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Symmetric Markov processes with mean field potentials

Dedicated to Professor Seizô Itô on his 60th birthday

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§ 0. Introduction.

Let M be a Polish space, μ be a probability measure on M and $\{P_x; x\in M\}$ be a family of probability measures on $D([0,\infty)\to M)$ which induces a μ-symmetric Markov process on M. We denote by $\{P_t\}_{t\geqq 0}$, $\mathcal{E}$ and $\mathcal{L}$ the induced semigroup, the Dirichlet form and the infinitesimal generator on $L^2(M; d\mu)$ respectively. Since P_μ is a stationary probability measure on $D([0,\infty)\to M)$, it is extensible to a stationary probability measure on W $(\equiv D(\boldsymbol{R}\to M))$. We denote it also by P_μ for simplicity of notation.

Now let $V: M\times M\to \boldsymbol{R}$ be a symmetric bounded continuous function. Let Φ_T, $T>0$, be a probability measure on W given by

$$\Phi_T(dw)=Z_T^{-1}\exp\Big(\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}V(w(t),w(s))dtds\Big)P_\mu(dw), \tag{0.1}$$

where

$$Z_T=\int_W \exp\Big(\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}V(w(t),w(s))dtds\Big)P_\mu(dw). \tag{0.2}$$

In this paper, we shall study whether Φ_T converges as $T\to\infty$ and, if so, what is the limit probability measure.

The basic assumption which we impose on the Markov process P_μ is the following.

(H.1) $(1-\mathcal{L})^{-1}$ is a compact operator in $L^2(M; d\mu)$.

(H.2) P_μ satisfies the Donsker-Varadhan's large deviation principle, i.e.,

$$\varliminf_{T\to\infty}\frac{1}{2T}\log P_\mu(\rho_T(w)\in G)\geqq -\inf\{\mathcal{E}(\phi,\phi);\ \phi(x)^2\mu(dx)\in G\} \tag{0.3}$$

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for any open set G in $\mathcal{P}(M)$, and

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \log P_\mu(\rho_T(w) \in K) \leqq -\inf\{\mathcal{E}(\phi,\phi)\,;\,\phi(x)^2\mu(dx) \in K\} \tag{0.4}$$

for any closed set K in $\mathcal{P}(M)$, where $\mathcal{P}(M)$ denotes the metric space consisting of all probability measures on M with Prohorov metric, and

$$\rho_T(w)(dx) \equiv \frac{1}{2T}\int_{-T}^{T} \delta_{w(t)}(x)dt \in \mathcal{P}(M), \qquad w \in W. \tag{0.5}$$

Then by virtue of Donsker-Varadhan [2], we have

$$\lim_{T\to\infty} \frac{1}{2T} \log Z_T = -\inf\{\tilde{F}(\phi)\,;\,\phi \in \mathcal{D}om(\mathcal{E})\}, \tag{0.6}$$

where

$$\tilde{F}(\phi) = -\int_{M\times M} V(x, y)\phi(x)^2\phi(y)^2\mu(dx)\mu(dy) + \mathcal{E}(\phi, \phi). \tag{0.7}$$

It is likely that if there is a unique ϕ minimizing the function $\tilde{F}$, then Φ_T converges to a $\phi(x)^2\mu(dx)$-symmetric stationary Markov process as $T\to\infty$. However, we do not know how to prove such a statement in general. Therefore we think of more restricted cases.

Let

$$f = -\lim_{T\to\infty} \frac{1}{2T} \log Z_T,\ \tilde{\mathcal{P}}_0 \equiv \{\phi \in \mathcal{D}om(\mathcal{E})\,;\ \tilde{F}(\phi) = f,\ \phi \geqq 0\}$$

and $\mathcal{P}_0 \equiv \{\phi(x)^2\mu(dx)\,;\,\phi \in \tilde{\mathcal{P}}_0\} \subset \mathcal{P}(M)$. We will add two more assumptions as follows.

(A.1) There are a compact metric space S, a signed measure on S with bounded total variation, and a bounded continuous function $g: M\times S \to \boldsymbol{R}$ such that $V(x, y) = \int_S g(x, s)g(y, s)\sigma(ds)$.

(A.2) $$\int_M \phi(x)^{-p}\mu(dx) < \infty \text{ for any } \phi \in \tilde{\mathcal{P}}_0 \text{ and } 1 < p < \infty.$$

For each $\phi \in \mathcal{D}om(\mathcal{E})$, let

$$\lambda(\phi) = \mathcal{E}(\phi, \phi) - 2\int_{M\times M} V(x, y)\phi(x)^2\phi(y)^2\mu(dx)\mu(dy) \tag{0.8}$$

and

(0.9) $$\bar{V}(\phi)(x)=2\int_M V(x,y)\phi(y)^2\mu(dy)+\lambda(\phi).$$

Let us define $\tilde{\mathcal{P}}_{00}\subset\tilde{\mathcal{P}}_0$ by

(0.10) $$\tilde{\mathcal{P}}_{00}=\Big\{\phi\in\tilde{\mathcal{P}}_0;\ \text{there is an } \varepsilon>0 \text{ such that}$$
$$4\int_{M\times M}V(x,y)\phi(x)\phi(y)\psi(x)\psi(y)\mu(dx)\mu(dy)$$
$$\leqq(1-\varepsilon)\Big(\mathcal{E}(\psi,\psi)-\int_M\bar{V}(\phi)(x)\psi(x)^2\mu(dx)\Big)$$
$$\text{for any } \psi\in\mathcal{D}om(\mathcal{E}) \text{ with } \int_M\psi(x)\phi(x)\mu(dx)=0\Big\},$$

and let $\mathcal{P}_{00}=\{\phi(x)^2\mu(dx)\,;\ \phi\in\tilde{\mathcal{P}}_{00}\}$.

Our main result is the following (Theorems (3.17) and (3.18)). Under the assumptions (H.1), (H.2), (A.1) and (A.2),

(0.11) $$0<\lim_{T\to\infty}e^{2Tf}Z_T\leqq\infty,$$

(0.12) $$\lim_{T\to\infty}e^{2Tf}Z_T<\infty,\ \text{iff}\ \mathcal{P}_0=\mathcal{P}_{00},$$

and

(0.13) if $\mathcal{P}_0=\mathcal{P}_{00}$, then $\sharp(\mathcal{P}_0)<\infty$ and Φ_T converges as $T\to\infty$ to a stationary probability measure on W which is a convex combination of ν-symmetric strongly mixing Markov process ($\nu\in\mathcal{P}_0$).

We will also discuss r-body potentials ($r\geqq 3$) in Section 4. This work is an extension of our previous work [4] on a sequence of i.i.d. random variables, and the technique of the proof is quite similar.

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§ 1. Preliminary Remark.

Let us think of the situation in Introduction. We assume the assumptions (H.1) and (H.2) in Introduction throughout this paper as we declared. Let d denote the Prohorov metric function on $\mathcal{P}(M)$. Then by the assumptions (H.1) and (H.2), we see that

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \log P_\mu[d(\rho_T(w), \mu)>\varepsilon]<0 \quad \text{for any } \varepsilon>0.$$

Therefore we see that the stationary Markov measure P_μ is ergodic.

