J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
34 (1987), 443-484.

Free energy and the convergence of distributions of diffusion processes of McKean type

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(Communicated by S. Kusuoka)

§1. Introduction.

In this paper, we investigate the convergence of the probability distribution $p(t)$ of a diffusion process of McKean type at time t to an invariant probability measure as t goes to ∞ by using the free energy function. The process we consider is given by the following stochastic differential equation of McKean type on $\boldsymbol{R}^d$:

$$(1.1)\qquad \begin{cases} dX(t)=dB(t)-\mathrm{grad}\,\Phi_1(X(t))dt+\mathrm{grad}\,\Phi_2[X(t),p(t)]dt, \\ p(t) \text{ is the probability distribution of } X(t), \\ \text{the initial distribution is } p_0, \end{cases}$$

where $\Phi_2[x,p]=\int_{\boldsymbol{R}^d}\Phi_2(x,y)p(dy)$ for any probability measure p on $\boldsymbol{R}^d$, $\{B(t);t\geqq 0\}$ is a standard Brownian motion. We assume that the potentials Φ_1 and Φ_2 satisfy the following:

(i) $\Phi_1\in C^\infty(\boldsymbol{R}^d)$ and there exist positive constants α, A and r such that $\Phi_1(x)=A|x|^\alpha$ for $|x|>r$.

(ii) $\Phi_2\in \mathcal{S}(\boldsymbol{R}^{2d})$ and $\Phi_2(x,y)=\Phi_2(y,x)$.

The probability distribution $p(t)$ of the solution $X(t)$ to SDE (1.1) is a weak solution of the following non-linear paraboric partial differential equation on $\boldsymbol{R}^d$.

$$(1.2)\qquad \begin{cases} \dfrac{\partial}{\partial t}p(t,x)=\dfrac{1}{2}\Delta_x p(t,x)+\mathrm{div}_x(p(t,x)\,\mathrm{grad}_x(\Phi_1(x)-\Phi_2[x,p(t)])) \\ \lim\limits_{t\to 0}p(t,x)dx=p_0 \qquad \text{(weakly)}, \end{cases}$$

*) This research was partially supported by Grant-in-Aid for Science Research (No. 60740123), Ministry of Education.

where p_0 is a given probability measure on $\boldsymbol{R}^d$.

In what follows the probability distribution $p(t)$ of the solution $X(t)$ of SDE (1.1) is denoted by $T_t p_0$.

This type to SDE was introduced by McKean in [9] where he proved propagation of chaos in the following sense. Consider the Markovian n-particle system with mean field interaction described by

$$(1.3)_n \quad \begin{cases} dX^{(n,i)}(t) = dB^{(n,i)}(t) - \operatorname{grad} \Phi_1(X^{(n,i)}(t))dt \\ \qquad\qquad + \dfrac{1}{n}\sum_{j=1}^{n} \operatorname{grad}_1 \Phi_2(X^{(n,i)}(t), X^{(n,j)}(t))dt, \quad (i=1, \cdots, n), \\ \text{the distribution of } (X^{(n,1)}(0), \cdots, X^{(n,n)}(0)) \text{ is } p_0^{\otimes n}. \end{cases}$$

Then for each fixed m, $(X^{(n,1)}(t), \cdots, X^{(n,m)}(t))$ converges in law to the m-tuple of independent copies of the solution to (1.1) as $n \to \infty$.

We denote by $\mathcal{P}(\boldsymbol{R}^d)$ the topological space of all probability measures on $\boldsymbol{R}^d$ with the usual weak topology. We also denote by $\mathcal{P}^{(m)}(\boldsymbol{R}^d)$ $(m>0)$ the subspace of $\mathcal{P}(\boldsymbol{R}^d)$ such that $\int_{\boldsymbol{R}^d} |x|^m p(dx) < \infty$ with the corresponding weak topology, i.e., $p_n \to p$ in $\mathcal{P}^{(m)}(\boldsymbol{R}^d)$ if and only if $\int_{\boldsymbol{R}^d} f(x)p_n(dx) \to \int_{\boldsymbol{R}^d} f(x)p(dx)$ for any continuous function f on $\boldsymbol{R}^d$ such that $\sup_x |f(x)|/(1+|x|^m) < \infty$.

Let us define the free energy function F on $\mathcal{P}(\boldsymbol{R}^d)$.

$$F(p) = \begin{cases} \displaystyle\int_{\boldsymbol{R}^d} (\log p(x) + 2\Phi_1(x))p(x)dx - \int_{\boldsymbol{R}^{2d}} \Phi_2(x, y)p(x)p(y)dxdy \\ \qquad\qquad \text{if } p \text{ has the density } p(x) \\ \infty, \qquad \text{otherwise.} \end{cases}$$

Then the following is our first result.

THEOREM 1.1. *Let q be an invariant probability measure of* SDE (1.1) *and suppose that there exists an open neighborhood U of q in $\mathcal{P}(\boldsymbol{R}^d)$ such that* (i) *q is the unique invariant probability measure in U and* (ii) *$F(p) \geqq F(q)$ for all $p \in U$. Then we have the following.* (1) *If $0 < \alpha \leqq 2$, then there exists an open neighborhood V of q in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ such that for any $p_0 \in V$ $T_t p_0 \to q$ in $\mathcal{P}(\boldsymbol{R}^d)$ as $t \to \infty$.* (2) *If $\alpha > 2$, then there exists an open neighborhood V of q in $\mathcal{P}(\boldsymbol{R}^d)$ such that for any $p_0 \in V$ $T_t p_0 \to q$ in $\mathcal{P}(\boldsymbol{R}^d)$ as $t \to \infty$.*

Next we consider the speed of the convergence with respect to L^1-norm and L^∞-norm. To do this we need the assumption that $\alpha \geqq 2$.

For any invariant probability measure q we introduce a symmetric compact linear operator K_q on $L_0^2(q^{-1}dx)$ by

$$K_q h(x)=2q(x)\int_{R^d}\Phi_2(x, y)h(y)dy-2q(x)\int_{R^{2d}}\Phi_2(z, y)q(z)h(y)dzdy \tag{1.4}$$

where $L_0^2(q^{-1}dx)=\{f\in L^2(q^{-1}dx);\ \int_{R^d}f(x)dx=0\}$. Formally, $I-K_q=D^2F(q)$ on $L_0^2(q^{-1}dx)$.

The following are our main results.

THEOREM 1.2. *Let q be an invariant probability measure which satisfies the following condition.*

(Cond. 1) $((I-K_q)h, h)_{L_0^2(q^{-1}dx)}\neq 0$ *for any non vanishing* $h\in L_0^2(q^{-1}dx)$.

(i) *Let $\alpha=2$. Then, there exists a positive constant λ with the following property: For any $p_0\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ with $T_tp_0\to q$ (weakly) there exists a positive constant M such that*

$$\|T_tp_0-q\|_{L^1(dx)}\leqq Me^{-\lambda t}$$

for any $t>0$.

(ii) *Let $\alpha>2$. Then, there exists a positive constant λ with the following property: For any $p_0\in\mathcal{P}(\boldsymbol{R}^d)$ with $T_tp_0\to q$ (weakly) there exists a positive constant M such that*

$$\left\|\frac{T_tp_0}{q}-1\right\|_{L^\infty}\leqq Me^{-\lambda t}$$

for any $t>1$.

REMARK. In Theorem 1.2, we need not to assume that $(I-K_q)$ is definite.

THEOREM 1.3. *Let q be an invariant probability measure which satisfies the following condition.*

(Cond. 2) *There exists an open neighborhood U of q in $\mathcal{P}(\boldsymbol{R}^d)$ such that*
(i) $F(p)\geqq F(q)$ *for any* $p\in U$ *and*
(ii) $((I-K_q)h, h)_{L_0^2(q^{-1}dx)}>0$ *for any non vanishing* $h\in L_0^2(q^{-1}dx)$.

(i) *Let $\alpha=2$. Then, there exist positive constants λ and M and an open neighborhood V of q in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ such that*

$$\|T_t p_0 - q\|_{L^1(dx)} \leqq M e^{-\lambda t}$$

for any $p_0 \in V$ *and any* $t>0$.
(ii) *Let* $\alpha>2$. *Then, there exist positive constants* λ *and* M' *and an open neighborhood* V' *of* q *in* $\mathcal{P}(\boldsymbol{R}^d)$ *such that*

$$\left\| \frac{T_t p_0}{q} - 1 \right\|_{L^\infty} \leqq M' e^{-\lambda t}$$

for any $p_0 \in V'$ *and* $t>1$.

We can take $1/(\gamma_3\gamma_4)$ as the exponent λ, where γ_3 is determined essentially by the minimum of $|1-\kappa_q|$ over all eigenvalues κ_q of K_q and γ_4 is determined essentially by the supremum of non vanishing maximum eigenvalues of $\tilde{L}_{e(p)}$ in $L_0^2(e(p)^{-1}dx)$ over all $p \in \mathcal{P}(\boldsymbol{R}^d)$, where $L_{e(p)} = \frac{1}{2}\Delta - \operatorname{grad}(\Phi_1(x) - \operatorname{grad}\Phi_2[x, p]) \cdot \operatorname{grad}$ and $\tilde{L}_{e(p)}$ is its Friedrichs' extension.

To prove the theorems in the case where $\alpha>2$, we use the ultra-contractive property introduced by Davies and Simon [2].

In [3] Davies and Simon showed that the spectrum of the Ornschtein-Uhlenbeck process on $\boldsymbol{R}$ with generator $L_0^* p = \frac{1}{2}\frac{d^2}{dx^2}p + \frac{d}{dx}(p\cdot x)$ $\left(\text{i.e., } \alpha=2,\ A=\frac{1}{2}\right)$ in $L^1(\boldsymbol{R}; dx)$ is equal to $\{z \in \boldsymbol{C}; \operatorname{Re}(z) \geqq 0\}$ and every $z \in \boldsymbol{C}$ with $\operatorname{Re}(z)>0$ is an eigenvalue of multiplicity 2. On the other hand, in the case where $\alpha=2$, $A=\frac{1}{2}$ and $\Phi_2=0$, our Theorem 1.3 (i) says that there exists a positive constant C such that for any $p \in \mathcal{P}^{(2)}(\boldsymbol{R})$ $\|T_t p - q_0\|_{L^1(\boldsymbol{R}; dx)} \leqq C\left(1+\int_{\boldsymbol{R}} |y|^2 p(dy)\right)^{1/2} e^{-(1/2)t}$ (Proposition B. 1 in Appendix B). So, we think the theorem is not trivial even in the linear case. We shall show that Theorems 1.2 and 1.3 can not be true in the case where $d=1$, $1<\alpha<2$ and $\Phi_2=0$ (Proposition B. 2 in Appendix B).

In § 2, we shall show the existence and the uniqueness of the solution of SDE (1.1) and the continuity of non-linear semigroup $\{T_t; t \geqq 0\}$ defined on $\mathcal{P}(\boldsymbol{R}^d)$. Then, we shall introduce the free energy function F and the function I following Donsker and Varadhan [4] (§ 3) and show the fundamental relation between F and I along the distribution of the solution of SDE (1.1) (Theorem 4.2). The proof of Theorem 1.1 will be given in § 5. In § 6 we shall show the exponential decay of free energy $F(T_t p_0) - F(q)$ in the case where $\alpha \geqq 2$ by using hypercontractivity. In § 7, we shall give the proof of Theorems 1.2 and 1.3. In Appendix A, we shall give some estimates for fundamental solutions of linear parabolic equations.

Acknowledgement.

The author expresses his gratitude to Professor S. Kusuoka who suggested the problem discussed in this paper and gave valuable advice. The author also expresses his gratitude to Professor H. Tanaka and Professor T. Ueno for their helpful remarks and encouragement.

§ 2. On the solution to SDE (1.1).

First we show the existence and uniqueness of the solution of SDE (1.1).

THEOREM 2.1. *There exists a solution to* SDE (1.1) *for any initial probability distribution* p_0 *on* $\boldsymbol{R}^d$, *and the pathwise uniqueness of solutions holds.*

PROOF (Existence). In the case where $0<\alpha\leqq 2$, this theorem is known (Funaki [6]). Then let assume $\alpha>2$. First we consider the case where the initial distribution $p_0\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$. The following argument is due to Tanemura [12]. Let us consider the following temporally homogeneous Markovian process

$$dY(t)=dB(t)-\mathrm{grad}\,\Phi_1(Y(t))dt. \tag{2.1}$$

Note that $x\cdot\mathrm{grad}\,|x|^\alpha\geqq 0$. Then by Ito's formula,

$$|Y(t)|^2=|Y(0)|^2+2\int_0^t Y(s)\,dB(s)-2\int_0^t Y(s)\cdot\mathrm{grad}\,\Phi_1(Y(s))ds+dt,$$

and hence, there exists a positive constant K such that

$$E[|Y(t)|^2]\leqq E[|Y(0)|^2]+2\int_0^t E[|Y(s)|^2]ds+(2K+d)t.$$

Then by Gronwall's inequality, for any $t>0$, $|Y(t)|<\infty$ with probability 1, and hence there exists a solution $Y(\cdot)$ with any initial distribution $p_0\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$. Here we consist a solution to SDE (1.1) by successive approximation as follows.

$$\begin{cases} X_0(t)=X(0) \\ X_{n+1}(t)=X(0)+B(t)-\int_0^t \operatorname{grad}\Phi_1(X_{n+1}(s))ds+\int_0^t \operatorname{grad}\Phi_2[X_{n+1}(s),\ p_n(s)]ds \end{cases}$$

where $p_n(t)$ is the probability distribution of $X_n(t)$.

