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Applications of the Malliavin calculus, Part III

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0. Introduction

The present article is a continuation of our earlier work in [5] and [6]. In so far as possible, we will adhere to the notation introduced in those papers; and, when the reader encounters unexplained notation here, he is advised to seek an explanation in one of those articles. Just to help the reader get started, we provided below a brief list of some of our most frequently used notation along with a reference to the place where it was introduced in [5] or [6].

i) $(\Theta, \mathcal{B}, \mathcal{W})$ is the standard Wiener space with paths in $\boldsymbol{R}^d$ ([5, Sec. 1]).

ii) $H=\{h\in\Theta; h'\in L^2([0,\infty); \boldsymbol{R}^d)\}$ with Hilbert norm $\|h\|_H=\|h'\|_{L^2([0,\infty);\ \boldsymbol{R}^d)}$ ([5, Sec. 1]).

iii) For functions Φ on Θ with values in a separable Hilbert space $E, D\Phi: \Theta\to H(E)$ $(\equiv H^*\otimes E)$ and $\mathcal{L}\Phi: \Theta\to E$ are the basic operations on which the Malliavin calculus is built; $\mathcal{G}(\mathcal{L}; E)$ is the space of E-valued Φ's to which D and $\mathcal{L}$ can be applied infinitely often; and for $\Phi, \Psi\in \mathcal{G}(\mathcal{L})$ $(\equiv\mathcal{G}(\mathcal{L}; \boldsymbol{R}^1))\langle\Phi, \Psi\rangle=(D\Phi, D\Psi)_{H(R)}$ ([5, Sec. 1]).

iv) Given vector fields $V_0, \cdots, V_d\in C_b^\infty(\boldsymbol{R}^N; \boldsymbol{R}^N)$, L is the second order (degenerate) elliptic operator $\frac{1}{2}\sum_{k=1}^{d} V_k^2+V_0$ and $X(t,x)$: $\Theta\to\boldsymbol{R}^N$ is the solution to the Stratonovich stochastic integral equation

$$(0.1)\qquad X(T,x)=x+\sum_{k=1}^{d}\int_0^T V_k(X(t,x))\circ d\theta_k(t)+\int_0^T V_0(X(t,x))dt,\quad (T,x)\in[0,\infty)\times\boldsymbol{R}^N;$$

$J(T,x)$ is the Jacobi matrix of $x\to X(T,x)$, $A(T,x)\equiv(\langle X_i(t,x), X_j(t,x)\rangle)_{1\leq i,j\leq N}$ is the Malliavin covariance matrix of $X(T,x)$, and

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$$\tilde{A}(T, x) \equiv J(T, x)^{-1} A(T, x)^t J(T, x)^{-1}$$

is equal to

$$(0.2) \qquad \sum_{k=1}^{d} \int_0^T J(t, x) V_k(X(t, x)) \otimes J(t, x)^{-1} V_k(X(t, x)) dt \qquad ([6, \text{Sec. } 2]).$$

v) $\mathcal{A} \equiv \{\phi\} \cup \bigcup_{l=1}^{\infty} (\{0, 1, \cdots, d\})^l$; for $\alpha \in \mathcal{A}$:

$$|\alpha| = \begin{cases} 0 & \text{if } \alpha = \phi \\ l & \text{if } \alpha \in (\{0, \cdots, d\})^l \end{cases};$$

$\|\alpha\| = |\alpha| + \text{card}\{1 \leqq j \leqq |\alpha|;\ \alpha_j = 0\}$, and $\theta^{(\alpha)}(T)$ is the multiple stochastic integral of order α ([6, Appendix]).

vi) Given vector fields $V_0, \cdots, V_d$ and V and an $\alpha \in \mathcal{A}$, $V_{(\alpha)} = V$ if $\alpha = \phi$ and $V_{(\alpha)} = [V_{\alpha_l}, V_{((\alpha_1, \cdots, \alpha_{l-1}))}]$ if $\alpha = (\alpha_1, \cdots, \alpha_l)$ with $l \geqq 1$; $\mathcal{J}_{V_0}(V_1, \cdots, V_d) \equiv \{(V_k)_{(\alpha)};\ 1 \leqq k \leqq d \text{ and } \alpha \in \mathcal{A}\}$ and

$$v(x) \equiv \inf \left\{ \sum_{k=1}^{d} \sum_{\|\alpha\| \leqq l-1} ((V_k)_{(\alpha)}(x), \eta)^2_{R^N};\ \eta \in S^{N-1} \right\} \qquad ([6, \text{Sec. } 2]).$$

vii) Given $c \in C_b^\infty(\mathbf{R}^N)$,

$$^cP(T, x, \cdot) \equiv \left[\exp\left(\int_0^T c(X(t, x))\, dt \right) \mathcal{W} \right] \circ X(T, x)^{-1}$$

and $\{^cP_t;\ t \geqq 0\}$ denotes the corresponding semigroup of operators on $C_b(\mathbf{R}^N)$; and when $c \equiv 0$, the superscript is left off ([6, Sec. 3]).

Sections 1) and 2) of the present article are devoted to the study of the smoothing properties of $\{^cP_t;\ t > 0\}$. Under the assumption that the Lie algebra $\mathcal{J}_{V_0}(V_1, \cdots, V_d)$ is finite dimensional as a $C_b^\infty(\mathbf{R}^N)$-module (cf. hypothesis (H) at the begining of section 2)), we show (cf. Corollary 2.19 below) that for every pair $\alpha = (\alpha_1, \cdots, \alpha_l)$ and $\beta = (\beta_1, \cdots, \beta_m)$ from $\mathcal{A}$:

$$\| V_{\alpha_1} \circ \cdots \circ V_{\alpha_l} \circ {}^cP_t \circ V_{\beta_1} \circ \cdots \circ V_{\beta_m} \|_{L^p(\mathbf{R}^N) \to L^p(\mathbf{R}^N)} \leqq C(\alpha, \beta) / t^{(\|\alpha\| + \|\beta\|)/2}$$

for all $p \in [1, \infty]$ and $0 < t \leqq 1$. Although this estimate is clearly related to the sub-ellipticity result of Rothschild and Stein [8], we do not know how to get from their result to ours or ours to theirs. In particular, our result does not contain any of the subtle cancellation on which theirs (involving singular integrals as it does) rests. Moreover, ours holds under much more general conditions than it appears likely that there is a chance of proving theirs. The technique with which we derive our result is a refinement of the ones which we used in [6, Sec. 2].

In the rest of this article we turn our attention to the detailed study of the transition function $P(T, x, \cdot)$. In order to carry out our program, we have to impose much more rigid conditions on the vector fields $V_0, \cdots, V_d$. For one thing, we must assume that there is an $l_0 \in N$ and an $\varepsilon>0$ such that $v_{l_0}(x) \geqq \varepsilon$ for all $x \in \boldsymbol{R}^N$. Secondly, and much more objectionable, we have had to assume that $V_0=\sum_{k=1}^{d} \sigma_k V_k$ for some choice of $\sigma_1, \cdots, \sigma_k \in C_b^\infty(\boldsymbol{R}^N)$. Under these hypotheses, we have shown (cf. Theorem 4.13 below) that $P(t, x, dy)=p(t, x, y)dy$ where $p(t, x, y)$ is bounded above and below by "Gaussian kernels" in which the Euclidean metric has been replaced by a "control metric" defined in terms of the vector fields $V_1, \cdots, V_d$. Such estimates, at least for x essentially equal to y, were obtained by A. Sanchez [9]. In collaboration with D. Jerison, Sanchez [4] has recently extended his estimates to cover x and y which lie within a unit (control theoretic) distance of one another. So far as we know, ours is the first time that the global estimate has been proved. (In both [4] and [9], the condition on V_0 is that it can be expressed in terms of the $\{V_1, \cdots, V_d\} \cup \{[V_k, V_l]; 1 \leqq k<l \leqq d\}$. It seems likely that our technique can be extended to cover this case also, but there would be quite a bit of work involved.) Once we have established the preceding estimate on $p(t, x, y)$, it is a relatively easy step to obtain both a quantitative Harnack principle as well as a Poincare inequality for the operator L. We have given the derivation of these corollaries in section 5). Using entirely different ideas, Jerison proved the same version of the Poincare inequality in [3].

1. Preliminary results

(1.1) DEFINITION. Let E be a separable real Hilbert space and n be an integer. We say that $f \in \mathcal{N}_n(\boldsymbol{R}^N; E)$, if f is a measurable map from $(0, \infty) \times \boldsymbol{R}^N \times \Theta$ into E such that

(1) $f(t, \cdot, \theta)$: $\boldsymbol{R}^N \to E$ is smooth for each $t \in (0, \infty)$ and $\mathcal{W}$-a.e.$\theta \in \Theta$,

(2) $f(\cdot, x, \cdot) : (0, \infty) \times \Theta \to E$ is progressively measurable for each $x \in \boldsymbol{R}^N$,

(3) $\dfrac{\partial^\alpha}{\partial x^\alpha} f(t, x, \cdot) \in \mathcal{G}(\mathcal{L}; E)$, and is continuous in $t \in (0, \infty)$ for any multi-index α and $x \in \boldsymbol{R}^N$, and

(4) $\sup_{0<t\leq T}\sup_{x\in R^N}\frac{1}{t^{n/2}}\left\|\frac{\partial^\alpha}{\partial^\alpha x}f(t,x,\theta)\right\|_{p;E}^{(m)}<\infty$ for any multi-index α, any integer $m\geq 1$, $T>0$ and $2\leq p<\infty$.

Then the following is obvious.

(1.2) LEMMA. (1) *Let* $f\in\mathcal{N}_n(\boldsymbol{R}^N;E)$, $n\in\boldsymbol{Z}$. *Then* $Df\in\mathcal{N}_n(\boldsymbol{R}^N;\mathcal{H}(E))$ *and* $\mathcal{L}f\in\mathcal{N}_n(\boldsymbol{R}^N;E)$.

(2) *Let* $f_i\in\mathcal{N}_{n_i}(\boldsymbol{R}^N;E_i)$, $n_i\in\boldsymbol{Z}$, $i=1,\cdots,m$, *and* $A:E_1\times\cdots\times E_m\to E$ *be a continuous multilinear operator. Then* $A(f_1,\cdots,f_m)\in\mathcal{N}_n(\boldsymbol{R}^N;E)$, *where* $n=\sum_{i=1}^m n_i$.

By Kusuoka-Stroock [6] Theorem 1.9, we have the following.

(1.3) LEMMA. (1) *Let* $f(t,x,\theta)=X(t,x;\theta)-x$. *Then* $f\in\mathcal{N}_0(\boldsymbol{R}^N;\boldsymbol{R}^N)$.

(2) $g(X(t,x;\theta))\in\mathcal{N}_0(\boldsymbol{R}^N;\boldsymbol{R})$, *for any* $g\in C_b^\infty(\boldsymbol{R}^N;\boldsymbol{R})$.

For any $f\in\mathcal{N}_0(\boldsymbol{R}^N;E)$, we define $H_if:[0,\infty)\times\boldsymbol{R}^N\times\Theta\to\mathcal{H}(E)$, $i=1,\cdots,d$, and $I_if:[0,\infty)\times\boldsymbol{R}^N\times\Theta\to E$, $i=0,\cdots,d$, by

$$H_if(t,x,\theta)(h)=\int_0^t f(s,x,\theta)\frac{d}{ds}h_i(s)ds,\qquad h\in H,\ i=1,\cdots,d, \tag{1.4}$$

$$I_0f(t,x,\theta)=\int_0^t f(s,x,\theta)ds, \tag{1.5}$$

and

$$I_if(t,x,\theta)=\int_0^t f(s,x,\theta)d\theta_i(s),\qquad i=1,\cdots,d. \tag{1.6}$$

Then we can easily prove the following by an induction argument based on Kusuoka-Stroock [5] Lemma 2.2.

(1.7) LEMMA. *If* $f\in\mathcal{N}_n(\boldsymbol{R}^N;E)$, $n\geq 0$, *then* $H_if\in\mathcal{N}_{n+1}(\boldsymbol{R}^N;\mathcal{H}(E))$, $I_if\in\mathcal{N}_{n+1}(\boldsymbol{R}^N;E)$, $i=1,\cdots,d$, *and* $I_0f\in\mathcal{N}_{n+2}(\boldsymbol{R}^N;E)$. *Moreover,*

$$D(H_if)=H_i(Df),\quad \mathcal{L}(H_if)=H_i(\mathcal{L}f),\quad i=1,\cdots,d, \tag{1.8}$$

$$D(I_0f)=I_0(Df),\quad \mathcal{L}(I_0f)=I_0(\mathcal{L}f),\quad \textit{and} \tag{1.9}$$

$$D(I_if)=I_i(Df)+H_if,\quad \mathcal{L}(I_if)=I_i(\mathcal{L}f)-\frac{1}{2}I_if,\quad i=1,\cdots,d. \tag{1.10}$$

For each $U\in C_b^\infty(\boldsymbol{R}^N;\boldsymbol{R}^N)$ we define $Y_U:[0,\infty)\times\boldsymbol{R}^N\times\Theta\to\boldsymbol{R}^N$ by

$$Y_U(t,x,\theta)=(X(t)_*^{-1}U)(x)=J(t,x;\theta)^{-1}U(X(t,x;\theta)). \tag{1.11}$$

Then we have the following.

(1.12) LEMMA. *For any* $U\in C_b^\infty(\boldsymbol{R}^N;\boldsymbol{R}^N)$ *and any integer* $n\geqq 0$,

$$Y_U(t,x,\theta)-\sum_{\|\alpha\|\leqq n-1}\theta^{(\alpha)}(t)\cdot U^{(\alpha)}(x)\in\mathcal{N}_n(\boldsymbol{R}^N;\boldsymbol{R}^N),$$

and

$$J(t,x;\theta)U(x)-\sum_{\|\alpha\|\leqq n-1}(-1)^{\|\alpha\|}\theta^{(\alpha)}(t)\cdot U^{(\alpha)}(X(t,x;\theta))\in\mathcal{N}_n(\boldsymbol{R}^N;\boldsymbol{R}^N).$$

PROOF. The first assertion is obvious from Lemma 1.4 and the argument [6] section 2. To prove the second, observe that

$$\begin{aligned}&J(t,x;\theta)\int_0^t J(s,x)^{-1}U^{(\alpha)}(X(s,x))\circ d\theta^{(\alpha)}(s)\\&=U^{(\alpha)}(X(t,x))\theta^{(\alpha)}(t)-J(t,x;\theta)\sum_{i=0}^d\int_0^t J(s,x)^{-1}[V_i,U^{(\alpha)}](X(s,x))\cdot\theta^{(\alpha)}(s)\circ d\theta_i(s)\\&=U^{(\alpha)}(X(t,x))\theta^{(\alpha)}(t)-J(t,x;\theta)\sum_{i=0}^d\int_0^t J(s,x)^{-1}U^{(\alpha,i)}(X(s,x))\circ d\theta^{(\alpha,i)}(s).\end{aligned}$$

Therefore, by induction, we have

$$\begin{aligned}J(t,x;\theta)U(x)&=U(X(t,x))-J(t,x;\theta)(Y_U(t,x,\theta)-U(x))\\&=U(X(t,x))-J(t,x;\theta)\sum_{i=0}^d\int_0^t J(s,x)^{-1}[V_i,U](X(s,x))\circ d\theta_i(s)\\&=\sum_{\|\alpha\|\leqq n-1}(-1)^{\|\alpha\|}\theta^{(\alpha)}(t)\cdot J(t,x;\theta)U^{(\alpha)}(X(t,x))\\&\quad+\sum_{\|\alpha\|=n}(-1)^n J(t,x;\theta)\int_0^t J(s,x;\theta)^{-1}U^{(\alpha)}(X(t,x))\circ d\theta^{(\alpha)}(s).\end{aligned}$$

Using this formula succesively, we obtain our results. Q.E.D.

Let $\tilde{A}(t,x)=\sum_{i=1}^d\int_0^t J(s,x)^{-1}V_i(X(s,x))\otimes J(s,x)^{-1}V_i(X(s,x))ds$, $0\leqq t\leqq 1$ and $x\in\boldsymbol{R}^N$. Also, let $\{h_j\}_1^\infty$ be an orthonormal basis in H.

(1.13) THEOREM. *For any* $\Phi\in\mathcal{G}(\mathcal{L})$, $f\in C_b^\infty(\boldsymbol{R}^N)$, $U\in C_b^\infty(\boldsymbol{R}^N;\boldsymbol{R}^N)$ *and* $\varepsilon>0$,

$$\begin{aligned}&E[\Phi\cdot U\{f(X(t,x))\}]\\&=-E\Big[\sum_{j=1}^\infty D\Phi(h_j)\cdot\Big(\sum_{i=1}^d H_i(Y_{V_i})(t,x)(h_j),\ (\tilde{A}(t,x)+\varepsilon I)^{-1}U(x)\Big)\cdot f(X(t,x))\Big]\\&\quad+E\Big[\Phi\sum_{j=1}^\infty\Big(\sum_{i=1}^d H_i(Y_{V_i})(h_j)(t,x),\ (\tilde{A}(t,x)+\varepsilon I)^{-1}\cdot D\tilde{A}(t,x)(h_j)\\&\qquad\times(\tilde{A}(t,x)+\varepsilon I)^{-1}U(x)\Big)f(X(t,x))\Big]\end{aligned}$$

$$-E\Big[\Phi\cdot\sum_{i=1}^{d}(I_i(Y_{V_i})(t,x),\ (\tilde{A}(t,x)+\varepsilon I)^{-1}U(x))\cdot f(X(t,x))\Big]$$
$$+\varepsilon E[\Phi\cdot(U(x),\ (\tilde{A}(t,x)+\varepsilon I)^{-1}\operatorname{grad}(f(X(t,x))))].$$

To prove this theorem we need to make some preparations.

(1.14) LEMMA.

(1) $$d(\mathcal{L}X(t,x))=\sum_{i=1}^{d}\Big[\mathcal{L}(V_i(X(t,x)))-\frac{1}{2}V_i(X(t,x))\Big]\circ d\theta_i(t)$$
$$+\Big[\mathcal{L}(V_0(X(t,x)))+\frac{1}{2}\sum_{i=1}^{d}\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\Big]dt.$$

(2) $$d\Big(\sum_{j=1}^{N}\langle(J(t,x)^{-1})_{\cdot j},X^j(t,x)\rangle\Big)$$
$$=2\Big(\sum_{i=1}^{d}J(t,x)^{-1}\Big\{\frac{\partial V_i}{\partial x}(X(t,x))\mathcal{L}(X(t,x))-\mathcal{L}(V_i(X(t,x)))\Big\}\Big)\circ d\theta_i(t)$$
$$+2J(t,x)^{-1}\Big\{\frac{\partial V_0}{\partial x}(X(t,x))\mathcal{L}(X(t,x))-\mathcal{L}(V_0(X(t,x)))$$
$$-\frac{1}{2}\sum_{i=1}^{d}\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\Big\}dt.$$

PROOF. Note that

$$dX(t,x)=\sum_{i=1}^{d}V_i(X(s,x))d\theta_i(t)+\Big\{V_0(X(t,x))+\frac{1}{2}\sum_{i=1}^{d}\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\Big\}dt.$$

Therefore by Kusuoka-Stroock [5] Lemma 2.2, we have

$$d(\mathcal{L}(X(t,x)))=\sum_{i=1}^{d}\Big[\mathcal{L}(V_i(X(s,x)))-\frac{1}{2}V_i(X(t,x))\Big]d\theta_i(t)$$
$$+\Big\{\mathcal{L}(V_0(X(t,x)))+\frac{1}{2}\sum_{i=1}^{d}\mathcal{L}\Big(\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\Big)\Big\}dt.$$

Since $d(V_i(X(t,x)))\cdot d\theta_i(t)=\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))dt$, we see that

$$d(\mathcal{L}(V_i(X(t,x))))\cdot d\theta_i(t)$$
$$=\Big[\mathcal{L}\Big(\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\Big)-\frac{1}{2}\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\Big]dt.$$

Hence

$$\Big[\mathcal{L}(V_i(X(t,x)))-\frac{1}{2}V_i(X(t,x))\Big]\circ d\theta_i(t)$$

$$=\left[\mathcal{L}(V_i(X(t,x)))-\frac{1}{2}V_i(X(t,x))\right]d\theta_i(t)$$
$$+\frac{1}{2}\left[\mathcal{L}\left(\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\right)-\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\right]dt.$$

Thus (1) is proved.

Note that

$$d(J(t,x)^{-1})=-\sum_{i=1}^{d}J(t,x)^{-1}\frac{\partial V_i}{\partial x}(X(t,x))\circ d\theta_i(t)-J(t,x)^{-1}\frac{\partial V_0}{\partial x}(X(t,x))dt.$$

Therefore we obtain

$$\begin{aligned}&d\langle J(t,x)^{-1},X(t,x)\rangle\\&=\sum_{i=1}^{d}\left[\langle J(t,x)^{-1},V_i(X(t,x))\rangle-\langle J(t,x)^{-1}\frac{\partial V_i}{\partial x}(X(t,x)),X(t,x)\rangle\right]\circ d\theta_i(t)\\&+\left[\langle J(t,x)^{-1},V_0(X(t,x))\rangle-\langle J(t,x)^{-1}\frac{\partial V_0}{\partial x}(X(t,x)),X(t,x)\rangle\right.\\&\left.-\sum_{i=1}^{d}J(t,x)^{-1}\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\right]dt.\end{aligned}$$

Observe that

$$\begin{aligned}&\langle J(t,x)^{-1},V(X(t,x))\rangle-\langle J(t,x)^{-1}\frac{\partial V}{\partial x}(X(t,x)),X(t,x)\rangle\\&=\langle(J^{-1})_{\cdot\nu},V^{\nu}(X)\rangle-\langle(J^{-1})_{\cdot\nu}\frac{\partial V^{\nu}}{\partial x^{\mu}}(X),X^{\mu}\rangle\\&=\langle(J^{-1})_{\cdot\nu},X^{\mu}\rangle\frac{\partial V^{\nu}}{\partial x^{\mu}}(X)-\langle(J^{-1})_{\cdot\nu},X^{\mu}\rangle\frac{\partial V^{\nu}}{\partial x^{\mu}}(X)-(J^{-1})_{\cdot\nu}\left\langle\frac{\partial V^{\nu}}{\partial x^{\mu}}(X),X^{\mu}\right\rangle\\&=-(J^{-1})_{\cdot\nu}\frac{\partial V^{\nu}}{\partial x^{\mu}\partial x^{\mu}}(X)\langle X^{\mu},X^{\mu}\rangle=-2(J^{-1})_{\cdot\nu}\left(\mathcal{L}V^{\nu}-\frac{\partial V^{\nu}}{\partial x^{\mu}}(X)\mathcal{L}X^{\mu}\right)\\&=-2J(t,x)^{-1}\left(\mathcal{L}(V(X(t,x)))-\frac{\partial V}{\partial x}(X(t,x))\mathcal{L}(X(t,x))\right).\end{aligned}$$

Thus (2) is proved. Q.E.D.