Suppose that $\phi \in \mathcal{D}om(\mathcal{L})$ and $\bar{V}: M \to \boldsymbol{R}$ is a bounded continuous function satisfying

$$\phi(x)>0 \quad \mu\text{-a. e. } x, \tag{1.1}$$

and

$$(\mathcal{L}+\bar{V})\phi=0. \tag{1.2}$$

Let ν be a probability measure on M given by $d\nu=\phi^2 d\mu$.

From (1.2), we see that

$$\phi=\exp(t\cdot(L+\bar{V}))\phi, \qquad t\geqq 0. \tag{1.3}$$

Let us define $U=U_\phi: L^2(M, d\nu)\to L^2(M; d\mu)$ by

$$U_\phi\psi=\phi\cdot\psi. \tag{1.4}$$

Then U is a unitary operator. By virtue of (1.3) and Feynman-Kac's formula, there is a probability measure Q_ν on W satisfying

$$\left.\frac{dQ_\nu}{dP_\mu}\right|_{\mathcal{F}_{-T}^T}(w)=\phi(w(-T))\phi(w(T))\exp\left(\int_{-T}^T \bar{V}(w(t))dt\right) \tag{1.5}$$

for any $T>0$, where $\mathcal{F}_{-T}^T=\sigma\{w(t); -T\leqq t\leqq T\}$. Actualy Q_ν is a ν-symmetric stationary Markov process and its induced semigroup $\{Q_t\}_{t\geqq 0}$ in $L^2(M; d\nu)$ is given by

$$Q_t=U^{-1}\cdot\exp(t(\mathcal{L}+\bar{V}))\cdot U. \tag{1.6}$$

(1.7) LEMMA. *There is a* $\lambda>0$ *such that*

$$|Q_t-\Pi|_{L^2\to L^2}\leqq e^{-\lambda t}, \qquad t\geqq 0.$$

Here Π *is a orthogonal projection in* $L^2(M; d\nu)$ *defined by*

$$(\Pi\psi)(x)=\int_M \psi d\nu. \tag{1.8}$$

In other words, Q_ν *is strongly mixing and* $\operatorname{Spec}(\mathcal{L})\subset(-\infty, -\lambda]\cup\{0\}$.

PROOF. Let Π' be a projection operator in $L^2(M; d\mu)$ given by

$$\Pi'\psi = (\psi, \phi)_{L^2}\cdot\phi. \tag{1.9}$$

Then it is obvious that

$$\Pi = U^{-1}\Pi' U. \tag{1.10}$$

Because the operator $\exp(t(\mathcal{L}+\bar{V}))$ is a compact, symmetric and positivity preserving operator for $t>0$ and P_μ is ergodic, by virtue of Perron-Frobenius' theorem, we see that ϕ is a unique ground state of $\mathcal{L}+\bar{V}$ and there is a $\lambda>0$ such that

$$\|\exp(t(\mathcal{L}+\bar{V}))-\Pi'\|_{L^2\to L^2} \leqq e^{-t\lambda}, \qquad t\geqq 0. \tag{1.11}$$

Then by (1.6) and (1.10), we have our assertion. Q.E.D.

Let R_ϕ be a linear operator in $L^2(M; d\mu)$ given by

$$R_\phi = 2\int_0^\infty (\exp(t(\mathcal{L}+\bar{V}))-\Pi')dt, \tag{1.12}$$

and let G_ν be a linear operator in $L^2(M; d\nu)$ given by

$$G_\nu = 2\int_0^\infty (Q_t - \Pi)dt. \tag{1.13}$$

Then we have

$$G_\nu = U^{-1}R_\phi U. \tag{1.14}$$

§ 2. Basic Lemmas.

Let us think of the situation in the previous section continuously. For each $a>0$, let

$$\Psi_a \equiv \left\{\psi\in L^\infty(M; d\nu);\ \|\psi\|_{L^\infty}\leqq a,\ (G_\nu\psi, \psi)_{L^2}\leqq 1 \text{ and } \int_M \psi\, d\nu = 0\right\}.$$

Then we have the following.

(2.1) LEMMA. *For any $a>0$ and $\varepsilon>0$, there is a $\delta>0$ such that*

$$\sup_{T>0}\sup_{\psi\in\Psi_a} E^{Q_\nu}\left[\exp\left(\frac{1-\varepsilon}{2}\frac{1}{2T}\left(\int_{-T}^{T}\psi(w(t))dt\right)^2\right),\ \left|\frac{1}{2T}\int_{-T}^{T}\psi(w(t))dt\right|<\delta\right]<\infty.$$

PROOF. Take a $\psi\in\Psi_a$. Then

$$E^{Q_\nu}\left[\exp\left(x\int_{-T}^{T}\phi(w(t))\,dt\right)\right]$$
$$=\sum_{n=0}^{\infty}x^n E^{Q_\nu}\left[\int_{-T<s_1<\cdots<s_n<T}\phi(w(s_1))\cdots\phi(w(s_n))\,ds_1\cdots ds_n\right].$$

Set $\tilde{Q}_t=Q_t-\Pi$ and $\bar{\phi}=$the multiplication operator $\phi\cdot$. Then they are bounded and symmetric, and also we see that $\Pi\bar{\phi}\Pi=0$ and $\tilde{Q}_t\Pi=0$. Therefore we have

$$E^{Q_\nu}\left[\exp\left(x\int_{-T}^{T}\phi(w(t))\,dt\right)\right]$$
$$=1+\sum_{n=2}^{\infty}x^n\int_{\substack{\tau_0,\cdots,\tau_{n-1}>0\\ \tau_0+\cdots+\tau_{n-1}<2T}}(1,Q_{\tau_0}\bar{\phi}Q_{\tau_1}\bar{\phi}\cdots Q_{\tau_{n-1}}\bar{\phi}1)_{L^2}\,d\tau_0\cdots d\tau_{n-1}$$

and for $n\geqq 2$,

$$(1,Q_{\tau_0}\bar{\phi}Q_{\tau_1}\bar{\phi}\cdots Q_{\tau_{n-1}}\bar{\phi}1)_{L^2}$$
$$=\sum_{l=1}^{[n/2]}\sum_{\substack{i_1+\cdots+i_l=n-l\\ i_1,\cdots,i_l\geqq 1}}(1,(\Pi\bar{\phi})(\tilde{Q}_{\tau_1}\bar{\phi})\cdots(\tilde{Q}_{\tau_{i_1}}\bar{\phi})(\Pi\bar{\phi})(\tilde{Q}_{\tau_{i_1+1}}\bar{\phi})\cdots$$
$$\times(\tilde{Q}_{\tau_{i_1+i_2}}\bar{\phi})(\Pi\bar{\phi})\cdots(Q_{\tau_{i_1+\cdots+i_l+l-1}}\bar{\phi})1)_{L^2}$$
$$=\sum_{l=1}^{[n/2]}\sum_{\substack{i_1+\cdots+i_l=n-l\\ i_1,\cdots,i_l\geqq 1}}\prod_{j=1}^{l}[(1,(\Pi\bar{\phi})(\tilde{Q}_{\tau_{i_1+\cdots+i_{j-1}+j}}\bar{\phi})\cdots(\tilde{Q}_{\tau_{i_1+\cdots+i_j+j-1}}\bar{\phi})1)_{L^2}].$$