Note that $(x-y)\cdot(\operatorname{grad}|x|^{\alpha}-\operatorname{grad}|y|^{\alpha})\geqq 0$ and $|\operatorname{grad}_x\Phi_2(x,y)-\operatorname{grad}_x\Phi_2(x',y')|^2\leqq K_2^2(|x-x'|^2+|y-y'|^2)$ for some $K_2>0$. Then, there exist positive constants K_1 and K_2 such that for any $x, y\in \boldsymbol{R}^d$ and $p, q\in \mathcal{P}(\boldsymbol{R}^d)$,

$$(x-y)\cdot(\operatorname{grad}\Phi_1(x)-\operatorname{grad}\Phi_1(y))\geqq -K_1|x-y|^2$$

and

$$|(x-y)\cdot(\operatorname{grad}\Phi_2[x,p]-\operatorname{grad}\Phi_2[y,q]|\leqq K_2\left\{|x-y|^2+\int_0^t|z_1-z_2|^2\mu_{p,q}(dz_1,dz_2)\right\}$$

where $\mu_{p,q}$ is any probability measure on $\boldsymbol{R}^{2d}$ that satisfies $\mu_{p,q}(dz_1,\boldsymbol{R}^d)=p(dz_1)$ and $\mu_{p,q}(\boldsymbol{R}^d, dz_2)=q(dz_2)$.

Then by Ito's formula,

$$\begin{aligned} &|X_{n+1}(t)-X_n(t)|^2 \\ &=-2\int_0^t(X_{n+1}(s)-X_n(s))\cdot(\operatorname{grad}\Phi_1(X_{n+1}(s))-\operatorname{grad}\Phi_1(X_n(s)))ds \\ &\quad+2\int_0^t(X_{n+1}(s)-X_n(s))(\operatorname{grad}\Phi_2[X_{n+1},\ p_n(s)]-\operatorname{grad}\Phi_2[X_n(s),\ p_{n-1}(s)])ds \\ &\leqq 2K_1\int_0^t|X_{n+1}(s)-X_n(s)|^2ds \\ &\quad+2K_2\int_0^t\left\{|X_{n+1}(s)-X_n(s)|^2+\int_{R^{2d}}|z_1-z_2|^2\mu_{p_n,p_{n-1}}(dz_1,\ dz_2)\right\} \\ &\leqq(2K_1+2K_2)\int_0^t|X_{n+1}(s)-X_n(s)|^2ds+2K_2\int_0^t E[|X_n(s)-X_{n-1}(s)|^2]ds. \end{aligned}$$

Then by Gronwall's inequality, we get

$$h_n(t)\leqq \exp((2K_1+2K_2)t)\cdot 2K_2\int_0^t h_{n-1}(s)ds,$$

where

$$h_n(t)=E\Big[\sup_{0\leqq s\leqq t}|X_{n+1}(s)-X_n(s)|^2\Big].$$

Then, we get

$$h_n(t)\leqq h_0(T)(2K_2\exp((2K_1+2K_2)T))^n\cdot\frac{t^n}{n!}$$

for any $t\in(0,T)$ and $n\in N$. Using Borel-Cantelli's lemma, we get a solution $X(\cdot)$ to SDE (1.1) with any initial distribution $p_0\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$. Next, let $p_0\in\mathcal{P}(\boldsymbol{R}^d)$. Clearly, for any $p_0\in\mathcal{P}(\boldsymbol{R}^d)$ there exists a sequence $\{p_0^{(n)}\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)\,;\,n\in N\}$ such that $p_0^{(n)}\to p_0$ weakly in $\mathcal{P}(\boldsymbol{R}^d)$. Let $P^{(n)}$ be the probability distribution of the solution $X(\cdot)$ to SDE (1.1) with initial distribution $p_0^{(n)}$. For any initial distribution $p_0\in\mathcal{P}(\boldsymbol{R}^d)$, there exists a unique solution to SDE (2.1) by Cameron-Martin-Maruyama-Girsanov's formula (cf. Theorem A.1 in Appendix). Let $Q^{(n)}$ and Q be the probability distributions of the solutions $Y(\cdot)$ to (2.1) with initial distribution $p_0^{(n)}$ and p_0, respectively. If $p_0^{(n)}\to p_0$ weakly in $\mathcal{P}(\boldsymbol{R}^d)$, then $Q^{(n)}\to Q$ weakly in $\mathcal{P}(W^d)$, where $W^d=C([0,\infty)\to\boldsymbol{R}^d)$ with the filtration $\{\mathcal{F}_t\,;\,t\geqq 0\}$. On the other hand, by Cameron-Martin-Maruyama-Girsanov's formula, we get

$$\text{(2.2)}\qquad \frac{dP^{(n)}}{dQ^{(n)}}\bigg|_{\mathcal{F}_t}(w)=\exp(\Phi_2[w(t),p^{(n)}(t)]-\Phi_2[w(0),p_0^{(n)}])$$
$$\times\exp\Big(\int_0^t ds\int_{W^d\otimes W^d}P^{(n)}(dw')P^{(n)}(dw'')\Psi(w(s),w'(s),w''(s))\Big),$$

where $p^{(n)}(t)=P^{(n)}\cdot\pi_t^{-1}$, π_t is the projection on W^d (i.e., $w\mapsto w(t)$) and

$$\text{(2.3)}\quad \Psi(x,x',x'')=\mathrm{grad}_1\,\Phi_2(x,x')\cdot\mathrm{grad}\,\Phi_1(x)-\frac{1}{2}\Delta_1\Phi_2(x,x')$$
$$-\frac{1}{2}\,\mathrm{grad}_1\,\Phi_2(x,x')\cdot\mathrm{grad}_1\,\Phi_2(x,x'')-\Big\{\frac{1}{2}\Delta_2\Phi_2(x,x')$$
$$-\mathrm{grad}_2\,\Phi_2(x,x')\cdot\mathrm{grad}\,\Phi_1(x')+\mathrm{grad}_2\,\Phi_2(x,x')\cdot\mathrm{grad}_1\,\Phi_2(x',x'')\Big\}.$$

By the assumptions on Φ_1 and Φ_2, $\Psi\in\mathcal{S}(\boldsymbol{R}^{3d})$. Then, there exists a constant $C<\infty$ such that $e^{-Cu}Q^{(n)}|_{\mathcal{F}_u}\leqq P^{(n)}|_{\mathcal{F}_u}\leqq e^{Cu}Q^{(n)}|_{\mathcal{F}_u}$ for any $t\in[0,u]$. Therefore, $\{P^{(n)}\,;\,n\in N\}$ is tight on $\mathcal{P}(W^d)$.

Let P' be a limit point of a subsequence $\{P^{(n')}\,;\,n\in N\}$. Since $P^{(n')}$ is the probability distribution of a solution of SDE (1.1), for any $f\in C_0^\infty(\boldsymbol{R}^d)$

$$f(w(t))-f(w(0))-\int_0^t L_{p^{(n')}(s)}f(w(s))ds$$

is a $(P^{(n')},\mathcal{F}_t)$-martingale, where

$$L_pf(x)=\frac{1}{2}\Delta f(x)-\mathrm{grad}\,\Phi_1(x)\cdot\mathrm{grad}f(x)+\mathrm{grad}\int_{R^d}\Phi_2(x,y)p(dy)\cdot\mathrm{grad}f(x)$$

for any $p\in\mathcal{P}(\boldsymbol{R}^d)$.

Since $p_0^{(n')}\to p_0$ weakly in $\mathcal{P}(\boldsymbol{R}^d)$, $P^{(n')}\otimes P^{(n')}\to P'\otimes P'$ weakly in

$\mathcal{P}(W^d \otimes W^d)$. On the other hand,

$$\int_0^t \operatorname{grad} \int_{R^d} \Phi_2(w(s), y) p^{(n')}(s)(dy) \cdot \operatorname{grad} f(w(s)) ds$$
$$= \int_{W^d} P^{(n')}(dw') \int_0^t \operatorname{grad}_1 \Phi_2(w(s), w'(s)) \cdot \operatorname{grad} f(w(s)) ds$$

and $\int_0^t \operatorname{grad}_1 \Phi_2(w(s), w'(s)) \cdot \operatorname{grad} f(w(s)) ds$ is a continuous function on $W^d \otimes W^d$. Thus, P' is the probability distribution of a solution to SDE (1.1) with initial distribution p_0.

(Uniqueness.) Following Shiga and Tanaka [10], we can prove the uniqueness of the solution to (1.1) in law sense with any initial probability distribution by using the propagation of chaos. The outline of the proof is as follows. Let P be the probability distribution of a solution to SDE (1.1) with initial distribution $p_0 \in \mathcal{P}(\boldsymbol{R}^d)$, and let P_n be the probability distribution of a solution to Markovian SDE $(1.3)_n$ with initial distribution $p_0^{\otimes n}$. Then by Cameron-Martin-Maruyama-Girsanov's formula,

$$\left.\frac{dP_n}{dP^{\otimes n}}\right|_{\mathcal{F}_t}(w_1, \cdots, w_n) = \exp\{H_n(w_1, \cdots, w_n)\},$$

where

$$\begin{aligned} &H_n(w_1, \cdots, w_n) \\ &= \exp\left\{ \sum_{i=1}^n \int_0^t \left\{ \frac{1}{n} \sum_{j=1}^n \operatorname{grad}_1 \Phi_2(w_i(s), w_j(s)) - \operatorname{grad} \Phi_2[w_i(s), p(s)] \right\} d\bar{w}_i(s) \right. \\ &\quad \left. - \frac{1}{2} \sum_{i=1}^n \int_0^t \left| \frac{1}{n} \sum_{j=1}^n \operatorname{grad}_1 \Phi_2(w_i(s), w_j(s)) - \operatorname{grad} \Phi_2[w_i(s), p(s)] \right|^2 ds \right\}, \end{aligned}$$

where

$$d\bar{w}(s) = dw(s) + \operatorname{grad} \Phi_1(w(s)) ds - \operatorname{grad} \Phi_2[w(s), p(s)] ds,$$

which is a Brownian motion under P. We set

$$a_2(w_1, w_2) = \int_0^t \{\operatorname{grad}_1 \Phi_2(w_1(s), w_2(s)) - \operatorname{grad} \Phi_2[w_1(s), p(s)]\} d\bar{w}(s)$$
$$\begin{aligned} a_3(w_1, w_2, w_3) = \int_0^t \{&(\operatorname{grad}_1 \Phi_2(w_1(s), w_2(s)) - \operatorname{grad} \Phi_2[w_1(s), p(s)]) \\ &\times (\operatorname{grad}_1 \Phi_2(w_1(s), w_3(s)) - \operatorname{grad} \Phi_2[w_1(s), p(s)])\} ds. \end{aligned}$$

Then,

$$\int_{W^d} a_2(w_1, w') P(dw') = \int_{W^d} a_2(w', w_2) P(dw') = \int_{W^d} a_2(w', w') P(dw') = 0,$$

$$\int_{W^d} a_3(w_1, w', w_3) P(dw') = \int_{W^d} a_3(w_1, w_2, w') P(dw') = 0.$$

Using the method of symmetric statistics, we see that

$$H_n \longrightarrow I_2(b_2 - b_2') - \frac{1}{2}\int_{W^d}\left(\int_{W^d} a_3(w', w'', w'') P(dw')\right) P(dw'')$$

in law as $n\to\infty$, where I_2 is the second order Wiener integral, $b_2(w_1, w_2)$ is the symmetrization of $a_2(w_1, w_2)$ and $b_2'(w_1, w_2)$ is the symmetrization of $\int_{W^d} a_3(w', w_1, w_2) P(dw')$. Therefore, we get

$$H_n \longrightarrow \frac{1}{2} I_2(b) - \frac{1}{2}\,\mathrm{trace}(A^*A)$$

in law as $n\to\infty$, where

$$b(w, w') = a(w, w') + a(w', w) - \int_{W^d \otimes W^d} a(w, w')^2 P(dw) P(dw')$$

$$Af(w) = \int_{W^d} a(w, w') f(w') P(dw').$$

Furthermore, we can show

$$E'\left[\exp\left(\frac{1}{2} I_2(b)\right)\right] = \exp\left(\frac{1}{2}\,\mathrm{trace}\, A^*A\right).$$

Then we can prove the law of large numbers as follows. Notice that $\exp(H_n(w_1, \cdots, w_n)) \geqq 0$ and $E^{\otimes\infty}[\exp(H_n(w_1, \cdots, w_n))] = 1$ for any n. Then, for any bounded measurable function f on W^d,

$$\begin{aligned}
&\lim_{n\to\infty} E\left[\exp\left(\sqrt{-1}\left\langle \frac{1}{n}\sum_{i=1}^{n} \delta_{X^{(n,i)}}, f\right\rangle\right)\right] \\
&= \lim_{n\to\infty} E^{\otimes\infty}\left[\exp\left(\sqrt{-1}\frac{1}{n}\sum_{i=1}^{n} f(w_i)\right)\exp(H_n(w_1, \cdots, w_n))\right] \\
&= E'\left[\exp(\sqrt{-1}\langle P, f\rangle)\exp\left(\frac{1}{2} I_2(b) - \frac{1}{2}\,\mathrm{trace}\, A^*A\right)\right] = \exp(\sqrt{-1}\langle P, f\rangle).
\end{aligned}$$

On the other hand, we know the uniqueness in law of the solution to SDE $(1.3)_n$ with initial distribution $p_0^{\otimes n}$. Then the uniqueness of the solution of SDE (1.1) with initial distribution p_0 follows immediately from the law of large numbers. For the probability distribution $p(t)$ of the solution $X(t)$, all the coefficients of SDE (1.1) are locally Lipschitz continuous. Then, the pathwise uniqueness of solutions holds. (Q.E.D.)

By Weyl's lemma, the probability distribution $p(t, dx; p_0)$ of the solution $X(t)$ to SDE (1.1) has a smooth density $p(t, x; p_0)$. We denote by $T_t p_0$ and $T_t p_0(x)$ the probability distribution of the solution $X(t)$ to SDE (1.1) and its density function in x, respectively. Then, by Theorem 2.1, $\{T_t; t \geqq 0\}$ is a non-linear semigroup on $\mathcal{P}(\boldsymbol{R}^d)$.

Next we consider the continuity of $T_t p_0$. As in the proof of Theorem 2.1, we can easily prove the following theorem.