(1.15) LEMMA. $d(2\cdot J(t,x)^{-1}\mathcal{L}(X(t,x))+\langle J(t,x)^{-1},X(t,x)\rangle)$

$$\begin{aligned}&=-\sum_{i=1}^{d}J(t,x)^{-1}V_i(X(s,x))d\theta_i(t)\\&=-\sum_{i=1}^{d}J(t,x)^{-1}V_i(X(s,x))\circ d\theta_i(t).\end{aligned}$$

PROOF. Using (1) in Lemma (1.14), we have

$$
\begin{aligned}
&d(J(t,x)^{-1}\mathcal{L}(X(t,x)))\\
&=d(J(t,x)^{-1})\circ\mathcal{L}(X(t,x))+J(t,x)^{-1}\circ d(\mathcal{L}(X(t,x)))\\
&=\sum_{i=1}^{d}J(t,x)^{-1}\Big\{-\frac{\partial V_i}{\partial x}(X(t,x))\mathcal{L}(X(t,x))+\mathcal{L}(V_i(X(t,x)))-\frac{1}{2}V_i(X(t,x))\Big\}\circ d\theta_i(t)\\
&\quad+J(t,x)^{-1}\Big\{-\frac{\partial V_0}{\partial x}(X(t,x))\mathcal{L}(X(t,x))+\mathcal{L}(V_0(X(t,x)))\\
&\qquad+\frac{1}{2}\sum_{i=1}^{d}\frac{\partial V_i}{\partial x}(X(t,x))V_i(X(t,x))\Big\}dt.
\end{aligned}
$$

Combining this with (2) in Lemma (1.14), we have our assertion. Q.E.D.

PROOF OF THEOREM (1.13). It is easy to see that

$$
\begin{aligned}
&(\tilde{A}(t,x)+\varepsilon I)J(t,x)^*(\operatorname{grad} f)(X(t,x))\\
&=J(t,x)^{-1}\langle f(X(t,x)),X(t,x)\rangle+\varepsilon J(t,x)^*(\operatorname{grad} f)(X(t,x)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E[\Phi\operatorname{grad}(f(X(t,x)))]&=E[\Phi J(t,x)^*(\operatorname{grad} f)(X(t,x))]\\
&=E[\Phi(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)^{-1}\langle f(X(t,x)),X(t,x)\rangle]\\
&\quad+\varepsilon E[\Phi\cdot(U(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}\operatorname{grad}(f(X(t,x))))]\\
&=-E[f(X(t,x))\langle\Phi(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)^{-1},X(t,x)\rangle]\\
&\quad-2E[f(X(t,x))(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)^{-1}\mathcal{L}(X(t,x))]\\
&\quad+\varepsilon E[\Phi\cdot(U(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}\operatorname{grad}(f(X(t,x))))].
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\langle\Phi(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)^{-1},X(t,x)\rangle\\
&=\Phi(\tilde{A}(t,x)+\varepsilon I)^{-1}\langle J(t,x)^{-1},X(t,x)\rangle\\
&\quad+\Phi\sum_{j=1}^{\infty}D((\tilde{A}(t,x)+\varepsilon I)^{-1})(h_j)J(t,x)^{-1}DX(t,x)(h_j)\\
&\quad+\sum_{j=1}^{\infty}D\Phi(h_j)(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)^{-1}DX(t,x)(h_j).
\end{aligned}
$$

Note that from [6] (2.5) we see that

$$J(t,x)^{-1}DX(t,x)=H_i(Y_{V_i})(t,x).$$

From Lemma 1.4, we have

$$2\cdot J(t,x)^{-1}\mathcal{L}(X(t,x))+\langle J(t,x)^{-1},X(t,x)\rangle=-I_i(Y_{V_i})(t,x).$$

Thus we have our theorem. Q.E.D.

(1.16) REMARK. If we introduce the dual operator ∂ of D, we can prove Theorem (1.13) easily in the following way. (For details about this operator ∂, see Meyer [7] and Sugita [10].) Note that

$$E[\Phi\cdot U\{f(X(t,x))\}]=E[\Phi(J(t,x)U(x),\nabla f(X(t,x)))],$$

$$J(t,x)^*\nabla f(X(t,x))=(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)(D(f(X(t,x))),DX(t,x))_H$$
$$+\varepsilon J(t,x)^*\nabla f(X(t,x))$$

and

$$DX(t,x)=J(t,x)\sum_{i=1}^{d}H_iY_{V_i}(t,x).$$

Thus we have

$$\begin{aligned}
&E[\Phi(U(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)^{-1}(D(f(X(t,x))),DX(t,x))_H)]\\
&=E[(D(f(X(t,x))),(\Phi\cdot U(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)^{-1}DX(t,x)))_H]\\
&=E[f(X(t,x)\partial(\Phi\cdot U(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)^{-1}DX(t,x)))]\\
&=\sum_{i=1}^{d}E[f(X(t,x))\partial(\Phi\cdot U(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}H_iY_{V_i}(t,x))]\\
&=-\sum_{i=1}^{d}E[f(X(t,x))((\tilde{A}(t,x)+\varepsilon I)^{-1}U(x),(D\Phi,H_iY_{V_i}(t,x))_H)]\\
&\quad+\sum_{i=1}^{d}E[f(X(t,x))\Phi(U(x),(D(\tilde{A}(t,x)+\varepsilon I)^{-1},H_iY_{V_i}(t,x))_H)]\\
&\quad-\sum_{i=1}^{d}E[f(X(t,x))\Phi((\tilde{A}(t,x)+\varepsilon I)^{-1}U(x),I_iY_{V_i}(t,x))],
\end{aligned}$$

where we have used

$$\partial(\phi\cdot\Psi)=-(D\phi,\Psi)_H+\phi\cdot\partial\Psi$$

and

$$\partial(H_if(t))=\int_0^t f(s)d\theta_i(s),\qquad i=1,\cdots,d.$$

Since

$$\begin{aligned}
&E[\Phi\cdot U\{f(X(t,x))\}]\\
&=E[\Phi(U(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}J(t,x)^{-1}(D(f(X(t,x))),DX(t,x))_H)]\\
&\quad+\varepsilon E[\Phi\cdot(U(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}\operatorname{grad}(f(X(t,x))))],
\end{aligned}$$

we get Theorem (1.13).

2. Precise regularity estimates for semigroups

In this section, we set for each multi-index $\alpha\in\mathcal{A}$ inductively, $V_{\langle\alpha\rangle}=0$ if $\alpha=\phi$ or 0, $V_{\langle\alpha\rangle}=V_k$ if $\alpha=(k)$ and $k=1,\cdots,d$, and $V_{\langle(\alpha,i)\rangle}=[V_i, V_{\langle\alpha\rangle}]$, $i=0,1,\cdots,d$, for simplicity of notation. Then it is easy to see that $V_{\langle(i,\alpha)\rangle}=(V_i)_{(\alpha)}$ if $i=1,\cdots,d$ and $V_{\langle(i,\alpha)\rangle}=0$ if $i=0$.

Throughout this section, we will always assume the following. (H) There is an $l_0\geqq 1$ such that for any multi-index α, $\|\alpha\|>l_0$, there are $a_{\alpha,\beta}\in C_b^\infty(\boldsymbol{R}^N;\boldsymbol{R})$, $\|\beta\|\leqq l_0$, satisfying $V_{\langle\alpha\rangle}(x)=\sum_{\|\beta\|\leqq l_0}a_{\alpha,\beta}(x)\cdot V_{\langle\beta\rangle}(x)$ for any $x\in\boldsymbol{R}^N$.

(2.1) REMARK. If there is an $l_0\geqq 1$, such that

$$\inf\{(\xi,\bar{A}_{l_0}(1,x)\xi);\ x\in\boldsymbol{R}^N,\ \xi\in\boldsymbol{R}^N,\ |\xi|=1\}>0,$$

then the hypothesis (H) holds. Here $\bar{A}_l(t,x)=\sum_{k=1}^{\infty}\sum_{\|\alpha\|\leqq l-1}t^{\|\alpha\|+1}(V_k)_{(\alpha)}(x)\otimes(V_k)_{(\alpha)}(x)$.

PROOF. Note that $\bar{A}_{l_0}(1,x)=\sum_{\|\alpha\|\leqq l_0}V_{\langle\alpha\rangle}(x)\otimes V_{\langle\alpha\rangle}(x)$, and $\bar{A}_{l_0}(1,\cdot)^{-1}\in C_b^\infty(\boldsymbol{R}^N;\boldsymbol{R}^N\otimes\boldsymbol{R}^N)$. Thus for any $U\in C_b^\infty(\boldsymbol{R}^N;\boldsymbol{R}^N)$, we have $U(x)=\sum_{\|\alpha\|\leqq l_0}(U(x),\bar{A}_{l_0}(1,x)^{-1}V_{\langle\alpha\rangle}(x))V_{\langle\alpha\rangle}(x)$. This proves our assertion. Q.E.D.

From the hypothesis (H), we see that there are $b_{\alpha,\beta}^{(i)}\in C_b^\infty(\boldsymbol{R}^N;\boldsymbol{R})$, $i=0,1,\cdots,d$, $\|\alpha\|,\|\beta\|\leqq l_0$, such that

$$[V_i,V_{\langle\alpha\rangle}](x)=\sum_{\|\beta\|\leqq l_0}b_{\alpha,\beta}^{(i)}(x)\cdot V_{\langle\beta\rangle}(x),\qquad i=0,1,\cdots,d,\ \|\alpha\|\leqq l_0.$$

(2.2) PROPOSITION. *Let $Y_{\alpha,\beta}(t,x)$, $\|\alpha\|,\|\beta\|\leqq l_0$, be the solution of the stochastic differential equation:*

$$(2.3)\qquad \begin{aligned} dY_{\alpha,\beta}(t,x)=&\sum_{i=1}^{d}\sum_{\|\gamma\|\leqq l_0}b_{\alpha,\gamma}^{(i)}(X(t,x))Y_{\gamma,\beta}(t,x)\circ d\theta_i(t)\\ &+\sum_{\|\gamma\|\leqq l_0}b_{\alpha,\gamma}^{(0)}(X(t,x))Y_{\gamma,\beta}(t,x)dt,\\ Y_{\alpha,\beta}(0,x)=1 &\text{ if } \alpha=\beta, \text{ and } Y_{\alpha,\beta}(0,x)=0 \text{ otherwise.}\end{aligned}$$

Then $(X(t)_*^{-1}V_{\langle\alpha\rangle})(x)=\sum_{\|\beta\|\leqq l_0}Y_{\alpha,\beta}(t,x)\cdot V_{\langle\beta\rangle}(x)$, $\|\alpha\|\leqq l_0$, $t\geqq 0$, $x\in\boldsymbol{R}^N$.

PROOF. Let $Z_\alpha(t,x)=\sum_{\|\beta\|\leqq l_0}Y_{\alpha,\beta}(t,x)\cdot V_{\langle\beta\rangle}(x)$. Then it is easy to see that

$$dZ_\alpha(t,x)=\sum_{i=1}^d\sum_{\|\beta\|\leqq l_0}b_{\alpha,\beta}^{(i)}(X(t,x))\cdot Z_\beta(t,x)\circ d\theta_i(t)+\sum_{\|\beta\|\leqq l_0}b_{\alpha,\beta}^{(0)}(X(t,x))\cdot Z_\beta(t,x)dt,$$

$Z_\alpha(0,x)=V_{\langle\alpha\rangle}(x)$. On the other hand, we have

$$\begin{aligned}d(X(t)_*^{-1}V_{\langle\alpha\rangle})(x)&=\sum_{i=1}^d(X(t)_*^{-1}[V_i,V_{\langle\alpha\rangle}])(x)\circ d\theta_i(t)+(X(t)_*^{-1}[V_0,V_{\langle\alpha\rangle}])(x)dt\\&=\sum_{i=1}^d\sum_{\|\beta\|\leqq l_0}b_{\alpha,\beta}^{(i)}(X(t,x))\cdot(X(t)_*^{-1}V_{\langle\beta\rangle})(x)\circ d\theta_i(t)\\&\quad+\sum_{\|\beta\|\leqq l_0}b_{\alpha,\beta}^{(0)}(X(t,x))\cdot(X(t)_*^{-1}V_{\langle\beta\rangle})(x)dt.\end{aligned}$$

This and the uniqueness of the solution of S.D.E. imply $Z_\alpha(t,x)=(X(t)_*^{-1}V_{\langle\alpha\rangle})(x)$. This completes the proof. Q.E.D.

(2.4) REMARK. Since $Y_{\alpha,\beta}(t,x)$, $\|\alpha\|$, $\|\beta\|\leqq l_0$, are given by the S.D.E. (2.3), it is easy to see that $Y_{\alpha,\beta}\in\mathcal{N}_0(\boldsymbol{R}^N;\boldsymbol{R})$ and that there are $C<\infty$ and $\lambda>0$ such that

$$\mathcal{W}\Big[\sup_{0\leqq t\leqq T}|Y_{\alpha,\beta}(t,x)-Y_{\alpha,\beta}(0,x)|>1\Big]\leqq C\exp\Big(-\frac{\lambda}{T}\Big)\tag{2.5}$$

for any $x\in\boldsymbol{R}^N$, $0<T\leqq 1$, $\|\alpha\|$, $\|\beta\|\leqq l_0$.

(2.6) PROPOSITION. *For each multi-index* α, $\|\alpha\|\leqq l_0$, *there are* $c_{\alpha,\beta}\in\mathcal{N}_{(\|\beta\|-\|\alpha\|)\$0}(\boldsymbol{R}^N;\boldsymbol{R})$, $\|\beta\|\leqq l_0$, *such that*

$$(X(t)_*^{-1}V_{\langle\alpha\rangle})(x)=\sum_{\|\beta\|\leqq l_0}c_{\alpha,\beta}(t,x)\cdot V_{\langle\beta\rangle}(x).$$

PROOF. Note that

$$\begin{aligned}(X(t)_*^{-1}V_{\langle\alpha\rangle})(x)=&\sum_{\|\beta\|\leqq l_0-\|\alpha\|}\theta^{(\beta)}(t)\cdot V_{\langle(\alpha,\beta)\rangle}(x)\\&+\sum_{\|\beta'\|=l_0-\|\alpha\|}S^{(\beta)}(t,(X(\cdot)_*^{-1}V_{\langle(\alpha,\beta)\rangle})(x))\\&+\sum_{\substack{\|\beta'\|=l_0-\|\alpha\|-1\\ \beta_*=0}}S^{(\beta)}(t,(X(\cdot)_*^{-1}V_{\alpha,\beta})(x)).\end{aligned}$$

From Proposition (2.2) and the hypothesis (H), we see that for any multi-index α, $\|\alpha\|>l_0$,

$$(X(t)_*^{-1}V_{\langle\alpha\rangle})(x)=\sum_{\|\beta\|\leqq l_0,\|\gamma\|\leqq l_0}a_{\alpha,\beta}(X(t,x))Y_{\beta,\gamma}(t,x)\cdot V_{\langle\gamma\rangle}(x).$$

Therefore we have our assertion from Lemmas (1.2), (1.3), (1.7), and Remark (2.4). Q.E.D.

The following is obvious from the definition of $\tilde{A}(t, x)$, Lemmas (1.2), (1.7), and Proposition (2.6).

(2.7) PROPOSITION. *There are* $e_{\alpha,\beta} \in \mathcal{N}_{\|\alpha\|+\|\beta\|}(\boldsymbol{R}^N; \boldsymbol{R})$, $\|\alpha\|, \|\beta\| \leqq l_0$, *such that* $\tilde{A}(t, x) = \sum_{\|\alpha\| \leqq l_0, \|\beta\| \leqq l_0} e_{\alpha,\beta}(t, x) \cdot V_{\langle\alpha\rangle}(x) \otimes V_{\langle\beta\rangle}(x)$.

(2.8) THEOREM. *Let* $\lambda(t, x) = \inf\{(\xi, \tilde{A}(t, x)\xi); \xi \in \boldsymbol{R}^N, (\xi, \bar{A}_{l_0}(t, x)\xi) = 1\}$. *Then there are* $C < \infty$, $c > 0$ *and* $\nu > 0$ *such that*

$$\mathcal{W}\left[\lambda(t, x) < \frac{1}{K}\right] \leqq C \exp(-cK^\nu) \tag{2.9}$$

for any $K > 1$, $x \in \boldsymbol{R}^N$ *and* $0 < t \leqq 1$.

PROOF. First observe that $\tilde{A}(t/K, x) \leqq \tilde{A}(t, x)$ and that

$$\begin{aligned}(\xi, \tilde{A}(t/K, x)\xi) &= \sum_{i=1}^{d} \int_0^{t/K} (\xi, (X(s)_*^{-1} V_i)(x))^2 ds \\ &\leqq \frac{1}{2} \sum_{i=1}^{d} \int_0^{t/K} \left\{ \sum_{\|\alpha\| \leqq l_0 - 1} \theta^{(\alpha)}(t)(\xi, (V_i)_{(\alpha)}(x)) \right\}^2 ds - \sum_{i=1}^{d} \int_0^{t/K} (\xi, R_i(s, x))^2 ds,\end{aligned}$$

where

$$\begin{aligned}R_i(s, x) &= (X(s)_*^{-1} V_i)(x) - \sum_{\|\alpha\| \leqq l_0 - 1} \theta^{(\alpha)}(s)(V_i)_{(\alpha)}(x) \\ &= \Sigma' S^{(\alpha)}(s, (X(\cdot)_*^{-1}(V_i)_{(\alpha)})(x)).\end{aligned}$$

Here Σ' denotes the summation on a multi-index α satisfying $\|\alpha\| \geqq l_0$ with $\|\alpha'\| \leqq l_0 - 1$. Then by the hypothesis (H) and Proposition (2.2), we have

$$R_i(s, x) = \sum_{\|\beta\| \leqq l_0} r_i^{(\beta)}(s, x) V_{\langle\beta\rangle}(x),$$

where

$$r_i^{(\beta)}(s, x) = \Sigma' \sum_{\|\gamma\| \leqq l_0} S^{(\alpha)}(s, a_{(i,\alpha),\gamma}(X(\cdot, x)) Y_{\gamma,\beta}(\cdot, x)).$$

Now, by virtue of [6] Theorem (A.5), we see that there are $C' < \infty$, $c' > 0$ and $\nu' > 0$ such that

$$\begin{aligned}&\mathcal{W}\left[\inf\left\{ \sum_{i=1}^{d} \int_0^{t/K} \left\{ \sum_{\|\alpha\| \leqq l_0 - 1} \theta^{(\alpha)}(s)(\xi, (V_i)_{(\alpha)}(x)) \right\}^2 ds; \right.\right. \\ &\qquad\qquad \left.\left. \xi \in \boldsymbol{R}^N, (\xi, \bar{A}_{l_0}(t, x)\xi) = 1 \right\} < 2K^{-(l_0+1/3)} \right] \\ &= \mathcal{W}\left[K^{l_0} \cdot \inf\left\{ \sum_{i=1}^{d} \int_0^{t/K} \left\{ \sum_{\|\alpha\| \leqq l_0 - 1} \frac{\theta^{(\alpha)}(s)}{t^{\|\alpha\|/2}} (\xi, t^{\|\alpha\|/2}(V_i)_{(\alpha)}(x)) \right\}^2 ds; \right.\right.\end{aligned}$$

$$\xi \in \boldsymbol{R}^N,\ (\xi, \bar{A}_{l_0}(t, x)\xi) = 1\Big\} < 2K^{-1/3}\Big]$$

$$= \mathcal{W}\Big[K^{l_0} \cdot \inf\Big\{\sum_{i=1}^{d} \int_0^{1/K} \Big\{\sum_{\|\alpha\| \leqq l_0 - 1} \theta^{(\alpha)}(s)(\xi, t^{\|\alpha\|/2}(V_i)_{(\alpha)}(x))\Big\}^2 ds;$$

$$\xi \in \boldsymbol{R}^N,\ (\xi, \bar{A}_{l_0}(t, x)\xi) = 1\Big\} < 2K^{-1/3}\Big]$$

$$\leqq C' \exp(-c' K^{\nu'})$$

for any $K \geqq 1$, $x \in \boldsymbol{R}^N$ and $0 < t \leqq 1$.

On the other hand, if $(\xi, \bar{A}_{l_0}(t, x)\xi) = 1$ and $0 \leqq s \leqq t \leqq 1$,

$$|(\xi, R_i(s, x))| \leqq \sum_{\|\alpha\| \leqq l_0} |r_i^{(\alpha)}(s, x)(\xi, V_{\langle\alpha\rangle}(x))| \leqq t^{-l_0} \sum_{\|\alpha\| \leqq l_0} |r_i^{(\alpha)}(s, x)|,$$

and so

$$\sum_{i=1}^{d} \int_0^{t/K} (\xi, R_i(s, x))^2 ds \leqq \sum_{i=1}^{d} \sum_{\|\alpha\| \leqq l_0} t^{-l_0} \cdot \frac{t}{K}\Big(\sup_{0 \leqq s \leqq t/K} |r_i^{(\alpha)}(s, x)|\Big)^2.$$

Therefore, taking into account Remark (2.4) and arging as in the proof of [6] Theorem 2.12, we see that there are $C'' < \infty$, $c'' > 0$ and $\nu'' > 0$ such that

$$\mathcal{W}\Big[\sup\Big\{\sum_{i=1}^{d} \int_0^{t/K} (\xi, R_i(s, x))^2 ds;\ \xi \in \boldsymbol{R}^N,\ (\xi, \bar{A}_{l_0}(t, x)\xi) = 1\Big\} > K^{-(l_0 + 2/3)}\Big]$$

$$\leqq \mathcal{W}\Big[\sum_{i=1}^{d} \sum_{\|\alpha\| \leqq l_0} \Big(\sup_{0 \leqq s \leqq t/K} |r_i^{(\alpha)}(s, x)|\Big)^2 > \Big(\frac{t}{K}\Big)^{(l_0 - 1/3)}\Big]$$

$$\leqq C'' \exp(-c'' K^{\nu''})$$

for any $K \geqq 1$, $x \in \boldsymbol{R}^N$ and $0 < t \leqq 1$. This completes the proof. Q.E.D.

(2.10) COROLLARY. *For any* $\alpha, \beta \in \mathcal{A}$ *with* $\|\alpha\|, \|\beta\| \leqq l_0$,

$$\sup\{t^{-(\|\alpha\| + \|\beta\|)/2} E[|(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x) + \varepsilon I)^{-1} V_{\langle\beta\rangle}(x))|^p]^{1/p};$$
$$x \in \boldsymbol{R}^N,\ 0 < t \leqq 1,\ \varepsilon > 0\} < \infty,$$

for any $1 < p < \infty$. *In particular,* $(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x) + \varepsilon I)^{-1} V_{\langle\beta\rangle}(x))$ *converges in* L^p *as* $\varepsilon \downarrow 0$ *for all* $t > 0$ *and* $1 < p < \infty$. *Finally, if we denote the limit by* $(V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1} V_{\langle\beta\rangle}(x))$, *then we have*

$$\text{(2.11)} \quad \sup\{t^{-(\|\alpha\| + \|\beta\|)/2} E[|(V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1} V_{\langle\beta\rangle}(x))|^p]^{1/p};\ x \in \boldsymbol{R}^N,\ 0 < t \leqq 1,\ \varepsilon > 0\} < \infty,$$

for any $1 < p < \infty$.

The proof is obvious, since

$$\begin{aligned}
&|(V_{\langle\alpha\rangle}(x), \bar{A}_{l_0}(t, x)^{-1}V_{\langle\beta\rangle}(x))| \\
&\leqq |(V_{\langle\alpha\rangle}(x), \bar{A}_{l_0}(t, x)^{-1}V_{\langle\alpha\rangle}(x))|^{1/2}|(V_{\langle\beta\rangle}(x), \bar{A}_{l_0}(t, x)^{-1}V_{\langle\beta\rangle}(x))|^{1/2} \\
&\leqq t^{-(\|\alpha\|+\|\beta\|)/2}.
\end{aligned}$$

(2.12) THEOREM. $(V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\beta\rangle}(x)) \in \mathcal{N}_{-(\|\alpha\|+\|\beta\|)}(\boldsymbol{R}^N; \boldsymbol{R})$ *for any* $\alpha, \beta \in \mathcal{A}$ *with* $\|\alpha\|, \|\beta\| \leqq l_0$.