Set $A_k=\int_{(0,\infty)^{k-1}}|(\phi,\tilde{Q}_{s_1}\bar{\phi}\cdots\tilde{Q}_{s_{k-1}}\bar{\phi}1)_{L^2}|\,ds_1\cdots ds_{k-1}$, $k\geqq 2$. Then, since $(\phi,\tilde{Q}_s\phi)_{L^2}\geqq 0$, we have

$$A_2=\int_0^{\infty}(\phi,\tilde{Q}_s\phi)_{L^2}ds=\frac{1}{2}(G_\nu\phi,\phi)_{L^2}\leqq\frac{1}{2}$$

and

$$A_k\leqq\|\phi\|_{L^\infty}^{k-2}\left(\frac{1}{\lambda}\right)^{k-1}\|\phi\|_{L^2}^2\leqq\left(\frac{a}{\lambda}\right)^{k-1}a,\qquad k\geqq 3.$$

Also, we have

$$\int_{\substack{\tau_0,\cdots,\tau_{n-1}\geqq 0\\ \tau_0+\cdots+\tau_{n-1}<2T}}d\tau_0\cdots d\tau_{n-1}\prod_{j=1}^{l}[(1,(\Pi\bar{\phi})(\tilde{Q}_{\tau_{i_1+\cdots+i_{j-1}+j}}\bar{\phi})\cdots$$
$$\times(\tilde{Q}_{\tau_{i_1+\cdots+i_j+j-1}}\bar{\phi})1)_{L^2}]\leqq\frac{(2T)^l}{l\,!}\prod_{j=1}^{l}A_{i_j+1}.$$

Therefore we obtain

$$E^{Q_\nu}\left[\exp\left(x\int_{-T}^{T}\phi(w(t))dt\right)\right]\leqq 1+\sum_{n=2}^{\infty}x^n\sum_{l=1}^{[n/2]}\sum_{\substack{i_1+\cdots+i_l=n-l\\ i_1,\cdots,i_l\geqq 1}}\frac{(2T)^l}{l!}\prod_{j=1}^{l}A_{i_j+1}$$

$$=\sum_{l=0}^{\infty}\frac{(2T)^l}{l!}x^l\left(\sum_{i=1}^{\infty}A_{i+1}x^i\right)^l=\exp\left((2T)x^2\left(\sum_{i=0}^{\infty}A_{i+2}x^i\right)\right)$$

$$\leqq\exp\left((2T)x^2\left(\frac{1}{2}+\sum_{i=0}^{\infty}x^{1+i}\left(\frac{a}{\lambda}\right)^{i+2}a\right)\right)$$

$$=\exp\left(Tx^2\left(1+x\cdot\frac{4a^3}{\lambda^2}\right)\right),\qquad \text{if}\quad 0\leqq x\leqq\lambda/2a.$$

Thus we see that

$$Q_\nu\left(\left|\frac{1}{2T}\int_{-T}^{T}\phi(w(t))dt\right|>x\right)$$

$$\leqq\exp(-2Tx^2)\left\{E^{Q_\nu}\left[\exp\left(x\int_{-T}^{T}\phi(w(t))dt\right)\right]+E^{Q_\nu}\left[\exp\left(x\int_{-T}^{T}-\phi(w(t))dt\right)\right]\right\}$$

$$\leqq 2\exp\left(-2Tx^2\left\{1-\frac{1}{2}\left(1+x\cdot\frac{4a^3}{\lambda^2}\right)\right\}\right)$$

$$\leqq 2\exp\left(-2Tx^2\left(\frac{1}{2}-\frac{\varepsilon}{4}\right)\right)$$

for $0\leqq x\leqq\dfrac{\lambda}{2a}\wedge\dfrac{\varepsilon\lambda^2}{8a^3}$.

Set $\delta=\dfrac{\lambda}{2a}\wedge\dfrac{\varepsilon\lambda^2}{8a^3}$. Then we have

$$E^{Q_\nu}\left[\exp\left(\frac{1-\varepsilon}{2}\cdot\frac{1}{2T}\left|\int_{-T}^{T}\phi(w(t))dt\right|^2\right),\ \left|\frac{1}{2T}\int_{-T}^{T}\phi(w(t))dt\right|<\delta\right]$$

$$=\int_0^{\delta-}\exp(T(1-\varepsilon)x^2)Q_\nu\left[\left|\frac{1}{2T}\int_{-T}^{T}\phi(w(t))dt\right|\in dx\right]$$

$$=\int_0^{\delta}2T(1-\varepsilon)\cdot\exp(T(1-\varepsilon)x^2)Q_\nu\left[x\leqq\left|\frac{1}{2T}\int_{-T}^{T}\phi(w(t))dt\right|<\delta\right]dx$$

$$-\left[\exp\,(T(1-\varepsilon)x^2)Q_\nu\left[x\leqq\left|\frac{1}{2T}\int_{-T}^{T}\phi(w(t))dt\right|<\delta\right]\right]_{x=0}^{x=\delta}$$

$$\leqq 1+4(1-\varepsilon)\int_0^{\delta}Tx\exp\left(-T\cdot\frac{\varepsilon}{2}x^2\right)dx$$

$$\leqq 1+\frac{4}{\varepsilon}.$$

This proves our assertion. Q.E.D.

Now let $\phi_i \in L^\infty(M; d\nu)$, $i=1, \cdots, m$, and suppose that $\int_M \phi_i \, d\nu = 0$, $i=1, \cdots, m$. Let $\{a_{ij}\}_{i,j=1,\cdots,m}$ be a symmetric matrix and let $U(x, y) = \sum_{i,j=1}^m a_{ij}\phi_i(x) \cdot \phi_j(y)$. Then we have the following.

(2.2) LEMMA. *If there is an $\varepsilon > 0$ such that*

$$\int_{M\times M} U(x, y)(G_\nu\phi)(x)(G_\nu\phi)(y)\nu(dx)\otimes\nu(dy) \leqq (1-\varepsilon)\cdot(G_\nu\phi, \phi)_{L^2} \tag{2.3}$$

for all $\phi \in L^2(M; d\nu)$, then there is a $\delta > 0$ such that

$$\sup_{T>0} E^{Q_\nu}\Big[\exp\Big(\frac{1}{2}\cdot\frac{1}{2T}\int_{-T}^T\int_{-T}^T U(w(t), w(s))dtds\Big), \\ \Big|\frac{1}{2T}\int_{-T}^T \phi_i(w(t))dt\Big| < \delta, \ i=1, \cdots, m\Big] < \infty.$$