THEOREM 2.2. *Let $P^{(n)}$ and P be the probability distributions of the solutions of* SDE (1.1) *on W^d with initial distributions $p_0^{(n)}$ and p_0, respectively. If $p_0^{(n)} \to p_0$ weakly in $\mathcal{P}(\boldsymbol{R}^d)$, then $P^{(n)} \to P$ weakly in $\mathcal{P}(W^d)$.*

THEOREM 2.3. *The mapping $T_t p_0$ on $[0, \infty) \times \mathcal{P}(\boldsymbol{R}^d)$ to $\mathcal{P}(\boldsymbol{R}^d)$ is continuous in $(t, p_0) \in [0, \infty) \times \mathcal{P}(\boldsymbol{R}^d)$.*

PROOF. Let $\{p_0^{(n)}; n \in N\}$ be a sequence of probability distributions on $\boldsymbol{R}^d$ convergent to p_0 weakly and let $\{t_n; n \in N\}$ be a sequence of times convergent to $t \in [0, \infty)$. By Theorem 2.2 and Skorohod's realization theorem of almost convergence, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and W^d-valued random variables $\{\tilde{X}^{(n)}; n \in N\}$ and $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that (i) the probability distributions of $\tilde{X}^{(n)}$ and $\tilde{X}$ under $\tilde{P}$ are $P^{(n)}$ and P $(n=1, 2, \cdots)$, respectively and (ii) $\tilde{X}^{(n)} \to \tilde{X}$ $\tilde{P}$-a.s., where $P^{(n)}$ and P are the same as in Theorem 2.2. For any bounded and uniformly continuous function f on $\boldsymbol{R}^d$,

$$
\begin{aligned}
&\left| \int_{\boldsymbol{R}^d} f(x) T_{t_n} p_0^{(n)}(dx) - \int_{\boldsymbol{R}^d} f(x) T_t p_0(dx) \right| \\
&= |\tilde{E}(f(\tilde{X}^{(n)}(t_n))) - \tilde{E}(f(\tilde{X}(t)))| \\
&\leqq \tilde{E}(|f(\tilde{X}^{(n)}(t_n)) - f(\tilde{X}(t_n))|) + \tilde{E}(|f(\tilde{X}(t_n)) - f(\tilde{X}(t))|) \longrightarrow 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

(Q.E.D.)

§ 3. The definitions and basic properties of F and I.

We define the free energy function F on $\mathcal{P}(\boldsymbol{R}^d)$ following Donsker and Varadhan [4].

DEFINITION 3.1. For $p \in \mathcal{P}(\boldsymbol{R}^d)$,

$$(3.1)\quad F(p)=\begin{cases}\int_{R^d}(\log p(x)+2\Phi_1(x))p(x)dx-\int_{R^{2d}}\Phi_2(x,y)p(x)p(y)dxdy, & \\ \qquad\text{if } p \text{ has the density } p(x) \text{ and} & \\ \qquad|\log p(x)+2\Phi_1(x)|\in L^1(\boldsymbol{R}^d;p(x)dx), & \\ \infty \qquad \text{otherwise.} & \end{cases}$$

THEOREM 3.1. (i) *The function* $F:\mathcal{P}(\boldsymbol{R}^d)\to\boldsymbol{R}\cup\{\infty\}$ *is lower semi-continuous.*

(ii) *Let* $K_r=\{p\in\mathcal{P}(\boldsymbol{R}^d); F(p)\leqq r\}$ *for any* $r\in\boldsymbol{R}$. *Then* K_r *is a compact set in* $\mathcal{P}(\boldsymbol{R}^d)$.

(iii) *If* F *attains its local minimum at* $q\in\mathcal{P}(\boldsymbol{R}^d)$, *then* q *is an invariant probability measure of McKean process defined by* (1.1).

PROOF. Since $\int_{R^{2d}}\Phi_2(x,y)p(dx)p(dy)$ is a bounded continuous function on $\mathcal{P}(\boldsymbol{R}^d)$, (i) and (ii) follow immediately from the results of Donsker and Varadhan [4]. (iii) has been shown in [11]. (Q.E.D.)

For a probability measure $p\in\mathcal{P}(\boldsymbol{R}^d)$, we set

$$(3.2)\qquad e(p)(x)=Z_p^{-1}\exp\Big(-2\Phi_1(x)+2\int_{R^d}\Phi_2(x,y)p(dy)\Big),$$

where

$$Z_p=\int_{R^d}\exp\Big(-2\Phi_1(x)+2\int_{R^d}\Phi_2(x,y)p(dy)\Big)dx.$$

Let $L_{e(p)}$ be an elliptic second order partial differential operator defined by

$$(3.3)\qquad L_{e(p)}f(x)=\frac{1}{2}e(p)(x)^{-1}\operatorname{div}(e(p)(x)\operatorname{grad}f(x))$$
$$=\frac{1}{2}\Delta f(x)-\operatorname{grad}\Big(\Phi_1(x)-\int_{R^d}\Phi_2(x,y)p(dy)\Big)\cdot\operatorname{grad}f(x),$$

for $f\in C_0^\infty(\boldsymbol{R}^d)$. Then $L_{e(p)}$ is a negative symmetric operator on $L^2(\boldsymbol{R}^d;e(p)dx)$. Let $\tilde{L}_{e(p)}$ be the Friedrichs' extension of $L_{e(p)}$ in $L^2(\boldsymbol{R}^d;e(p)dx)$. Following Donsker and Varadhan, we define I function on $\mathcal{P}(\boldsymbol{R}^d)$.

DEFINITION 3.2. (i) For any fixed $q\in\mathcal{P}(\boldsymbol{R}^d)$, we define $I_{e(q)}:\mathcal{P}(\boldsymbol{R}^d)\to[0,\infty)\cup\{\infty\}$ by

$$(3.4)\qquad I_{e(q)}(p)=\begin{cases} \|(-\tilde{L}_{e(q)})^{1/2}\varphi\|^2_{L^2(e(q)dx)}, \\ \quad \text{if } p \text{ has the density } p(x) \text{ and } \varphi\in\mathcal{D}((-\tilde{L}_{e(q)})^{1/2}), \\ \quad \text{where } \varphi(x)=(p(x)/e(q)(x))^{1/2}, \\ \infty, \qquad \text{otherwise.} \end{cases}$$

(ii) We define the function $I:\mathcal{P}(\boldsymbol{R}^d)\to[0,\infty)\cup\{\infty\}$ by

$$(3.5)\qquad I(p)=I_{e(p)}(p).$$

THEOREM 3.2. (i) *The function* $I:\mathcal{P}(\boldsymbol{R}^d)\to[0,\infty)\cup\{\infty\}$ *is lower semicontinuous.*

(ii) $I(p)=0$ *if and only if* p *is an invariant probability measure of the McKean process* (1.1).

(iii) *If* $p\in\mathcal{P}(\boldsymbol{R}^d)$ *has the density* $p(x)$ *such that* $p(x)>0$ *(a.e.) and has* $\operatorname{grad} p(x)$ *in the distribution sense and the right hand side of the following* (3.6) *is finite, then*

$$(3.6)\qquad I(p)=\frac{1}{8}\int_{\boldsymbol{R}^d}\left\{\left|\operatorname{grad} p(x)+2p(x)\operatorname{grad}\left(\Phi_1(x)-\int_{\boldsymbol{R}^d}\Phi_2(x,y)p(y)dy\right)\right|^2\right\}/p(x)dx.$$

PROOF. (i) Let q be any probability measure on $\boldsymbol{R}^d$, and $L_{e(q)}$ be the elliptic operator given by (3.3). Let $Y(t,y)$ be the solution to the following SDE.

$$Y(t,y)=y+B(t)-\int_0^t \operatorname{grad}\Phi_1(Y(s,y))ds+\int_0^t \operatorname{grad}\Phi_2[Y(s,y),e(q)]ds$$

and let $p(t,y,x)$ be the transition density of the diffusion process $\{Y(t,y);\ t\geqq 0\}$. We define a semigroup $\{S_t;\ t\geqq 0\}$ on $B(\boldsymbol{R}^d)=\{f;$ a bounded measurable function on $\boldsymbol{R}^d\}$ by

$$(3.7)\qquad S_t f(y)=\int_{\boldsymbol{R}^d} p(t,y,x)f(x)dx.$$

We set

$\mathcal{C}^+=\{f\in C^\infty(\boldsymbol{R}^d)$; there exist positive constants c_1, c_2, c_3 and c_4 such that $c_1\leqq f(x)\leqq c_2$, $\left|\dfrac{\partial f}{\partial x_i}(x)\right|\leqq c_3$ $(1\leqq i\leqq d)$ and $|L_{e(q)}f(x)|\leqq c_4$ for any $x\in\boldsymbol{R}^d\}$.

Set $J_i^j(t)=J_i^j(t, y)=(\partial/\partial y_i)Y^j(t, y)$. Then $J_i^j(t)$ satisfies the following equation:

$$J_i^j(t)=\delta_i^j-\int_0^t \sum_{k=1}^d \frac{\partial^2}{\partial y_k \partial y_j}\Phi(Y(s, y))J_i^k(s)ds \qquad (1\leqq i, j\leqq d),$$

where $\Phi(y)=\Phi_1(y)-\Phi_2[y, q]$.

Set $J(t, y)=\sum_{i=1}^d \sum_{j=1}^d |J_i^j(t, y)|^2$. Then by the assumptions on Φ_1 and Φ_2, there exists a positive constant C such that

$$\begin{aligned} J(t, y) &= d-2\sum_{i=1}^d \int_0^t \sum_{j,k=1}^d J_i^j(s, y)\frac{\partial^2}{\partial y_k \partial y_j}\Phi(Y(s, y))J_i^k(s, y)ds \\ &\leqq d+2C\int_0^t J(s, y)ds. \end{aligned}$$

By Gronwall's inequality, there exists a positive constant $C=C(t_0)$ such that for any $t\leqq t_0$ and any $y\in \boldsymbol{R}^d$

$$0\leqq J(t, y)\leqq C.$$

Then, for $f\in \mathcal{C}^+$

$$\left|\frac{\partial}{\partial y_i}E[f(Y(t, y))]\right|\leqq\left(\sum_{k=1}^d \|\partial f/\partial y_k\|_\infty^2\right)^{1/2}J(t, y)^{1/2}<\infty.$$

By this and $L_{e(q)}S_t f(x)=S_t L_{e(q)}f(x)$, we see that $S_t\mathcal{C}^+\subset\mathcal{C}^+$ for any $t\geqq 0$ and any $q\in\mathcal{P}(\boldsymbol{R}^d)$.

Then, following Donsker and Varadhan [4], we get the following expression of the function $I_{e(q)}$.

$$I_{e(q)}(p)=-\inf_{f\in\mathcal{C}^+}\int_{\boldsymbol{R}^d}\frac{L_{e(q)}f(y)}{f(y)}p(dy) \tag{3.8}$$

for any p and $q\in\mathcal{P}(\boldsymbol{R}^d)$. Since $\Phi_2\in\mathcal{S}(\boldsymbol{R}^{2d})$, we get that if $p_n\to p$ weakly in $\mathcal{P}(\boldsymbol{R}^d)$ then $L_{e(p_n)}f(x)=\frac{1}{2}\Delta f(x)-\operatorname{grad}\Phi_1(x)\cdot\operatorname{grad}f(x)+\int_{\boldsymbol{R}^d}\operatorname{grad}\Phi_2(x, y)\times\operatorname{grad}f(x)dp_n(y)$ converges to $L_{e(p)}f(x)$ with respect to supremum-norm for any $f\in\mathcal{C}^+$. Therefore, the map $p\mapsto\int_{\boldsymbol{R}^d}\frac{L_{e(p)}f(x)}{f(x)}p(dx)$ from $p\in\mathcal{P}(\boldsymbol{R}^d)$ into $\boldsymbol{R}$ is continuous for each $f\in\mathcal{C}^+$. Then by (3.8) $I(p)$ is lower semi-continuous. In Donsker and Varadhan [4] it was proved that $I_{e(q)}(p)=0$

if and only if p is an invariant probability measure for $\{S_t; t\geqq 0\}$ (i.e., $pS_t=p$), and it is known that $\{S_t; t\geqq 0\}$ has the unique invariant probability measure $e(q)(x)dx$. On the other hand, p is an invariant probability measure of McKean process defined by (1.1) if and only if p has the density $p(x)$ that satisfies $p(x)=e(p)(x)$ ([11]), which implies (ii). (iii) follows from the following lemma. (Q.E.D.)

Denote by $H^1(\boldsymbol{R}^d; e(p)dx)$ the class of functions $f\in L^2(\boldsymbol{R}^d; e(p)dx)$ for which there exist $g_i\in L^2(\boldsymbol{R}^d; e(p)dx)$, $1\leqq i\leqq d$, such that $\int_{\boldsymbol{R}^d} f(x)\frac{\partial}{\partial x_i}\varphi(x)dx= -\int_{\boldsymbol{R}^d} g_i(x)\varphi(x)dx$ for any $\varphi\in C_0^\infty(\boldsymbol{R}^d)$. Then the following lemma is known (Prop. 4.4S in Davies and Simon [2]).

LEMMA 3.1. *$\mathcal{D}((-\tilde{L}_{e(p)})^{1/2})=H^1(\boldsymbol{R}^d; e(p)dx)$ and*

$$\|(-\tilde{L}_{e(p)})^{1/2}f\|^2_{L^2(e(p)dx)}=\frac{1}{2}\|\,|\mathrm{grad} f|\,\|^2_{L^2(e(p)dx)} \tag{3.9}$$

for any $f\in\mathcal{D}((-\tilde{L}_{e(p)})^{1/2})$.

§ 4. The behavior of the function F along $\{T_t\}$.

In this section we investigate the behavior of the free energy function F along the non-linear semigroup $\{T_t; t\geqq 0\}$ on $\mathcal{P}(\boldsymbol{R}^d)$ determined by the solution to SDE (1.1) and show the fundamental relation between F and I along the semigroup $\{T_t; t\geqq 0\}$ (Theorem 4.2).