PROOF. Note that $(\tilde{A}(t, x)+\varepsilon I)^{-1} \in \mathcal{G}(\mathcal{L}, \boldsymbol{R}^N \otimes \boldsymbol{R}^N)$, $t>0$, $x \in \boldsymbol{R}^N$, $\varepsilon>0$. Therefore we have

$$\begin{aligned}
&D(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\beta\rangle}(x)) \\
&=(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}D\tilde{A}(t, x)(\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\beta\rangle}(x)) \\
&=\sum_{\|\gamma_1\|\leqq l_0}\sum_{\|\gamma_2\|\leqq l_0} De_{\gamma_1,\gamma_2}(t, x)(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\gamma_1\rangle}(x)) \\
&\quad\times(V_{\langle\gamma_2\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\beta\rangle}(x)),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}((V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\beta\rangle}(x))) \\
&=(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}\mathcal{L}\tilde{A}(t, x)(\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\beta\rangle}(x)) \\
&\quad+\sum_{i=1}^{\infty}(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}D\tilde{A}(t, x)(h_i)(\tilde{A}(t, x)+\varepsilon I)^{-1} \\
&\quad\times D\tilde{A}(t, x)(h_i)(\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\beta\rangle}(x)) \\
&=\sum_{\|\gamma_1\|\leqq l_0}\sum_{\|\gamma_2\|\leqq l_0}\mathcal{L}e_{\gamma_1,\gamma_2}(t, x)(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\gamma_1\rangle}(x)) \\
&\quad\times(V_{\langle\gamma_2\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\beta\rangle}(x)) \\
&\quad+\sum_{\|\gamma_1\|\leqq l_0}\sum_{\|\gamma_2\|\leqq l_0}\sum_{\|\gamma_3\|\leqq l_0}\sum_{\|\gamma_4\|\leqq l_0}(De_{\gamma_1,\gamma_2}(t, x), De_{\gamma_3,\gamma_4}(t, x)) \\
&\quad\times(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\gamma_1\rangle}(x))(V_{\langle\gamma_2\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\gamma_3\rangle}(x)) \\
&\quad\times(V_{\langle\gamma_3\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\beta\rangle}(x)).
\end{aligned}$$

Thus, as $\varepsilon \downarrow 0$, $D(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon)^{-1}V_{\langle\beta\rangle}(x))$ converges in $L^p(\Theta; \mathcal{H}(\boldsymbol{R}), \mathcal{W})$ and $\mathcal{L}((V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon)^{-1}V_{\langle\beta\rangle}(x)))$ converges in $L^p(\Theta, \mathcal{W})$ for each $1<p<\infty$. Therefore $(V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\beta\rangle}(x)) \in \mathcal{K}(\mathcal{L}; \boldsymbol{R})$ and

$$\begin{aligned}
&(2.13)\quad D(V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\beta\rangle}(x)) \\
&=\sum_{\|\gamma_1\|\leqq l_0}\sum_{\|\gamma_2\|\leqq l_0} De_{\gamma_1,\gamma_2}(t, x)(V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\gamma_1\rangle}(x))(V_{\langle\gamma_2\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\beta\rangle}(x)),
\end{aligned}$$

and

(2.14) $\mathcal{L}((V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\beta\rangle}(x)))$

$$= \sum_{\|\gamma_1\|\leqq l_0}\sum_{\|\gamma_2\|\leqq l_0} \mathcal{L}e_{\gamma_1,\gamma_2}(t, x)(V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\gamma_1\rangle}(x))(V_{\langle\gamma_2\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\beta\rangle}(x))$$
$$+ \sum_{\|\gamma_1\|\leqq l_0}\sum_{\|\gamma_2\|\leqq l_0}\sum_{\|\gamma_3\|\leqq l_0}\sum_{\|\gamma_4\|\leqq l_0} (De_{\gamma_1,\gamma_2}(t, x), De_{\gamma_3,\gamma_4}(t, x))$$
$$\times (V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\gamma_1\rangle}(x))(V_{\langle\gamma_2\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\gamma_3\rangle}(x))$$
$$\times (V_{\langle\gamma_3\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\beta\rangle}(x)).$$

Also, by Propositions (2.7) and (2.11), we see that

$$\sup\{t^{-(|\alpha|+|\beta|)/2}E[\|D(V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\beta\rangle}(x))\|^p_{\mathcal{H}(R)}]^{1/p};\ x\in \boldsymbol{R}^N,\ 0<t\leqq 1\}<\infty$$

and

$$\sup\{t^{-(|\alpha|+|\beta|)/2}E[|\mathcal{L}(V_{\langle\alpha\rangle}(x), \tilde{A}(t, x)^{-1}V_{\langle\beta\rangle}(x))|^p]^{1/p};\ x\in \boldsymbol{R}^N,\ 0<t\leqq 1\}<\infty.$$

Because of (2.13) and (2.14), it is now clear that we can proceed by induction to get our conclusion. Q.E.D.

(2.15) THEOREM. *For any $\Phi\in\mathcal{N}_n(\boldsymbol{R}^N; \boldsymbol{R})$, $n\in \boldsymbol{Z}$, and $\alpha\in\mathcal{A}$, there are $\Phi_\alpha, \Phi'_\alpha\in\mathcal{N}_{n-\|\alpha\|}(\boldsymbol{R}^N; \boldsymbol{R})$ such that*

$$E[\Phi(t, x)V_{\langle\alpha\rangle}f(X(t, x))]=E[\Phi_\alpha(t, x)f(X(t, x))]$$

and

$$V_{\langle\alpha\rangle}(E[\Phi(t, x)f(X(t, x))])=E[\Phi'_\alpha(t, x)f(X(t, x))]$$

for any $f\in C_b^\infty(\boldsymbol{R}^N; \boldsymbol{R})$, $t>0$, $x\in \boldsymbol{R}^N$.

PROOF. It is sufficient to prove our assertion for $\alpha\in\mathcal{A}$ with $\|\alpha\|\leqq l_0$. From Theorem (1.13), we have

$$E[\Phi(t, x)V_{\langle\alpha\rangle}\{f(X(t, x))\}]$$
$$= -\sum_{i=1}^d\sum_{\|\beta\|\leqq l_0} E[(D\Phi(t, x), H_i(c_{(i),\beta})(t, x))_H$$
$$\times (V_{\langle\beta\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\alpha\rangle}(x))_{\boldsymbol{R}^N}f(X(t, x))]$$
$$+\sum_{i=1}^d\sum_{\|\beta\|\leqq l_0} E[(H_i(c_{(i),\beta})(t, x), D\{(V_{\langle\beta\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\alpha\rangle}(x))_{\boldsymbol{R}^N}\})_H$$
$$\times\Phi(t, x)f(X(t, x))]$$
$$-\sum_{i=1}^d\sum_{\|\beta\|\leqq l_0} E[\Phi(t, x)I_i(c_{(i),\beta})(t, x)$$
$$\times (V_{\langle\beta\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}V_{\langle\alpha\rangle}(x))_{\boldsymbol{R}^N}f(X(t, x))]$$
$$+\varepsilon E[\Phi(t, x)(V_{\langle\alpha\rangle}(x), (\tilde{A}(t, x)+\varepsilon I)^{-1}\operatorname{grad}(f(X(t, x))))_{\boldsymbol{R}^N}].$$

However, it is easy to see that

$$\begin{aligned}&\varepsilon E[|\Phi(t,x)(V_{\langle\alpha\rangle}(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}\operatorname{grad}(f(X(t,x))))_{R^N}|]\\ &\leqq \varepsilon^{1/2}E[|\Phi(t,x)|(V_{\langle\alpha\rangle}(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}V_{\langle\alpha\rangle}(x))_{R^N}^{1/2}\\ &\quad\times\{\varepsilon(\operatorname{grad}(f(X(t,x))),(\tilde{A}(t,x)+\varepsilon I)^{-1}\operatorname{grad}(f(X(t,x))))_{R^N}\}^{1/2}]\\ &\leqq \varepsilon^{1/2}E[|\Phi(t,x)|(V_{\langle\alpha\rangle}(x),(\tilde{A}(t,x)+\varepsilon I)^{-1}V_{\langle\alpha\rangle}(x))_{R^N}^{1/2}|\operatorname{grad}(f(X(t,x)))|]\\ &\longrightarrow 0 \quad \text{as } \varepsilon\downarrow 0.\end{aligned}$$

Thus we have

$$\begin{aligned}(2.16)\quad & E[\Phi(t,x)V_{\langle\alpha\rangle}\{f(X(t,x))\}]\\ &=-\sum_{i=1}^{d}\sum_{\|\beta\|\leqq l_0}E[(D\Phi(t,x),H_i(c_{(i),\beta})(t,x))_H(V_{\langle\beta\rangle}(x),\tilde{A}(t,x)^{-1}V_{\langle\beta\rangle}(x))_{R^N}f(X(t,x))]\\ &\quad+\sum_{i=1}^{d}\sum_{\|\beta\|\leqq l_0}E[(H_i(c_{(i),\beta})(t,x),D\{(V_{\langle\beta\rangle}(x),\tilde{A}(t,x)^{-1}V_{\langle\alpha\rangle}(x))\})_H\Phi(t,x)f(X(t,x))]\\ &\quad-\sum_{i=1}^{d}\sum_{\|\beta\|\leqq l_0}E[\Phi(t,x)I_i(c_{(i),\beta})(t,x)(V_{\langle\beta\rangle}(x),\tilde{A}(t,x)^{-1}V_{\langle\alpha\rangle}(x))_{R^N}f(X(t,x))].\end{aligned}$$

Note that

$$V_{\langle\alpha\rangle}\{E[\Phi(t,x)f(X(t,x))]\}=E[V_{\langle\alpha\rangle}(\Phi(t,x))f(X(t,x))+\Phi(t,x)V_{\langle\alpha\rangle}(f(X(t,x)))].$$

Therefore, from Theorem (2.12), we see that Φ_α exists.

To find Φ'_α, observe that

$$\begin{aligned}E[\Phi(t,x)V_{\langle\alpha\rangle}f(X(t,x))]&=E[\Phi(t,x)((X(t)_*^{-1}V_{\langle\alpha\rangle})(x),(X(t)^*df)(x))]\\ &=\sum_{\|\beta\|\leqq l_0}E[\Phi(t,x)c_{\alpha,\beta}(t,x)V_{\langle\beta\rangle}(f(X(t,x)))].\end{aligned}$$

Thus from (2.16) and Theorem (2.12), we see that there is a Φ'_α with the required properties. Q.E.D.

By induction, we can easily see the following.

(2.17) COROLLARY. *For any $\Phi\in\mathcal{N}_n(R^N;R)$, $n\in Z$, and $\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_k\in\mathcal{A}$, there is a $\Phi'\in\mathcal{N}_{n-l}(R^N;R)$ such that*

$$V_{\langle\alpha_1\rangle}\cdots V_{\langle\alpha_m\rangle}\{E[\Phi(t,x)V_{\langle\beta_1\rangle}\cdots V_{\langle\beta_k\rangle}f(X(t,x))]\}=E[\Phi'(t,x)f(X(t,x))]$$

for any $f\in C_b^\infty(R^N;R)$, $t>0$, $x\in R^N$, where $l=\sum_{j=1}^{m}\|\alpha_j\|+\sum_{j=1}^{k}\|\beta_j\|$.

(2.18) THEOREM. *For any $\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_k\in\mathcal{A}$ and $1<p<\infty$, there is a constant $C<\infty$ such that*

$$|(V_{\langle\alpha_1\rangle}\cdots V_{\langle\alpha_m\rangle}{}^cP_tV_{\langle\beta_1\rangle}\cdots V_{\langle\beta_k\rangle}f)(x)|\leqq Ct^{-l/2}\,{}^cP_t(|f|^p)(x)^{1/p}$$

for any $f \in C_b^\infty(\boldsymbol{R}^N; \boldsymbol{R})$, $0<t\leqq 1$, $x \in \boldsymbol{R}^N$. *Here* $l=\sum_{j=1}^{m}\|\alpha_j\|+\sum_{j=1}^{k}\|\beta_j\|$.

PROOF. Let

$$\Phi_1(t, x)=\exp\left(\int_0^t c(X(s, x))ds\right) \text{ and } \Phi_2(t, x)=\exp\left(-\int_0^t c(X(s, x))ds\right).$$

Then $\Phi_1, \Phi_2 \in \mathcal{N}_0(\boldsymbol{R}^N; \boldsymbol{R})$, by Lemmas (1.3) and (1.7). Therefore from Corollary (2.17), there is a $\Phi' \in \mathcal{N}_{-\|\alpha\|-\|\beta\|}(\boldsymbol{R}^N; \boldsymbol{R})$ and

$$\begin{aligned}
&|(V_{\alpha_1}\cdots V_{\alpha_m}{}^cP_tV_{\beta_1}\cdots V_{\beta_k}f)(x)|\\
&=|V_{\alpha_1}\cdots V_{\alpha_m}\{E[\Phi_1(t, x)V_{\beta_1}\cdots V_{\beta_k}f(X(t, x))]\}|\\
&=|E[\Phi'(t, x)f(X(t, x))]|\\
&\leqq E[|\Phi'(t, x)|^q\Phi_2(t, x)^{q/p}]^{1/q}E[\Phi_1(t, x)|f(X(t, x))|^p]^{1/p}\\
&\leqq E[|\Phi'(t, x)|^q\Phi_2(t, x)^{q/p}]^{1/q}P_t^c(|f|^p)(x)^{1/p}.
\end{aligned}$$

Therefore we have our assertion. Q.E.D.

(2.19) COROLLARY. *For any* $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_k \in \mathcal{A}$, *there is a constant* $C<\infty$ *such that*

$$\|V_{\langle\alpha_1\rangle}\cdots V_{\langle\alpha_m\rangle}{}^cP_tV_{\langle\beta_1\rangle}\cdots V_{\langle\beta_k\rangle}f\|_{L^p(\boldsymbol{R}^N)}\leqq Ct^{-l/2}\|f\|_{L^p(\boldsymbol{R}^N)}$$

for any $f \in C_0^\infty(\boldsymbol{R}^N; \boldsymbol{R})$, $1\leqq p\leqq\infty$ *and* $0<t\leqq 1$. *Here* $l=\sum_{j=1}^{m}\|\alpha_j\|+\sum_{j=1}^{k}\|\beta_j\|$.

PROOF. By Theorem 2.18, it is obvious that our assertion is true for $p=\infty$. Let $\tilde{V}_i\equiv V_i$, $i=1, \cdots, d$, $\tilde{V}_0\equiv -V_0+\frac{1}{2}\sum_{k=1}^{d}\operatorname{div}(V_k)V_k$, and $\tilde{c}\equiv c-\operatorname{div}(V_0)+\frac{1}{2}\sum_{k=1}^{d}V_k(\operatorname{div}(V_k))+\frac{1}{2}\sum_{k=1}^{d}(\operatorname{div}(V_k))^2$. Then it is easy to see that the family of vector fields $\{\tilde{V}_0, \cdots, \tilde{V}_d\}$ satisfies the hypothesis (H). Let $\{{}^c\tilde{P}_t;\ t\geqq 0\}$ be the semigroup associated with $\{\tilde{V}_0, \cdots, \tilde{V}_d\}$ and $\tilde{c}$. Then by Theorem 3.14 in [6], ${}^c\tilde{P}_t$ is the formal adjoint of ${}^cP_t$. Since our assertion holds for ${}^c\tilde{P}_t$ in the case of $p=\infty$, we see that our theorem is true for $p=1$. Then, by the interpolation theorem, we see that our theorem is true for all p. Q.E.D.

3. The distribution of the approximating process

Until further notice, we will be assuming that

(3.1) i) $V_0=0$ and $\{V_1, \cdots, V_d\}\subset C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$
ii) $\nu_{l_0}(x)\geqq\varepsilon$, $x\in\mathbf{R}^N$ for some $l_0\in\mathbf{N}$ and $\varepsilon>0$.

Under these conditions we will (cf. Theorem 4.9) obtain some reasonably precise estimates on the density $p(t, x, \cdot)$ of $\mathcal{W}\circ X(t, x)^{-1}$. Our method will be to expand $X(t, x, \theta)$ in a stochastic Taylor expansion at $t=0$ and to make a careful analysis of the Taylor approximations. This procedure can be viewed as a stochastic analogue of the ideas introduced by L. Rothschild and E. Stein in [8] and sharped by C. Fefferman and D. Phong in [1] and A. Sanchez in [9].

Given $\alpha\in\mathcal{A}$, define $V^{(\alpha)}$ so that

$$V^{(\alpha)}=\begin{cases} V_k & \text{if } \alpha=(k) \\ \sum_{i=1}^N V_k^i\dfrac{\partial V^{(\beta)}}{\partial x_i} & \text{if } \alpha=(\beta, k). \end{cases}$$

For $l\in\mathbf{N}$, set

$$X_l(t, x)=x+\sum_{1\leqq|\alpha|\leqq l}\theta^{(\alpha)}(t)V^{(\alpha)}(x), \quad (t, x)\in[0, \infty)\times\mathbf{R}^N. \tag{3.2}$$

(Note that, since $V_0=0$, we might as well take $\mathcal{A}=\{\phi\}\cup\bigcup_{l=1}^\infty\{1, \cdots, d\}^l$ and we can ignore the distinction between $|\alpha|$ and $\|\alpha\|$.) The process $X_l(\cdot, x)$ is, of course, the l-th order Taylor's approximation of $X(\cdot, x)$ which we will be studying in this section (cf. Theorem 3.36). In order to get the kind of refined estimates on $\mathcal{W}\circ X_l(t, x)^{-1}$ which we will need, we must begin by studying the distribution of $\{\theta^{(\alpha)}(t)\colon 1\leqq|\alpha|\leqq l\}$.

Set $\mathcal{A}_l=\bigcup_{j=1}^l\{1, \cdots, d\}^j\subset\mathcal{A}$ and for $i=1, \cdots, d$, define

$$W_i=\sum_{0\leqq\|\alpha\|\leqq l-1}z_\alpha\frac{\partial}{\partial z_{\alpha,i}}, \tag{3.3}$$

where $z_\phi\equiv 1$. It is then easy to check that, for all $z\in\mathbf{R}^{\mathcal{A}_l}$, there is a unique solution $Z(\cdot, z)$ to

$$Z(T, z; \theta)=z+\sum_{i=1}^d\int_0^T W_i(Z(t, z; \theta))\circ d\theta_i(t), \qquad T\geqq 0. \tag{3.4}$$

Moreover, it is easy to check that $\theta^{(\alpha)}(\cdot)=Z_\alpha(\cdot)$, $\alpha\in\mathcal{A}_l$, where $Z(\cdot)=Z(\cdot, 0)$. Thus, we would be well-advised to study the family of vector fields $\{W_1, \cdots, W_d\}$. The following result summarizes some facts proved in the appendix A.

(3.5) THEOREM. *Let* $\mathcal{G}=\mathrm{Lie}(W_1, \cdots, W_d)(0)$ *and think of* $\mathcal{G}$ *as a subspace of* $\mathbf{R}^{\mathcal{A}_l}$. *Then* $\mathcal{G}=\bigoplus_{\lambda=1}^{l}\mathcal{G}_\lambda$ *where* $\mathcal{G}_\lambda=\mathcal{G}\cap\mathrm{span}\left\{\frac{\partial}{\partial z_\alpha}; |\alpha|=\lambda\right\}$. *Moreover, there is a mapping* $(n, v)\in\mathcal{G}\times\mathcal{G}\to u\times v\in\mathcal{G}$ *and vector fields* $\tilde{W}_1, \cdots, \tilde{W}_d$ *on* $\mathcal{G}$ *such that*

i) $(\mathcal{G}, \times)$ *is a nilpotent Lie group with* $u^{-1}=-u$,

ii) *Lebesgue measure is* $\mathcal{G}$*-invariant,*

iii) $\mathrm{Lie}(\tilde{W}_1, \cdots, \tilde{W}_d)$ *is the Lie algebra of right* $\mathcal{G}$*-invariant vector fields,*

iv) *for all* $a>0$, $(S_a)_*\tilde{W}_k=a\tilde{W}_k$, $1\leqq k\leqq d$, *where* $S_a:\mathcal{G}\to\mathcal{G}$ *is the linear map such that* $S_a u=a^{|\lambda|}u$ *for* $u\in\mathcal{G}_\lambda$ *and* $1\leqq\lambda\leqq l$.

Finally, there is a proper immersion $\Psi:\mathcal{G}\to\mathbf{R}^{\mathcal{A}_l}$ *such that* $|\Psi(u)-u|\leqq C|u|^2$ *for some* $C<\infty$ *and all* $u\in\mathcal{G}$ *with* $|u|\leqq 1$ *and* $Z(\cdot)=\Psi\circ U(\cdot)$ (a.s.$\mathcal{W}$) *where*

$$U(T)=\sum_{k=1}^{d}\int_0^T\tilde{W}_k(U(t))\circ d\theta_k(t), \qquad T\geqq 0. \tag{3.6}$$

Because of Theorem (3.5), we now transfer our attention to the process $U(\cdot)$ in (3.6).

(3.7) LEMMA. *There is a smooth map* $(t, x)\in(0, \infty)\times\mathcal{G}\to\rho_t(u)\in(0, \infty)$ *such that*

i) $(\mathcal{W}\circ U(t)^{-1})(du)=\rho_t(u)du$, $t>0$,

ii) *for all* $(s, t, u)\in(0, \infty)\times(0, \infty)\times\mathcal{G}$:

$$\rho_{s+t}(u)=\int_{\mathcal{G}}\rho_t(uv^{-1})\rho_s(v)dv,$$

iii) *for all* $(t, u)\in(0, \infty)\times\mathcal{G}$:

$$\rho_t(u)=t^{-\nu/2}\rho_1(S_{t^{-1/2}}u), \quad \textit{where} \quad \nu=\sum_{\lambda=1}^{l}\lambda\dim(\mathcal{G}_\lambda),$$

and

iv) *there exist* $C\in(0, \infty)$ *and* $\delta>0$ *such that* $\rho_t(u)\leqq Ce^{-\delta/t}$ *for all* $(t, u)\in(0, \infty)\times\mathcal{G}$ *with* $|u|=1$.

PROOF. Since the $\tilde{W}_k$'s are right $\mathcal{G}$-invariant

$$U(T)\times u=u+\sum_{k=1}^{d}\int_0^T\tilde{W}_k(U(t)\times u)\circ d\theta_k(t), \qquad T\geqq 0.$$

Hence, if $Q_t=\mathcal{W}\circ U(t)^{-1}$, then $(t, v)\in(0, \infty)\times\mathcal{G}\to Q_t\circ R_v^{-1}$ is a transition probability function $(R_v(u)=u\times v,\ u\in\mathcal{G})$. In particular:

$$Q_{s+t}(\Gamma)=\int_{\mathcal{G}} Q_t(\Gamma v^{-1})Q_s(dv),\ \Gamma\in\mathcal{B}_{\mathcal{G}} \quad\text{and}\quad s,t>0. \tag{3.8}$$

Also, because $(S_a)_*\tilde{W}_k=a\tilde{W}_k$, we see that $\mathcal{W}\circ(S_aU(\cdot))^{-1}=\mathcal{W}\circ U(a\cdot)^{-1}$, and so

$$Q_a=Q_1\circ(S_{a^{1/2}})^{-1},\qquad a>0. \tag{3.9}$$

Next, observe that, because $\mathrm{Lie}(\tilde{W}_1,\cdots,\tilde{W}_d)(0)=\mathcal{G}$, Theorem (4.5) of [6] tells us that there is a smooth $(t,u)\in(0,\infty)\times\mathcal{G}\rightarrow\rho_t(u)\in[0,\infty)$ such that $Q_t(du)=\rho_t(u)du$, $t>0$, and that

$$\rho_t(u)\leqq Ce^{-\delta|u|^2/t} \tag{3.10}$$

for some $C<\infty$ and $\delta>0$ and for all $(t,u)\in(0,1]\times\mathcal{G}$ such that $|u|\leqq 1$. Clearly iii) follows from (3.9), and so (3.10) holds for all $t>0$ and $u\in\mathcal{G}$ with $|u|=1$. That is, iv) holds. Finally to see the $\rho_t(u)>0$ for all $(t,u)\in(0,\infty)\times\mathcal{G}$, simply note that $\mathrm{supp}\,Q_t=\mathcal{G}$, $t>0$, and therefore that $\{\rho_t>0\}$ is a dense open set for all $t>0$. Clearly $\rho_T(u)>0$ follows from this and ii) with $s=t=T/2$. Q.E.D.