PROOF. Let $E \equiv \sum_{i=1}^m \boldsymbol{R}\phi_i \subset L^\infty(M; d\nu)$. Then $(G_\nu\cdot, *)|_{E\times E}$ is a strictly positive definite symmetric bilinear form. Therefore there are $\tilde{\phi}_j \in E$ and $\mu_i \in \boldsymbol{R}$, $i=1, \cdots, d (= \dim E)$ such that

$$(G_\nu\tilde{\phi}_i, \tilde{\phi}_j) = \delta_{i,j}, \qquad i, j = 1, \cdots, d, \tag{2.4}$$

and

$$U(x,y) = \sum_{i=1}^d \mu_i\tilde{\phi}_i(x)\tilde{\phi}_i(y). \tag{2.5}$$

Then we see that $\mu_i \leqq 1-\varepsilon$, $i=1, \cdots, d$ from the assumption (2.3). Therefore

$$\int_{-T}^T\int_{-T}^T U(w(t),w(s))dtds \leqq (1-\varepsilon)\sum_{i=1}^d\Big(\int_{-T}^T \tilde{\phi}_i(w(t))dt\Big)^2.$$

Since $\Big\{x \in \boldsymbol{R}^d; \|x\| \leqq \frac{1}{1+\varepsilon}\Big\} = \cap\Big\{\Big\{x \in \boldsymbol{R}^d; (x, \xi) \leqq \frac{1}{1+\varepsilon}\Big\} \|\xi\| = 1\Big\}$, there are $\xi_i = (\xi_i^1, \cdots, \xi_i^d) \in \boldsymbol{R}^d$, $i=1, \cdots, N$ with $\|\xi_i\| = 1$ such that

$$\bigcap_{i=1}^N \Big\{x \in \boldsymbol{R}^d; (x, \xi_i) \leqq \frac{1}{1+\varepsilon}\Big\} \subset \{x \in \boldsymbol{R}^d; \|x\| < 1\}.$$

This implies that

$$\|x\|^2 \leqq (1+\varepsilon)\max_{i=1,\cdots,N}(x, \xi_i)^2, \qquad x \in \boldsymbol{R}^d. \tag{2.6}$$

Set $\tilde{\phi}_i \equiv \sum_{j=1}^{d} \xi_i^j \tilde{\phi}_j \in E$, $i=1, \cdots, N$. Then we see that

$$(G_\nu \tilde{\phi}_i, \tilde{\phi}_i)_{L^2}=1, \tag{2.7}$$

and

$$\int_M \tilde{\phi}_i(x)\nu(dx)=0, \quad i=1, \cdots, N. \tag{2.8}$$

Also we have

$$\sum_{i=1}^{d}\left(\int_{-T}^{T} \tilde{\phi}_i(w(t))dt\right)^2 \leqq (1+\varepsilon) \max_{i=1,\cdots,N}\left(\int_{-T}^{T} \tilde{\phi}_i(w(t))dt\right)^2. \tag{2.9}$$

It is obvious that there is an $a>0$ for which $\{\tilde{\phi}_i\}_{i=1}^{N} \subset \Psi_a$. Therefore by Lemma (2.1) for sufficiently small $\delta'>0$,

$$\begin{aligned}
&\sup_{T>0} E^{Q_\nu}\Big[\exp\Big(\frac{1}{2}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} U(w(t), w(s))dtds\Big), \\
&\qquad \max_{i=1,\cdots,N}\Big|\frac{1}{2T}\int_{-T}^{T} \tilde{\phi}_i(w(t))dt\Big|<\delta'\Big] \\
&\leqq \sup_{T>0}\max_{i=1,\cdots,N} E^{Q_\nu}\Big[\exp\Big(\frac{1-\varepsilon^2}{2}\frac{1}{2T}\Big(\int_{-T}^{T} \tilde{\phi}_i(w(t))dt\Big)^2\Big), \\
&\qquad \Big|\frac{1}{2T}\int_{-T}^{T} \tilde{\phi}_i(w(t))dt\Big|<\delta'\Big] \\
&<\infty.
\end{aligned}$$

Since $\tilde{\phi}_i$'s are linear combinations of ϕ_j, we have our assertion. Q.E.D.

Now let $U: M\times M \to \boldsymbol{R}$ be a function satisfying the assumption (A.1) in Introduction as $V=U$.

(2.10) LEMMA. *If there is an $\varepsilon>0$ such that*

$$\int_{M\times M} U(x, y)(G_\nu\phi)(x)(G_\nu\phi)(y)\nu(dx)\otimes\nu(dy) \leqq (1-\varepsilon)\cdot(G_\nu\phi, \phi)_{L^2} \tag{2.11}$$

for all $\phi \in L^2(M; d\nu)$, then there is a $\delta>0$ such that

$$\begin{aligned}
&\sup_{T>0} E^{Q_\nu}\Big[\exp\Big(T\int_{M\times M} U(y_1, y_2)(\rho_T(w)-\nu)^{\otimes 2}(dy_1\otimes dy_2)\Big), \\
&\qquad \sup_{s\in S}\Big|\int_M g(y, s)(\rho_T(w)-\nu)(dy)\Big|<\delta\Big]<\infty.
\end{aligned}$$

PROOF. Let $g_0: M\times S \to \boldsymbol{R}$ be a continuous function given by

$$g_0(x, s)=g(x, s)-\int_M g(y, s)\nu(dy). \tag{2.12}$$

Then we have

$$\int_{M\times M} U(y_1,y_2)(\rho_T(w)-\nu)^{\otimes 2}(dy_1\otimes dy_2)=\int_S \sigma(ds)\left(\int_M g_0(y,s)\rho_T(w)(dy)\right)^2, \tag{2.13}$$

and

$$\int_M g(y,s)(\rho_T(w)-\nu)(dy)=\int_M g_0(y,s)\rho_T(w)(dy). \tag{2.14}$$

Let $K_{T,\delta}$ denote the set $\left\{w\in W;\sup_{s\in S}\left|\int_M g_0(y,s)\rho_T(w)(dy)\right|<\delta\right\}$. Let $\{S_k^{(m)}\}_{k=1}^m$, $m=1,2,\cdots$, be decompositions of S satisfying

$$\lim_{m\to\infty}\max\{\text{diameter}(S_k^{(m)});\ k=1,\cdots,m\}=0. \tag{2.15}$$

Choose an element $s_k^{(m)}\in S_k^{(m)}$ for each $j=1,\cdots,m$ and $m=1,2,\cdots$, and let

$$g_0^{(m)}(y,s)=\sum_{k=1}^m \chi_{S_k^{(m)}}(s)g_0(y,s_k^{(m)}),\qquad y\in M,\ s\in S. \tag{2.16}$$

Then we have

$$d_m\equiv\sup_{s\in S}|g_0(y,s)-g_0^{(m)}(y,s)|\longrightarrow 0,\qquad m\to\infty. \tag{2.17}$$

Let $U_0: M\times M\to \boldsymbol{R}$ and $U_0^{(m)}: M\times M\to\boldsymbol{R}$ be functions given by

$$U_0(x,y)=\int_S g_0(x,s)g_0(y,s)\sigma(ds), \tag{2.18}$$

and

$$U_0^{(m)}(x,y)=\int_S g_0^{(m)}(x,s)g_0^{(m)}(y,s)\sigma(ds)=\sum_{k=1}^m \sigma(S_k^{(m)})g_0(x,s_k^{(m)})g_0(y,s_k^{(m)}). \tag{2.19}$$