First we prepare some lemmas on the continuity of the probability density $T_tp_0(x)$ of the solution $X(t)$ to SDE (1.1) in the initial distribution $p_0\in\mathcal{P}(\boldsymbol{R}^d)$. Let $q(t, y, x)$ be the transition probability density of the diffusion process governed by (2.1). Then, by Cameron-Martin-Maruyama-Girsanov's formula, we get

LEMMA 4.1. *$T_tp_0(x)$ has the following expression:*

$$T_tp_0(x)=\int_{\boldsymbol{R}^d} p_0(dy)q(t, y, x)r(t, y, x; p_0), \tag{4.1}$$

where

$$\begin{aligned} &r(t, y, x; p_0)\\ &=\exp\left(\int_{\boldsymbol{R}^d}\Phi_2(x, z)T_tp_0(z)dz-\int_{\boldsymbol{R}^d}\Phi_2(y, z)p_0(dz)\right)\end{aligned} \tag{4.2}$$

$$\times E_y^Y\left[\exp\left(\int_0^t ds\int_{R^{2d}}\Psi(Y(s),z',z'')T_sp_0(z')T_sp_0(z'')dz'dz''\right)\Big| Y(t)=x\right]$$

$$=\exp\left(\int_{W^d}(\Phi_2(x,w(t))-\Phi_2(y,w(o))P(dw))\right)$$

$$\times E_y^Y\left[\exp\left(\int_{W^d}P(dw')P(dw'')\int_0^t\Psi(Y(s),w'(s),w''(s))ds\Big| Y(t)=x\right)\right],$$

$\Psi(y,y',y'')\in\mathcal{S}(R^{3d})$ *is defined by* (2.3), $Y(\cdot)$ *is the solution of* (2.1) *starting at* y *and* P *is the probability distribution on* W^d *of the solution of* SDE (1.1) *with initial distribution* p_0.

REMARK. Since the frozen temporally inhomogeneous Markov process of the McKean process of (1.1) has smooth density $p(0,y,t,x;p_0)$ by Weyl's lemma and $q(t,y,x)$ is smooth and positive, it follows from (4.1) and (4.2) that the map $y\mapsto r(t,y,x;p_0)$ from R^d into $[0,\infty)$ is bounded and continuous.

LEMMA 4.2. *The map* $p_0\mapsto T_tp_0(x)$ *from* $\mathcal{P}(R^d)$ *into* R *is continuous for each* $t>0$ *and* $x\in R^d$.

PROOF. Let $t>0$ and $x\in R^d$ be fixed and let $\{p_0^{(n)};n\in N\}$ be a sequence of probability measures on R^d which converges weakly to a probability measure p on R^d. By Remark after Lemma 4.1, to prove the lemma, it will suffice to show that $r(t,\cdot,x;p_0^{(n)})$ converges to $r(t,\cdot,x;p_0)$ with supremum norm as $n\to\infty$. Let $P^{(n)}$ and P be the probability distributions of the solutions $X^{(n)}(\cdot)$ and $X(\cdot)$ to SDE (1.1) with initial distributions $p_0^{(n)}$ and p_0, respectively. Note that $P^{(n)}\otimes P^{(n)}$ converges to $P\otimes P$ weakly in $\mathcal{P}(W^d\otimes W^d)$ by Theorem 2.2. Then, by Skolohod's realization theorem of almost convergence, there exist a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ and $W^d\otimes W^d$-valued random variables $\{(\tilde{X}_1^{(n)},\tilde{X}_2^{(n)});n\in N\}$ and $(\tilde{X}_1,\tilde{X}_2)$ on the probability space such that (i) the probability distribution of $(\tilde{X}_1^{(n)},\tilde{X}_2^{(n)})$ and $(\tilde{X}_1,\tilde{X}_2)$ under $\tilde{P}$ are $P^{(n)}\otimes P^{(n)}$ and $P\otimes P$, respectively, (ii) $(\tilde{X}^{(n)},\tilde{X}^{(n)})\to(\tilde{X}_1,\tilde{X}_2)$ $\tilde{P}$-a.s. Since $\Psi\in\mathcal{S}(R^{3d})$, there exists a constant $M<\infty$ such that

$$|\Psi(y,y',y'')|\leqq M \text{ for any } y,y' \text{ and } y''\in R^d,$$
$$|\Psi(y,y_1',y_1'')-\Psi(y,y_2',y_2'')|\leqq M\{1\wedge(|y_1'-y_2'|+|y_1''-y_2''|)\}$$

for any $y\in R^d$. Then,

$$\sup_w\Big|\int_{W^{2d}}P^{(n)}(dw')\otimes P^{(n)}(dw'')\int_0^t\Psi(w(s),w'(s),w''(s))ds$$
$$-\int_{W^{2d}}P(dw')\otimes P(dw'')\int_0^t\Psi(w(s),w'(s),w''(s))ds\Big|$$

$$\leqq \sup_w \int_0^t \tilde{E}[|\Psi(w(s), \tilde{X}_1^{(n)}(s), \tilde{X}_2^{(n)}(s)) - \Psi(w(s), \tilde{X}_1(s), \tilde{X}_2(s))|]ds$$
$$\leqq \sup_w M \int_0^t \tilde{E}[1 \wedge (|\tilde{X}_1^{(n)}(s) - \tilde{X}_1(s)| + |\tilde{X}_2^{(n)}(s) - \tilde{X}_2(s)|)]ds$$
$$\leqq Mt \cdot \tilde{E}(1 \wedge \|(\tilde{X}_1^{(n)}, \tilde{X}_2^{(n)}) - (\tilde{X}_1, \tilde{X}_2)\|_{W^{2d}}) \longrightarrow 0 \quad \text{as } n \to \infty,$$

which completes the lemma. (Q.E.D.)

Let us remind the following convergence theorem.

LEMMA 4.3. *Let* $(X, \mathcal{B}, m)$ *be a measure space. Let* f_n $(n \in N_0)$ $(N_0 = N \cup \{0\})$ *be measurable functions on* $(X, \mathcal{B})$ *and let* h_n $(n \in N_0)$ *be positive summable functions on* $(x, \mathcal{B}, m)$. *If the following conditions* (i)-(iii) *are satisfied, then* f_n *converges to* f_0 *in* $L^1(X, \mathcal{B}, m)$.

(i) $f_n(x) \longrightarrow f_0(x)$, $h_n(x) \longrightarrow h_0(x)$ m-a.s. *as* $n \to \infty$,

(ii) $|f_n(x)| \leqq h_n(x)$ m-a.s. $(n \in N_0)$

(iii) $\int_X h_n(x) m(dx) \longrightarrow \int_X h_0(x) m(dx)$ *as* $n \to \infty$.

THEOREM 4.1. (i) *For* α *with* $0 < \alpha \leqq 2$ *and for any* $t > 0$, *the map* $p_0 \mapsto F(T_t p_0)$ *from* $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ *into* $\boldsymbol{R}$ *is continuous.*

(ii) *For* α *with* $\alpha > 2$ *and for any* $t > 0$, *the map* $p_0 \mapsto F(T_t p_0)$ *from* $\mathcal{P}(\boldsymbol{R}^d)$ *into* $\boldsymbol{R}$ *is continuous.*

PROOF. Let $0 < t_0 < t_1$ be fixed. First consider the case where $0 < \alpha \leqq 2$. Let $q(t, y, x)$ be a transition probability density of the diffusion process defined by (2.1). Let $\{p_0^{(n)} \in \mathcal{P}^{(2)}(\boldsymbol{R}^d); n \in N_0\}$ be a convergent sequence to $p_0^{(0)}$ weakly in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ sense. By Lemma 4.1, there exists a positive constant $M_1 < \infty$ such that

$$e^{-M_1 t_1} \int_{R^d} p_0^{(n)}(dy) q(t, y, x) \leqq T_t p_0^{(n)}(x) \leqq e^{M_1 t_1} \int_{R^d} p_0^{(n)}(dy) q(t, y, x).$$

Then by Theorem A.2, there exist positive constants C_i $(i = 1, \cdots, 5)$ such that for any $t \in [t_0, t_1]$ and $n \in N$,

$$C_1 \int_{R^d} p_0^{(n)}(dy) e^{-C_2(|x|^2 + |y|^2)} \leqq T_t p_0^{(n)}(x) \leqq C_3 \int_{R^d} p_0^{(n)}(dy) e^{C_4|x|^2 - C_5|x-y|^2} \leqq C_3 e^{C_4|x|^2}.$$

Then, by Jensen's inequality and the monotoneness of $\log(\cdot)$ we see that for any $t \in [t_0, t_1]$ and $n \in N$,

$$|\log T_t p_0^{(n)}(x) + 2\Phi_1(x)| T_t p_0^{(n)}(x) \leqq h_n(x),$$

where

$$h_n(x)=C_6\Big(1+|x|^2+\sup_n\int_{R^d}|y|^2p_0^{(n)}(dy)\Big)\int_{R^d}p_0^{(n)}(dy)q(t,y,x),$$

here C_6 is a positive constant which depends only on C_i $(i=1,2,3,4)$.

Since $p_0^{(n)}\to p_0$ in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$, by Theorem A.2 $h_n(x)\to h_0(x)$. We set $f_n(x)=(\log T_tp_0^{(n)}(x)+2\Phi_1(x))T_tp_0^{(n)}(x)$. Since the drift part of SDE (2.1) is Lipschitz continuous in the case where $0<\alpha\leqq 2$, by Gronwall's inequality there exists a constant $M_2<\infty$ such that

$$\int_{R^d}|x|^2q(t,y,x)dx\leqq\exp(M_2t_1)(1+|y|^2)$$

for any $t\leqq t_1$. Therefore, $\int_{R^d}(1+|x|^2)q(t,y,x)dx\in C_b^{(2)}(\boldsymbol{R}^d)$ as a function of $y\in\boldsymbol{R}^d$, where $C_b^{(2)}(\boldsymbol{R}^d)$ is the set of continuous functions f on $\boldsymbol{R}^d$ such that $\sup_{x\in R^d}(|f(x)|/(1+|x|^2))<\infty$. Since $p_0^{(n)}\to p_0$ in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$, $\int_{R^d}dx(1+|x|^2)\times\int_{R^d}p_0^{(n)}(dy)q(t,y,x)\to\int_{R^d}dx(1+|x|^2)\int_{R^d}p_0^{(0)}(dy)q(t,y,x)$ as $n\to\infty$, which implies that $\int_{R^d}h_n(x)dx\to\int_{R^d}h_0(x)dx$ as $n\to\infty$. On the other hand by Lemma 4.2, $f_n(x)\to f_0(x)$, it follows from Lemma 4.3 that $\int_{R^d}dx(\log T_tp_0^{(n)}(x)+2\Phi_1(x))T_tp_0^{(n)}(x)\to\int_{R^d}dx(\log T_tp_0^{(0)}(x)+2\Phi_1(x))T_tp_0^{(0)}(x)$ as $n\to\infty$. Since $\Phi_2\in C_b(\boldsymbol{R}^d)$ by Theorem 2.3,

$$\text{(4.3)}\qquad \int_{R^{2d}}\Phi_2(x,y)T_tp_0^{(n)}(x)T_tp_0^{(n)}(y)dxdy \longrightarrow\int_{R^{2d}}\Phi_2(x,y)T_tp_0^{(0)}(x)T_tp_0^{(0)}(y)dxdy\quad\text{as } n\to\infty.$$

This completes the proof of (i). Let $\alpha>2$ and let $p_0^{(n)}\to p_0^{(0)}$ weakly in $\mathcal{P}(\boldsymbol{R}^d)$ as $n\to\infty$. Then, by Theorem A.2, there exist positive constants C_7 and C_8 such that for any $t\in[t_0,t_1]$ and any $x,y\in\boldsymbol{R}^d$,

$$C_7e^{-2\Phi_1(x)}\leqq q(t,y,x)\leqq C_8e^{-2\Phi_1(x)}.$$

Then, there exists a positive constant C_9 such that

$$|\log T_tp_0^{(n)}(x)+2\Phi_1(x)|\,T_tp_0^{(n)}(x)\leqq C_9e^{-\Phi_1(x)},$$

for any $t\in[t_0,t_1]$, $x,y\in\boldsymbol{R}^d$ and $n\in N_0$. Then, by Lebesgue's dominated convergence theorem, $\int_{R^d}(\log T_tp_0^{(n)}(x)+2\Phi_1(x))T_tp_0^{(n)}(x)dx\to\int_{R^d}(\log T_tp_0^{(0)}(x)$

$+2\Phi_1(x))T_tp_0^{(0)}(x)dx$ as $n\to\infty$. This and (4.3) imply (ii). (Q.E.D.)

We can prove the following proposition in a similar way to prove Theorem 4.1.

PROPOSITION 4.1. (i) *In the case where* $0<\alpha\leqq 2$, *for any* $t>0$, *the mapping* $T_t:\mathcal{P}^{(2)}(\boldsymbol{R}^d)\to\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ *is continuous with respect to* $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$*-weak topology.*

(ii) *In the case where* $\alpha>2$, *for any* $t>0$ *and any* $m>0$, *the mapping* $T_t:\mathcal{P}(\boldsymbol{R}^d)\to\mathcal{P}^{(m)}(\boldsymbol{R}^d)$ *is continuous.*

PROPOSITION 4.2. *Let* $\alpha>2$ *and let* q *be an invariant probability measure of* SDE (1.1). *Then there exists a positive constant* C *such that*

$$\begin{aligned}&\|(T_{t+s}p_1-T_{t+s}p_2)/q\|_{L^\infty}\\&\leqq C\Big(\|T_tp_1-T_tp_2\|_{L^1}+\|T_{t+s}p_1-T_{t+s}p_2\|_{L^1}+\int_t^{t+s}\|T_up_1-T_up_2\|_{L^1}du\Big)\end{aligned}$$

for any $t>0$, $s\in[1/2,1]$ *and any* p_1 *and* $p_2\in\mathcal{P}(\boldsymbol{R}^d)$.

PROOF. Let $q(t,y,x)$ be a transition probability density of the diffusion with generator $L_0=\dfrac{1}{2}\Delta-\operatorname{grad}\Phi_1(x)\cdot\operatorname{grad}$. By Theorem 4.1,

$$T_{t+s}p_0(x)=\int_{\boldsymbol{R}^d}T_tp_0(y)q(s,y,x)r(t,y,t+s,x;p_0)dy,$$

where

$$\begin{aligned}&r(t,y,t+s,x;p_0)\\&=\exp\Big(\int_{\boldsymbol{R}^d}\Phi_2(x,z)T_{t+s}p_0(z)dz-\int_{\boldsymbol{R}^d}\Phi_2(y,z)T_tp_0(z)dz\Big)\\&\quad\times E_y\Big[\exp\Big(\int_t^{t+s}du\int_{\boldsymbol{R}^{2d}}\Psi(Y(u-t),z',z'')T_up_0(z')T_up_0(z'')dz'dz''\Big)\Big|\,Y(s)=x\Big],\end{aligned}$$

$\{Y(t);t\geqq 0\}$ is a diffusion with generator L_0 starting at y and $\Psi\in\mathcal{S}(\boldsymbol{R}^{3d})$ is defined by (2.3).