In order to complete our discussion of $\mathcal{W}\circ U(t)^{-1}$, we define $\bar{U}(\cdot,u,h)\in C([0,\infty);G)$ for $(u,h)\in\mathcal{G}\times H$ by

$$\bar{U}(T,u,h)=u+\sum_{k=1}^{d}\int_0^T\tilde{W}_k(\bar{U}(t,u,h))\dot{h}_k(t)dt,\qquad T\geqq 0,$$

and for $u,v\in\mathcal{G}$ we set

$$e(u,v)=\inf\{\|h\|_H;\ \bar{U}(1,u,h)=v\}.$$

Because $\mathrm{Lie}(\tilde{W}_1,\cdots,\tilde{W}_d)(u)=\mathcal{G}$ for all $u\in\mathcal{G}$, e is a metric on $\mathcal{G}$ which has all the properties described in Lemma (B.1). In addition, if $\bar{U}(\cdot,h)\equiv\bar{U}(\cdot,0,h)$ and $e(u)\equiv e(u,0)$, then $\bar{U}(\cdot,u,h)=\bar{U}(\cdot,h)\times u$, $S_a\bar{U}(\cdot,h)=\bar{U}(\cdot,ah)$, and so:

$$\begin{cases} e(u,v)=e(uv^{-1})\\ e(S_au)=a\cdot e(u).\end{cases} \tag{3.11}$$

(3.12) THEOREM. *There is an $M\in[1,\infty)$ such that*

$$\frac{1}{Mt^{\nu/2}}\exp(-Me(u)^2/t)\leqq\rho_t(u)\leqq\frac{M}{t^{\nu/2}}\exp(-e(u)^2/Mt)$$

for all $(t, u) \in (0, \infty) \times \mathcal{G}$.

PROOF. In view of property iii) in Theorem (3.7) and the scaling property of e in (3.10), it suffices to treat $t=1$ and $u \neq 0$.

To prove the upper bound, define $\omega(u), u \in \mathcal{G}\backslash\{0\}$, so that $\omega(u)=a$ if $|S_{1/a}u|=1$. Clearly $\omega(S_a u)=a\cdot\omega(u)$ and $\omega(u)\geqq 1$ if and only if $|u|\geqq 1$. Also, $\sigma \equiv \sup\left\{\frac{e(u)}{\omega(u)};\ u \in \mathcal{G}\backslash\{0\}\right\} = \max\left\{\frac{e(u)}{\omega(u)};\ |u|=1\right\}<\infty$. Hence, the required upper bound follows from the corresponding upper bound with ω replacing e. That is, we must check that $\rho_1(u) \leqq M\exp(-\omega(u)^2/M)$ for some $M<\infty$ and all $u \in \mathcal{G}\backslash\{0\}$. Obviously, there is no problem when $|u|\leqq 1$. On the other hand, if $|u|\geqq 1$ and $1/t=\omega(u)^2$, then $0<t\leqq 1$ and, by iv) and v) of Theorem (3.7):

$$\rho_1(u)=t^{\nu/2}\rho_t(S_{t^{1/2}}u)\leqq Ce^{-\delta/t}=Ce^{-\delta\omega(u)^2}$$

since $|S_{t^{1/2}}u|=1$. The upper bound follows from this.

The lower bound is proved as follows. Set $B_e(u, r)=\{v \in \mathcal{G};\ e(u, v)<r\}$ for $u \in \mathcal{G}$ and $r>0$, and note that $\gamma=\inf\{\rho_1(u);\ u \in B_e(0, 3)\}>0$. Given $u \in \mathcal{G}$ with $e(u)\geqq 3$, let $n=[e(u)^2]+1$. Then

$$\rho_1(u)=\int dv_1 \cdots \int dv_{n-1}\rho_{1/n}(v_1)\rho_{1/n}(v_2v_1^{-1}) \cdots \rho_{1/n}(uv_{n-1}^{-1}).$$

Using Lemma (B.2), choose $u_0, \cdots, u_n$ so that $u_0=0$, $u_n=u$, and $e(u_{m-1}, u_m)=e(u)/n$ for $1\leqq m\leqq n$. Then for $v_m \in B_e(u_m, n^{-1/2})$, $1\leqq m\leqq n-1$: $e(v_{m+1}v_m^{-1})\leqq 3n^{-1/2}$ for $0\leqq m\leqq n-1$ where $v_0\equiv 0$ and $v_n\equiv u$. Hence, by ii) and iii) of Theorem (3.7):

$$\begin{aligned}\rho_1(u) &\geqq \int_{B_e(u_1, n^{-1/2})} dv_1 \cdots \int_{B_e(u_{n-1}, n^{-1/2})} dv_{n-1}\rho_{1/n}(v_1)\rho_{1/n}(v_2v_1^{-1}) \cdots \rho_{1/n}(uv_{n-1}^{-1})\\ &\geqq (\gamma n^{\nu/2})^n |B_e(0, n^{-1/2})|^{n-1}.\end{aligned}$$

Noting that $|B_e(0, r)|=r^\nu|B_e(0, 1)|$ and recalling that $n=[e(u)^2]+1$, one can easily derive the lower bound from here. Q.E.D.

Define $\bar{F}: \boldsymbol{R}^{\mathcal{A}_l}\times\boldsymbol{R}^N\to\boldsymbol{R}^N$ and $F: \mathcal{G}\times\boldsymbol{R}^N\to\boldsymbol{R}^N$ by $\bar{F}(z; x)=\sum_{1\leqq\|\alpha\|\leqq l-1} z_\alpha V^{(\alpha)}(x)$ and $F(u; x)=\bar{F}(\Psi(u); x)$. Denote by $JF(u, x)$ the Jacobian transformation determined by $u\to F(u, x)$.

(3.13) LEMMA. *There is an* $\varepsilon'>0$ *such that* $JF(0, x)^t JF(0, x)\geqq\varepsilon' I_{\boldsymbol{R}^N}$, $x \in \boldsymbol{R}^N$, *so long as* $l\geqq l_0$.

PROOF. Clearly it suffices to treat the case when $l=l_0$. As in the proof of (A.9), introduce the quantities e_α and C_α, $\alpha\in\mathcal{A}_{l_0}$, in $\bigoplus_{\lambda=1}^{l_0}(\boldsymbol{R}^d)^{\otimes\lambda}$, and define Ξ: $\bigoplus_{\lambda=1}^{l_0}(\boldsymbol{R}^d)^{\otimes\lambda}\to C^\infty(\boldsymbol{R}^N;\boldsymbol{R}^N)$ to be the linear map such that $\Xi(e_\alpha)=V^{(\alpha)}$. Then $\Xi(C_{(k,\beta)})=V_{(k,\beta)}$ for $1\leqq k\leqq d$ and $0\leqq|\beta|\leqq l_0$. Next, let $\{D_\mu\}_1^m$ be an orthonormal basis for $\{C_\alpha;\alpha\in\mathcal{A}_{l_0}\}$ and define $B_{\mu,\alpha}=(D_\mu, e_\alpha)_{\bigoplus_{\lambda=1}^{l_0}(\boldsymbol{R}^d)^{\otimes\lambda}}$. Set $V_\mu=\sum_{\alpha\in\mathcal{A}_{l_0}}B_{\mu,\alpha}V^{(\alpha)}$ and $u_\mu=\sum_{\alpha\in\mathcal{A}_{l_0}}B_{\mu,\alpha}z_\alpha$ for $1\leqq\mu\leqq m$. Then $\{V_\mu\}_1^m$ spans $\{V_\beta;\beta\in\mathcal{A}_{l_0}\}$ and $\left\{\frac{\partial}{\partial u_\mu}\right\}_1^m$ is an orthonormal basis in $\mathcal{G}$. In particular, by ii) of (3.1),

$$\sum_1^m V_\mu(x)\otimes V_\mu(x)\geqq\varepsilon' I_{\boldsymbol{R}^N}\quad\text{for some } \varepsilon'>0 \text{ and all } x\in\boldsymbol{R}^N.$$

Noting that $\Psi(u)-u=O(|u|^2)$ as $|u|\to 0$ in $\mathcal{G}$, we see that $\frac{\partial\Psi_\alpha}{\partial u_\mu}(0)=\frac{\partial z_\alpha}{\partial u_\mu}(0)=B_{\alpha,\mu}$. Hence

$$\frac{\partial F(\cdot,x)}{\partial u_\alpha}(0)=\sum_{\alpha\in\mathcal{A}_{l_0}}B_{\mu,\alpha}V^{(\alpha)}(x)=V_\mu(x)$$

and so

$$JF(0,x)^t JF(0,x)=\sum_1^m V_\mu(x)\otimes V_\mu(x)\geqq\varepsilon' I_{\boldsymbol{R}^N}.\qquad\text{Q.E.D.}$$

In view of Lemma (3.13) and i) in (3.1), the following statement is a standard application of the implicit function theorem.

(3.14) LEMMA. *Let $l\geqq l_0$ be given and set $D=\dim\mathcal{G}-N$. There exist $r_1, r_2\in(0,\infty)$ and a bounded mapping $x\in\boldsymbol{R}^N\to F'_x\in C_b^\infty(\{u\in\mathcal{G};|u|<r_1\}\times\{\xi\in\boldsymbol{R}^N;|\xi-x|<r_1\};\boldsymbol{R}^D)$ such that $u\to(F_x(u,\xi),F'_x(u,\xi))$ is a diffeomorphism of $\{u\in\mathcal{G};|u|<r_1\}$ onto an open set $O(x,\xi)\supset\{v\in\boldsymbol{R}^{N+D};|v|<r_2\}$. In addition, $x\in\boldsymbol{R}^N\to F_x^{-1}\in C_b^\infty(\{(v,\xi)\in\boldsymbol{R}^{N+D}\times\boldsymbol{R}^N;|\xi-x|<r_1$ and $v\in O(x,\xi)\};\mathcal{G})$ is a bounded mapping. In particular, if $M_{x,y}=\{u\in\mathcal{G};|u|<r_1$ and $x+F(u,x)=y\}$ for $(x,y)\in\boldsymbol{R}^N\times\boldsymbol{R}^N$, then:*

i) *either $M_{x,y}=\phi$ or $M_{x,y}$ is a smooth open sub-manifold of $\mathcal{G}$ having codimension N,*

ii) *if $x,y\in\boldsymbol{R}^N$, $|\xi-x|<r_1$ and $M_{\xi,y}\neq\phi$, then $F_x(\cdot,\xi)|_{M_{\xi,y}}$ is diffeomorphic onto $\{\omega\in\boldsymbol{R}^D;(y-\xi,\omega)\in O(x,\xi)\}$,*

iii) *if $(x,y)\in\boldsymbol{R}^N\times\boldsymbol{R}^N\to m_{x,y}\in C_0(\mathcal{G})^*$ is defined so that $m_{x,y}=0$ when $M_{x,y}=\phi$ and $m_{x,y}$ is the Riemannian volume on $M_{x,y}$ (as a submanifold* *of* $\mathcal{G}$) *when* $M_{x,y}\neq\phi$, *then for each* $n\geqq 0$ *the mapping* $\psi\in C_0^n(\{u\in\mathcal{G};|u|<r_1\})\rightarrow\int_{M_{\cdot,*}}\psi(u)m_{\cdot,*}(du)\in C_b^n(\boldsymbol{R}^N\times\boldsymbol{R}^N)$ *is bounded, and*

iv) $(u,x)\rightarrow\det(JF(u,x)^tJF(u,x))$ *is a uniformally positive element of* $C_b^\infty(\{u\in\mathcal{G};|u|,r_1\}\times\boldsymbol{R}^N)$ *and for each* $\psi\in C_0(\{u\in\mathcal{G};|u|<r_1\})$:

$$\int_{\mathcal{G}}\psi(u)du=\int_{\boldsymbol{R}^N}dy\int_{M_{x,y}}\psi(u)K(u,x)m_{x,y}(du),\ x\in\boldsymbol{R}^N,$$

where $K(u,x)=(\det(JF(u,x)^tJF(u,x)))^{-1/2}$.

(3.15) LEMMA. *Let* $l\geqq l_0$. *Then there is an* $r_3\in(0,r_1\wedge r_2)$ *and a* $\Phi\in C_b^\infty(\{u\in\mathcal{G};|u|<r_3\}\times\boldsymbol{R}^N\times\{\eta\in\boldsymbol{R}^N;|\eta|<r_3\};\{v\in\mathcal{G};|v|<r_1\})$ *such that*

i) $\Phi(u,x,0)=u$ *for* $(u,x)\in\mathcal{G}\times\boldsymbol{R}^N$ *with* $|u|<r_3$ *and*

ii) $F(\Phi(u,x,\eta),x)=F(u,x)+\eta$ *for* $(u,x,\eta)\in\mathcal{G}\times\boldsymbol{R}^N\times\boldsymbol{R}^N$ *with* $|u|<r_3$ *and* $|\eta|<r_3$.

In particular, $\Phi(\cdot,x,\eta)$ *maps* $M_{x,y}$ *into* $M_{x,y+\eta}$ *for all* $(x,y,\eta)\in\boldsymbol{R}^N\times\boldsymbol{R}^N\times\boldsymbol{R}^N$ *with* $|\eta|<r_3$.

PROOF. Given $(u,x)\in\mathcal{G}\times\boldsymbol{R}^N$, define $\Xi_{u,x}(\xi)=F(u+{}^tJF(0,x)\xi,x)$ for $\xi\in\boldsymbol{R}^N$. Then, $(\Xi_{0,x})^{(1)}(0)=JF(0,x)^tJF(0,x)\geqq\varepsilon' I_{\boldsymbol{R}^N}$. Hence, we can choose an $r_3\in(0,r_1\wedge r_2)$ so that, for all $(u,x)\in\mathcal{G}\times\boldsymbol{R}^N$ with $|u|<r_3$, $\Xi_{u,x}$ maps a open neighborhood of 0 in $\boldsymbol{R}^N$ diffeomorphically onto $\{\xi\in\boldsymbol{R}^N;|\xi|<r_3\}$ and the map $(u,x,\eta)\rightarrow\Phi(u,x,\eta)\equiv u+{}^tJF(0,x)(\Xi_{u,x})^{-1}(\eta)$ is an element of $C_b^\infty(\{u\in\mathcal{G};|u|<r_3\}\times\boldsymbol{R}^N\times\{\eta\in\boldsymbol{R}^N;|\eta|<r_3\};\{v\in\mathcal{G};|v|<r_1\})$. Clearly Φ has the desired properties. Q.E.D.

Warning. From now on, we will be assuming that $l\geqq l_0$.

Recall the process $X_l(\cdot,x)$ defined in (3.2) and note that $X_l(\cdot,x)=x+F(U(\cdot),x)$. Given $\eta\in C_0^\infty(\{u\in\mathcal{G};|u|<r_3\})$, set

$$P_l^\eta(t,x,\cdot)=(\eta(U(t))\mathcal{W})\circ X_l(t,x)^{-1},\quad(t,x)\in(0,\infty)\times\boldsymbol{R}^N.$$

Then, by Lemmas (3.7) and (3.14), $P_l^\eta(t,x,dy)=p_l^\eta(t,x,y)dy$ where

$$p_l^\eta(t,x,y)=\int_{M_{x,y}}\eta(u)\rho_t(u)K(u,x)m_{x,y}(du).\tag{3.16}$$

Our goal is to use (3.16) in conjunction with what we already know about $\rho_t(u)$ to get precise local estimates on $p_l^\eta(t,x,y)$ as $t\downarrow 0$.

Given $(x,h)\in\boldsymbol{R}^N\times H$, define $\bar{X}(\cdot,x,h)\in C([0,\infty);\boldsymbol{R}^N)$ by

$$\bar{X}(T,x,h)=x+\sum_{i=1}^d\int_0^T V_i(\bar{X}(t,x,h))\dot{h}^i(t)dt,\quad T\geqq 0.$$

For $x, y \in \boldsymbol{R}^N$, define

$$d(x, y)=\inf\{\|h\|_H;\ \bar{X}(1, x, h)=y\}.$$

By Lemma (B.2), d is a metric on $\boldsymbol{R}^N$ which is compatible with the Euclidean topology.

(3.17) LEMMA. *For each* $l \geqq l_0$ *there exists a* $C_l \in [1, \infty)$ *such that*

$$(3.18)\quad |\bar{X}(T, x, h)-(x+F(\bar{U}(T; h), x))| \leqq C_l\|h\|_H^{l+1},\quad (T, x, h) \in [0, 2]\times \boldsymbol{R}^N \times H,$$

$$(3.19)\quad \frac{1}{C_l}|u| \leqq e(u) \leqq C_l|u|^{1/l},\ u \in \mathcal{G} \text{ with } |u| \wedge e(u) \leqq 1$$

$$(3.20)\quad |e(u)-e(v)| \leqq C_l|u-v|^{1/l},\ u, v \in \mathcal{G} \text{ with } |u|, |v|<1,$$

and

$$(3.21)\quad d(x, x+F(u, x)) \leqq C_l e(u),\ (u, x) \in \mathcal{G} \times \boldsymbol{R}^N \text{ with } e(u) \leqq 1.$$

Moreover, there is a $K \in [1, \infty)$ *such that*

$$(3.22)\quad \frac{1}{K}|x-y| \leqq d(x, y) \leqq K|x-y|^{1/l_0},\ x, y \in \boldsymbol{R}^N \text{ with } |x-y| \wedge d(x, y) \leqq 1.$$

PROOF. The existence of a C_l satisfying (3.18) is an easy consequence of Taylor's theorem. The left hand side of (3.19) is obvious and the right hand side follows by homogeneity. Given (3.19), (3.20) follows from $|e(u)-e(v)| \leqq e(u, v)=e(uv^{-1})$ and $|uv^{-1}|=e(uv^{-1}, vv^{-1}) \leqq C_l|u-v|$ for $u, v \in \mathcal{G}$ with $|u|, |v| \leqq 1$.

The left hand side of (3.22) is again obvious. To prove the right hand side, take $l=l_0$. Because $\Phi(0, x, 0)=0$, we can choose $B \in (0, \infty)$ so that $|\Phi(0, x, \eta)| \leqq B|\eta|$ for $(x, \eta) \in \boldsymbol{R}^N \times \boldsymbol{R}^N$ with $|\eta|<r_3$. Now choose $0<\delta<r_3$ so that $(C_l^{l+2}B\cdot\delta^{1/l}) \vee B\delta \leqq \frac{1}{2} \wedge r_3$. Given $x, y \in \boldsymbol{R}^N$ with $|x-y|<\delta$, we choose $\{(x_m, u_m, h_m)\}_1^\infty \subset \{\xi \in \boldsymbol{R}^N;\ |\xi-y|<\delta\} \times \{u \in \mathcal{G};\ |u| \leqq 1\} \times H$ inductively as follows. First, $x_0=x$. Secondly given x_m, take $u_m=\Phi(0, x_m, y-x_m)$ and choose $h_m \in H$ so that $\|h_m\|_H=e(u_m)$ and $u_m=\bar{U}(1, h_m)$. Then $|u_m| \leqq B|x_m-y| \leqq B\delta \leqq \frac{1}{2} \wedge r_3$. Thirdly, take $x_{m+1}=\bar{X}(1, x_m, h_m)$. Then

$$\begin{aligned}|x_{m+1}-y| &= |x_m+F(u_m, x_m)-\bar{X}(1, x_m, h_m)| \leqq C_l\|h_m\|_H^{l+1}=C_l e(u_m)^{l+1}\\ &\leqq C_l^{l+2}|u_m|^{1+1/l} \leqq C_l^{l+2}B|x_m-y|^{1+1/l}\end{aligned}$$

$$\leqq (C_l^{l+2}B\cdot\delta^{1/l})|x_m-y|\leqq\frac{1}{2}|x_m-y|\leqq\delta.$$

Note that

$$d(x_m, x_{m+1})\leqq\|h_m\|_H=e(u_m)\leqq C_l|u_m|^{1/l}\leqq C_lB^{1/l}|x_m-y|^{1/l}$$
$$\leqq C_lB^{1/l}\left(\frac{1}{2}\right)^{m/l}|y-x|^{1/l}.$$

Hence, since $x_m\to y$,

$$d(x, y)=\lim_{m\to\infty}d(x_0, x_m)\leqq\left(C_lB^{1/l}\sum_0^\infty\left(\frac{1}{2}\right)^{m/l}\right)|x-y|^{1/Q}.$$

Clearly the right hand side of (3.22) follows from this.

To prove (3.21), let $u\in\mathcal{G}$ with $C_le(u)^{l+1}\leqq 1$ be given and choose $h\in H$ so that $\|h\|_H=e(u)$ and $\bar{U}(1, h)=u$. Then

$$d(x, x+F(u, x))\leqq d(x, \bar{X}(1, x, h))+d(\bar{X}(1, x, h), x+F(u, x))$$
$$\leqq e(u)+d(\bar{X}(1, x, h), x+F(u, x)).$$

Note that $|\bar{X}(1, x, h)-(x+F(u, x))|=|\bar{X}(1, x, h)-(x+F(\bar{U}(1, h), x)|\leqq C_l\|h\|_H^{l+1}=C_le(u)^{l+1}\leqq 1$. Thus by (3.22),

$$d(x, x+F(u, x))\leqq e(u)+K(C_le(u)^{l+1})^{1/l_0}.\qquad\text{Q.E.D.}$$

(3.23) LEMMA. *There is an $r_4\in(0, r_3]$ and a $\gamma\in(0, 1]$ such that for all $x\in\boldsymbol{R}^N$ and $y\in Y_x\equiv\{x+F(u, x); |u|<r_4\}$, there is a diffeomorphism $S_{x,y}$ from $M_{x,y}\cap\{u\in\mathcal{G}; |u|<r_4\}$ into $M_{x,x}\cap\{v\in\mathcal{G}; |v|<r_3\}$ with the property that*

(3.24) $$\gamma m_{x,x}\leqq(K(\cdot, x)m_{x,y})\circ S_{x,y}^{-1}\leqq\frac{1}{\gamma}m_{x,x}\ \text{on}\ \{S_{x,y}(u); u\in M_{x,y}\ \text{and}\ |u|<r_4\}$$

and

(3.25) $$\gamma(e(S_{x,y}(u))+d(x, y))\leqq e(u)\leqq\frac{1}{\gamma}(e(S_{x,y}(u))+d(x, y)),\quad u\in M_{x,y}\ \text{with}\ |u|<r_4,$$

for all $x\in\boldsymbol{R}^N$ and $y\in Y_x$.

PROOF. Choose $\rho_1\in(0, r_3)$ so that

$$|v|+|\bar{U}(1, v, h)|+|\bar{X}(1, x, h)-x|+|F(\bar{U}(1, v, h), x)-F(v, x)|<r_3$$

for all $(x, v, h)\in\boldsymbol{R}^N\times\mathcal{G}\times H$ with $|v|\vee\|h\|_H\leqq\rho_1$, and for such (x, v, h) set

$$\tilde{\Phi}(v, x, h)=\Phi(\bar{U}(1, v, h), x, \bar{X}(1, xh)-x-F(\bar{U}(1, x, h), x)-F(v, x)).$$

Next, choose $\rho_2\in(0, \rho_3]$ so that $|\tilde{\Phi}(v, x, h)|\leqq 1/C_l$ for all $(v, x, h)\in \boldsymbol{R}^N\times\mathcal{G}\times H$ with $|v|\vee\|h\|_H\leqq\rho_2$.