Then we see that

$$\begin{aligned}&\int_{M\times M} U(x,y)(G_\nu\phi)(x)(G_\nu\phi)(y)\nu(dx)\nu(dy)\\ &=\int_{M\times M} U_0(x,y)(G_\nu\phi)(x)(G_\nu\phi)(y)\nu(dx)\nu(dy),\qquad \phi\in L^2(M;d\nu),\end{aligned} \tag{2.20}$$

and

$$\int_{M\times M}|U_0(x,y)-U_0^{(m)}(x,y)|^2\nu(dx)\nu(dy)\longrightarrow 0,\qquad m\to\infty. \tag{2.21}$$

Note that $G_\nu^{1/2}$ is a bounded linear operator. Then from the assumption (2.11) and (2.20), we have

$$\int_{M\times M} U_0(x, y)(G_\nu^{1/2}\phi)(x)(G_\nu^{1/2}\phi)(y)\nu(dx)\nu(dy) \leqq (1-\varepsilon)(\phi, \phi)_{L^2} \tag{2.22}$$

for any $\phi\in L^2(M; d\nu)$. Therefore, by (2.21) and (2.22), there is an m_1 such that

$$\int_{M\times M} U_0^{(m)}(x, y)(G_\nu\phi)(x)(G_\nu\phi)(y)\nu(dx)\nu(dy) \leqq \left(1-\frac{\varepsilon}{2}\right)(G_\nu\phi, \phi)_{L^2} \tag{2.23}$$

for any $m\geqq m_1$ and $\phi\in L^2(M; d\nu)$. Observe that for any $\delta>0$,

$$\begin{aligned} &\log E^{Q_\nu}\left[\exp\left(T\int_{M\times M} U_0(y_1, y_2)\rho_T(w)^{\otimes 2}(dy_1\otimes dy_2)\right), K_{T,\delta}\right] \\ &\leqq I_{T,m,\delta}^{(1)} + I_{T,m,\delta}^{(2)}, \end{aligned} \tag{2.24}$$

where

$$I_{T,m,\delta}^{(1)} = \frac{1}{p}\log E^{Q_\nu}\left[\exp\left(p\cdot T\int_{M\times M} U_0^{(m)}(y_1, y_2)\rho_T(w)^{\otimes 2}(dy_1\otimes dy_2)\right), K_{T,\delta}\right], \tag{2.25}$$

$$\begin{aligned} I_{T,m,\delta}^{(2)} = \frac{1}{q}\log E^{Q_\nu}\Big[\exp\Big(q\cdot T\int_{M\times M} &(U_0(y_1, y_2) - U_0^{(m)}(y_1, y_2)) \\ &\times\rho_T(w)^{\otimes 2}(dy_1\otimes dy_2)\Big), K_{T,\delta}\Big], \end{aligned} \tag{2.26}$$

$p=\dfrac{1-\varepsilon/3}{1-\varepsilon/2}$ and $q=\dfrac{p}{p-1}$.

Let $\sigma=\sigma_1-\sigma_2$ be the Jordan decomposition of the signed measure σ and let $\sigma_0=\sigma_1+\sigma_2$. Then σ_0, σ_1 and σ_2 are finite measures on S. Since we have

$$\begin{aligned} &\int_{M\times M}(U_0(y_1, y_2) - U_0^{(m)}(y_1, y_2))\rho_T(w)^{\otimes 2}(dy_1\otimes dy_2) \\ &= \int_S \sigma(ds)\left\{\left(\int_M g_0(y, s)\rho_T(w)(dy)\right)^2 - \left(\int_M g_0^{(m)}(y, s)\rho_T(w)(dy)\right)^2\right\} \\ &\leqq \int_S \sigma_0(ds)\left[\frac{1}{2}d_m\left\{\int_M (g_0(y, s)+g_0^{(m)}(y, s))\rho_T(w)(dy)\right\}^2\right. \\ &\qquad\left. + \frac{1}{2}d_m^{-1}\left\{\int_M (g_0(y, s)-g_0^{(m)}(y, s))\rho_T(w)(dy)\right\}^2\right]. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &I_{T,m,\delta}^{(2)} \\ &\leqq \frac{1}{q}\int_S \frac{1}{r}\sigma_0(ds)\left\{\frac{1}{2}\log E^{Q_\nu}\left[\exp\left(Tqr\cdot d_m\left\{\int_M (g_0(y, s)\right.\right.\right.\right. \end{aligned} \tag{2.27}$$

$$+g_0^{(m)}(y, s))\rho_T(w)(dy)\Big\}^2\Big), K_{T,\delta}\Big]$$

$$+\frac{1}{2}\log E^{Q_\nu}\Big[\exp\Big(Tqr\cdot d_m^{-1}\Big\{\int_M (g_0(y, s)-g_0^{(m)}(y, s))\rho_T(w)(dy)\Big\}^2\Big), K_{T,\delta}\Big]\Big\},$$

where $r=\sigma_0(S)$. Observe that

$$\sup_{s\in S}\{qr\cdot d_m(g_0(y, s)+g_0^{(m)}(y, s))^2\}\longrightarrow 0,$$

$$\sup_{s\in S}\{qr\cdot d_m^{-1}(g_0(y, s)-g_0^{(m)}(y, s))^2\}\longrightarrow 0, \qquad m\to\infty,$$

and

$$\int_M (g_0(y, s)\pm g_0^{(m)}(y, s))\rho_T(w)(dy)=\frac{1}{2T}\int_{-T}^{T}(g_0(w(t), s)\pm g_0^{(m)}(w(t), s))dt.$$

Then by virtue of Lemma (2.1) and (2.27), we see that there exist an integer m_2 and positive numbers δ_m, $m\geqq m_2$, such that

$$\sup_{T>0} I^{(2)}_{T,m,\delta_m}<\infty, \qquad m\geqq m_2. \tag{2.28}$$

On the other hand, since $p\cdot\left(1-\frac{\varepsilon}{2}\right)\leqq 1-\frac{\varepsilon}{4}$, by Lemma (2.2), (2.23) and (2.25), we see that for each $m\geqq m_1$, there is a $\delta'_m>0$ such that

$$\sup_{T>0} I^{(1)}_{T,m,\delta'_m}<\infty, \qquad m\geqq m_1. \tag{2.29}$$

Therefore from (2.24), (2.28) and (2.29) we have our assertion. Q.E.D.

§ 3. Main results.

Let us think of the situation in Introduction. We assume the assumptions (H.1), (H.2), (A.1) and (A.2) in Introduction. From the assumption (H.1), we see that the set $\{\phi\in\mathcal{D}om(\mathcal{E}); \mathcal{E}(\phi, \phi)\leqq K\}$ is compact in $L^2(M; d\mu)$. Therefore $\mathcal{P}_0\neq\emptyset$ and

$$\overline{\lim_{T\to\infty}}\frac{1}{2T}\log E^{P_\mu}\Big[\exp\Big(2T\int_{M\times M}V(x, y)\rho_T(w)^{\otimes 2}(dx\otimes dy)\Big), d(\rho_T(w), \mathcal{P}_0)>\delta\Big]<-f \tag{3.1}$$

for any $\delta>0$, where d is the Prohorov metric of $\mathcal{P}(M)$.