Let $q_0(x)=Z_0^{-1}\exp(-2\Phi_1(x))$, $Z_0=\int_{\boldsymbol{R}^d}\exp(-2\Phi_1(z))dz$. Then q_0 is the invariant probability measure of $Y(\cdot)$ and $\|q_0/q\|_\infty\vee\|q/q_0\|_\infty<\infty$. Then,

$$\begin{aligned}&\|(T_{t+s}p_1-T_{t+s}p_2)/q\|_{L^\infty}\\&\leqq\|q_0/q\|_\infty\sup_x\Big[q_0(x)^{-1}\Big|\int_{\boldsymbol{R}^d}T_tp_1(y)q(s,y,x)r(t,y,t+s,x;p_1)dy\end{aligned}$$

$$-\int_{R^d} T_t p_2(y) q(s, y, x) r(t, y, t+s, x; p_2) dy \Big| \Big]$$

$$\leqq \|q_0/q\|_\infty \sup_x \Big[q_0(x)^{-1} \int_{R^d} r(t, y, t+s, x; p_1) q(s, y, x) |T_t p_1(y) - T_t p_2(y)| dy \Big]$$

$$+ \|q_0/q\|_\infty \sup_x \Big[q_0(x)^{-1} \int_{R^d} T_t p_2(y) q(s, y, x) |r(t, y, t+s, x; p_1)$$

$$- r(t, y, t+s, x; p_2)| dy \Big]$$

$$= I_1 + I_2.$$

Since the semigroup $\{e^{tL_0}; t \geqq 0\}$ has ultracontractivity, there exists a positive constant C such that

$$\sup_x \Big[q_0(x)^{-1} \Big| \int_{R^d} q(s, y, x) f(y) dy \Big| \Big] \leqq C \int_{R^d} |f(y)| dy$$

for any $f \in L^1(R^d; dx)$ and any $s \in [1/2, 1]$ (Theorem 3.2 in Davies and Simon [2]). Then, there exists a positive constant C_1 such that

$$I_1 \leqq C_1 \| T_t p_1 - T_t p_2 \|_{L^1}$$

for any $s \in [1/2, 1]$.

On the other hand, by a simple calculation we see that there exists a positive constant C such that

$$|r(t, y, t+s, x; p_1) - r(t, y, t+s, x; p_2)|$$
$$\leqq C \Big(\| T_{t+s} p_1 - T_{t+s} p_2 \|_{L^1} + \| T_t p_1 - T_t p_2 \|_{L^1} + \int_t^{t+s} \| T_s p_1 - T_s p_2 \|_{L^1} du \Big)$$

for any $x, y \in R^d$, $t > 0$ and $s \leqq 1$. Then, there exists a positive constant C_2 such that

$$I_2 \leqq C_2 \Big(\| T_t p_1 - T_t p_2 \|_{L^1} + \| T_{t+s} p_1 - T_{t+s} p_2 \|_{L^1} + \int_t^{t+s} \| T_u p_1 - T_u p_2 \|_{L^1} du \Big),$$

which completes the proof. (Q.E.D.)

Now we shall prove the following theorem which plays a key role in the following sections.

THEOREM 4.2. (i) *Let p_0 be a probability measure on R^d with a compact support. Then $F(T_t p_0)$ is differentiable in $t \in (0, \infty)$ and*

$$\frac{d}{dt} F(T_t p_0) = -4 I(T_t p_0). \tag{4.4}$$

(ii) *Let* $0<\alpha\leqq 2$. *Then*

$$F(T_tp_0)-F(T_sp_0)\leqq -\int_s^t 4I(T_up_0)du, \tag{4.5}$$

for any $0<s<t$ *and any* $p_0\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$. *In the case where* $\alpha>2$, (4.5) *holds for any* $0<s<t$ *and any* $p_0\in\mathcal{P}(\boldsymbol{R}^d)$.

REMARK. Since $I(p)$ is non-negative, $F(T_tp_0)$ is a non-increasing function in $t\in(0,\infty)$ by this theorem.

PROOF. Let p_0 be a probability measure on $\boldsymbol{R}^d$ with $\mathrm{supp}(p_0)\subset\{y\in\boldsymbol{R}^d; |y|\leqq a\}$ for some positive a, and let $0<t_0<t_1$ be fixed. Let $p(s, y, t, x; p_0)$ be a transition probability density of the frozen Markovian process for the McKean process (1.1) defined by the following SDE:

$$\begin{cases} dY(t)=dB(t)-\mathrm{grad}(\Phi_1(Y(t))-\Phi_2[Y(t),\ T_tp_0])dt, \\ \text{the distribution of } Y(0) \text{ is } p_0, \\ \Phi_2[y,\ T_tp_0]=\int_{\boldsymbol{R}^d}\Phi_2(x, z)T_tp_0(z)dz. \end{cases} \tag{4.6}$$

Then, T_tp_0 is expressed by

$$T_tp_0(x)=\int_{\boldsymbol{R}^d}p_0(dy)p(0, y, t, x; p_0).$$

Formally, (4.4) is given as follows:

$$\frac{d}{dt}F(T_tp_0)$$

$$=\frac{d}{dt}\left\{\int_{\boldsymbol{R}^d}(\log T_tp_0(x)+2\Phi_1(x))T_tp_0(x)dx+\int_{\boldsymbol{R}^{2d}}\Phi_2(x, y)T_tp_0(x)T_tp_0(y)dxdy\right\}$$

$$=\int_{\boldsymbol{R}^d}(\log T_tp_0(x)+2\Phi_1(x)-2\Phi_2[x,\ T_tp_0])\frac{\partial}{\partial t}T_tp_0(x)dx+\int_{\boldsymbol{R}^d}\frac{\partial}{\partial t}T_tp_0(x)dx \tag{4.7}$$

$$=-\frac{1}{2}\int_{\boldsymbol{R}^d}|\mathrm{grad}(\log T_tp_0(x)+2\Phi_1(x)-2\Phi_2[x,\ T_tp_0])|^2T_tp_0(x)dx \tag{4.8}$$

$$=-\frac{1}{2}\int_{\boldsymbol{R}^d}|\mathrm{grad}\, T_tp_0(x)+2T_tp_0(x)\,\mathrm{grad}(\Phi_1(x)-2\Phi_2[x,\ T_tp_0])|^2/T_tp_0(x)dx \tag{4.9}$$

$$=-4I(T_tp_0). \tag{4.10}$$

We justify each step. By Theorem A.2 in Appendix A, there exist positive constants C_1, C_2, C_3 and C_4 such that for any $x\in \boldsymbol{R}^d$ and $t\in[t_0, t_1]$

$$(4.11)\qquad T_tp_0(x)\geqq C_1\int_{\boldsymbol{R}^d} p_0(dy)\exp(-C_2|x|^2-C_3|y|^2)\geqq C_1\exp(C_3a^2-C_2|x|^2)>0$$

if $0<\alpha\leqq 2$, and

$$(4.12)\qquad T_tp_0(x)\geqq C_4\int_{\boldsymbol{R}^d} p_0(dy)\exp(-2\Phi_1(x))\geqq C_4\exp(-2\Phi_1(x))>0,\ \text{if}\ \alpha>2.$$

Since

$$\frac{\partial}{\partial t}T_tp_0(x)=\frac{1}{2}\operatorname{div}\{\operatorname{grad}T_tp_0(x)+2T_tp_0(x)\operatorname{grad}(\Phi_1(x)-\Phi_2[x, T_tp_0])\},$$

by Theorem A.2, there exist positive constants C_5, C_6 and C_7 such that for any $x\in\boldsymbol{R}^d$, $t\in[t_0, t_1]$ and $\alpha>0$

$$\begin{aligned}\left|\frac{\partial}{\partial t}T_tp_0(x)\right| &\leqq C_5\int_{\boldsymbol{R}^d} p_0(dy)\exp(C_6|y|^\alpha-C_7|x-y|^2)\\ &\leqq C_5\exp(C_6a^\alpha)\exp(-C_7(|x|^2-a^2)\wedge 0).\end{aligned}$$

By Theorem A.2, (4.11) and (4.12), there exist positive constants C_8 and C_9 such that for any $x\in\boldsymbol{R}^d$ and $t\in[t_0, t_1]$

$$(4.13)\qquad |\log T_tp_0(x)+2\Phi_1(x)-2\Phi_2[x, T_tp_0]|\leqq\begin{cases}C_8(1+a^2+|x|^2) & \text{if } 0<\alpha\leqq 2,\\ C_9(1+|x|^\alpha) & \text{if } \alpha>2.\end{cases}$$

By Theorem A.2 and (4.11), there exist positive constants C_{10}, C_{11} and C_{12} such that for any $x\in\boldsymbol{R}^d$, $t\in[t_0, t_1]$ and $\alpha>0$

$$\begin{aligned}(4.14)\qquad &|\log T_tp_0(x)+2\Phi_1(x)-2\Phi_2[x, T_tp_0]|\\ &\times\left|\frac{\partial}{\partial x_i}T_tp_0(x)+2T_tp_0(x)\frac{\partial}{\partial x_i}(\Phi_1(x)-\Phi_2[x, T_tp_0])\right|\\ &\leqq C_{10}(1+a^{2\vee\alpha}+|x|^{2\vee\alpha})(1+|x|)\exp(C_{11}a^\alpha-C_{12}((|x|^2-a^2)\wedge 0)).\end{aligned}$$

Then (4.7) is justified by (4.11), (4.12) and (4.13). (4.8) is justified by integration by parts using (4.14) and $\int_{\boldsymbol{R}^d}T_tp_0(x)dx=1$. (4.9) follows from (4.8) by simple calculation with $T_tp_0(x)>0$ $(t>0)$. (4.10) follows from (4.9) and Theorem 3.2 (iii). For any $p_0\in\mathcal{P}(\boldsymbol{R}^d)$, we get a sequence $\{p_0^{(n)}\in\mathcal{P}(\boldsymbol{R}^d); n\in\boldsymbol{N}\}$ such that $\operatorname{supp}(p_0^{(n)})\subset\{x\in\boldsymbol{R}^d; |x|\leqq n\}$ and $p_0^{(n)}\to p_0$ weakly with $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ topology. Then, by Theorem 4.1, Theorem 3.2 (i) and Fatou's lemma we get (4.5). (Q.E.D.)

§ 5. Convergence to an invariant probability measure.

First let us notice that in general the invariant probability measures for SDE (1.1) of McKean type are not unique (Theorem 4.2 in [11]). For a probability measure p_0 on $\boldsymbol{R}^d$, we set

$$l(p_0)=\bigcap_{s>0}\overline{\bigcup_{t\geqq s}\{T_t p_0\}}, \tag{5.1}$$

here the closure is taken with respect to the weak topology in $\mathcal{P}(\boldsymbol{R}^d)$.

Theorem 5.1. *Let $p_0\in\mathcal{P}(\boldsymbol{R}^d)$. In the case where $0<\alpha\leqq 2$ let assume $p_0\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ furthermore. Then the following hold.*

(i) *$l(p_0)$ is a compact connected set in $\mathcal{P}(\boldsymbol{R}^d)$ with respect to the weak topology.*

(ii) *If $q\in l(p_0)$, then q is an invariant probability measure and $F(q)\leqq\lim_{t\to\infty}F(T_t p_0)$.*

Remark. Moreover, in the case where $\alpha\geqq 2$ by using logarithmic Sobolev inequality, we see that $F(q)=\lim_{t\to\infty}F(T_t p_0)$ for any $q\in l(p_0)$.

Proof. (i) follows immediately from Theorem 2.3, Theorem 3.1 (i) and (ii). Let $q\in l(p_0)$. We put $F_\infty=\lim_{t\to\infty}F(T_t p_0)$. Then, there exists a sequence $\{t_n; n\in N\}$ such that $T_{t_n}p_0\to q$ weakly. By Theorem 3.1 and Theorem 4.2 (ii), we get $F(q)\leqq F_\infty$. Let $\{\varepsilon_n; n\in N\}$ be a sequence of strictly decreasing positive numbers converging to 0 with $\varepsilon_1<1$. Then, we can choose a subsequence $\{s_n; n\in N\}$ of $\{t_n; n\in N\}$ such that $s_{n+1}-s_n\geqq 1$ and $F(T_{s_{n-1}}p_0)-F_\infty\leqq 4\varepsilon_n^2$. By Theorem 4.2 (ii), $4\int_{s_n-\varepsilon_n}^{s_n}I(T_u p_0)du\leqq F(T_{s_{n-1}}p_0)-F(T_{s_n}p_0)\leqq F(T_{s_{n-1}}p_0)-F_\infty\leqq 4\varepsilon_n^2$. Then, there exists a sequence $\{\delta_n; n\in N\}$ of positive numbers such that $0\leqq\delta_n\leqq\varepsilon_n$ and $I(T_{s_n-\delta_n}p_0)\leqq\varepsilon_n$. Since $\{T_{s_n-\delta_n}p_0; n\in N\}$ is precompact in $\mathcal{P}(\boldsymbol{R}^d)$ by Theorem 3.1 (ii), there exists a subsequence $\{T_{s_{n'}-\delta_{n'}}p_0; n\in N\}$ of $\{T_{s_n-\delta_n}p_0; n\in N\}$ and $q'\in\mathcal{P}(\boldsymbol{R}^d)$ such that $T_{s_{n'}-\delta_{n'}}p_0\to q'$ weakly in $\mathcal{P}(\boldsymbol{R}^d)$. Because of the semicontinuity of I on $\mathcal{P}(\boldsymbol{R}^d)$, $0\leqq I(q')\leqq\varliminf_{n\to\infty}I(T_{s_{n'}-\delta_{n'}}p_0)=0$, then $I(q')=0$, which implies that q' is an invariant probability measure of (1.1) by Theorem 3.2 (ii). By Theorem 2.3, $q=\lim_{n\to\infty}T_{t_n}p_0=\lim_{n\to\infty}T_{s_{n'}}p_0=\lim_{n\to\infty}T_{\delta_{n'}}T_{s_{n'}-\delta_{n'}}p_0=T_0q'=q'$, which completes the theorem. (Q.E.D.)