Given $x\in\boldsymbol{R}^N$ and $(v, h)\in M_{x,x}\times H$ with $|v|\vee\|h\|_H\leqq\rho_2$, note that $x+F(\tilde{\Phi}(v, x, h), x)=\bar{X}(1, x, h)$. Hence, by (3.21):

$$d(x, \bar{X}(1, x, h))\leqq C_l e(\tilde{\Phi}(v, x, h)). \tag{3.26}$$

Next, choose $h_v\in H$ so that $\|h_v\|_H=e(v)$ and $v=\bar{U}(1, h_v)$, and define g to be the element of H such that

$$g(t)=\begin{cases}h_v(t), & t\in[0, 1]\\ h_v(1)+h(t-1), & t\in[1, \infty).\end{cases}$$

Then, $\bar{U}(2, g)=\bar{U}(1, v, h)$. Thus, if $A\in(0, \infty)$ is chosen so that $|\Phi(u, y, \eta)-u|\leqq A|u|$ for all $(u, y, \lambda)\in\mathcal{G}\times\boldsymbol{R}^N\times\boldsymbol{R}^N$ with $|u|\vee|\lambda|<r_3$, then

$$\begin{aligned}&|\tilde{\Phi}(v, x, h)-\bar{U}(2, g)|\\ &\leqq A|\bar{X}(1, x, h)-x-F(\bar{U}(2, g), x)|\\ &\leqq A|\bar{X}(1, x, h)-\bar{X}(2, x, g)|+A|\bar{X}(2, x, g)-x-F(\bar{U}(2, g), x)|.\end{aligned}$$

By (3.18), $|\bar{X}(2, x, g)-x-F(\bar{U}(2, g), x)|\leqq C_l\|g\|_H^{l+1}$. On the other hand, there is a $B\in(0, \infty)$ such that $|\bar{X}(1, x', f)-\bar{X}(1, x'', f)|\leqq B|x'-x''|$ for all x', $x''\in\boldsymbol{R}^N$ and $f\in H$ with $\|f\|_H\leqq\rho_2$. Thus, since $\bar{X}(2, x, g)=\bar{X}(1, \bar{X}(1, x, h_v), h)$,

$$|\bar{X}(1, x, h)-\bar{X}(2, x, g)|\leqq B|\bar{X}(1, x, h_v)-x|.$$

But $v\in M_{x,x}$ and therefore, by (3.17):

$$\begin{aligned}|\bar{X}(1, x, h_v)-x|&=|\bar{X}(1, x, h_v)-x-F(v, x)|\\ &=|\bar{X}(1, x, h_v)-x-F(\bar{U}(1, h_v), x)|\leqq C_l e(v)^{l+1}.\end{aligned}$$

We have therefore shown that there is a $C\in(0, \infty)$ such that

$$|\tilde{\Phi}(v, x, h)-\bar{U}(2, g)|\leqq C(e(v)^{1+1/l}+\|g\|_H^{1+1/l})^l.$$

Thus, by (3.20):

$$|e(\tilde{\Phi}(v, x, h))-e(\bar{U}(2, g))|\leqq C'(e(v)^{1+1/l}+\|g\|_H^{1+1/l}),$$

where $C'=C_lC$. In particular, since $e(\bar{U}(2, g))\leqq 2^{1/2}\|g\|_H\leqq 2^{1/2}(e(v)+\|h\|_H)$, we see that there is a $C''\in(0, \infty)$ such that

$$e(\tilde{\Phi}(v, x, h))\leqq C''(e(v)+\|h\|_H)\quad\text{for any }(x, v, h)\in\boldsymbol{R}^N\times\mathcal{G}\times H \tag{3.27}$$

with $|v|\vee\|h\|_H\leqq\rho_2$.

At the same time, $e(v)=e(0,v)\leqq e(0,\bar{U}(2,g))+e(\bar{U}(2,g),v)=e(\bar{U}(2,g))+e(\bar{U}(1,h))\leqq e(\bar{U}(2,g))+\|h\|_H$; and so

$$\begin{aligned} e(\tilde{\Phi}(v,x,h)) &\geqq e(v)-\|h\|_H-C'(e(v)^{1+1/l}+\|g\|_H^{1+1/l}) \\ &\geqq e(v)-\|h\|_H-C'(e(v)^{1+1/l}+(2^{1/2}(e(v)+\|h\|_H))^{1+1/l}). \end{aligned}$$

Hence, there is a $\rho_3\in(0,\rho_2]$ such that

$$e(\tilde{\Phi}(v,x,h))\geqq\frac{1}{2}e(v)-2\|h\|_H \tag{3.28}$$

for all $x\in\boldsymbol{R}^N$ and $(v,h)\in M_{x,x}\times H$ with $|v|\vee\|h\|_H\leqq\rho_3$.

Note that $\tilde{\Phi}(v,x,0)=v$ for $x\in\boldsymbol{R}^N$ and $v\in\mathcal{G}$ with $|v|\leqq\rho_3$. Hence, by the implicit function theorem, there exist $\rho\in(0,\rho_3]$ and $r\in(0,r_3]$ such that $\tilde{\Phi}(\cdot,x,h)$ is a diffeomorphism from $\{v\in\mathcal{G};\,|v|<\rho\}$ onto an open set containing $\{u\in\mathcal{G};\,|u|<r\}$ for each $(x,h)\in\boldsymbol{R}^N\times H$ with $\|h\|_H<\rho$. In addition, we may assume that the maps $(x,h)\to\tilde{\Phi}(\cdot,x,h)\in C_b^\infty(\{v\in\mathcal{G};\,|v|<\rho\};\,\mathcal{G})$ and $(x,h)\to\tilde{\Phi}(\cdot,x,h)^{-1}\in C_b^\infty(\{u\in\mathcal{G};\,|u|<r\};\,\{v\in\mathcal{G};\,|v|<\rho\})$ are bounded on $\boldsymbol{R}^N\times\{h\in H;\,\|h\|_H<\rho\}$. In particular, (since for $(v,x,h)\in\mathcal{G}\times\boldsymbol{R}^N\times H$ with $|v|\vee\|h\|_H<\rho$, $\tilde{\Phi}(v,x,h)\in M_{x,\bar{X}(1,x,h)}$ if and only if $v\in M_{x,x}$), $\tilde{\Phi}(\cdot,x,h)^{-1}$ is a diffeomorphism on $M_{x,\bar{X}(1,x,h)}\cap\{u\in\mathcal{G};\,|u|<r\}$ into $M_{x,x}\cap\{v\in\mathcal{G};\,|v|<\rho\}$ and there is a $\delta\in(0,1]$ such that

$$\delta m_{x,\bar{X}(1,x,h)}\leqq m_{x,x}\circ\tilde{\Phi}(\cdot,x,h)^{-1}\leqq\frac{1}{\delta}m_{x,\bar{X}(1,x,h)} \tag{3.29}$$

on $M_{x,\bar{X}(1,x,h)}\cap\{u\in\mathcal{G};\,|u|<r\}$ for all $(x,h)\in\boldsymbol{R}^N\times\{h\in H;\,\|h\|_H<\rho\}$.

Finally, using (3.19) and (3.21), choose $r_4\in(0,r]$ so that $d(x,x+F(u,x))<\rho$ for all $x\in\boldsymbol{R}^N$ and $u\in\mathcal{G}$ with $|u|<r_4$. Given $x\in\boldsymbol{R}^N$ and $y\in Y_x$, choose $h_{x,y}\in H$ so that $d(x,y)=\|h_{x,y}\|_H$ and $y=\bar{X}(1,x,h_{x,y})$. Clearly, $\|h_{x,y}\|_H<\rho$; and so

$$S_{x,y}(\cdot)\equiv\tilde{\Phi}(\cdot,x,h)^{-1}|_{M_{x,y}\cap\{u\in\mathcal{G};|u|<r_4\}}$$

is a diffeomorphism into $M_{x,x}\cap\{v\in\mathcal{G};\,|v|<\rho\}$. Also, because $(u,x)\to K(u,x)$ is uniformly bounded and uniformly positive on $\{u\in\mathcal{G};\,|u|<r_4\}\times\boldsymbol{R}^N$, the existence of a $\gamma\in(0,1]$ satisfying (3.24) follows immediately from (3.29). To prove the right hand side of (3.25), take $v=S_{x,y}(u)$ and $h=h_{x,y}$ in (3.27). To prove the left hand side, combine (3.26) and (3.28) with the same choice of v and h. Q.E.D.

Now choose and fix $\eta\in C_b^\infty(\{u\in\mathcal{G};\,|u|<r_4\})$ and $\psi\in C_0^\infty((-1,1))$ satisfying $0\leqq\eta,\psi\leqq1$, $\eta(u)=1$ for $|u|\leqq r_4/2$, and $\psi(\xi)=1$ for $|\xi|\leqq1/2$. For $\lambda>0$, define $\psi_\lambda(\cdot)=\psi\left(\frac{\cdot}{\lambda^{1/2}}\right)$. Given $a>0$, let $P_l^a(t,x,\cdot)=(\eta(U(t))\psi_{at}(e(U(t)))\mathcal{W})\circ(X_l(t,x))^{-1}$. Then, by (3.16) and Lemma (3.23), $P_l^a(t,x,dy)=p_l^a(t,x,y)dy$, where:

$$\text{(3.30)}\qquad \gamma\int_{M_{x,x}}\eta(S_{x,y}^{-1}(v))\psi_{at}(e(S_{x,y}^{-1}(v)))\rho_t(S_{x,y}(v))m_{x,x}(dv)$$
$$\leqq p_l^a(t,x,y)\leqq\frac{1}{\gamma}\int_{M_{x,x}}\eta(S_{x,y}^{-1}(v))\psi_{at}(e(S_{x,y}^{-1}(v)))\rho_t(S_{x,y}(v))m_{x,x}(dv).$$

For $r>0$ and $x\in\boldsymbol{R}^N$, set $B_d(x,r)=\{y\in\boldsymbol{R}^N;\, d(x,y)<r\}$ and for $\Gamma\in\mathcal{B}_{\boldsymbol{R}^N}$ let $|\Gamma|$ denote the Lebesgue measure of Γ.

(3.31) LEMMA. *For each $\beta\in(0,1)$ there is a $\delta(\beta)\in(0,1)$ such that for all $a,b\in\left[\beta,\frac{1}{\beta}\right]$ and $(t,x)\in(0,1]\times\boldsymbol{R}^N$:*

$$\text{(3.32)}\qquad t^{-\nu/2}m_{x,x}(\{v\in M_{x,x};\, e(v)<(at)^{1/2}\})|B_d(x,(bt)^{1/2})|\in\left[\delta(\beta),\frac{1}{\delta(\beta)}\right].$$

In particular, there is a $K_l\in[1,\infty)$ such that

$$\text{(3.33)}\qquad m_{x,x}(\{v\in M_{x,x};\, e(v)<(2t)^{1/2}\})\leqq K_l m_{x,x}(\{v\in M_{x,x};\, e(v)<t^{1/2}\})$$

for all $(t,x)\in(0,1]\times\boldsymbol{R}^N$ and a $C\in[1,\infty)$ such that

$$\text{(3.34)}\qquad |B_d(x,(2t)^{1/2})|\leqq C|B_d(x,t^{1/2})|$$

for all $(t,x)\in(0,1]\times\boldsymbol{R}^N$.

PROOF. First, take $a=\left(\frac{4}{\gamma}\right)^2$. Then, by (3.30), (3.25) and (3.19) together with Theorem (3.12):

$$p_l^a(t,x,y)\geqq\frac{\gamma e^{-Ma/2}}{Mt^{\nu/2}}m_{x,x}(\{v\in M_{x,x};\, |S_{x,y}^{-1}(v)|<r_4/2 \text{ and } e(v)+d(x,y)<2t^{1/2}\})$$
$$\geqq\frac{\delta'}{t^{\nu/2}}m_{x,x}(\{v\in M_{x,x};\, e(v)<t^{1/2}\})$$

for any $y\in B_d(x,t^{1/2})$ and $t\in(0,T_0]$, where $\delta'=\gamma e^{-Ma/2}/M$ and $T_0=(r_4/2C_l)^2$. Hence, since $\int p_l^a(t,x,y)dy\leqq 1$:

$$t^{-\nu/2}m_{x,x}(\{v\in M_{x,x};\, e(v)<t^{1/2}\})|B_d(x,t^{1/2})|\leqq\frac{1}{\delta},$$

for all $(t, x) \in (0, T] \times \boldsymbol{R}^N$.

Second, take $a=\gamma^2$. Then, by the same estimates as we used in the preceding paragraph:

$$p_l^a(t, x, y) \leqq \frac{M}{\gamma t^{\nu/2}} m_{x,x}(\{v \in M_{x,x};\, e(v) < t^{1/2}\}).$$

At the same time, by (3.19) and (3.21):

$$\begin{aligned}&\int_{B_d(x,t^{1/2})} p_l^a(t, x, y)dy \\ &\geqq \mathcal{W}(|U(t)| \leqq r_4/2,\ e(U(t)) \leqq \gamma t^{1/2},\ d(x, x+F(U(t), x)) \leqq t^{1/2}) \\ &\geqq \mathcal{W}(e(U(t)) \leqq \beta t^{1/2})\end{aligned}$$

for some $\beta > 0$ and all $(t, x) \in (0, T_0] \times \boldsymbol{R}^N$. Since $t^{-1/2}e(U(t))$ is distributed under $\mathcal{W}$ in the same way as $e(U(1))$ is, we conclude from this and the preceding that:

$$t^{-\nu/2} m_{x,x}(\{v \in M_{x,x};\, e(v) < t^{1/2}\}) |B_d(x, t^{1/2})| \in \left[\delta, \frac{1}{\delta}\right] \tag{3.35}$$

for all $(t, x) \in (0, T_0] \times \boldsymbol{R}^N$ and some $\delta \in (0, 1]$.
From (3.35) it follows that

$$\begin{aligned}&m_{x,x}(\{v \in M_{x,x};\, e(v) < (2t)^{1/2}\}) |B_d(x, (2t)^{1/2})| \\ &\leqq \frac{2^{\nu/2}}{\delta^2} m_{x,x}(\{v \in M_{x,x};\, e(v) < t^{1/2}\}) |B_d(x, t^{1/2})|\end{aligned}$$

for all $(t, x) \in (0, T_0/2] \times \boldsymbol{R}^N$, and from this it is obvious that for every $T > 0$ there is a $C(T)$ such that

$$\begin{aligned}&m_{x,x}(\{v \in M_{x,x};\, e(v) < (2t)^{1/2}\}) \leqq C(T) m_{x,x}(\{v \in M_{x,x};\, e(v) < t^{1/2}\}) \\ &|B_d(x, (2t)^{1/2})| \leqq C(T) |B_d(x, t^{1/2})|\end{aligned}$$

for all $(t, x) \in (0, T] \times \boldsymbol{R}^N$. Finally, given $\beta \in (0, 1)$, set $T = 1/\beta T_0$ and use the preceding in conjunction with (3.35) to find a $\delta(\beta) \in (0, 1)$ for which (3.32) holds. Q.E.D.

REMARK. The inequality in (3.34) is known as the "doubling condition" and was first proved for the metric d by A. Sanchez [9]. Although Sanchez's proof also involves the use of a nilpotent group related to G, our method differs substantially from his.

Let $P_l(t, x, \cdot) = (\eta(U(t))\mathcal{W}) \circ (X_l(t, x))^{-1}$, $(t, x) \in (0, \infty) \times \boldsymbol{R}^N$, where η is the same as it was in the definition of $P_l^{\alpha}(t, x, \cdot)$. (Note that $P_l(t, x, \cdot)=P_l^{\infty}(t, x, \cdot)$.)

(3.36) THEOREM. *There is a* $p_l \in C^{\infty}((0, \infty)\times \boldsymbol{R}^N \times \boldsymbol{R}^N)$ *such that* $P_l(t, x, dy)=p(t, x, y)dy$ *for all* $(t, x) \in (0, \infty)\times \boldsymbol{R}^N$. *Moreover, for each* $l \geqq l_0$ *there exists an* $R_l \in (0, \infty)$ *such that* $p_l(t, x, \cdot) \in C_0^{\infty}(\{y \in \boldsymbol{R}^N; |y-x|<R_l\})$ *for all* $(t, x) \in (0, \infty)\times \boldsymbol{R}^N$; *and for each* $n \geqq 0$ *there exists a* $\nu_n \in (0, \infty)$ *such that for all* $l \geqq l_0$:

$$(3.37) \qquad \|p_l(t, x, \cdot)\|_{C_b^n(\boldsymbol{R}^N)} \leqq C(l, n)t^{-\nu_n/2}, \quad (t, x) \in (0, 1]\times \boldsymbol{R}^N$$

for some $C(l, n) \in (0, \infty)$. *Finally, for each* $l \geqq l_0$ *there exist* $r_l \in (0, 1]$ *and* $M_l \in [1, \infty)$ *such that*

$$(3.38) \qquad \frac{1}{M_l|B_d(t, x)|^{1/2}} \exp(-M_l d(x, y)^2/t) \leqq p_l(t, x, y) \leqq \frac{M_l}{|B_d(t, x)|^{1/2}} \exp(-d(x, y)^2/M_l t)$$

for all $(t, x) \in (0, 1]\times \boldsymbol{R}^N$ *and* $y \in \boldsymbol{R}^N$ *with* $d(x, y)<r_l$.

PROOF. The existence as well as the support property of p_l are obvious. In proving (3.37), the only problem is to show that ν_n can be chosen independent of $l \geqq l_0$. To this end, set $A_l(t, x)=\langle\langle X_l(t, x), X_l(t, x)\rangle\rangle_{\mathcal{L}}$ (the Malliavin covariance of $X_l(t, x)$) for $(t, x) \in (0, \infty)\times \boldsymbol{R}^N$. Then desired estimate will follow from Theorem (1.31) in [5] once we show the existence of a $\mu \in (0, \infty)$ such that for each $p \in [1, \infty)$ there is a $C_p \in (0, \infty)$ with the property that

$$(3.39) \qquad E^{\mathcal{W}}[(1/\det(A_l(t, x)))^p, \ |U(t)|<r_4]^{-1/p} \leqq C_p t^{-\mu}$$

for all $l \geqq l_0, p \in [1, \infty)$ and $(t, x) \in (0, 1]\times \boldsymbol{R}^N$. Noting that

$$\begin{aligned} A_l(t, x) &= JF(U(t), x)\langle\langle U(t), U(t)\rangle\rangle {}^t JF(U(t), x) \\ &\geqq JF(U(t), x)\pi_0\langle\langle U(t), U(t)\rangle\rangle \pi_0{}^t JF(U(t), x) \geqq A_{l_0}(t, x), \end{aligned}$$

where $\pi_0 : \bigoplus_{\lambda=1}^{l} \mathcal{G}_\lambda \to \bigoplus_{\lambda=1}^{l_0} \mathcal{G}_\lambda$ denotes orthogonal projection, we see that $A_l(t, x) \geqq A_{l_0}(t, x)$. Therefore we need only produce μ and $\{C_p; p \in [1, \infty)\}$ for $l=l_0$. But $\mathrm{Lie}(\pi_* W_1, \cdots, \pi_* W_d)(0)=\mathcal{G}$ implies that $\|1/\det(\langle\langle U(t), U(t)\rangle\rangle)_{\mathcal{L}}\|_{L^p(\mathcal{W})} \leqq B_p t^{-\mu}, t \in (0, 1]$, for some $\mu \in (0, \infty)$ and $B_p \in (0, \infty)$. In addition, by Lemmas (3.13) and (3.14), $JF(u, x){}^t JF(u, x) \geqq \varepsilon I_{\boldsymbol{R}^N}, |u|<r_1$, for some $\varepsilon \in (0, \infty)$. This implies (3.39) for $l=l_0$.

To prove (3.38), first note that by (3.29) (with $a=\infty$):

$$p_l(t, x, y) \geqq \gamma \int_{\{v \in M_{x,x}; e(v) < (\beta t)^{1/2}\}} \eta(S_{x,y}^{-1}(v)) \rho_t(S_{x,y}^{-1}(v)) m_{x,x}(dv)$$

for any $\beta > 0$. Thus, by (3.19) and (3.25), if $d(x, y) \leqq \gamma r_4/4C_l$ and β is small enough, then

$$p_l(t, x, y) \geqq \gamma \int_{\{v \in M_{x,x}; e(v) < (\beta t)^{1/2}\}} \rho_t(S_{x,y}^{-1}(v)) m_{x,x}(dv)$$

for $t \in (0, 1]$. Hence the left hand side of (3.38) follows from (3.25), (3.32), and Theorem (3.12). To prove the right hand side of (3.38), we again start with (3.25) and thereby obtain:

$$\begin{aligned} p_l(t, x, y) \leqq & \frac{1}{\gamma} \int_{\{v \in M_{x,x}; e(v) < 1\}} (\eta \rho_t)(S_{x,y}^{-1}(v)) m_{x,x}(dv) \\ & + \frac{1}{\gamma} \int_{\{v \in M_{x,x}; e(v) < 1\}} (\eta \rho_t)(S_{x,y}^{-1}(v)) m_{x,x}(dv). \end{aligned}$$

Note that, by (3.25), if $d(x, y) \leqq 1/2$ and $e(v) \geqq 1$, then $(\eta \rho_t)(S_{x,y}^{-1}(v)) \leqq M \exp(-\lambda/t)$, $t \in (0, 1]$, for some $\lambda > 0$. Hence the second term on the right can be ignored. At the same time:

$$\begin{aligned} & \int_{\{v \in M_{x,x}; e(v) < 1\}} (\eta \rho_t)(S_{x,y}^{-1}(v)) m_{x,x}(dv) \\ & \leqq M t^{-\nu/2} \exp(-(\gamma d(x, y))^2/Mt) \times \int_{\{v \in M_{x,x}; e(v) < 1\}} \exp(-(\gamma e(v))^2/Mt) m_{x,x}(dv) \end{aligned}$$

and

$$\begin{aligned} & \int_{\{v \in M_{x,x}; e(v) < 1\}} \exp(-(\gamma e(v))^2/Mt) m_{x,x}(dv) \\ & = \frac{2\gamma^2}{Mt} \int_0^1 s \exp(-(\gamma s)^2/Mt) m_{x,x}(\{v \in M_{x,x}; e(v) < s\}) ds \\ & = \frac{2\gamma^2}{M} \int_0^{t^{-1/2}} s \exp(-(\gamma s)^2/M) m_{x,x}(\{v \in M_{x,x}; e(v) < t^{1/2} s\}) ds. \end{aligned}$$

Finally, by (3.33), $m_{x,x}(\{v \in M_{x,x}; e(v) < t^{1/2} s\}) \leqq K^{s+1} m_{x,x}(\{v \in M_{x,x}; e(v) < t^{1/2}\})$ for some $K \in (0, \infty)$ and for all $s \in (0, t^{-1/2}]$. Hence, the right hand side of (3.38) now follows from (3.32). Q.E.D.

4. Upper and lower bounds for the fundamental solution

We continue to work under the conditions stated at the beginning of section 3) (cf. (3.1)).