Let $\lambda(\phi)$ and $\overline{V}(\phi)$ be as in Introduction ((0.8), (0.9)). Then we have the following.

(3.2) PROPOSITION. *$\tilde{\mathcal{P}}_0 \subset \mathcal{D}om(\mathcal{L})$, and for each $\phi \in \tilde{\mathcal{P}}_0$*

$$(\mathcal{L}+\bar{V}(\phi)(x))\phi=0, \tag{3.3}$$

and

$$\begin{aligned} 4\int_{M\times M} V(x, y)\phi(x)\phi(y)\psi(x)\psi(y)\mu(dx)\mu(dy) \\ \leqq \mathcal{E}(\psi, \psi)-\int_M \bar{V}(\phi)(x)\psi(x)^2\mu(dx) \end{aligned} \tag{3.4}$$

for any $\psi \in \mathcal{D}om(\mathcal{E})$ with $\int_M \psi(x)\phi(x)\mu(dx)=0$.

PROOF. Take an arbitrary $\psi \in \mathcal{D}om(\mathcal{E})$ with $\int_M \psi(x)\phi(x)\mu(dx)=0$. Then $G(t) \equiv \tilde{F}((\phi+t\cdot\psi)/\|\phi+t\cdot\psi\|_{L^2(M;\, d\mu)}) \geqq f$ for $t \in \boldsymbol{R}$ near 0. Since $G(0)=f$ and $G(t)$ is smooth in t near 0, we have

$$\frac{dG}{dt}(0)=0, \tag{3.5}$$

and

$$\frac{d^2G}{dt^2}(0) \geqq 0. \tag{3.6}$$

From (3.5), we get

$$4\int_{M\times M} V(x,y)\phi(y)^2\phi(x)\psi(x)\mu(dx)\mu(dy)-2\mathcal{E}(\phi, \psi)=0. \tag{3.7}$$

This implies that

$$2\int_{M\times M} V(x, y)\phi(y)^2\phi(x)\tilde{\psi}(x)\mu(dx)\mu(dy)-\mathcal{E}(\phi, \tilde{\psi})+\lambda(\phi)(\phi, \tilde{\psi})_{L^2}=0 \tag{3.8}$$

for all $\tilde{\psi} \in \mathcal{D}om(\mathcal{E})$. This proves the first statement. The latter one also follows from (3.6). Q.E.D.

Now take a $\phi \in \tilde{\mathcal{P}}_0$ and fix it. By Proposition (3.2), we see that we can apply the results in Sections 1 and 2 to ϕ. Let $\nu, Q_\nu, G_\nu, U_\phi$ and R_ϕ be as in Section 1.

Then we have the following.

(3.9) LEMMA. *Suppose that $\phi \in \tilde{\mathcal{P}}_{00}$, i.e., there is an $\varepsilon>0$ such that*

(3.10)
$$4\int_{M\times M}V(x,y)\phi(x)\phi(y)\psi(x)\psi(y)\mu(dx)\mu(dy)$$
$$\leqq(1-\varepsilon)\left\{\mathcal{E}(\psi,\psi)-\int_M\bar{V}(\phi)(x)\psi(x)^2\mu(dx)\right\}$$

for any $\psi\in\mathcal{D}om(\mathcal{E})$ *with* $\int_M\psi(x)\phi(x)\mu(dx)=0$. *Then for any* $p\in\left[1,\frac{1}{1-\varepsilon}\right)$, *there is a* $\delta>0$ *such that*

(3.11)
$$\sup_{T>0}E^{Q_\nu}\left[\exp\left(p\cdot 2T\int_{M\times M}V(x,y)\,(\rho_T(w)-\nu)^{\otimes 2}(dx\otimes dy)\right),\right.$$
$$\left.d(\rho_T(w),\nu)<\delta\right]<\infty.$$

PROOF. From the assumption (3.10), we see that

(3.12)
$$2\int_{M\times M}V(x,y)\phi(x)\phi(y)\,(R_\phi\psi)(x)\,(R_\phi\psi)(y)\mu(dx)\mu(dy)\leqq(1-\varepsilon)\,(R_\phi\psi,\psi)_{L^2}$$

for any $\psi\in L^2(M;d\mu)$. Let $U(x,y)=2pV(x,y)$. Then from (1.14) and (3.12), we have

(3.13)
$$\int_{M\times M}U(x,y)\,(G_\nu\psi)(x)\,(G_\nu\psi)(y)\nu(dx)\nu(dy)\leqq p(1-\varepsilon)\,(G_\nu\psi,\psi)_{L^2}$$

for all $\psi\in L^2(M;d\nu)$. Noting that $R\to\sup_{s\in S}\left|\int_M g(x,s)\,(R-\nu)\,(dx)\right|$ is a continuous function from $\mathcal{P}(M)$ into $\boldsymbol{R}$, we have our assertion from Lemma (2.10) and (3.13). Q.E.D.

(3.14) THEOREM. *Suppose that* $\phi\in\tilde{\mathcal{P}}_{00}$. *Then there is a* $\delta_0>0$ *such that*

$$e^{2fT}E^{P_\mu}\left[\Phi(w)\cdot\exp\left(\sqrt{-1}\cdot\sqrt{2T}\int u(x)\,(\rho_T(w)-\nu)\,(dx)\right)\right.$$
$$\left.\times\exp\left(\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}V(w(t),w(s))dsdt\right),\ d(\rho_T(w),\nu)<\delta\right]$$
$$\longrightarrow E^{Q_\nu}[\Phi(w)]\left(\int_M\phi(x)\mu(x)\right)^2\cdot\det(I-G_\nu^{1/2}V_\nu G_\nu^{1/2})^{-1/2}$$
$$\times\exp\left(-\frac{1}{2}(G_\nu^{1/2}(I-G_\nu^{1/2}V_\nu G_\nu^{1/2})^{-1}G_\nu^{1/2}u,u)_{L^2(M;d\nu)}\right)$$

as $T\to\infty$ *for any bounded* $\mathcal{F}_{-T}^{T}$*-measurable function* $\Phi:W\to\boldsymbol{R}$, $u\in L^\infty(M;d\mu)$ *and* $0<\delta<\delta_0$. *Here* V_ν *is a nuclear operator in* $L^2(M;d\nu)$ *given by*

(3.15) $$V_{\nu}\phi(x)=2\int_{M}V(x,\ y)\phi(y)\nu(dy).$$

PROOF. Note that

$$e^{2fT}E^{P_{\mu}}\Big[\Phi(w)\cdot\exp\Big(\sqrt{-1}\cdot\sqrt{2T}\int u(x)(\rho_{T}(w)-\nu)(dx)\Big)$$
$$\times\exp\Big(\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}V(w(t),\ w(s))dsdt\Big),\ d(\rho_{T}(w),\ \nu)<\delta\Big]$$
$$=E^{Q_{\nu}}\Big[\Phi(w)\phi(w(-T))^{-1}\phi(w(T))^{-1}\exp\Big(\sqrt{-1}\cdot\sqrt{2T}\int u(x)(\rho_{T}(w)-\nu)(dx)\Big)$$
$$\times\exp\Big(2T\int_{M\times M}V(x,\ y)(\rho_{T}(w)-\nu)^{\otimes 2}(dx\otimes dy)\Big),\ d(\rho_{T}(w),\nu)<\delta\Big].$$