COROLLARY. *Suppose that the set of all invariant probability measures of* SDE (1.1) *is a discrete subset of* $\mathcal{P}(\boldsymbol{R}^d)$. *If* $0<\alpha\leqq 2$, *then for any* $p_0\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ *there exists an invariant probability measure* q *such that* $T_t p_0\to q$ *weakly in* $\mathcal{P}(\boldsymbol{R}^d)$ *as* $t\to\infty$. *If* $\alpha>2$, *then for any* $p_0\in\mathcal{P}(\boldsymbol{R}^d)$ *the same assertion holds.*

REMARK. Assume that there is a unique invariant probability measure, say q. Then from this corollary we see that for any $p\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ $T_t p_0\to q$ weakly in $\mathcal{P}(\boldsymbol{R}^d)$ as $t\to\infty$. As shown in [11], if q is not bifurcation point of invariant probability measures, then it is not difficult to prove $T_t p_0\to q$ by semigroup method. But if not, probably it is hard to prove this by semigroup method.

Now let us prove Theorem 1.1. First note that $F(q)<F(p)$ for any $p\in U, p\neq q$. Indeed, if $F(p)=F(q)$ then by Theorem 3.1 (iii), p is an invariant probability measure of SDE (1.1), which contradicts the assumption. Let U_1 be an open neighborhood of q in $\mathcal{P}(\boldsymbol{R}^d)$ such that $\overline{U_1}\subset U$. Let $K_\varepsilon=\{p\in\mathcal{P}(\boldsymbol{R}^d); F(p)<F(q)+\varepsilon\}$. Then, there exists $\varepsilon_0>0$ such that $K_{\varepsilon_0}\cap\partial U_1=\emptyset$. Actually, if not, by Theorem 3.1 (ii), there exists $p_0\in\partial U_1$ such that $F(p_0)\leqq F(q)$, which is contradiction.

Set $K_{\varepsilon_0}(q)=K_{\varepsilon_0}\cap U_1$ and $V=\{p\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ [resp. $\mathcal{P}(\boldsymbol{R}^d)$]; $T_1 p\in K_{\varepsilon_0}(q)\}$ in the case where $0<\alpha\leqq 2$ [resp. $\alpha>2$]. In view of Theorems 2.3 and 4.2, $T_t p_0\in K_{\varepsilon_0}(q)$ for any $p_0\in V$ and any $t\geqq 1$. Then by Theorem 5.1, $T_t p_0\to q$ in $\mathcal{P}(\boldsymbol{R}^d)$ for any $p_0\in V$.

Note that $T_1^{-1}(K_{\varepsilon_0})=\{p\in\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ [resp. $\mathcal{P}(\boldsymbol{R}^d)$]; $F(T_1 p)<F(q)+\varepsilon_0\}$ in the case where $0<\alpha\leqq 2$ [resp. $\alpha>2$]. By Theorem 4.1, $T_1^{-1}(K_{\varepsilon_0})$ is an open set in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ [resp. $\mathcal{P}(\boldsymbol{R}^d)$] if $0<\alpha\leqq 2$ [resp. $\alpha>2$]. Then, $V=T_1^{-1}(K_{\varepsilon_0})\cap T_1^{-1}(U_1)$ is an open neighborhood of q in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ [resp. $\mathcal{P}(\boldsymbol{R}^d)$] in the case where $0<\alpha\leqq 2$ [resp. $\alpha>2$], which completes the proof.

(Q.E.D.)

§6. Exponential decay of free energy.

In the rest of this paper, we assume that α is greater than or equal to 2.

In this section we shall show that the free energy F decreases exponentially along the probability distribution of McKean process (1.1). For any fixed $q\in\mathcal{P}(\boldsymbol{R}^d)$, let H_q be the entropy on $\mathcal{P}(\boldsymbol{R}^d)$, i.e.,

$$(6.1) \qquad H_q(p)=\begin{cases} \int_{R^d}(dp/dq)(x)\log(dp/dq)(x)q(dx), & \text{if } p \text{ is absolutely continuous relative to } q, \\ \infty & \text{otherwise.} \end{cases}$$

Then, $F(p)=H_{q_0}(p)-\int_{R^{2d}}\Phi_2(x, y)p(dx)p(dy)-\log Z_0$, where

$$(6.2) \qquad q_0(x)=Z_0^{-1}\exp(-2\Phi_1(x)),$$
$$Z_0=\int_{R^d}\exp(-2\Phi_1(x))dx.$$

First we quote the following lemma (Csiszár [1]).

LEMMA 6.1. *Let $p(x)$ and $q(x)$ be probability densities on R^d. Then,*

$$\|p-q\|^2_{L^1(dx)}\leqq 2H_q(p).$$

PROOF. Set $\varphi(x)=p(x)/q(x)$, $f(x)=x\log x$, $A=\{x\in R^d; p(x)\leqq q(x)\}$ and let $\mathcal{B}_0$ be a σ-field given by $\mathcal{B}_0=\{A, A^c, R^d, \phi\}$. Then by Jensen's inequality

$$\begin{aligned} H_q(p) &= E^q[f(\varphi(X))] \\ &= E^q[E^q[f(\varphi(X))\,|\,\mathcal{B}_0]] \\ &\geqq E^q[f(E^q[\varphi(X)\,|\,\mathcal{B}_0])] \\ &= p(A)\log(p(A)/q(A))+(1-p(A))\log((1-p(A))/(1-q(A))). \end{aligned}$$

On the other hand,

$$\|p-q\|_{L^1(R^d;\, dx)}\leqq 2(q(A)-p(A)).$$

For any $0\leqq u\leqq v\leqq 1$,

$$2(v-u)\leqq\sqrt{2}\,(u\log(u/v)+(1-u)\log((1-u)/(1-v)))^{1/2}.$$

This completes the proof. (Q.E.D.)

LEMMA 6.2. *For any probability distributions p and q on R^d,*

$$(6.3) \qquad F(p)-F(e(q))=H_{e(q)}(p)-\int_{R^{2d}}\Phi_2(x, y)(p-q)(dx)(p-q)(dy)$$
$$+\int_{R^{2d}}\Phi_2(x, y)(q-e(q))(dx)(q-e(q))(dy).$$

PROOF. Since there exist positive constants C_1 and C_2 such that for any $q\in\mathcal{P}(R^d)$

$$C_1 q_0 \leqq e(q) \leqq C_2 q_0, \tag{6.4}$$

it will suffice to show (6.3) for $p \in \mathscr{P}(\boldsymbol{R}^d)$ such that $H_{q_0}(p) < \infty$. Since by the definition,

$$\begin{aligned} F(p) &= H_{e(q)}(p) + \int_{\boldsymbol{R}^d} p(x) \log(e(q)(x)/q_0(x))dx \\ &\quad - \int_{\boldsymbol{R}^{2d}} \Phi_2(x, y)p(x)p(y)dxdy + \log Z_0, \\ F(e(q)) &= \int_{\boldsymbol{R}^{2d}} e(q)(x) \log(e(q)(x)/q_0(x))dx \\ &\quad - \int_{\boldsymbol{R}^{2d}} \Phi_2(x, y)e(q)(x)e(q)(y)dxdy + \log Z_0, \end{aligned}$$

we get (6.3) by a simple calculation. (Q.E.D.)

LEMMA 6.3. *Let q be an invariant probability measure. Then for any $p \in \mathscr{P}(\boldsymbol{R}^d)$*

$$F(p) - F(q) \leqq (1 + 2\|\Phi_2\|_{L^\infty})(H_{e(p)}(p) + H_q(e(p))).$$

PROOF. It is known that q is an invariant probability measure if and only if $q = e(q)$ ([11]). Then,

$$\begin{aligned} F(p) - F(q) &= F(p) - F(e(p)) + F(e(p)) - F(e(q)) \\ &= H_{e(p)}(p) + H_q(e(p)) + \int_{\boldsymbol{R}^{2d}} \Phi_2(x, y)(p - e(p))(dx)(p - e(p))(dy) \\ &\quad - \int_{\boldsymbol{R}^{2d}} \Phi_2(x, y)(e(p) - q)(dx)(e(p) - q)(dy) \qquad \text{(by Lemma 6.2)} \\ &\leqq H_{e(p)}(p) + H_q(e(p)) + \|\Phi_2\|_{L^\infty}(\|p - e(p)\|^2 + \|e(p) - q\|^2) \\ &\leqq H_{e(p)}(p) + H_q(e(p)) + \|\Phi_2\|_{L^\infty} 2(H_{e(p)}(p) + H_q(e(p))) \\ &\qquad \text{(by Lemma 6.1).} \end{aligned}$$

(Q.E.D.)

LEMMA 6.4. *Let q be an invariant probability measure. Then e defined in* (3.2) *is continuous at q as a functional from $\mathscr{P}(\boldsymbol{R}^d)$ to $L^\infty(q^{-1})$.*

PROOF. $|(e(p)(x) - q(x))/q(x)|$

$$\begin{aligned} &= \left| Z_q Z_p^{-1} \exp\left(2\int_{\boldsymbol{R}^d} \Phi_2(x, y)p(dy) - 2\int_{\boldsymbol{R}^d} \Phi_2(x, y)q(y)dy\right) - 1 \right| \\ &\leqq e^{4\|\Phi_2\|_{L^\infty}} |Z_q Z_p^{-1} - 1| \\ &\quad + \left| \exp\left(2\int_{\boldsymbol{R}^d} \Phi_2(x, y)p(dy) - 2\int_{\boldsymbol{R}^d} \Phi_2(x, y)q(y)dy\right) - 1 \right| \end{aligned}$$

$$\leqq 2e^{8\|\Phi_2\|_{L^\infty}}\int_{R^d} q_0(z)dz \left|\int_{R^d}\Phi_2(x, y)p(dy)-\int_{R^d}\Phi_2(x, y)q(dy)\right|$$
$$+2e^{8\|\Phi_2\|_{L^\infty}}\left|\int_{R^d}\Phi_2(x, y)p(dy)-\int_{R^d}\Phi_2(x, y)q(dy)\right|.$$

Since $\Phi_2\in\mathcal{S}(\boldsymbol{R}^{2d})$, there exists a positive constant C such that $|\Phi_2(x, y)-\Phi_2(x, z)|\leqq C(1\wedge|y-z|)$. Let $\{p_n; n\in N\}$ be a sequence of probability distributions on $\boldsymbol{R}^d$ such that $p_n\to q$ weakly. Then by Skorohod's realization theorem of almost sure convergence, there exist a probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P})$ and $\boldsymbol{R}^d$ valued random variables $\tilde{X}_n$ $(n\in N)$ and $\tilde{X}$ defined on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P})$ such that (i) the probability distributions of $\tilde{X}_n$ and $\tilde{X}$ under $\tilde{P}$ are p_n and q, respectively and (ii) $\tilde{X}_n\to\tilde{X}$ ($\tilde{P}$-a.s.) as $n\to\infty$. Then, $|e(p)(x)/q(x)-1|\leqq\text{const.}\,\tilde{E}[1\wedge|\tilde{X}_n-\tilde{X}|]\to 0$ as $n\to\infty$. (Q.E.D.)

Let q be an invariant probability measure and let K_q be the compact linear operator defined in (1.4). We denote by $\mathrm{Spec}(K_q)$ the set of eigenvalues of K_q in $L_0^2(q^{-1}dx)$. By a formal calculation we get $D^2H_q(q+\cdot)[h][h]=\frac{1}{2}\|h\|^2_{L_0^2(q^{-1}dx)}$ and $D^2H_{e(q+\cdot)}(q+\cdot)[h][h]=\frac{1}{2}\|(I-K_q)h\|^2_{L_0^2(q^{-1}dx)}$.

LEMMA 6.5. *Let q be an invariant probability measure such that $1\notin\mathrm{Spec}(K_q)$. Then there exist positive constants γ_1 and δ such that*

$$H_q(p)\leqq\gamma_1 H_{e(p)}(p)$$

holds for any $p\in\mathcal{P}(\boldsymbol{R}^d)$ with $\|p/q-1\|_{L^\infty}<\delta$.

