum_{n=n_0}^{\infty}\|R_n(t, \omega, \omega_1)-R_{n-1}(t, \omega, \omega_1)\|\\ &\leq\frac{2\sqrt{3}}{\varepsilon}\sum_{n=n_0}^{\infty}\{|X_n(t-, \omega)-X_{n-1}(t-, \omega)|+|X_n(t-, \omega_1)-X_{n-1}(t-, \omega_1)|\}<\infty.\end{aligned}$$

Next we define $R\in\mathcal{R}$ by

$$R(t, \omega, \omega_1)=\begin{cases}\lim_{n\to\infty} R_n(t, \omega, \omega_1), & (t, \omega, \omega_1)\in\Gamma\\ \text{identity}, & \text{otherwise,}\end{cases}$$

and claim that $\{X(t, \omega), R\}$ is a solution of (3.3). For this it is enough to prove that

$$(3.5) \qquad E\left[\left|\int_{S_t}\{b(X_n(s-, \omega), X_n(s-, \omega_1), \theta, R_n\sigma) -b(X(s-, \omega), X(s-, \omega_1), \theta, R\sigma)\}dM\right|^2\right]$$

tends to 0 as $n\to\infty$. If $\tilde{R}_n=R(X_n(s-,\omega), X_n(s-,\omega_1), X(s-,\omega), X(s-,\omega_1))$, then (3.5) is dominated by

$$\begin{aligned}&2E\left\{\int_{S_t}|b(X_n(s-,\omega), X_n(s-,\omega_1), \theta, R_n\sigma)\right.\\&\qquad\qquad\left.-b(X(s-,\omega), X(s-,\omega_1), \theta, \tilde{R}_nR_n\sigma)|^2d\lambda\right\}\\&+2E\left\{\int_{S_t}|b(X(s-,\omega)-X(s-,\omega_1), \theta, \tilde{R}_nR_n\sigma)\right.\\&\qquad\qquad\left.-b(X(s-,\omega), X(s-,\omega_1), \theta, R\sigma)|^2d\lambda\right\}\\&\leq \text{const.}\int_0^t E\{|X_n(s)-X(s)|^2\}ds+2E\left\{\int_{S_t}b_n(s,\omega,\omega_1,\theta,\sigma)d\lambda\right\},\end{aligned}$$

where

$$\begin{aligned}b_n(s,\omega,\omega_1,\theta,\sigma)=|b(X(s-,\omega), X(s-,\omega_1), \theta, R_n'\sigma)\\-b(X(s-,\omega), X(s-,\omega_1), \theta, \sigma)|^2, \quad R_n'=\tilde{R}_nR_nR^{-1}.\end{aligned}$$

Moreover, we can easily prove that $R_n'(s,\omega,\omega_1)\to I$ as $n\to\infty$ for each fixed $(s,\omega,\omega_1)\in\Gamma$, and hence for each fixed $(s,\omega,\omega_1,\theta)\in(0,\infty)\times\tilde{\Omega}\times\tilde{\Omega}_1\times(0,\pi)$ we have

$$\lim_{n\to\infty} b_n(s,\omega,\omega_1,\theta,\sigma)=0$$

for almost all σ with respect to $\hat{d}\sigma$. Since we also have the bound $b_n(s,\omega,\omega_1,\theta,\sigma)\leq|X(s-,\omega)-X(s-,\omega_1)|^2\cdot\theta^2$, an application of Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} E\left\{\int_{S_t}b_n(s,\omega,\omega_1,\theta,\sigma)d\lambda\right\}=0,$$

which implies that (3.5) tends to 0 as $n\to\infty$. Thus the existence proof is finished.

To prove the law uniqueness let $h_n(t)=k2^{-n}$ for $k2^{-n}<t\leq(k+1)2^{-n}$ and $h(0)=0$. Let $X(t,\omega)$ be any square integrable solution of (3.3) with auxiliary process $R=R(t,\omega,\omega_1)\in\mathfrak{R}$ and consider the SDE

$$\begin{aligned}(3.6)\qquad Y_n(t,\omega)=X(0,\omega)+\int_{S_t}b(Y_n(h_n(s),\omega), Y_n(h_n(s),\omega_1), \theta, R_n\sigma)dM\\-c\int_0^t\{Y_n(h_n(s))-\overline{Y_n(h_n(s))}\}ds,\end{aligned}$$

where

$$\begin{aligned}R_n&=R_n(s,\omega,\omega_1)\\&=R(X(s-,\omega), X(s-,\omega_1), Y_n(h_n(s),\omega), Y_n(h_n(s),\omega_1))R(s,\omega,\omega_1).\end{aligned}$$

(3.6) can be solved easily; in fact, once we know $Y_n(t,\omega)$ for $0\leq t\leq k2^{-n}$, we can define $Y_n(t,\omega)$ for $k2^{-n}<t\leq(k+1)2^{-n}$ by the right hand side of (3.6). We can prove that

$$E\{|Y_n(t)-Y_n(s)|^2\}\leq \text{const.}\,|t-s|,\quad 0\leq s<t\leq T,$$

and also, by making use of Lemma 2, that

$$E\{|Y_n(t)-X(t)|^2\}\leq \text{const.}\int_0^t E\{|Y_n(s)-X(s)|^2\}ds+\text{const.}\,\Delta_n(t),$$

where

$$\Delta_n(t)=\sup\{E(|Y_n(u)-Y_n(s)|^2):0\leq s<u\leq t,\ u-s\leq 2^{-n}\}\leq \text{const.}\,2^{-n},$$