Since Q_{ν} has the strong mixing property, we see that $\Phi(w)$, $\phi(w(-T))$, $\phi(w(T))$, $\Big\{\sqrt{2T}\int v(x)(\rho_{T}(w)-\nu)(dx),\ v\in L^{2}(M\,;d\nu)\Big\}$ are asymptotically independent as $T\rightarrow\infty$ under $Q_{\nu}(dw)$ and that

(3.16) $$E^{Q_{\nu}}\Big[\exp\Big(\sqrt{-1}\cdot\sqrt{2T}\int v(x)(\rho_{T}(w)-\nu)(dx)\Big)\Big]$$
$$\longrightarrow\exp\Big(-\frac{1}{2}(G_{\nu}v,\ v)_{L^{2}}\Big),\qquad T\rightarrow\infty,$$

for any $v\in L^{2}(M;d\nu)$. Observe that

$$E^{Q_{\nu}}\Big[\Big|\Phi(w)\phi(w(-T))^{-1}\phi(w(T))^{-1}$$
$$\times\exp\Big(2T\int_{M\times M}V(x,\ y)(\rho_{T}(w)-\nu)^{\otimes 2}(dx\otimes dy)\Big)\Big|^{r},\ d(\rho_{T}(w),\nu)<\delta_{0}\Big]$$
$$\leqq\|\Phi\|_{L^{\infty}}\cdot\Big(\int_{M}\phi(x)^{-2rq+1}\mu(dx)\Big)^{2}$$
$$|\times E^{Q_{\nu}}\Big[\exp\Big(pr\cdot 2T\int_{M\times M}V(x,\ y)(\rho_{T}(w)-\nu)^{\otimes 2}(dx\otimes dy)\Big),\ d(\rho_{T}(w),\nu)<\delta_{0}\Big]$$

for $p, q, r>1$ with $p^{-1}+q^{-1}=1$. Then by Lemma (3.9), if we take p and r sufficiently close to 1 and if we take sufficiently small $\delta_{0}>0$, we have

$$\sup_{T>0}E^{Q_{\nu}}\Big[\Big|\Phi(w)\phi(w(-T))^{-1}\phi(w(T))^{-1}$$
$$\times\exp\Big(2T\int_{M\times M}V(x,\ y)(\rho_{T}(w)-\nu)^{\otimes 2}(dx\otimes dy)\Big)\Big|^{r},\ d(\rho_{T}(w),\nu)<\delta_{0}\Big]<\infty.$$

Therefore our assertion follows from this, the asymptotic independence and (3.15). Q.E.D.

(3.17) THEOREM. *If* $\mathcal{P}_0\backslash\mathcal{P}_{00}\neq\emptyset$, *then* $\lim_{T\to\infty} e^{2fT}Z_T=\infty$.

PROOF. Take a $\phi\in\tilde{\mathcal{P}}_0\backslash\tilde{\mathcal{P}}_{00}$. Then we have

$$e^{2fT}Z_T=E^{Q_\nu}\Big[\phi(w(-T))^{-1}\phi(w(T))^{-1} \times\exp\Big(2T\int_{M\times M}V(x,y)(\rho_T(w)-\nu)^{\otimes 2}(dx\otimes dy)\Big)\Big].$$

$\phi(w(-T))$, $\phi(w(T))$, $\left\{\sqrt{2T}\int v(x)(\rho_T(w)-\nu)(dx),\ v\in L^2(M;d\nu)\right\}$ are asymptotically independent as $T\to\infty$ under $Q_\nu(dw)$, since Q_ν is strongly mixing. The assumption that $\phi\in\tilde{\mathcal{P}}_0\backslash\tilde{\mathcal{P}}_{00}$ implies that the symmetric bounded operator $I-G_\nu^{1/2}V_\nu G_\nu^{1/2}$ has zero spectrum. Therefore we see that

$$\varliminf_{T\to\infty} E^{Q_\nu}\Big[\exp\Big(2T\int_{M\times M}V(x,y)(\rho_T(w)-\nu)^{\otimes 2}(dx\otimes dy)\Big)\Big]=\infty.$$

Thus we have our assertion from Fatou's lemma. Q.E.D.

(3.18) THEOREM. *Suppose that* $\mathcal{P}_0=\mathcal{P}_{00}$. *Then*

$$\lim_{T\to\infty}e^{2fT}Z_T=\sum_{\nu\in\mathcal{P}_0}\Big(\int_M(d\nu/d\mu)^{1/2}d\mu\Big)^2\det(I-G_\nu^{1/2}V_\nu G_\nu^{1/2})^{-1/2}, \tag{3.19}$$

and Φ_T *converges to* $\sum_{\nu\in\mathcal{P}_0}a_\nu Q_\nu$ *as* $T\to\infty$ *weakly as probability measures on* W. *Here*

$$a_\nu=z^{-1}\Big(\int_M(d\nu/d\mu)^{1/2}d\mu\Big)^2\det(I-G_\nu^{1/2}V_\nu G_\nu^{1/2})^{-1/2}, \tag{3.20}$$

and

$$z=\sum_{\nu\in\mathcal{P}_0}\Big(\int_M(d\nu/d\mu)^{1/2}d\mu\Big)^2\det(I-G_\nu^{1/2}V_\nu G_\nu^{1/2})^{-1/2}. \tag{3.21}$$

PROOF. Let $T_0>0$ and Φ be a bounded $\mathcal{F}^{T_0}_{-T_0}$-measurable function on W. Then by (3.1) and Theorem (3.14), we have

$$E^{P_\mu}\Big[\Phi(w)\cdot\exp\Big(\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}V(w(t),w(s))dtds\Big)\Big] \longrightarrow\sum_{\nu\in P_0}\Big(\int_M(d\nu/d\mu)^{1/2}d\mu\Big)^2\det(I-G_\nu^{1/2}V_\nu G_\nu^{1/2})^{-1/2}E^{Q_\nu}[\Phi(w)].$$

This proves our assertion. Q.E.D.

(3.22) REMARK. The assumptions (H.1), (H.2) and (A.2) hold if the semigroup $\{P_t\}_{t\geq 0}$ in $L^2(M; d\mu)$ has the ultraboundedness property (cf. Davies-Simon [1], Kusuoka-Stroock [3] and Stroock [5]).

§ 4. Remarks on r-body potentials.

In this section, we consider r-body potentials, $r\geqq 3$. We will state results without proof, because the proof is almost the same as in the case of pair potentials. Let M, W, $\{P_x; x\in M\}$, μ, $\{P_t\}_{t\geqq 0}$, $\mathcal{L}$, $\mathcal{E}$ be as in Introduction, and we assume the assumptions (H.1) and (H.2). Let $V: M^r\to \boldsymbol{R}$ be a symmetric bounded continuous function satisfying the following assumption.