PROOF. By a simple calculation we see that there exist positive constants C_1, C_2 and C_3 such that

$$|e(q+h)(x)-q(x)-K_qh(x)|\leqq C_1q(x)\left\|\int_{Rd}\Phi_2(\cdot, y)h(y)dy\right\|^2_{L^\infty} \tag{6.5}$$

$$\left|H_q(q+h)-\frac{1}{2}\|h\|^2_{L_0^2(q^{-1}dx)}\right|\leqq C_2\left\|\frac{h}{q}\right\|_{L^\infty}\|h\|^2_{L_0^2(q^{-1}dx)} \tag{6.6}$$

and

$$\left|H_{e(q+h)}(q+h)-\frac{1}{2}\|(I-K_q)h\|^2_{L_0^2(q^{-1}dx)}\right|\leqq C_3\left\|\frac{h}{q}\right\|_{L^\infty}\|h\|^2_{L_0^2(q^{-1}dx)} \tag{6.7}$$

for any $h\in L_0^\infty(q^{-1})=\left\{h:\boldsymbol{R}^d\to\boldsymbol{R};\ \|h/q\|<\infty,\ \int_{R^d}h(x)dx=0\right\}$ which satisfies $\|h/q\|_{L^\infty}\leqq\frac{1}{2}$. Set

$$\delta=\frac{1}{4C_3}\|I-K_q\|_{\mathcal{L}(L_0^2(q^{-1}dx))}\wedge\frac{1}{2},$$
$$C_4=2C_3(1+C_2)\|(I-K_q)^{-1}\|_{\mathcal{L}(L_0^2(q^{-1}dx))}$$
$$\gamma_1=(1+(C_2+C_4))\|(I-K_q)^{-1}\|_{\mathcal{L}(L_0^2(q^{-1}dx))}.$$

Then by (6.6) and (6.7), for any $h\in L_0^\infty(q^{-1})$ such that $\|h/q\|_\infty\leqq\delta$

$$\begin{aligned}
H_q(q+h)&\leqq\frac{1}{2}\|h\|^2_{L_0^2(q^{-1}dx)}+C_2\|h/q\|_\infty\|h\|^2_{L_0^2(q^{-1}dx)}\\
&=(1+2(C_2+C_4)\|h/q\|_\infty)\frac{1}{2}\|h\|^2_{L_0^2(q^{-1}dx)}-C_4\|h/q\|_\infty\|h\|^2_{L_0^2(q^{-1}dx)}\\
&\leqq(1+2(C_2+C_4)\|h/q\|_\infty)\|(I-K_q)^{-1}\|_{\mathcal{L}(L_0^2(q^{-1}dx))}\\
&\quad\times\frac{1}{2}\|(I-K_q)^{-1}h\|^2_{L_0^2(q^{-1}dx)}-C_4\|h/q\|_\infty\|h\|^2_{L_0^2(q^{-1}dx)}\\
&\leqq(1+2(C_2+C_4)\|h/q\|_\infty)\|(I-K_q)^{-1}\|_{\mathcal{L}(L_0^2(q^{-1}dx))}H_{e(q+h)}(q+h)\\
&\quad-\|h/q\|_\infty\|h\|^2_{L_0^2(q^{-1}dx)}(C_4-C_3(1+2(C_2+C_4)\|h/q\|_\infty))\\
&\quad\times\|(I-K_q)^{-1}\|_{L_0^2(q^{-1}dx)}\\
&\leqq\gamma_1H_{e(q+h)}(q+h).
\end{aligned}$$

(Q.E.D.)

LEMMA 6.6. *There exists a positive constant γ_2 such that for any $p\in\mathcal{P}(\boldsymbol{R}^d)$*

$$H_{e(e(p))}(e(p))\leqq\gamma_2H_{e(p)}(p).$$

PROOF. We have

$$\begin{aligned}
&H_{e^2(p)}(e(p))\\
&=\int_{R^d}(e(p)(x)/e^2(p)(x))\log(e(p)(x)/e^2(p)(x))e^2(p)(x)dx\\
&=\int_{R^d}((e(p)(x)/e^2(p)(x))\log(e(p)(x)/e^2(p)(x))-(e(p)(x)/e^2(p)(x)-1))e^2(p)(x)dx\\
&\leqq\frac{1}{2}\|e(p)/e^2(p)\|_\infty^{-1}\int_{R^{2d}}(e(p)(x)/e^2(p)(x)-1)^2e^2(p)(x)dx.
\end{aligned}$$

Since $\|e(p)/e^2(p)\|_\infty\geqq e^{-8\|\Phi_2\|_\infty}$ and $|e(p)(x)/e^2(p)(x)-1|\leqq\text{const.}\|p-e(p)\|_{L^1(dx)}$, there exists a positive constant C such that $H_{e^2(P)}(e(p))\leqq C\|p-e(p)\|^2_{L^1(dx)}$ $\leqq 2CH_{e(p)}(p)$ (by Lemma 6.1). (Q.E.D.)

LEMMA 6.7. *Let q be an invariant probability measure such that $1\notin\operatorname{Spec}(K_q)$. Then there exist a positive constant γ_3 and an open neighborhood V_1 of q in $\mathcal{P}(\boldsymbol{R}^d)$ such that for any $p\in V_1$*

$$|F(p)-F(q)|\leqq\gamma_3 H_{e(p)}(p).$$

PROOF. We set $\gamma_3=(1+2\|\Phi_2\|_\infty)(1+\gamma_1\gamma_2)$ and $V_1=\{p\in\mathcal{P}(\boldsymbol{R}^d);\ \|e(p)/q-1\|_\infty<\delta\}$, where γ_1 and δ are the same ones as in Lemma 6.5 and γ_2 is the same in Lemma 6.6. Then by Lemma 6.4, V_1 is an open neighborhood of q in $\mathcal{P}(\boldsymbol{R}^d)$. For any $p\in V_1$

$$\begin{aligned}|F(p)-F(q)|&\leqq(1+2\|\Phi_2\|_\infty)(H_{e(p)}(p)+H_q(e(p))) &&\text{(by Lemma 6.3)}\\ &\leqq(1+2\|\Phi_2\|_\infty)(H_{e(p)}(p)+\gamma_1 H_{e^2(p)}(e(p))) &&\text{(by Lemma 6.5)}\\ &\leqq(1+2\|\Phi_2\|_\infty)(1+\gamma_1\gamma_2)H_{e(p)}(p) &&\text{(by Lemma 6.6).}\end{aligned}$$

(Q.E.D.)

Let $L_{e(p)}$ be an elliptic operator defined in (3.3) and let denote by $\tilde{L}_{e(p)}$ the Friedrichs' extension of $L_{e(p)}$ in $L^2(e(p)dx)$. Then by the assumption that $\alpha\geqq 2$, the semigroup $\{e^{t\tilde{L}_{e(p)}};t\geqq 0\}$ has a hypercontractive property (Davies-Simon [2]). Thus by (6.4) we get

LEMMA 6.8. *There exist positive constants t_0 and C ($\geqq 1$) such that*

$$\|e^{t_0\tilde{L}_{e(p)}}f\|_{L^4(e(p)dx)}\leqq C\|f\|_{L^2(e(p)dx)}$$

for any $p\in\mathcal{P}(\boldsymbol{R}^d)$ and any $f\in L^2(e(p)dx)$.

LEMMA 6.9. (i) *For any $p\in\mathcal{P}(\boldsymbol{R}^d)$ the dimension of the eigenspace of $\tilde{L}_{e(p)}$ corresponding to eigenvalue 0 is one.*

(ii) *There exists a positive constant κ_0 such that for any $p\in\mathcal{P}(\boldsymbol{R}^d)$*

$$\mathrm{Spec}(\tilde{L}_{e(p)})\subset(-\infty,-\kappa_0]\cup\{0\}.$$

PROOF. By Lemma 3.1, $\mathcal{D}((-\tilde{L}_{e(p)})^{1/2})=H^1(\boldsymbol{R}^d;e(p)dx)$ for any $p\in\mathcal{P}(\boldsymbol{R}^d)$. Since $H^1(\boldsymbol{R}^d;e(p_1)dx)=H^1(\boldsymbol{R}^d;e(p_2)dx)$ as a set for any p_1 and $p_2\in\mathcal{P}(\boldsymbol{R}^d)$, we denote it by $\mathcal{H}$. Then to prove (ii) of the lemma, it will suffice to show

$$\text{(6.8)}\qquad \inf_{p\in\mathcal{P}(\boldsymbol{R}^d)}\ \inf_{f\in\mathcal{H},f\neq\text{const.}}\frac{\frac{1}{2}\||\mathrm{grad}\,f|\|^2_{L^2(e(p)dx)}}{\|f-(f,1)_{L^2(e(p)dx)}\|_{L^2(e(p)dx)}}>0.$$

Let assume that the left hand side of (6.8)$=0$. Then there exists a sequence $\{p_n;n\in N\}$ in $\mathcal{P}(\boldsymbol{R}^d)$ such that

$$\text{(6.9)}\qquad \inf_{f\in\mathcal{H},f\neq\text{const.}}\frac{\frac{1}{2}\||\mathrm{grad}\,f|\|^2_{L^2(e(p_n)dx)}}{\|f-(f,1)_{L^2(e(p_n)dx)}\|_{L^2(e(p_n)dx)}}\longrightarrow 0.$$

Since $\Phi_2 \in \mathcal{S}(\boldsymbol{R}^{2d})$, $\left\{\int_{\boldsymbol{R}^d} \Phi_2(\cdot, y)p(dy); p \in \mathcal{P}(\boldsymbol{R}^d)\right\}$ is precompact in $\mathcal{S}(\boldsymbol{R}^d)$. Then there exist a subsequence $\{p_{n'}; n \in N\}$ of $\{p_n; n \in N\}$ and $\varphi_2 \in \mathcal{S}(\boldsymbol{R}^d)$ such that $\int_{\boldsymbol{R}^d} \Phi_2(\cdot, y)p_{n'}(dy) \to \varphi_2(\cdot)$ in $\mathcal{S}(\boldsymbol{R}^d)$. Set $q_1(x) = Z_{q_1}^{-1}\exp(-2\Phi_1(x) + 2\varphi_2(x))$, $L_{q_1}f(x) = \frac{1}{2}q_1(x)^{-1}\operatorname{div}(q_1(x)\operatorname{grad} f(x))$ and let $\tilde{L}_{q_1}$ be the Friedrichs' extension of L_{q_1} in $L^2(q_1dx)$. Then, since $\tilde{L}_{q_1}$ has a discrete spectrum and the eigenvalue 0 is simple, there exists a positive constant δ such that

$$
(6.10) \qquad \frac{\frac{1}{2}\| |\operatorname{grad} f| \|^2_{L^2(q_1dx)}}{\| f - (f, 1)_{L^2(q_1dx)}\|_{L^2(q_1dx)}} \geqq \delta.
$$

On the other hand, since $\|q_1/e(p_{n'}) - 1\|_\infty \to 0$ and $\|e(p_{n'})/q_1 - 1\|_\infty \to 0$ as $n \to \infty$, for any $\varepsilon > 0$ there exists n_1 such that for any $n \geqq n_1$ and for any non-constant $f \in \mathcal{H}$

$$
(6.11) \qquad \frac{\frac{1}{2}\| |\operatorname{grad} f| \|^2_{L^2(q_1dx)}}{\| f - (f, 1)_{L^2(q_1dx)}\|_{L^2(q_1dx)}} \leqq (1+\varepsilon)\frac{\frac{1}{2}\| |\operatorname{grad} f| \|^2_{L^2(e(p_{n'})dx)}}{\| f - (f, 1)_{L^2(e(p_{n'})dx)}\|_{L^2(e(p_{n'})dx)}}.
$$

This and (6.9) imply that for any $\varepsilon > 0$ there exist $n \in N$ and non-constant $f \in \mathcal{H}$ such that

$$
(6.12) \qquad \frac{\frac{1}{2}\| |\operatorname{grad} f| \|^2_{L^2(q_1dx)}}{\| f - (f, 1)_{L^2(q_1dx)}\|_{L^2(q_1dx)}} \leqq \varepsilon.
$$

This contradicts (6.10). (Q.E.D.)

LEMMA 6.10. *There exists a positive constant* τ *such that*

$$
\|e^{\tau L_{e(p)}} f\|_{L^4(e(p)dx)} \leqq \|f\|_{L^2(e(p)dx)}
$$

for any $p \in \mathcal{P}(\boldsymbol{R}^d)$ *and any* $f \in L^2(e(p)dx)$.

PROOF. The following argument is due to J. Glimm. Let C $(\geqq 1)$ and t_0 be constants as in Lemma 6.8 and κ_0 be a constant as in Lemma 6.9. Let $\tau = t_0 + \frac{2}{\kappa_0}\log C$. For any $p \in \mathcal{P}(\boldsymbol{R}^d)$ and any $f \in L^2(e(p)dx)$ there exists a constant c such that $f = \bar{f} + c$ and $(\bar{f}, c)_{L^2(e(p)dx)} = 0$. Then,

$$
\begin{aligned}
\|e^{tL_{e(p)}}f\|^4_{L^4(e(p)dx)} &\leqq |c|^4+6|c|^2\|e^{tL_{e(p)}}\bar{f}\|_{L^2(e(p)dx)} \\
&\quad +4|c|C^3\|e^{tL_{e(p)}}\bar{f}\|^3_{L^2(e(p)dx)}+C^4\|e^{tL_{e(p)}}\bar{f}\|^4_{L^2(e(p)dx)} \qquad \text{(by Lemma 6.8)}\\
&\leqq |c|^4+6|c|^2\|\bar{f}\|^2_{L^2(e(p)dx)}e^{-2\kappa_0 t}+4|c|C^3\|\bar{f}\|^3_{L^2(e(p)dx)}e^{-3\kappa_0(t-t_0)} \\
&\quad +C^4\|\bar{f}\|^4_{L^2(e(p)dx)}e^{-4\kappa_0(t-t_0)} \qquad \text{(by Lemma 6.9)}\\
&\leqq |c|^4+|c|^2\|\bar{f}\|^2_{L^2(e(p)dx)}(6e^{-2\kappa_0 t}+4C^3e^{-3\kappa_0(t-t_0)}) \\
&\quad +\|\bar{f}\|^4_{L^2(e(p)dx)}(C^4e^{-4\kappa_0(t-t_0)}+4C^3e^{-3\kappa_0(t-t_0)}) \\
&\leqq (|c|^2+\|\bar{f}\|^2_{L^2(e(p)dx)})^2, \qquad \text{for } t\geqq\tau \\
&= \|f\|^4_{L^2(e(p)dx)}. \qquad \text{(Q.E.D.)}
\end{aligned}
$$

Let us recall that $I(p)=\frac{1}{2}\||\mathrm{grad}\,\varphi|\|^2_{L^2(e(p)dx)}$, $\varphi(x)=(p(x)/e(p)(x))^{1/2}$ if p has the density $p(x)$. Then by virtue of Gross' result [7], the equivalence of hypercontractivity and logarithmic Sobolev inequality, we have the following by Lemma 6.10.

LEMMA 6.11. *There exists a positive constant γ_4 such that*

$$H_{e(p)}(p)\leqq\gamma_4 I(p)$$

for any $p\in\mathcal{P}(\boldsymbol{R}^d)$.

The following lemmas are our main results in this section.