Given $x \in \boldsymbol{R}^N$, let $X^0(\cdot, x)$ be the solution to

$$(4.1) \qquad X^0(T, x) = x + \sum_{k=1}^{d} \int_0^T V_k(X^0(t, x)) \circ d\theta_k(t), \quad T \geqq 0,$$

and set $P^0(t, x, \cdot) = \mathcal{W} \circ X^0(t, x)^{-1}$. Then (cf. [6]) there is a $p^0 \in C^\infty((0, \infty) \times \boldsymbol{R}^N \times \boldsymbol{R}^N)$ such that $P^0(t, x, dy) = p^0(t, x, y)dy$, $(t, x) \in (0, \infty) \times \boldsymbol{R}^N$. Moreover, if $L^0 = \frac{1}{2} \sum_{k=1}^{d} V_k^2$ and $(L^0)^*$ is the formal adjoint of L^0, then

$$(4.2) \qquad \frac{\partial p^0}{\partial t}(t, x, y) = [L^0 p^0(t, \cdot, y)](x) = [(L^*) p^0(t, x, \cdot)](y),$$

$$(t, x, y) \in (0, \infty) \times \boldsymbol{R}^N \times \boldsymbol{R}^N,$$

$$\lim_{t \downarrow 0} p^0(t, x, y) = \delta(x - y).$$

In addition, for each $n \geqq 0$, there exist $C_n \in (0, \infty)$, $\nu_n \in (0, \infty)$, and $\mu_n \in (0, \infty)$ such that

$$(4.3) \qquad \max_{|\alpha| + |\beta| \leqq n} \left| \frac{\partial^{|\alpha + \beta|} p^0}{\partial x^\alpha \partial x^\beta}(t, x, y) \right| \leqq C_n t^{-\nu_n} \exp(-\mu_n |y - x|^2 / t)$$

for $(t, x, y) \in (0, 1] \times \boldsymbol{R}^N \times \boldsymbol{R}^N$. Our goal in this section is to provide more precise estimates for $p^0(t, x, y)$ (cf. Theorem (4.9) below.)

(4.4) LEMMA. *For each $l \geqq 1$ there exists a $B_l \in (0, \infty)$ such that*

$$\mathcal{W}(|X^0(t, x) - X_l(t, x)| \geqq R) \leqq B_l[\exp(-R^{2/(l+1)}/B_l t) + \exp(-R^{2/(l+2)}/B_l t)]$$

for all $t \in (0, 1]$ and $R \in (0, \infty)$.

PROOF. After transforming Stratonovich integrals to their Itô equivalents, one sees that $X^0(t, x) - X_l(t, x)$ can be written as a finite linear conbination of terms, each of which has the form $I^{(\alpha)}\eta(t) = (I_{\alpha_1} \circ \cdots \circ I_{\alpha_n} \eta)$ where $\eta : [0, \infty) \times \Theta \to \boldsymbol{R}$ is a progressively measurable function uniformly bounded by 1 and $\alpha \in \mathcal{A}$ with $\|\alpha\| \in \{l+1, l+2\}$. Thus, we need only show that for each $\alpha \in \mathcal{A} \backslash \{\phi\}$ there is a $C_\alpha \in (0, \infty)$ such that

$$(4.5) \qquad \mathcal{W}\Big(\sup_{0 \leqq t \leqq T} |I^{(\alpha)}\eta(t)| \geqq R\Big) \leqq C_\alpha \exp(-R^{2/\|\alpha\|}/2T), \quad T \in (0, 1] \text{ and } R \in (0, \infty).$$

If $\alpha=(k)$, $1\leqq k\leqq d$, then (4.5) with $C_\alpha=2$ is a standard estimate on stochastic integrals. If $\alpha=(0)$, then $\mathcal{W}(\sup_{0\leqq t\leqq T}|I^{(\alpha)}\eta(t)|\geqq R)=0$ for $R>T$ and so (4.5) holds with $C_\alpha=e^{1/2}$. Next, assume that (4.5) holds for α and let $\beta=(\alpha,k)$. If $1\leqq k\leqq d$, then (by the same estimate as we used above)

$$\mathcal{W}\Big(\sup_{0\leqq t\leqq T}|I^{(\beta)}\eta(t)|\geqq R\Big)\leqq 2\exp(-R^2/2AT)+\mathcal{W}\Big(\sup_{0\leqq t\leqq T}|I^{(\alpha)}\eta(t)|\geqq A\Big)$$
$$\leqq 2\exp(-R^2/2AT)+\exp(-R^{2/\|\alpha\|}/2T)$$

for all $A\in(0,\infty)$. Taking $A=R^{\|\alpha\|/(\|\alpha\|+1)}$, we see that (4.5) holds for β with $C_\beta=2+C_\alpha$. If $k=0$, then

$$\mathcal{W}\Big(\sup_{0\leqq t\leqq T}|I^{(\beta)}\eta(t)|\geqq R\Big)\leqq\mathcal{W}\Big(\sup_{0\leqq t\leqq T}|I^{(\alpha)}\eta(t)|\geqq R/T\Big)\leqq C_\alpha\exp(-R^{2/\|\alpha\|}/2T^{1+2/\|\alpha\|}).$$

Since $R^{2/\|\alpha\|}/T^{1+2/\|\alpha\|}\geqq R^{2/\|\beta\|}/T$ for $R\geqq T^{\|\beta\|/2}$, we see that (4.5) again holds with $C_\beta=C_\alpha+e^{1/2}$. Q.E.D.

(4.6) LEMMA. *There exists a* $\delta\in(0,1]$ *and* $M\in[1,\infty)$ *such that*

$$\frac{1}{M|B_d(x,t^{1/2})|}\exp(-M\cdot d(x,y)^2/t)-Mt\leqq p^0(t,x,y)$$
$$\leqq\frac{M}{|B_d(x,t^{1/2})|}\exp(-d(x,y)^2/Mt)+Mt$$

for all $(t,x,y)\in(0,1]\times\boldsymbol{R}^N\times\boldsymbol{R}^N$ *with* $d(x,y)<\delta$.

PROOF. In view of (3.38), it suffices to show that there is an $l\geqq l_0$ such that

$$|p^0(t,x,y)-p_l(t,x,y)|\leqq Ct$$

for some $C\in(0,\infty)$ and all $(t,x,y)\in(0,1]\times\boldsymbol{R}^N\times\boldsymbol{R}^N$.

Define $\hat{p}^0(t,x,\xi)=\int\exp(i(\xi,y)_{\boldsymbol{R}^N})p^0(t,x,y)dy$ and $\hat{p}_l(t,x,\xi)=\int\exp(i(\xi,y)_{\boldsymbol{R}^N})\times p_l(t,x,y)dy$ for $(t,x,\xi)\in(0,\infty)\times\boldsymbol{R}^N\times\boldsymbol{R}^N$. By (4.3) together with the results in Lemma (3.36), we see that there is a $B_l\in(0,\infty)$ such that

$$|\xi|^{N+1}|\hat{p}^0(t,x,\xi)-\hat{p}_l(t,x,\xi)|\leqq B_lt^{-\mu}$$

for $(t,x,\xi)\in(0,\infty)\times\boldsymbol{R}^N\times\boldsymbol{R}^N$, where $\mu\in(0,\infty)$ does not depend on $l\geqq l_0$. At the same time, by Lemma (4.4):

$$\begin{aligned}&|\hat{p}^0(t, x, \xi)-\hat{p}_l(t, x, \xi)|\\ &\leqq E^{\mathcal{W}}[1-\eta(U(t))]+E^{\mathcal{W}}[|\exp(i(X^0(t, x)-X_l(t, x), \xi)_{R^N})-1|]\\ &\leqq A_l(1+|\xi|)t^{(l+1)/2}\end{aligned}$$

for some $A_l \in (0, \infty)$ and all $(t, x, \xi) \in (0, 1]\times \boldsymbol{R}^N \times \boldsymbol{R}^N$. Hence

$$\begin{aligned}&|p^0(t, x, y)-p_l(t, x, y)| \leqq \|\hat{p}^0(t, x, \cdot)-\hat{p}_l(t, x, \cdot)\|_{L^1(R^N)}\\ &\leqq A_l t^{(l+1)/2}\int_{|\xi|\leqq R}(1+|\xi|)d\xi + B_l t^{-\mu}\int_{|\xi|\geqq R}|\xi|^{-N-1}d\xi\\ &\leqq C_l(t^{(l+1)/2}R^{N+1}+t^{-\mu}R^{-1})\end{aligned}$$

for some $C_l \in (0, \infty)$ and all $(t, R) \in (0, 1]\times[1, \infty)$. Taking $R=t^{-\lambda_l}$, where $\lambda_l=(l+1+2\mu)/2(N+2)$, we conclude that

$$|p^0(t, x, y)-p_l(t, x, y)| \leqq 2C_l t^{\gamma_l},\ (t, x, y) \in (0, 1]\times \boldsymbol{R}^N \times \boldsymbol{R}^N,$$

where $\gamma_l=(l+1-2(N+1)\mu)/2(N+1)\to\infty$ as $l\to\infty$. Q.E.D.

(4.7) LEMMA. *There is an $M \in [1, \infty)$ and a $\sigma \in (0, \infty)$ such that*

$$p^0(t, x, y) \leqq \frac{M}{t^\sigma}\exp(-d(x, y)^2/Mt)$$

for $(t, x, y) \in (0, 1]\times \boldsymbol{R}^N \times \boldsymbol{R}^N$ with $d(x, y) \leqq 1$.

PROOF. Choose $\phi \in C_0^\infty(\boldsymbol{R}^N)^+$ so that $\phi=1$ on $\{y \in \boldsymbol{R}^N; |y| \leqq 1/2\}$ and $\phi=0$ on $\{y \in \boldsymbol{R}^N; |y| \geqq 1\}$. Given $(t, x, y) \in (0, 1]\times \boldsymbol{R}^N \times \boldsymbol{R}^N$, set $q(t, x, y; \xi)=\int \exp(i(\xi, \zeta)_{R^N})\phi((\zeta-\xi)t^{-l_0})p^0(t, x, \zeta)d\zeta$, $\zeta \in \boldsymbol{R}^N$. Then, by (4.3), there is a $B \in (0, \infty)$ and a $\lambda \in (0, \infty)$ such that

$$p^0(t, x, y) \leqq \|q(t, x, y;\ \cdot)\|_{L^1(R^N)} \leqq B[R^N \mathcal{W}(|X^0(t, x)-y| \leqq t^{l_0})+t^{-\lambda}R^{-1}]$$

for all $R \in (0, \infty)$, and so

$$p^0(t, x, y) \leqq 2Bt^{-\lambda N/(N+1)}\mathcal{W}(|X^0(t, x)-y| \leqq t^{l_0})^{1/(N+1)}.$$

Hence, we need only check that there exist $A \in [1, \infty)$ and $\mu \in (0, \infty)$ such that

$$\mathcal{W}(|X^0(t, x)-y| \leqq t^{l_0}) \leqq \frac{A}{t^\mu}\exp(-d(x, y)^2/At) \tag{4.8}$$

for $(t, x, y) \in (0, 1]\times \boldsymbol{R}^N \times \boldsymbol{R}^N$ with $d(x, y) \leqq 1$.

Let $K \in [1, \infty)$ be the constant appearing in (3.21). If $d(x, y) \leqq 3Kt^{1/2}$, then (4.8) is trivial. Thus we will assume that $3Kt^{1/2}\leqq t\leqq 1$. Since, by (3.21),

$$|X^0(t,x)-y|^{1/l_0}\geqq\frac{1}{K}d(X^0(t,x),y)$$
$$\geqq\frac{1}{K}[d(x,y)-d(x,X_{l_0}(t,x))-d(X_{l_0}(t,x),X^0(t,x))]$$

so long as $|X^0(t,x)-y|\leqq 1$, we then have

$$\mathcal{W}(|X^0(t,x)-y|\leqq t^{l_0})\leqq\mathcal{W}(d(x,X_{l_0}(t,x))\geqq d(x,y)/3)$$
$$+\mathcal{W}(d(X_{l_0}(t,x),X^0(t,x))\geqq d(x,y)/3).$$

By (3.29) (with $a=\infty$), (3.24), and Theorem (3.11), we see that

$$\mathcal{W}(d(x,X_{l_0}(t,x))\geqq d(x,y)/3)\leqq\frac{C}{t^{\nu/2}}\exp(-\alpha d(x,y)^2/t)$$

or some $C\in(0,\infty)$ and $\alpha\in(0,\infty)$. At the same time, by Lemma (4.4) and (3.21):

$$\mathcal{W}(d(X_{l_0}(t,x),X^0(t,x))\geqq d(x,y)/3)\leqq\mathcal{W}(|X^0(t,x)-X_{l_0}(t,x)|\geqq(d(x,y)/3K)^{l_0})$$
$$\leqq B_{l_0}\exp(-(d(x,y)/3K)^{2l_0/(l_0+1)}/2t)\leqq B_{l_0}\exp(-d(x,y)^2/18K^2t).$$
Q.E.D.

(4.9) THEOREM. *There is an $M\in[1,\infty)$ such that*

$$\frac{1}{M|B_d(x,t^{1/2})|}\exp(-Md(x,y)^2/t)\leqq p^0(t,x,y)$$
$$\leqq\frac{M}{|B_d(x,t^{1/2})|}\exp(-d(x,y)^2/Mt)$$

for all $(t,x,y)\in(0,1]\times\boldsymbol{R}^N\times\boldsymbol{R}^N$.

PROOF. We first prove the right hand side. Assuming that $d(x,y)\leqq 1$, the desired estimate follows from Lemma (4.7) when either $1\geqq t\geqq\exp(-d(x,y)^2/2\mu At)$ or $d(x,y)\geqq\delta$. On the other hand, if both $d(x,y)\leqq\delta$ and $t\leqq\exp(-d(x,y)/2\mu At)$, then the estimate follows from Lemma (4.6). Hence we need only be concerned with the case when $d(x,y)\geqq 1$ and $0<t\leqq 1$. To handle this case, set $n=[|y-x|]+1$ and $\xi_m=x+\frac{m}{n}\cdot(y-x)$. Then, by (3.21):

$$d(x, y)\leqq \sum_{m=0}^{n-1} d(\xi_m, \xi_{m+1})\leqq K \sum_{m=0}^{n-1} |\xi_{m+1}-\xi_m|^{1/l_0}=Kn\left(\frac{|x-y|}{n}\right)^{1/l_0}$$
$$\leqq Kn\leqq K(1+K^{1/l_0})|y-x|.$$

Hence, the case when $d(x, y)\geqq 1$ and $0<t\leqq 1$ follows from (4.3).

The proof of the left hand side is very much like that of the lower bound in Theorem (3.11). By Lemma (4.6), we can find a $\gamma\in(0, \delta]$ such that $p^0(t, x, y)\geqq \frac{1}{2M|B_d(x, t^{1/2})|}$ for all $(t, x, y)\in(0, \gamma]\times \boldsymbol{R}^N\times \boldsymbol{R}^N$ with $d(x, y)^2\leqq 2t$. Now let $(t, x, y)\in(0, 1]\times \boldsymbol{R}^N\times \boldsymbol{R}^N$ be given and set $n=2\left(\left[\frac{d(x, y)^2}{t}\vee\frac{1}{\gamma}\right]+1\right)$. Using Lemma (B.2), choose $x_0, \cdots, x_n\in \boldsymbol{R}^N$ so that $x_0=x$, $x_n=y$, and $d(x_{m-1}, x_m)=d(x, y)/n$ for $1\leqq m\leqq n$. Then $\frac{t}{n}\leqq\gamma$, and, for $y_m\in B_d(x_{m-1}, (t/8n)^{1/2})$, $1\leqq m\leqq n-1$,

$$d(y_{m-1}, y_m)\leqq 2\left(\frac{t}{8n}\right)^{1/2}+\frac{d(x, y)}{n}\leqq\left(\frac{2t}{n}\right)^{1/2}, \quad 1\leqq m\leqq n,$$

where $y_0=x$ and $y_n=y$. Hence:

$$\begin{aligned}
&p^0(t, x, y)\\
&\geqq \frac{1}{(2M)^n}\frac{1}{|B_d(x, t^{1/2})|}\int_{B_d(x_0, (t/8n)^{1/2})}\frac{1}{|B_d(y_1, (t/n)^{1/2})|}dy_1\\
&\quad\cdots\int_{B_d(x_{n-2}, (t/8n)^{1/2})}\frac{1}{|B_d(y_{n-1}, (t/n)^{1/2})|}dy_{n-1}\\
&\geqq \frac{1}{(2M)^n}\frac{1}{|B_d(x, t^{1/2})|}\frac{|B_d(x_0, (t/8n)^{1/2})|}{|B_d(x_0, (2t/n)^{1/2})|}\cdots\frac{|B_d(x_{n-2}, (t/8n)^{1/2})|}{|B_d(x_{n-2}, (2t/n)^{1/2})|}\\
&\geqq \frac{1}{|B_d(x, t^{1/2})|}\frac{1}{(2MC^3)^n},
\end{aligned}$$

where C is the constant in (3.33). Clearly the desired lower bound follows from this. Q.E.D.

The estimate in Theorem (4.9) can be extended quite easily to cover a slight larger class of operators. To be precise, let $\sigma_1, \cdots, \sigma_d\in C_b^\infty(\boldsymbol{R}^N)$ be given and set $V_0=\sum_{k=1}^d \sigma_k V_k$. For $x\in \boldsymbol{R}^N$, let $X(t, x)$ be the solution to

$$X(T, x)=x+\sum_{k=1}^d\int_0^T V_k(X(t, x))\circ d\theta_k(t)+\int_0^T V_0(X(t, x))dt, \tag{4.10}$$

$T \geqq 0$. Given $c \in C_b^\infty(\boldsymbol{R}^N)$, set

$${}^cP(T, x, \cdot) = \left(\exp\left[\int_0^T c(X(t, x))\,dt\right] \cdot \mathcal{W}\right) \circ X(T, x)^{-1}.$$

Then, for $(t, x) \in (0, \infty) \times \boldsymbol{R}^N$, ${}^cP(T, x, dy) = {}^cp(T, x, y)dy$ where ${}^cp \in C^\infty((0, \infty) \times \boldsymbol{R}^N \times \boldsymbol{R}^N)$. Moreover, if $L = \frac{1}{2}\sum_{k=1}^d V_k^2 + V_0$, then

$$\frac{\partial\, {}^cp}{\partial t}(t, x, y) = [(L + c)\,{}^cp(t, \cdot, y)](x) = [(L^* + c)\,{}^cp(t, x, \cdot)](y), \quad (t, x, y) \in (0, \infty) \times \boldsymbol{R}^N \times \boldsymbol{R}^N, \tag{4.11}$$

$$\lim_{t \downarrow 0} {}^cp(t, x, y) = \delta(x - y).$$

In addition, for each $n \geqq 0$ there exist $C_n \in (0, \infty)$, $\mu_n \in (0, \infty)$, and $\nu_n \in (0, \infty)$ such that:

$$\left|\frac{\partial^{|\alpha+\beta|}\, {}^cp}{\partial x^\alpha \partial y^\beta}(t, x, y)\right| \leqq C_n t^{-\nu_n} \exp(-\mu_n |y - x|^2/t), \quad (t, x, y) \in (0, \infty) \times \boldsymbol{R}^N \times \boldsymbol{R}^N. \tag{4.12}$$

(4.13) THEOREM. *For each choice of* $\sigma_1, \cdots, \sigma_d$, *and* c *from* $C_b^\infty(\boldsymbol{R}^N)$, *there exists an* $M \in [1, \infty)$ *such that*

$$\frac{1}{M|B_d(x, t^{1/2})|} \exp(-Md(x, y)^2/t) \leqq {}^cp(t, x, y) \leqq \frac{M}{|B_d(x, t^{1/2})|} \exp(-d(x, y)^2/Mt)$$

for $(t, x, y) \in (0, 1] \times \boldsymbol{R}^N \times \boldsymbol{R}^N$.

PROOF. As in the proof of Theorem (4.9), the desired estimate for all $(t, x, y) \in (0, 1] \times \boldsymbol{R}^N \times \boldsymbol{R}^N$ follows readily from (4.12) once it has been proved in the case when $(t, x, y) \in (0, 1] \times \boldsymbol{R}^N \times \boldsymbol{R}^N$ with $d(x, y) \leqq 1$. Thus, we will restrict our attension to this case.

Choose $\psi' \in C_b^\infty(\boldsymbol{R})$ so that $0 \leqq \psi'(\xi) \leqq 1$ for all $\xi \in \boldsymbol{R}$, $\psi'(\xi) = 1$ for all $|\xi| \leqq 1$, and $\psi'(\xi) = 0$ for all $|\xi| \geqq 3$. Given $\lambda \in [1, \infty]$, define $\psi_\lambda(\eta) = \int_0^\eta \psi'(\xi/\lambda)d\xi$ ($= \eta$ when $\lambda = \infty$) for $\eta \in \boldsymbol{R}$ and set

$$R_\lambda(t. x) = \exp\left[\sum_{k=1}^d \left\{\psi_\lambda\left(\int_0^t \sigma_k(X^0(s, x))d\theta_k(s)\right) - \frac{1}{2}\int_0^t \sigma_k(X^0(s, x))^2 ds\right\} + \int_0^t c(X^0(s, x))ds\right].$$

Then ${}^{c}P(T, x, \cdot)=(R_\infty(t, x)\mathcal{W})\circ X^0(t, x)^{-1}$. Hence, if $f\in C_0^\infty(\boldsymbol{R}^N)^+$ with $\int_{\boldsymbol{R}^N} f(\xi)d\xi=1$ and if $f_\varepsilon(\xi)=\varepsilon^{-N}f(\xi/\varepsilon)$ for $\varepsilon>0$, then for all $\lambda\in[1, \infty)$ and $(t, x, y)\in(0, \infty)\times\boldsymbol{R}^N\times\boldsymbol{R}^N$:

$$\begin{aligned}{}^{c}p(t, x, y)&=\lim_{\varepsilon\downarrow 0} E^{\mathcal{W}}[R_\infty(t, x)f_\varepsilon(X^0(t, x)-y)]\\ &=\lim_{\varepsilon\downarrow 0} E^{\mathcal{W}}[q_{\lambda,\varepsilon}(t, x, y)+r_{\lambda,\varepsilon}(t, x, y)]\end{aligned}$$

where $q_{\lambda,\varepsilon}(t, x, y)=E^{\mathcal{W}}[R_\lambda(t, x)f_\varepsilon(X^0(t, x)-y)]$ and

$$r_{\lambda,\varepsilon}(t, x, y)=E^{\mathcal{W}}[(R_\infty(t, x)-R_\lambda(t, x))f_\varepsilon(X^0(t, x)-y)].$$

Clearly there is a $K\in[1, \infty)$ such that

$$\frac{e^{-2d\lambda}}{K}p^0(t, x, y)\leqq \varliminf_{\varepsilon\downarrow 0} q_{\lambda,\varepsilon}(t, x, y)\leqq \varlimsup_{\varepsilon\downarrow 0} q_{\lambda,\varepsilon}(t, x, y)\leqq Ke^{2d\lambda}p^0(t, x, y)$$

for all $\lambda\in[1, \infty)$ and $(t, x, y)\in(0, 1]\times\boldsymbol{R}^N\times\boldsymbol{R}^N$. To estimate $r_{\lambda,\varepsilon}(t, x, y)$, define $F_\varepsilon(\eta)=\int_{(-\infty,\eta_1]\times\cdots\times(-\infty,\eta_N]} f_\varepsilon(\xi)d\xi$ for $\varepsilon>0$ and $\eta\in\boldsymbol{R}^N$. Then $|F_\varepsilon(\eta)|\leqq 1$ and

$$r_{\lambda,\varepsilon}(t, x, y)=E^{\mathcal{W}}\left[(R_\infty(t, x)-R_\lambda(t, x))\frac{\partial^N F_\varepsilon}{\partial\eta_1\cdots\partial\eta_N}(X^0(t, x)-y)\right].$$

Recall that there is a $\mu\in(0, \infty)$ such that for each $p\in[1, \infty)$

$$\|1/\det\langle\langle X^0(t, x), X^0(t, x)\rangle\rangle_{\mathcal{L}}\|_{L^p(\mathcal{W})}\leqq A_p t^{-\mu},\quad (t, x)\in(0, 1]\times\boldsymbol{R}^N,$$

for some $A_p\in(0, \infty)$. Thus, by Theorem (1.20) in [5], there is a $\Psi_\lambda(t, x)\in\bigcap_{p\in[1,\infty)} L^p(\mathcal{W})$ for all $\lambda\in[1, \infty)$ and $(t, x, y)\in(0, 1]\times\boldsymbol{R}^N\times\boldsymbol{R}^N$ such that:

$$\begin{aligned}&\Psi_\lambda(t, x)=0\quad\text{on}\ \bigcap_{k=1}^{d}\left\{\left|\int_0^t \sigma_k(X^0(t, x))d\theta_k(s)\right|\leqq\lambda\right\},\\ &\|\Psi_\lambda(t, x)\|_{L^2(\mathcal{W})}\leqq Bt^{-\sigma},\ \lambda\in[1, \infty)\quad\text{and}\ (t, x)\in(0, 1]\times\boldsymbol{R}^N,\end{aligned}$$

for some $B\in(0, \infty)$ and $\sigma\in(0, \infty)$, and

$$r_{\lambda,\varepsilon}(t, x, y)=E^{\mathcal{W}}[\Psi_\lambda(t, x)F_\varepsilon(X^0(t, x)-y)].$$

Hence,

$$|r_{\lambda,\varepsilon}(t, x, y)|\leqq Bt^{-\sigma}\left(\sum_{k=1}^{d}\mathcal{W}\left(\left|\int_0^t \sigma_k(X^0(s, x))d\theta_k(s)\right|\geqq\lambda\right)\right)^{1/2}$$

for all $\lambda\in[1,\infty)$, $\varepsilon>0$, and $(t,x,y)\in(0,1]\times R^N\times R^N$, and so, by standard estimates, there is a $C\in(0,\infty)$ and a $\gamma\in(0,\infty)$ such that

$$|r_{\lambda,\varepsilon}(t,x,y)|\leqq Ce^{-\gamma\lambda^2/t}$$

for all $\lambda\in[1,\infty)$, $\varepsilon>0$, and $(t,x,y)\in(0,\infty]\times R^N\times R^N$.