(A.1′) There are a compact metric space S, a singed measure σ with finite total variation, and bounded continuous functions $f_i: M\to\boldsymbol{R}$, $i=1, \cdots, r$, such that

$$V(x_1, \cdots, x_r) = \sum_{\{i_1,\cdots,i_r\}=\{1,\cdots,r\}} \int_S \prod_{j=1}^r f_{i_j}(x_j, s)\sigma(ds),$$

for any $(x_1, \cdots, x_r)\in M^r$.

Let Φ_T, $T>0$, be a probability measure on W given by

(4.1)
$$\Phi_T(dw) = Z_T^{-1}\exp\left(-\frac{1}{(2T)^{r-1}}\int_{-T}^T\cdots\int_{-T}^T V(w(t_1), \cdots, w(t_r))dt_1\cdots dt_r\right)P_\mu(dw),$$

where

(4.2)
$$Z_T = \int_W \exp\left(-\frac{1}{(2T)^{r-1}}\int_{-T}^T\cdots\int_{-T}^T V(w(t_1), \cdots, w(t_r))dt_1\cdots dt_r\right)P_\mu(dw).$$

Let $\tilde{F}: \mathcal{D}om(\mathcal{E})\to\boldsymbol{R}$ be a function given by

$$\tilde{F}(\phi) = \mathcal{E}(\phi, \phi) - \int_{M^r} V(x_1, \cdots, x_r)\phi(x_1)^2\cdots\phi(x_r)^2\mu(dx_1)\cdots\mu(dx_r) \tag{4.3}$$

for each $\phi\in\mathcal{D}om(\mathcal{E})$. Then by Donsker-Varadhan [2], we have

$$-f\equiv \lim_{T\to\infty}\frac{1}{2T}\log Z_T = -\inf\{\tilde{F}(\phi); \Phi\in\mathcal{D}om(\mathcal{E})\}. \tag{4.4}$$

Let $\tilde{\mathcal{P}}_0=\{\phi\in\mathcal{D}om(\mathcal{E}); \phi\geqq 0, \tilde{F}(\phi)=f\}$ and $\mathcal{P}_0=\{\phi(x)^2\mu(dx); \phi\in\tilde{\mathcal{P}}_0\}$. For each $\phi\in\mathcal{D}om(\mathcal{E})$, let

$$(4.5)\qquad \lambda(\phi)=\mathcal{E}(\phi,\phi)-r\int_{M^r}V(x_1,\cdots,x_r)\mu(dx_1)\cdots\mu(dx_r)$$

and

$$(4.6)\qquad \bar{V}(\phi)(x)=\lambda(\phi)+r\int_{M^{r-1}}V(x,y_1,\cdots,y_{r-1})\phi(y_1)^2\cdots\phi(y_r)^2 \times\mu(dy_1)\cdots\mu(dy_{r-1}),\qquad x\in M.$$

Then we have the following.

(4.7) PROPOSITION. *$\tilde{\mathcal{P}}_0\subset\mathcal{D}om(\mathcal{L})$, and for each $\phi\in\tilde{\mathcal{P}}_0$*

$$(4.8)\qquad (\mathcal{L}+\bar{V}(\phi)(x))\phi=0.$$

Therefore we can apply the argument in Section 1 to each element ϕ in $\tilde{\mathcal{P}}_0$. For each $\nu\in\mathcal{P}_0$, let V_ν be a nuclear operator in $L^2(M;d\nu)$ given by

$$(4.9)\qquad V_\nu\psi(x)=r(r-1)\int_{M^{r-1}}V(x,z_0,\cdots,z_{r-1})\psi(z_0)\nu(dz_0)\cdots\nu(dz_{r-2})$$

for $\psi\in L^2(M;d\nu)$. And let

$$(4.10)\qquad \mathcal{P}_{00}=\{\nu\in\mathcal{P}_0;\,I-G_\nu^{1/2}V_\nu G_\nu^{1/2}\text{ is strictly positive definite in }L^2(M;d\nu)\}.$$

Furthermore we assume the following.

$$(\mathrm{A.2}')\qquad \int_M\phi(x)^{-p}\mu(dx)<\infty\ \text{for any}\ \phi\in\tilde{\mathcal{P}}_{00}\ \text{and}\ p\in(1,\infty).$$

Then we have the following.

(4.11) THEOREM. *Suppose that $\phi\in\tilde{\mathcal{P}}_{00}$. Then there is a $\delta_0>0$ such that*

$$e^{2fT}E^{P_\mu}\Big[\Phi(w)\cdot\exp\Big(\sqrt{-1}\cdot\sqrt{2T}\int u(x)(\rho_T(w)-\nu)(dx)\Big)\times\exp\Big(-\frac{1}{(2T)^{r-1}}\int_{-T}^{T}\cdots\int_{-T}^{T}V(w(t_1),\cdots,w(t_r))dt_1\cdots dt_r\Big),\ d(\rho_T(w),\nu)<\delta\Big]$$

$$\longrightarrow E^{Q_\nu}[\Phi(w)]\Big(\int_M\phi(x)\mu(x)\Big)^2\cdot\det(I-G_\nu^{1/2}V_\nu G_\nu^{1/2})^{-1/2}\times\exp\Big(-\frac{1}{2}(G_\nu^{1/2}(I-G_\nu^{1/2}V_\nu G_\nu^{1/2})^{-1}G_\nu^{1/2}u,u)_{L^2(M;d\nu)}\Big)$$

as $T\to\infty$ for any bounded $\mathcal{F}^T_{-T}$-measurable function $\Phi:W\to\boldsymbol{R}$, $u\in L^\infty(M;d\mu)$ and $0<\delta<\delta_0$.

(4.12) THEOREM. (1) *If* $\mathcal{P}_0 \backslash \mathcal{P}_{00} \neq \emptyset$, *then* $\lim_{T\to\infty} e^{2fT} Z_T = \infty$.

(2) *If* $\mathcal{P}_0 = \mathcal{P}_{00}$, *then* $\sharp(\mathcal{P}_0) < \infty$ *and*

$$\lim_{T\to\infty} e^{2fT} Z_T = \sum_{\nu\in\mathcal{P}_0} \left(\int_M (d\nu/d\mu)^{1/2} d\mu\right)^2 \det(I - G_\nu^{1/2} V_\nu G_\nu^{1/2})^{-1/2}, \tag{4.13}$$

and Φ_T *converges to* $\sum_{\nu\in\mathcal{P}_0} a_\nu Q_\nu$ *as* $T\to\infty$ *weakly as probability measures on* W. *Here*

$$a_\nu = z^{-1}\left(\int_M (d\nu/d\mu)^{1/2} d\mu\right)^2 \det(I - G_\nu^{1/2} V_\nu G_\nu^{1/2})^{-1/2}, \tag{4.14}$$

and

$$z = \sum_{\nu\in\mathcal{P}_0} \left(\int_M (d\nu/d\mu)^{1/2} d\mu\right)^2 \det(I - G_\nu^{1/2} V_\nu G_\nu^{1/2})^{-1/2}. \tag{4.15}$$

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