const. being independent of n. Therefore

$$E\{|Y_n(t)-X(t)|^2\}\longrightarrow 0,\qquad n\to\infty. \tag{3.7}$$

On the other hand, let $k2^{-n}<t\leq(k+1)2^{-n}$, $x\in \boldsymbol{R}^3$ and set

$$\begin{aligned}\Phi_n(t)=ix\cdot Y_n(k2^{-n},\omega)&+\int_{S_t-S_{k2^{-n}}}(e^{ix\cdot b}-1-ix\cdot b)d\lambda\\ &-ic(t-k2^{-n})x\cdot\{Y_n(k2^{-n},\omega)-\overline{Y_n(k2^{-n},\omega)}\}\\ &\text{(where } b=b(Y_n(k2^{-n},\omega),\ Y_n(k2^{-n},\omega_1),\theta,\sigma)).\end{aligned}$$

Then, for $k2^{-n}<t\leq(k+1)2^{-n}$ we have

$$E\{e^{ix\cdot Y_n(t)}|\mathcal{F}_{k2^{-n}}\}=e^{\Phi_n(t)},\qquad \text{a.s.},$$

and hence the probability measure on W induced by the process $Y_n(t,\omega)$, $t\geq 0$, is uniquely determined by u_0. This combined with (3.7) proves the law uniqueness of square integrable solutions of (3.3).

In the rest of this section let $\{\Omega,P\}$ and $\{\tilde{\Omega},\tilde{P}\}$ be the same as in 2.2 and consider the SDE

$$\begin{aligned}X(t)=X(0)&+\int_{S_t}b(X(s-),\ Y(s,\tilde{\omega}),\theta,R(s,\omega,\tilde{\omega})\sigma)dM\\ &-c\int_0^t\{X(s)-\overline{Y(s,\tilde{\omega})}\}ds\end{aligned} \tag{3.8}$$

where $Y(t,\tilde{\omega})$ is a given square integrable process defined on $\{\tilde{\Omega},\tilde{P}\}$ and $R=R(t,\omega,\tilde{\omega})$ is similar to one in (3.3).

PROPOSITION 2. *Let the condition* (II) *be satisfied and* $X(0,\omega)$ *be a given* $\mathcal{F}_0$*-measurable random variable with* $E\{|X(0,\omega)|^2\}<\infty$. *Then for* *any given square integrable process* $Y(t, \tilde{\omega})$ *there exists a square integrable solution of* (3.8). *Also the law uniqueness holds in the same sense as in Proposition* 1.

The above proposition can be proved by a method similar to Proposition 1. Only point one has to be careful is that the SDE (2.8) is now replaced by

$$X_n(t)=X(0)+\int_{S_t} b(X_n(h_n(s)),\ Y(h_n(s), \tilde{\omega}),\ \theta,\ R_n\sigma)dM \\ -c\int_0^t \{X_n(h_n(s))-\overline{Y(h_n(s))}\}ds,$$

where

$$R_n=R_n(s, \omega, \tilde{\omega}) \\ =R(X(s-, \omega),\ Y(s, \tilde{\omega}),\ X_n(h_n(s), \omega),\ Y(h_n(s), \tilde{\omega}))R(s, \omega, \tilde{\omega}).$$

Next we consider the SDE

$$(3.9)\quad X(t)=X(0)+\int_{S_t} b(X(s-),\ \tilde{X}(s, \tilde{\omega}),\ \theta,\ R(s, \omega, \tilde{\omega})\sigma)dM-c\int_0^t \{X(s)-\overline{X(s)}\}ds,$$

where $\tilde{X}(t, \tilde{\omega})$ satisfies the same conditions as stated in (2.14b). Then the following theorem can be proved in the same spirit as in Theorem 2.

THEOREM 4. *Under the condition* (II) *the probability measure on* W *induced by any square integrable solution of* (3.9) *is uniquely determined by the probability distribution* u_0 *of* $X(0, \omega)$.

REMARK 4. The martingale problems corresponding to the SDE's (2), (3), (2.14), (3.2), (3.3) and (3.9) have the same form and the probability distribution $u(t)$, at time t, of a solution to any one of these SDE's is a weak solution of the Boltzmann equation (1).

References

[1] Itô, K., On stochastic differential equations, Mem. Amer. Math. Soc. **4** (1951).
[2] McKean, H. P., A class of Markov processes associated with nonlinear parabolic equations, Proc. Nat. Acad. Sci. USA **56** (1966), 1907–1911.
[3] Murata, H., Propagation of chaos for Boltzmann-like equation of non-cutoff type in the plane, Hiroshima Math. J. **7** (1977), 479–515.
[4] Sznitman, A. S., Équations de type de Boltzmann, spatialement homogènes, Z. Wahrsch. Verw. Gebiete **66** (1984), 559–592.

[5] Tanaka, H., On Markov process corresponding to Boltzmann's equation of Maxwellian gas, Proc. 2nd Japan-USSR Symp. on Probab. Theory, Lecture Notes in Math. vol. 330, Springer-Verlag, Berlin-Heidelberg-New York, 1973, 478-489.
[6] Tanaka, H., Stochastic differential equation associated with the Boltzmann equation of Maxwellian molecules in several dimensions, Proc. of Conf. on Stochastic Analysis (edited by A. Friedman and M. Pinsky), Academic Press, New York-San Fancisco-London, 1978, 301-314.
[7] Tanaka, H., Probabilistic treatment of the Boltzmann equation of Maxwellian molecules, Z. Wahrsch. Verw. Gebiete **46** (1978), 67-105.

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