After combining the preceding with Theorem (4.9), one sees that the desired estimate for $(t,x,y)\in(0,1]\times R^N\times R^N$ with $d(x,y)\leqq 1$ follows upon taking $\lambda\in[1,\infty)$ to be sufficiently large. Q.E.D.

(4.14) THEOREM. *There is an $M<\infty$ such that for any $n,m\geqq 0$, $i_1,\cdots,i_n,j_1,\cdots,j_m=1,\cdots,d$, there is a $C<\infty$ such that*

$$|V_{i_1,x}\cdots V_{i_n,x}V_{j_1,y}\cdots V_{j_m,y}{}^cp(t,x,y)|\leqq\frac{C}{t^{(n+m)/2}|B_d(x,t^{1/2})|}\exp(-d(x,y)^2/Mt)$$

for any $0<t\leqq 1$, $x,y\in R^N$.

We will give the proof of this theorem in the end of the next section.

5. Harnack's principle and the Poincare inequality

First let us remind results in (3.22), (3.34) and Theorem (4.13) as follows:
There are constants K', C and $M<\infty$ such that

$$\frac{1}{K'}\leqq|B_d(x,1)|\leqq K',\tag{5.1}$$

$$|B_d(x,2r)|\leqq C|B_d(x,r)|\tag{5.2}$$

and

$$\frac{1}{M|B_d(x,t^{1/2})|}\exp(-M\cdot d(x,y)^2/t)\leqq{}^cp(t,x,y)\leqq\frac{M}{|B_d(x,t^{1/2})|}\exp(-d(x,y)^2/Mt)\tag{5.3}$$

for all $r\in(0,\infty)$, $t\in(0,1]$ and $x,y\in R^N$.

We mainly think of the case where $c\equiv 0$.

(5.4) REMARK. By (5.2), we see that there are $K<\infty$ and $\beta,\beta'\in(0,\infty)$ such that

$$|B_d(x, \lambda r)| \leqq K\lambda^\beta |B_d(x, r)|$$

for any $x \in \boldsymbol{R}^N$, $r \in (0, \infty)$ and $\lambda \in [1, \infty)$.

(5.5) LEMMA. *For any $k \in (0, 1)$, there exists an $\alpha > 0$ such that for any $x_0 \in \boldsymbol{R}$ and $r \in (0, 1]$,*

$$p_{B_d(x_0,r)}(t, x, y) \geqq \frac{1}{2M|B_d(x, t^{1/2})|} \exp(-M \cdot d(x, y)^2/t)$$

for all $t \in (0, (\alpha r)^2]$ and $x, y \in B(x_0, kr)$ with $d(x, y) \leqq \alpha r$. Here $p_{B_d(x_0,r)}(t, x, y)dy = P_{x_0}[\zeta > t, X(t) \in dy]$ and $\zeta = \inf\{t > 0; X(t) \notin B_d(x_0, r)\}$.

PROOF. Let $x, y \in B(x_0, kr)$ with $d(x, y) \leqq \alpha r$ and $t \in (0, (\alpha r)^2]$. Then,

$$\begin{aligned}
&p_{B_d(x_0,r)}(t, x, y) \\
&= p(t, x, y) - E^{P_x}[p(t-\zeta, X(\zeta), y), \zeta < t] \\
&\geqq \frac{1}{M|B_d(x, t^{1/2})|} \exp(-M \cdot d(x, y)^2/t)\Big[1 - M^2 \exp\Big(-\frac{r^2}{t}\Big(\frac{(1-k)^2}{2M} - \alpha^2 M\Big)\Big) \\
&\quad \times E^{P_x}\Big[\frac{|B_d(x, t^{1/2})|}{|B_d(X(\zeta), (t-\zeta)^{1/2})|} \exp\Big(-\frac{r^2}{2M(t-\zeta)}\Big), \zeta < t\Big]\Big].
\end{aligned}$$

Note that:

$$\frac{|B_d(x, t^{1/2})|}{|B_d(X(\zeta), (t-\zeta)^{1/2})|} \leqq \frac{|B_d(X(\zeta), 4r)|}{|B_d(X(\zeta), (t-\zeta)^{1/2})|} \leqq 4^\beta K\Big(\frac{r}{(t-\zeta)^{1/2}}\Big)^\beta$$

and so

$$E^{P_x}\Big[\frac{|B_d(x, t^{1/2})|}{|B_d(X(\zeta), (t-\zeta)^{1/2})|} \exp\Big(-\frac{r^2}{2M(t-\zeta)}\Big), \zeta < t\Big] \leqq Ce^{-\nu/\alpha^2}$$

for some $C < \infty$ and $\nu > 0$. Thus, for $0 < \alpha \leqq \dfrac{1-k}{2M}$:

$$\begin{aligned}
&p_{B_d(x_0,r)}(t, x, y) \\
&\geqq \frac{1}{M|B_d(x, t^{1/2})|} \exp(-M \cdot d(x, y)^2/t)\Big[1 - C \cdot \exp\Big(-\Big(\frac{(k-1)^2}{4M} + \nu\Big)/\alpha^2\Big)\Big].
\end{aligned}$$

Choose $0 < \alpha < 1$ so that $C \cdot \exp\Big(-\Big(\dfrac{(k-1)^2}{4M} + \nu\Big)/\alpha^2\Big) \leqq \dfrac{1}{2}$. Q.E.D.

(5.6) LEMMA. *For any $k \in (0, 1)$ and $\delta \in (0, 1)$, there exists a $\nu \in (0, 1)$ such that for each $x_0 \in \boldsymbol{R}$ and $0 < r \leqq 1$,*

$$p_{B(x_0,r)}(t, x, y) \geqq \frac{\nu}{|B_d(x_0, kr)|}$$

for all $x, y \in B_d(x_0, kr)$ *and* $\delta r^2 \leqq t \leqq r^2$.

PROOF. Take a $k' \in (0, 1)$ with $k' > k$. Then, by Lemma 5.5, there is an $\alpha' \in (0, 1]$ such that

$$p_{B_d(x_0,r)}(\tau, x, y) \geqq \frac{1}{2M|B_d(x, \tau^{1/2})|} \exp(-M \cdot d(x, y)^2/\tau)$$

for any $x_0 \in \boldsymbol{R}^N$, $0 < \tau \leqq (\alpha' r)^2$ and $x, y \in B_d(x_0, k'r)$ with $d(x, y) \leqq \alpha' r$. Let $\varepsilon = (k'-k)/2\Lambda\alpha'$. Then we have

$$p_{B_d(x_0,r)}(\tau, x, y) \geqq \frac{1}{2M\varepsilon^{-\beta}K|B_d(x, \varepsilon r)|} \exp(-M \cdot d(x, y)^2/\tau)$$

for any $x_0 \in \boldsymbol{R}^N$, $0 < \tau \leqq (\varepsilon r)^2$ and $x, y \in B_d(x_0, k'r)$ with $d(x, y) \leqq \varepsilon r$.

Let $x, y \in B_d(x_0, kr)$ and $\delta r^2 \leqq \tau \leqq r^2$, and set

$$n = 16([r^2/\varepsilon^2\tau] + 1).$$

Then $16\varepsilon^{-2} \leqq 16r^2/\tau\varepsilon^2 \leqq n \leqq 17r^2/\tau\varepsilon^2 \leqq 17\varepsilon^{-2}\delta$. Choose $\xi_0, \cdots, \xi_{2n}$ such that $\xi_0 = x$, $\xi_n = x_0$, $\xi_{2n} = y$, $d(\xi_k, \xi_{k+1}) \leqq r/n$ and $\xi_0, \cdots, \xi_{2n} \in B_d(x_0, kr)$. Then $d(\xi_k, \xi_{k+1}) \leqq r/n^{1/2} \leqq \varepsilon r/4$. Set $\sigma = \tau/n$. Then $\sigma^{1/2} \leqq \varepsilon r/4$. Hence, if $\eta_k \in B(\xi_k, \sigma^{1/2})$, then $d(\eta_k, \eta_{k+1}) \leqq \varepsilon r$ and so

$$\begin{aligned} p_{B_d(x_0,r)}(\sigma, \eta_k, \eta_{k+1}) &\geqq \frac{1}{2M|B_d(\eta_k, \sigma^{1/2})|} \exp(-M \cdot d(\eta_k, \eta_{k+1})^2/\sigma) \\ &\geqq \frac{1}{2MC|B_d(\xi_k, \sigma^{1/2})|} \exp(-M \cdot d(\eta_k, \eta_{k+1})^2/\sigma). \end{aligned}$$

Also, since $n \leqq 17r^2/\varepsilon^2\tau$, $\sigma = \tau/n \leqq 17\varepsilon^{-2}(r/n)^2$, we have

$$d(\eta_k, \eta_{k+1}) \leqq \frac{r}{n} + 2(17)^{1/2}\varepsilon^{-1}\frac{r}{n} \leqq (1 + 10\varepsilon^{-1})\frac{r}{n}.$$

Since

$$\begin{aligned} p_{B_d(x_0,r)}(\tau, x, y) \geqq &\int_{B_d(\xi_1,\sigma^{1/2})} \cdots \int_{B_d(\xi_{2n-1},\sigma^{1/2})} p_{B_d(x_0,r)}(t, x, \eta_1) \\ &\times p_{B_d(x_0,r)}(t, \eta_1, \eta_2) \cdots p_{B_d(x_0,r)}(t, \eta_{2n-1}, y) d\eta_1 \cdots d\eta_{2n-1}, \end{aligned}$$

we now have:

$$p_{B_d(x_0,r)}(\tau, x, y) \geqq \left(\frac{1}{2MC}\right)^{2n} \frac{1}{|B_d(x, \sigma^{1/2})|} \exp(-M(1+10\varepsilon^{-1})^2(r^2/\tau n))$$
$$\geqq \left(\frac{1}{2MC}\right)^{2n} \frac{1}{|B_d(x_0, r)|} \exp(-M(1+10\varepsilon^{-1})^2(r^2/\tau n))$$
$$\geqq \left(\frac{1}{2MC}\right)^{34/\varepsilon^2\delta} \frac{1}{Kk^{-\beta}|B_d(x_0, r)|} \exp(-M(1+10\varepsilon^{-1})^2/\delta).$$

This proves our lemma. Q.E.D.

(5.7) THEOREM. *For any $k \in (0, 1)$, there is a $\rho \in (0, 1)$ such that for any $r \in (0, 1]$ and $(s_0, x_0) \in \mathbf{R} \times \mathbf{R}^N$, if $u \in C(\mathbf{R} \times \mathbf{R}^N)$ satisfies $\partial_t u + Lu = 0$ on $(s_0, s_0+r^2) \times B_d(x_0, r)$, then*

$$\mathrm{Osc}(u; s_0, x_0, kr) \leqq \rho\, \mathrm{Osc}(u; s_0, x_0, r).$$

Here $\mathrm{Osc}(u, s, x, r) \equiv \sup\{|u(\sigma, \xi) - u(\tau, \eta)|; s \leqq \sigma, \tau \leqq s+r^2, \xi, \eta \in B_d(x, r)\}$.

PROOF. Define

$$\begin{aligned}
\Sigma &= \sup\{u(\tau, \eta); 0 \leqq \tau - s_0 \leqq r^2, \eta \in B_d(x_0, r)\},\\
\sigma &= \inf\{u(\tau, \eta); 0 \leqq \tau - s_0 \leqq r^2, \eta \in B_d(x_0, r)\},\\
\Sigma' &= \sup\{u(\tau, \eta); 0 \leqq \tau - s_0 \leqq (kr)^2, \eta \in B_d(x_0, kr)\}
\end{aligned}$$

and

$$\sigma' = \inf\{u(\tau, \eta); 0 \leqq \tau - s_0 \leqq (kr)^2, \eta \in B_d(x_0, kr)\}.$$

Set $\Gamma = \{x \in B_d(x_0, kr); u(s_0+r^2, x) \geqq (\Sigma+\sigma)/2\}$.
If $|\Gamma|/|B_d(x_0, kr)| \geqq 1/2$, then $P_x(X(t) \in \Gamma, \zeta < t) \geqq \nu/2$ for all $(1-k^2)r^2 \leqq t \leqq r^2$ and $x \in B_d(x_0, kr)$. Hence, if $(\tau, \eta) \in [s_0, s_0+k^2r^2] \times B_d(x_0, kr)$, then

$$\begin{aligned}
u(\tau, \eta) - \sigma &= E^{P_{\tau,x}}[u((s_0+r^2)\Lambda\zeta, X((s_0+r^2)\Lambda\zeta)) - \sigma]\\
&\geqq (\Sigma - \sigma)/2 P_x(X(t\Lambda\zeta) \in \Gamma, \zeta < t)\\
&\geqq (\Sigma - \sigma)/4,
\end{aligned}$$

where $t = s_0 + r^2 - \tau$. Hence: $\sigma' - \sigma \geqq (\Sigma - \sigma)\nu/4$, and so

$$\Sigma' - \sigma' \leqq \Sigma - \sigma' \leqq (1-\nu/4)(\Sigma - \sigma).$$

If $|\Gamma|/|B_d(x_0, kr)| < 1/2$, replace u by $-u$. Q.E.D.

(5.8) COROLLARY. *For any $k \in (0, 1)$, there exist $C < \infty$ and $\beta \in (0, 1]$ such that for all (s_0, x_0), $0 < R \leqq 1$ and u satisfying $\partial_t u + Lu = 0$ on $[s_0, s_0+R^2] \times B(x_0, R)$:*

$$|u(s, x)-u(t, y)| \leqq C\|u\|\left(\frac{|t-s|^{1/2} d(x, y)}{R}\right)^{\beta}$$

for all $(s, x), (t, y) \in [s_0, s_0+k^2R^2] \times B_d(x_0, kR)$.

PROOF. Let $(s, x), (t, y) \in [s_0, s_0+k^2R^2] \times B_d(x_0, kR)$ with $s \leqq t$ be given and set $\delta = |t-s|^{1/2} \vee d(x, y)$. If $\delta \geqq kR$, then there is nothing to do. If $\delta < kR$, choose $n \geqq 1$ such that $k^{-(n-1)}\delta \leqq \delta < k^{-n}\delta$. Then $[s_0, s_0+(k^m\delta)^2] \times B_d(x_0, k^m\delta) \subset [s_0, s_0+R^2] \times B_d(x_0, R)$ for $0 \leqq m \leqq n$. Hence, since $(t, y) \in [s, s+\delta^2] \times B_d(x, \delta)$:

$$|u(s, x)-u(t, x)| \leqq 2\|u\| \rho^n.$$

Thus, if β is defined by $\rho = k^{\beta}$, then, since $k^{-n} > R/\delta$:

$$|u(s, x)-u(t, x)| \leqq 2\|u\|(\delta/R)^{\beta}. \qquad \text{Q.E.D.}$$

(5.9) THEOREM. *There is an* $M \in (0, \infty)$ *such that for all* $x, y \in \boldsymbol{R}^N$, $s < t$ *with* $R = d(x, y) \vee (t-s)^{1/2}$ *and all* $u \geqq 0$ *with* $\partial_\tau u + Lu = 0$ *in* $[s, s+R^2] \times B_d(x, R)$:

$$u(t, y) \leqq u(s, x) \exp[M(1+d(x, y)^2/(t-s))].$$

PROOF. We may assume that $s=0$. First assume that $d(x, y) \leqq t^{1/2}$ and that $1 = \min_{B_d(x, t^{1/2})} u(0, \cdot) = u(0, x_1)$ where $d(x, x_1) \leqq t^{1/2}$.

Given $\tau > 0$ and $l > 0$, define $S(\tau, l) = \{\xi \in B_d(x, 4t^{1/2}); u(\tau, \xi) \geqq l\}$. Then, by Lemma 5.6, for $t \leqq \tau \leqq (4t^{1/2})^2$:

$$1 = u(0, x_1) \geqq \nu l |S(\tau, l)| / |B_d(x, 4t^{1/2})|; \text{ and so } |S(\tau, l)| / |B_d(x, 4t^{1/2})| \leqq 1/\nu l.$$

Next, given $R \in (0, 1]$ and $\eta \in B_d(x, 4t^{1/2})$, note that

$$\frac{|B_d(\eta, Rt^{1/2})|}{|B_d(x, 4t^{1/2})|} \geqq \frac{|B_d(\eta, Rt^{1/2})|}{|B_d(\eta, 5t^{1/2})|} \geqq \frac{1}{K}\left(\frac{R}{5}\right)^{\beta}.$$

Thus if $R(l) \equiv 5\left(\frac{2K}{\nu l}\right)^{1/\beta} \leqq 1$, then for $(\tau, \eta) \in [t, (4t^{1/2})^2] \times B_d(x, 4t^{1/2})$:

$$B_d(\eta, R(l)t^{1/2}) \not\subset S(\tau, l).$$

Now define σ be $1-\sigma = \frac{1+\rho}{2}$ and set $\lambda = \frac{1-\sigma}{\rho} = \frac{1+1/\rho}{2} \in (1, \infty)$. Let $R(\sigma l) \leqq 1$ and $[\tau, \tau+(4R(\sigma l)t^{1/2})^2] \times B_d(\eta, 4R(\sigma l)t^{1/2}) \subset [t, (4t^{1/2})^2] \times B_d(x, 4t^{1/2})$.

Then there exists a $\xi\in B_d(\eta, R(\sigma l)t^{1/2})$ such that $u(\tau, \xi)\leqq\sigma l$. Hence, if $u(\tau, \xi)\geqq l$, then

$$\min_{[\tau,\tau+(4R(\sigma l)t^{1/2})^2]\times B_d(\eta,4R(\sigma l)t^{1/2})} u\geqq\mathrm{Osc}(u; \tau, \eta, 4R(\sigma l)t^{1/2})$$
$$\geqq\frac{1}{\rho}\mathrm{Osc}(u; \tau, \eta, R(\sigma l)t^{1/2})\geqq\frac{1}{\rho}(u(\tau, \eta)-u(\tau, \xi))\geqq\lambda l.$$

That is, $R(\sigma l)\leqq 1$, $[\tau, \tau+(4R(\sigma l)t^{1/2})^2]\times B_d(\eta, 4R(\sigma l)t^{1/2})\subset[t, (4t^{1/2})^2]\times B_d(x, 4t^{1/2})$, and $u(\tau, \xi)\geqq l$ imply the existence of $(\tau', \xi')\in[\tau, \tau+(4R(\sigma l)t^{1/2})^2]\times B_d(\eta, 4R(\sigma l)t^{1/2})$ with $u(\tau', \xi')\geqq\lambda l$.

Finally, choose $l_0>0$ so that $1+4\sum_{m=0}^{\infty}R(\sigma\lambda^m l_0)\leqq 4$. Then

$$t^{1/2}+\sum_{m=0}^{\infty}4R(\sigma\lambda^m l_0)t^{1/2}\leqq 4t^{1/2}$$

and

$$t+\sum_{m=0}^{\infty}(4R(\sigma\lambda^m l_0)t^{1/2})^2\leqq(4t^{1/2})^2.$$

Thus, if $u(t, y)\geqq l_0$, then there is $\{(\tau_m, \eta_m)\}_0^\infty\subset[t, (4t^{1/2})^2]\times B_d(x, 4t^{1/2})$, $\tau_0=t$, $\eta_0=y$, and $u(\tau_m, \eta_m)\geqq\lambda^m l_0$, and this leads to the contradiction. Hence, $u(t, y)<l_0$. Thus, the proof is complete when $d(x, y)\leqq t^{1/2}$.

If $d(x, y)>t^{1/2}$, choose n such that $n-1\leqq\frac{d(x, y)^2}{t}<n$ and $\xi_0, \cdots, \xi_n$ with $\xi_0=x$, $\xi_n=y$ and $d(\xi_k, \xi_{k+1})\leqq\frac{d(x, y)}{n}\leqq(t/n)^{1/2}$. Then $u(kt/n, \xi_k)\leqq l_0 u((k-1)t/n, \xi_{k-1})$ and so:

$$u(t, y)\leqq l_0^n u(0, x)\leqq\exp\Big(\Big(\frac{d(x, y)^2}{t}+1\Big)\log l_0\Big)u(0, x). \qquad \text{Q.E.D.}$$

(5.10) THEOREM. *There is a* $C\in(0, \infty)$ *such that*

$$\int_{B_d(\xi,r/2)}(f(x)-f_{(\xi,r/2)})^2dx\leqq Cr^2\sum_{k=1}^{d}\int_{B_d(\xi,r)}(V_kf)(x)^2dx$$

for all $\xi\in\boldsymbol{R}^N$, $r\in(0, 1]$ *and* $f\in C^\infty(\boldsymbol{R}^N)$. *Here* $f_{(\xi,r/2)}=\frac{1}{B_d(\xi, r/2)}\int_{B_d(\xi,r/2)}f(y)dy$.

PROOF. Given $\xi\in\boldsymbol{R}^N$ and $r\in(0, 1]$, define

$$\tilde{\mathcal{E}}^{(\xi,r)}(\phi, \phi)=\sum_{k=1}^{d}\int_{B_d(\xi,r)}(V_k\phi)(x)^2dx, \quad \phi\in C^\infty(\boldsymbol{R}^N).$$

Then $\tilde{\mathcal{E}}^{(\xi,r)}$ is a closable Dirichlet form whose closure we again denote by $\tilde{\mathcal{E}}^{(\xi,r)}$. Denote by $\{\tilde{P}_t^{(\xi,r)}; t\geqq 0\}$ the symmetric Markov semigroup determined by $\tilde{\mathcal{E}}^{(\xi,r)}$. Then $\{\tilde{P}_t^{(\xi,r)}; t\geqq 0\}$ is conservative and the associated transition probability function $\tilde{p}^{(\xi,r)}(t, x, dy)$ dominates $p_{B_d(\xi,r)}(t, x, dy)dy$. In particular, there is an $\varepsilon>0$ such that

$$\chi_{B_d(\xi,r/2)}(y)\tilde{p}^{(\xi,r)}(r^2, x, dy)\geqq\frac{\varepsilon}{B_d(\xi, r/2)}\chi_{B_d(\xi,r/2)}(y)dy$$

for all $x\in B_d(\xi, r/2)$, $\xi\in \boldsymbol{R}^N$ and $r\in(0, 1]$.

Now let $f\in\mathrm{Dom}(\tilde{\mathcal{E}}^{(\xi,r)})$. Then for $x\in B_d(\xi, r/2)$:

$$\begin{aligned}[\tilde{P}_{r^2}^{(\xi,r)}(f-[\tilde{P}_{r^2}^{(\xi,r)}f](x))^2](x)&\geqq\frac{\varepsilon}{B_d(\xi, r/2)}\int_{B_d(\xi,r/2)}(f(y)-[\tilde{P}_{r^2}^{(\xi,r)}f](x))^2dy\\&\geqq\frac{\varepsilon}{B_d(\xi, r/2)}\int_{B_d(\xi,r/2)}(f(y)-f_{(\xi,r/2)})^2dy.\end{aligned}$$

Hence

$$\begin{aligned}\varepsilon\int_{B_d(\xi,r/2)}(f(y)-f_{(\xi,r/2)})^2dy&\leqq\int_{B_d(\xi,r)}[\tilde{P}_{r^2}^{(\xi,r)}(f-[\tilde{P}_{r^2}^{(\xi,r)}f](x))^2](x)dx\\&\leqq\int_{B_d(\xi,r)}(f(x)-[\tilde{P}_{r^2}^{(\xi,r)}f](x)^2)dx\\&=2\int_0^{r^2}\tilde{\mathcal{E}}^{(\xi,r)}(\tilde{P}_t^{(\xi,r)}f, \tilde{P}_t^{(\xi,r)}f)\leqq 2r^2\tilde{\mathcal{E}}^{(\xi,r)}(f,f).\end{aligned}$$

Q.E.D.

(5.11) PROPOSITION. *There are $C'<\infty$ and $\nu<\infty$ such that*

$$\frac{|B_d(y, t)|}{|B_d(x, t)|}\leqq C'\exp\left(\nu\frac{d(x, y)}{t}\right)$$

for all $t\in(0, \infty)$ and $x, y\in\boldsymbol{R}^N$.

PROOF. For $t\geqq 2d(x, y)$, since $B_d(x, t)\subset B_d(y, t/2)$, by (5.1) we have

$$\frac{|B_d(y, t)|}{|B_d(x, t)|}\leqq C.$$

Suppose that $t<2d(x, y)$. Set $n=1+[\log(2d(x, y)/t)/\log 2]$. Then $nt\leqq 2^{n-1}t\leqq 2d(x, y)\leqq 2^n t$. Therefore

$$\begin{aligned}|B_d(x, t)|&\geqq K^{-1}2^{-n\beta}|B_d(x, 2^n t)|\geqq K^{-1}|B_d(y, 2^{n-1}t)|\exp(-n\beta\cdot\log 2)\\&\geqq K^{-1}|B_d(y, t)|\exp\left(-(\beta\cdot\log 2)\frac{2d(x, y)}{t}\right).\end{aligned}$$

This proves our assertion. Q.E.D.

Proof of Theorem 4.14. Because of semigroup property and the argument in the proof of Corollary (2.19), it is sufficient to prove that

$$|V_{i_1,x}\cdots V_{i_n,x}{}^c p(2t,x,y)|\leqq Ct^{-n/2}|B_d(x,t^{1/2})|^{-1}\exp\left(-\frac{d(x,y)}{M't}\right)$$

for all $t\in(0,1]$ and $x,y\in \boldsymbol{R}^N$.

Set $f(z)=f_{t,y}(z)={}^c p(t,z,y)$. Then by Theorem (2.18)

$$|V_{i_1,x}\cdots V_{i_n,x}{}^c p(2t,x,y)|=|(V_{i_1}\cdots V_{i_n}{}^c P_t f)(x)|\leqq Ct^{-n/2}|{}^c P_t(f^2)(x)|^{1/2}.$$

However, by (5.3), Remark 5.4 and Proposition (5.11),

$$\begin{aligned}
&{}^c P_t(f^2)(x)\\
&=\int_{R^N} dz\,{}^c p(t,x,z)\,{}^c p(t,z,y)^2\\
&\leqq \int_{R^N} dz\frac{M^3}{|B_d(x,t^{1/2})|\,|B_d(z,t^{1/2})|^2}\exp\left(-\frac{d(x,z)^2}{Mt}-\frac{2d(z,y)^2}{Mt}\right)\\
&\leqq \frac{C''}{|B_d(x,t^{1/2})|}\int_{R^N} dz\frac{1}{|B_d(x,t^{1/2})|}\exp\left(-\frac{d(x,z)^2}{2Mt}\right)\\
&\qquad\qquad\times\frac{1}{|B_d(z,t^{1/2})|}\exp\left(-\frac{2d(z,y)^2}{Mt}\right)\\
&\leqq \frac{\tilde{C}}{|B_d(x,t^{1/2})|}\int_{R^N} dz\,{}^c p(2M^2t,x,z)\,{}^c p(M^2t,z,y)\\
&\leqq \frac{\tilde{C}}{|B_d(x,t^{1/2})|}\,{}^c p(3M^2t,x,y)\\
&\leqq \frac{\tilde{C}M}{|B_d(x,t^{1/2})|^2}\exp\left(-\frac{d(x,y)^2}{3M^3t}\right).
\end{aligned}$$

This proves our assertion. Q.E.D.

A. The Lie algebra Lie$(W_1,\cdots,W_d)$

The purpose of this appendix is to prove Theorem (3.5). Before beginning, we introduce some notion.

Given $\alpha,\beta\in\mathcal{A}_l$ with $|\alpha+\beta|\leqq l$, set $(\alpha,\beta)=(\alpha_1,\cdots,\alpha_\lambda,\beta_1,\cdots,\beta_\mu)$ if $\alpha=(\alpha_1,\cdots,\alpha_\lambda)$ and $\beta=(\beta_1,\cdots,\beta_\mu)$. Let $z=(z_\alpha;\alpha\in\mathcal{A}_l)$ denote the standard coordinate system on $E=\boldsymbol{R}^{\mathcal{A}_l}$ and set $\mathcal{T}=\mathrm{span}\left\{\frac{\partial}{\partial z_\alpha};\alpha\in\mathcal{A}_l\right\}$. For $\alpha\in\mathcal{A}_l$, define L_α, $R_\alpha \in \mathrm{Hom}(\mathcal{T};\mathcal{T})$ so that

$$L_\alpha \frac{\partial}{\partial z_\beta} = \begin{cases} 0 & \text{if} \quad |\alpha|+|\beta|>l \\ \dfrac{\partial}{\partial z_{(\alpha,\beta)}} & \text{if} \quad |\alpha|+|\beta|\leqq l \end{cases},$$

and

$$R_\alpha \frac{\partial}{\partial z_\beta} = L_\beta \frac{\partial}{\partial z_\alpha}.$$

Also, for $\alpha \in \mathcal{A}_l$, define $W_\alpha = W_k$ if $\alpha=(k)$ and $W_\alpha=[W_k, W_\beta]$ if $\alpha=(k,\beta)$.

(A.1) LEMMA. *If $\alpha=(k,\beta)$ for some $1\leqq k\leqq d$ and $\beta\in\mathcal{A}_{l-1}$, then*

$$W_\alpha(0) = L_k W_\beta(0) - R_k W_\beta(0). \tag{A.2}$$

Also, for all $\alpha\in\mathcal{A}_l$:

$$W_\alpha = W_\alpha(0) + \sum_{\beta\in\mathcal{A}_{l-1}} z_\beta L_\beta W_\alpha(0). \tag{A.3}$$

In particular, $\hat{\mathcal{G}} \equiv \mathrm{Lie}(W_1, \cdots, W_d) = \mathrm{span}\{W_\alpha; \alpha\in\mathcal{A}_l\}$, $W\in\hat{\mathcal{G}} \to W(0)\in\mathcal{G}$ is an isomorphism, and $\mathcal{G} = \bigoplus_{\lambda=1}^{l} \mathcal{G}_\lambda$.

PROOF. (A.2) and (A.3) are proved by an easy induction on $|\alpha|$; and, given (A.2) and (A.3), the rest is obvious. Q.E.D.

For $a>0$, define $\hat{S}_a \in \mathrm{Hom}(E;E)$ so that $\hat{S}_a z_\alpha = a^{|\alpha|} z_\alpha$, $\alpha\in\mathcal{A}_l$. Clearly the $S_a\in\mathrm{Hom}(\mathcal{G};\mathcal{G})$ in Theorem (3.5) is presicely $\hat{S}_a|_{\mathcal{G}}$. Next, define $u\in\mathcal{G}\to W^u\in\hat{\mathcal{G}}$ to be the inverse of $W\in\hat{\mathcal{G}}\to W(0)\in\mathcal{G}$. Then, by (A.3), one sees that

$$(\hat{S}_a)_* W^u = W^{S_a u}. \tag{A.4}$$

(A.5) LEMMA. *Define $\Phi(u)=\exp(W^u)(0)$, $u\in\mathcal{G}$. Then $\Phi\in C^\infty(\mathcal{G};E)$ and*

$$\hat{S}_a\circ\Phi = \Phi\circ S_a, \qquad a>0. \tag{A.6}$$

Next, set $G=\Phi(\mathcal{G})$. Then G is a $\{\exp(W); W\in\hat{\mathcal{G}}\}$ invariant, embedded submanifold of E. Finally, let $\pi: E\to\mathcal{G}$ denote orthogonal projection. Then $S_a\circ\pi = \pi\circ\hat{S}_a$, $a>0$, and $\pi|_G$ is a diffeomorphism from G onto $\mathcal{G}$.

PROOF. Obviously $\Phi\in C^\infty(\mathcal{G};E)$, and (A.6) is an immediate consequence of (A.4). In addition, since, by Lemma (A.1) $\hat{\mathcal{G}}$ is nilpotent, the invariance of G under $\{\exp(W); W\in\hat{\mathcal{G}}\}$ is a consequence of the Baker-Campbell-

Hausdorff formula. To prove that G is an embedded submanifold of E, it suffices to check Φ is a proper immersion; and this, as well as the fact that $\pi|_G$ is diffeomorphic onto $\mathcal{G}$, will follow once we show that $\Xi\equiv\pi\circ\Phi$ is a diffeomorphism from $\mathcal{G}$ onto itself.

First note that, since $\mathcal{G}$ is $\{S_a; a>0\}$-invariant and $S_a\circ\pi=\pi\circ\hat{S}_a$, $a>0$, $S_a\circ\Xi=S_a\circ\Xi$. In particular, to show that Ξ is a diffeomorphism of $\mathcal{G}$ onto itself, it suffices to check that the Jacobian of Ξ does not vanish at 0. But $|\Xi(u)-u|\leqq|\Phi(u)-u|$ and there is a $C<\infty$ such that $|\Phi(u)-u|\leqq C|u|^2$ for $|u|\leqq 1$. Hence

$$|\Xi(u)-u|\leqq|\Phi(u)-u|\leqq C|u|^2,\quad |u|\leqq 1, \tag{A.7}$$

from which the required result is immediate. Q.E.D.

We next define $\Psi=(\pi|_G)^{-1}$. Noting that $\Psi=\Phi\circ\Xi^{-1}$, we conclude from (A.7) that $|\Psi(u)-u|\leqq C|u|^2$, $|u|\leqq 1$, for some $C<\infty$. Next, given $u, v\in\mathcal{G}$, set $u\times v=\pi(\exp(W^{\Xi^{-1}(u)})(\Psi(v)))$.

(A.8) LEMMA. *$(\mathcal{G},\times)$ is a nilpotent Lie group with 0 as its identity. Moreover, if $\tilde{W}\equiv(\pi|_G)_*W$, $W\in\hat{\mathcal{G}}$, then $\tilde{W}$ is right $\mathcal{G}$-invariant. In particular,* Lie$(\tilde{W}_1,\cdots,\tilde{W}_d)$ *is the Lie algebra of right $\mathcal{G}$-invariant vector fields on $\mathcal{G}$.*

PROOF. Clearly $(\mathcal{G},\times)$ is a Lie group. To prove that $\tilde{W}$ is right $\mathcal{G}$-invariant, let $v\in\mathcal{G}$ be given and define $R_v(u)-u\times v$, $u\in\mathcal{G}$. Then, for $f\in C^\infty(\mathcal{G})$:

$$\begin{aligned}(R_v)_*W(0)f&=\frac{d}{dt}f(\pi(e^{tW}(0))\times v)|_{t=0}=\frac{d}{dt}f(\pi(e^{tW}(\Psi(v))))|_{t=0}\\&=W(\Psi(u))f\circ\pi=\tilde{W}(v)f.\end{aligned}$$

Q.E.D.

(A.9) LEMMA. *For all $u\in\mathcal{G}$, $u^{-1}=-u$.*

PROOF. What we must show is that

$$\pi(\exp(-W)(0))=-\pi(\exp(W)(0)),\qquad W\in\hat{\mathcal{G}}. \tag{A.10}$$

To prove (A.10), think of W as a mapping from $\boldsymbol{R}^{\mathcal{A}_l}$ into itself and define $W^{(n)}(z)=W(W^{(n-1)}(z))$, $n\geqq 1$, where $W^{(0)}(z)=z$. Then, by (A.3):

$$\exp(W)(0)=\sum_{m=1}^{l}\frac{1}{m!}[W^{(m)}(0)-W^{(m-1)}(0)].$$

Thus, what we must show is that

(A.11) $\pi((-W)^{(m)}(0)-(-W)^{(m-1)}(0))=-\pi(W^{(m)}(0)-W^{(m-1)}(0)),\quad 1\leqq m\leqq l$

for all $W\in\hat{\mathcal{G}}$.

To prove (A.11), let $(e_1, \cdots, e_d)$ be an orthogonal basis in $\boldsymbol{R}^d$; and, for $\alpha=(\alpha_1, \cdots, \alpha_\lambda)\in\mathcal{A}_l$ set $e_\alpha=e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_\lambda}$ and define

$$C_\alpha=\begin{cases} e_k & \text{if } \alpha=k \\ e_k\otimes e_\beta-e_\beta\otimes e_k & \text{if } \alpha=(k, \beta). \end{cases}$$

Next, let $A:\mathcal{T}\to\bigoplus_{m=1}^{l}(\boldsymbol{R}^d)^{\otimes m}$ be the isometry given by $A\left(\frac{\partial}{\partial z_\alpha}\right)=e_\alpha$, $\alpha\in\mathcal{A}_l$. Then, by (A.2), $A(W_\alpha(0))=C_\alpha$, and, by (A.3):

(A.12) $$A(W^{(m)}(0)-W^{(m-1)}(0))=\sum_{|\alpha^1|+\cdots+|\alpha^m|\leqq l} a_{\alpha^1}\cdots a_{\alpha^m}C_{\alpha^1}\otimes\cdots\otimes C_{\alpha^m}$$

if $W=\sum_{\alpha\in\mathcal{A}_l} a_\alpha W_\alpha$. In particular,

$$(-W)^{(m)}(0)-(-W)^{(m-1)}(0)=(-1)^m[W^{(m)}(0)-W^{(m-1)}(0)];$$

and so (A.11) is obvious for odd m and reduces to proving that

(A.13) $$\pi(W^{(m)}(0)-W^{(m-1)}(0))=0$$

when m is even. In view of (A.12), (A.13) is equivalent to

(A.14) $$(D(\alpha^1, \cdots, \alpha^m), C_\beta)=0$$

for $\alpha^1, \cdots, \alpha^m, \beta\in\mathcal{A}_l$ with $|\beta|=|\alpha^1|+\cdots+|\alpha^m|$, where

$$D(\alpha^1, \cdots, \alpha^m)=\sum_{\pi\in\Pi_m} C_{\alpha^{\pi(1)}}\otimes\cdots\otimes C_{\alpha^{\pi(m)}}$$

and Π_m is the permutation group on $\{1, \cdots, m\}$. To prove (A.14), we use induction on $|\alpha|$ to show that C_α is a linear combination of terms $e_\rho-(-1)^{|\rho|}e_{\tilde{\rho}}$, where $\rho=(\rho_1, \cdots, \rho_\lambda)$ is a permutation of $\alpha=(\alpha_1, \cdots, \alpha_\lambda)$ and $\tilde{\rho}=(\rho_\lambda, \cdots, \rho_1)$. Since m is even, it follows that $D(\alpha^1, \cdots, \alpha^m)$ is a linear combination of terms $e_\rho+(-1)^{|\rho|}e_{\tilde{\rho}}$, where $\rho=(\rho^{\pi(1)}, \cdots, \rho^{\pi(m)})$ for some $\pi\in\Pi_m$ and ρ^μ is some permutation of α^μ for each $1\leqq\mu\leqq m$. Hence, when m is even, the left hand side of (A.14) is a linear combination of terms

$$(e_\rho+(-1)^{|\rho|}e_{\tilde{\rho}},\ e_\sigma-(-1)^{|\sigma|}e_{\tilde{\sigma}}),$$

each of which is clearly 0. Q.E.D.

We are at least ready to complete the proof of Theorem (3.5). First note that, by (A.2), (A.3), and the fact that $\mathcal{G}=\bigoplus_{\lambda=1}^{l}\mathcal{G}_\lambda$, the $\tilde{W}$ is divergence free for all $W\in\hat{\mathcal{G}}$. Hence, Lebesgue measure is left $\mathcal{G}$-invariant. At the same time, since $u^{-1}=u$, it Lebesgue measure is also invariant under $\mathcal{G}$-inversion, and so it is $\mathcal{G}$-invariant. Thus, it remains to check that $Z(\cdot)=\Psi\circ U(\cdot)$ (a.s., $\mathcal{W}$), where $U(\cdot)$ satisfies (3.6). To this end, observe that, since $W|_G$ is a tangent field to G for all $W\in\hat{\mathcal{G}}$, $Z(\cdot)\in C([0,\infty);G)$ (a.s., $\mathcal{W}$). Thus, if $U(\cdot)\equiv\pi\circ Z(\cdot)$, then $Z(\cdot)=\Psi\circ U(\cdot)$ (a.s., $\mathcal{W}$) and, (3.6) follows immediately from (3.4) with $z=0$.

B. Some elementary properties of control metrics

Let $\{Y_1,\cdots,Y_d\}\subset C^\infty(\boldsymbol{R}^N;\boldsymbol{R}^N)$ be a set of vector fields with the properties

i) $\max_{1\leq k\leq d}\max_{1\leq j\leq N}\left\|\frac{\partial}{\partial y^j}Y_k\right\|_{C_b(\boldsymbol{R}^N,\boldsymbol{R}^N)}<\infty$

ii) $\mathrm{Lie}(Y_1,\cdots,Y_d)(y)=\boldsymbol{R}^N$, $y\in\boldsymbol{R}^N$.

Given $y\in\boldsymbol{R}^N$ and $h\in H$, define $Y(\cdot,y;h)\in C([0,\infty);\boldsymbol{R}^N)$ by

$$Y(T,y;h)=y+\sum_{k=1}^{d}\int_0^T Y_k(Y(t,y;h))\dot{h}_k(t)dt,\qquad T\geqq 0.$$

The following lemma summarizes some facts from basic control theory.

(B.1) LEMMA. *For all $y,y'\in\boldsymbol{R}^N$ there is an $h\in H$ such that $Y(1,y;h)=y'$. Moreover, for each $y\in\boldsymbol{R}^N$ and $\delta>0$, $\{Y(1,y;h);\|h\|_H<\delta\}$ is a neighborhood of y.*

For $h\in H$ and $0\leqq s<t<\infty$, let $h_{[s,t]}\in H$ be defined by $\dot{h}_{[s,t]}=\chi_{[s,t]}\dot{h}$. Given $y,y'\in\boldsymbol{R}^N$, define

$$d(y,y')=\inf\{\|h_{[0,1]}\|_H;\ Y(1,y;h)=y'\}.$$

(B.2) LEMMA. *d is a metric on $\boldsymbol{R}^N$ and the topology determined by d is the same as the Euclidean topology. Moreover, for $y,y'\in\boldsymbol{R}^N$ there exists an $h\in H$ such that $Y(1,y,h)=y'$ and*

$$d(Y(t,y;h),\ Y(s,y;h))=(t-s)d(y,y'),\qquad 0\leqq s<t\leqq 1.$$

PROOF. The only assertions in the first part which requires comment is the triangle inequality for d. To prove the triangle inequality as well as the second part, we note that for any $T>0$:

$$d(y, y')=T^{1/2}\inf\{\|h_{[0,T]}\|_H;\ Y(T, y; h)=y'\}$$

and that the infimum on the right is achieved.

Now let y, y' and y'' be distinct elements of $\boldsymbol{R}^N$. Set $s=d(y, y')$, $t=d(y', y'')$ and choose $h', h''\in H$ so that $Y(s, y; h')=y'$, $Y(t, y'; h'')=y''$, $\|h'_{[0,s]}\|_H=d(y, y')^{1/2}$, and $\|h''_{[0,t]}\|_H=d(y', y'')^{1/2}$. Define $h\in H$ so that

$$\dot{h}(\tau)=\dot{h}'_{[0,s]}(\tau)+\chi_{[s,s+t]}(\tau)\dot{h}''(\tau-s), \qquad \tau\geqq 0.$$

Clearly $Y(t+s, y; h)=y''$, and so

$$\begin{aligned} d(y, y'')&\leqq(t+s)^{1/2}\|h_{[0,t+s]}\|_H \\ &=(t+s)^{1/2}(\|h'_{[0,s]}\|_H^2+\|h''_{[0,t]}\|_H^2) \\ &=d(y, y')+d(y', y''). \end{aligned}$$

That is, d satisfies the triangle inequality.

To prove the second part, let y and y' be distinct elements of $\boldsymbol{R}^N$ and choose $h\in H$ so that $Y(1, y; h)=y'$ and $\|h_{[0,1]}\|_H=d(y, y')$. Set $y_t=Y(t, y; h)$. Given $0\leqq s<t\leqq 1$, it is clear that $(t-s)^{1/2}\|h_{[s,t]}\|_H\geqq d(y_s, y_t)$. On the other hand, if $(t-s)^{1/2}\|h_{[s,t]}\|_H>d(y_s, y_t)$, then we could find an $h'\in H$ such that $\dot{h}'|_{[0,1]\setminus[s,t]}=\dot{h}|_{[0,1]\setminus[s,t]}$, $\|h'_{[s,t]}\|_H<\|h_{[s,t]}\|_H$ and $Y(1, y; h')=y'$. But this would mean that $d(y, y')\leqq\|h'_{[0,1]}\|_H<\|h_{[0,1]}\|_H=d(y, y')$. In other words,

$$d(y_s, y_t)=(t-s)^{1/2}\|h_{[s,t]}\|_H, \quad 0\leqq s<t\leqq 1. \tag{B.3}$$

In particular, by the triangle inequality:

$$(t-s)^{1/2}\|h_{[s,t]}\|_H=d(y_s, y_t)\geqq d(y_t, y)-d(y_s, y)=t^{1/2}\|h_{[0,t]}\|_H-s^{1/2}\|h_{[0,s]}\|_H.$$

Since $\|h_{[s,t]}\|_H^2=\|h_{[0,t]}\|_H^2-\|h_{[0,s]}\|_H^2$, it follows easily from this that

$$s^{1/2}\|h_{[0,t]}\|_H=t^{1/2}\|h_{[0,s]}\|_H.$$

and so

$$s^{1/2}d(y, y')=\|h_{[0,s]}\|_H.$$

Plugging this into (B.3), one gets the required result. Q.E.D.

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