LEMMA 6.12. *Let q be an invariant probability measure satisfying* (Cond. 1) *in Theorem* 1.2. *Then there exists a positive constant λ such that for any $p_0\in\mathcal{P}(\boldsymbol{R}^d)$ with $T_tp_0\to q$ (weakly) there exists a positive time t_0 such that for any $t\geqq t_0$*

$$F(T_tp_0)-F(q)\leqq(F(T_{t_0}p_0)-F(q))e^{-\lambda(t-t_0)}. \tag{6.13}$$

LEMMA 6.13. *Let q be an invariant probability measure satisfying* (Cond. 2) *in Theorem* 1.3. (i) *If $\alpha=2$, then there exist a positive constant λ and an open neighborhood V of q in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ such that for any $p_0\in V$ and any $t\geqq 1$*

$$F(T_tp_0)-F(q)\leqq(F(T_1p_0)-F(q))e^{-\lambda(t-1)}. \tag{6.14}$$

(ii) *If $\alpha>2$, then there exist a positive constant λ and an open neighborhood V of q in $\mathcal{P}(\boldsymbol{R}^d)$ such that* (6.14) *holds for any $p_0\in V$ and any $t\geqq 1$.*

PROOF. Since the proof of Lemma 6.12 is similar, we give only the proof of Lemma 6.13. First we show that under (Cond. 2) the invariant probability measure q is isolated in $\mathcal{P}(\boldsymbol{R}^d)$. Let $\{q_n; n\in N\}$ be a sequence of invariant probability measures that converge to q weakly. Then since $q_n=e(q_n)$, by Lemma 6.4, $q_n\to q$ in $L^\infty(q^{-1})$ as $n\to\infty$. Then by (6.5), for any $\varepsilon>0$ there exists q_n such that $((I-K_q)(q_n-q), q_n-q)_{L^2(qdx)}<\varepsilon\|q_n-q\|^2_{L^2(qdx)}$, $(q_n-q\neq 0)$. This contradicts that $(I-K_q)$ is strictly positive definite. If q satisfies (Cond. 1), then by Lemmas 6.7 and 6.11 there exist positive constants γ_3 and γ_4 and an open neighborhood V_1 of q in $\mathcal{P}(\boldsymbol{R}^d)$ such that for any $p\in V_1$

$$F(p)-F(q)\leqq\gamma_3\gamma_4 I(p). \tag{6.15}$$

By Theorem 1.1, there exists an open neighborhood V_2 of q in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ [resp. $\mathcal{P}(\boldsymbol{R}^d)$] such that for any $p_0\in V_2$ $T_tp_0\to q$ (weakly) as $t\to\infty$ if $\alpha=2$ [resp. $\alpha>2$]. Set $V=V_2\cap T_1^{-1}V_1$. Then Theorem 4.2 and (6.15), we get

$$(F(T_sp_0)-F(q))-(F(T_tp_0)-F(q))\geqq\int_s^t\frac{4}{\gamma_3\gamma_4}(F(T_up_0)-F(q))du \tag{6.16}$$

for any $p_0\in V$ and $t\geqq s$. Thus setting $\lambda=\dfrac{4}{\gamma_3\gamma_4}$, we get (6.14). (Q.E.D.)

§ 7. Proof of Theorems 1.2 and 1.3.

LEMMA 7.1. *Let q be an invariant probability measure and p_0 be a probability measure on $\boldsymbol{R}^d$ such that $T_tp_0\to q$ weakly and there exist positive constants λ, M and t_0 such that $F(T_tp_0)-F(q)\leqq Me^{-\lambda(t-t_0)}$ for any $t\geqq t_0$. Then there exists a sequence of times $\{t_n; n\in N\}$ such that* (i) *$t_0+n\leqq t_n\leqq t_0+n+1$ and* (ii) *$I(T_{t_n}p_0)\leqq\dfrac{M}{4}e^{-\lambda n}$.*

PROOF. Since $F(T_tp_0)$ is monotone decreasing, $F(T_tp_0)\geqq F(q)$. Then by Theorem 4.2,

$$Me^{-\lambda(s-t_0)}\geqq F(T_sp_0)-F(q)\geqq F(T_sp_0)-F(T_tp_0)\geqq 4\int_s^t I(T_up_0)du$$

for any $t_0\leqq s<t$. Set $s=t_0+n$ and $t=t_0+n+1$. Then clearly there exists a sequence $\{t_n; n\in N\}$ which satisfies (i) and (ii) of the lemma. (Q.E.D.)

In the following of this section we shall give only the proof of Theorem 1.3, since the proof of Theorem 1.2 is similar. Let q be an invariant probability measure which satisfies (Cond. 2). First we show the exponential order convergence of T_tp_0 to q with respect to L^1-

norm. By Ito's formula for the McKean process (1.1), we see that $\int_{R^{2d}}\Phi_2(x,y)(T_tp_0(x)-q(x))(T_tp_0(y)-q(y))dxdy$ is differentiable in t (>0) and

$$\text{(7.1)}\qquad \frac{d}{dt}\int_{R^{2d}}\Phi_2(x,y)(T_tp_0(x)-q(x))(T_tp_0(y)-q(y))dxdy$$
$$=2\int_{R^{2d}}L_{e(T_tp_0),x}\Phi_2(x,y)T_tp_0(x)(T_tp_0(y)-q(y))dxdy.$$

Then by Lemma 6.2,

$$\text{(7.2)}\qquad H_q(T_tp_0)=F(T_tp_0)-F(q)$$
$$+\int_{R^{2d}}\Phi_2(x,y)(T_sp_0(x)-q(x))(T_sp_0(y)-q(y))dxdy$$
$$+2\int_s^t du\int_{R^{2d}}L_{e(T_up_0),x}\Phi_2(x,y)T_up_0(x)(T_up_0(y)-q(y))dxdy$$

for any $t\geqq s>0$. By Lemmas 6.1, 6.3, 6.5, 6.6 and 6.11, for q there exist positive constants γ_1, γ_2 and γ_4 such that

$$\text{(7.3)}\qquad \|T_tp_0-q\|_{L^1}^2\leqq 4(1+\gamma_1\gamma_2)\gamma_4 I(T_tp_0).$$

Since $\Phi_2\in\mathcal{S}(\boldsymbol{R}^{2d})$, by (3.3) there exist positive constants C_1 and C_2 such that

$$\text{(7.4)}\qquad H_q(T_tp_0)\leqq F(T_tp_0)-F(q)+C_1I(T_sp_0)+C_2\sqrt{t-s}\left(\int_s^t I(T_up_0)du\right)^{1/2}$$
$$\leqq F(T_tp_0)-F(q)+C_1I(T_sp_0)+\frac{C_2\sqrt{t-s}}{4}(F(T_sp_0)-F(T_tp_0))^{1/2}$$
$$\text{(by Theorem 4.2).}$$

Then by Lemmas 6.1, 6.13 and 7.1 there exist a positive constant λ, an open neighborhood V_0 of q in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ [resp. $\mathcal{P}(\boldsymbol{R}^d)$] if $\alpha=2$ [resp. $\alpha>2$] and a sequence $\{t_n; n\in N\}$ with $n+1<t_n<n+3$ such that

(i) $F(T_tp_0)-F(q)\leqq(F(T_1p_0)-F(q))e^{-\lambda(t-1)}$

and

(ii) $I(T_{t_n}p_0)\leqq\frac{1}{4}(F(T_1p_0)-F(q))e^{-\lambda n}$.

Letting $s=t_n$, we have from (7.4),

(7.5) $$\|T_t p_0 - q\|_{L^1}^2 \leqq 2(F(T_1 p_0) - F(q))e^{-\lambda(t-1)} + \frac{1}{2}C_1(F(T_1 p_0) - F(q))e^{-\lambda n} + C_2(F(T_1 p_0) - F(q))^{1/2} e^{-(\lambda/2)(t_n - 1)}$$

for any $p_0 \in V_0$ and any $t \in [t_n, t_{n+1})$. Since $F(T_1 \cdot)$ is continuous as a function on $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ [resp. $\mathcal{P}(\boldsymbol{R}^d)$] if $\alpha = 2$ [resp. $\alpha > 2$] (Theorem 4.1), there exist positive constants λ and M and an open neighborhood V of q in $\mathcal{P}^{(2)}(\boldsymbol{R}^d)$ [resp. $\mathcal{P}(\boldsymbol{R}^d)$] if $\alpha = 2$ [resp. $\alpha > 2$] such that for any $t > 1$

(7.6) $$\|T_t p_0 - q\|_{L^1(dx)} \leqq M e^{-(\lambda/4)t}.$$

Next we prove exponential order convergence with respect to $L^\infty(q^{-1})$-norm in the case where $\alpha > 2$. By Proposition 4.2, there exists a positive constant C such that

$$\begin{aligned}\|(T_{t+1}p_0 - q)/q\|_{L^\infty} &= \|(T_1 T_t p_0 - T_1 q)/q\|_{L^\infty} \\ &\leqq C\left(\|T_t p_0 - q\|_{L^1} + \|T_{t+1}p_0 - q\|_{L^1} + \int_t^{t+1} \|T_u p_0 - q\|_{L^1} du\right) \\ &\leqq M' e^{-(\lambda/4)t},\end{aligned}$$

where $M' = C(M + Me^{-\lambda/4} + 4M\lambda^{-1})$. This completes the proof of the theorem. (Q.E.D.)

Appendix A.

In this appendix, we shall give some estimates on the transition probability densities of the diffusion processes on $\boldsymbol{R}^d$ with the generators of the form $L = \frac{1}{2}\Delta_x - \mathrm{grad}_x \Phi(x, t) \cdot \mathrm{grad}_x$ by Cameron-Martin-Maruyama-Girsanov's formula. We assume that $\Phi \in C^\infty(\boldsymbol{R}^d \times [0, \infty))$, and we denote by $p(s, y, t, x)$ the transition probability density of the diffusion governed by

(A.1) $$dX(t) = dB(t) - \mathrm{grad}_x \Phi(X(t), t)dt, \qquad X(s) = y.$$

First notice Cameron-Martin-Maruyama-Girsanov's formula (Ershov [5]) for the diffusion of

(A.2) $$dX(t) = dB(t) + b(X(t), t)dt, \qquad X(s) = y.$$

LEMMA A.1. *Let P^B and P^X be the probability distributions on $C([s, T] \to \boldsymbol{R}^d)$ of the d-dimensional Brownian motion and the diffusion* *process of* (A.2), *respectively. Suppose that the following two conditions are satisfied*:

(i) $P^B\left(\int_s^T |b(w(u), u)|^2 du<\infty\right)=1$

and

(ii) $E^B\left[\exp\left(-\int_s^T b(w(u), u)dw(u) -\frac{1}{2}\int_s^T |b(w(u), u)|^2 du\right)\right]=1.$

Then, there exists a unique solution of (A.2) *and* P^X *and* P^B *are mutually absolutely continuous with the density*

$$\frac{dP^X}{dP^B}(w)=\exp\left(-\int_s^T b(w(u), u)dw(u) -\frac{1}{2}\int_s^T |b(w(u), u)|^2 du\right).$$

We set

$$g_0(t, x)=(2\pi)^{-d/2}\exp(-|x|^2/2t). \tag{A.3}$$

THEOREM A.1. *Suppose that there exists a constant* $C\geqq 0$ *such that*

$$\begin{cases} -\Phi(x, u)\leqq C \\ -\frac{1}{2}|\mathrm{grad}_x \Phi(x, u)|^2+\frac{1}{2}\Delta_x\Phi(x, u)+\frac{\partial}{\partial u}\Phi(x, u)\leqq C \end{cases} \tag{A.4}$$

for any $u\in[s, T]$ *and* $x\in \boldsymbol{R}^d$. *Then,*

$$\begin{aligned} p(s, y, t, x)=g_0(t-s, x-y)\exp(-\Phi(x, t)+\Phi(y, s)) \\ \times E\left[\exp\left(-\int_s^t \Psi(B_{s,y}^{t;x}(u), u)du\right)\right], \end{aligned} \tag{A.5}$$

where

$$\Psi(x, u)=\frac{1}{2}|\mathrm{grad}_x \Phi(x, u)|^2-\frac{1}{2}\Delta_x\Phi(x, u)-\frac{\partial}{\partial u}\Phi(x, u) \tag{A.6}$$

and

$$B_{s,y}^{t;x}(u)=\frac{u-s}{t-s}x+\frac{t-u}{t-s}y+B(u-s)-\frac{u-s}{t-s}B(t-s) \tag{A.7}$$

(=*the law of pinned Brownian motion starting at* y *at time* s, *where* $B(\cdot)$ *is a standard Brownian motion*).

Furthermore,

$$(\text{A.8}) \quad \begin{cases} g_0(t-s, x-y)\exp\left\{-C-\Phi(x,t)-\int_s^t E[\Psi(B_{s,y}^{t,x}(u),u)]du\right\} \\ \leqq p(s,y,t,x) \\ \leqq g_0(t-s,x-y)\exp(\Phi(y,s)+C(1+T)). \end{cases}$$

PROOF. To prove (A.5), it will suffice to check the condition (ii) in Lemma A.1. We put $\sigma_n=\inf\{u>s;\ |\mathrm{grad}_x\Phi(w(u),u)|>n\}$. Then, it is easy to check that $M(t)=\exp\left(-\int_s^t \mathrm{grad}_x\Phi(w(u),u)dw(u)-\frac{1}{2}\int_s^t|\mathrm{grad}_x\Phi(w(u),u)|^2du\right)$ is a local martingale under P^B and $E[M(t\wedge\sigma_n)]=1$. On the other hand $\int_s^t \mathrm{grad}_x\Phi(w(u),u)dw(u)=\Phi(w(t),t)-\Phi(w(s),s)-\int_s^t\left\{\frac{1}{2}\Delta_x\Phi(w(u),u)+\frac{\partial}{\partial u}\times\Phi(w(u),u)\right\}du$ (a.s.) under P^B. Then, $M(t)\leqq\exp(\Phi(y,s)+C(1+T)$ (a.s.) under P^B. By Lebesgue's bounded convergence theorem, we get $E^B[M(t)]=1$. The estimate (A.8) from bellow is given by Jensen's inequality. (Q.E.D.)

Throughout the rest of this section, we assume that $\Phi(x,t)$ has the form

(Φ.0) $\Phi(x,t)=\Phi_1(x)+\Phi_2(x,t)$ and they satisfy

(Φ.1) $\Phi_1\in C^\infty(\boldsymbol{R}^d)$ and there exists positive constants α and A such that $\Phi_1(x)=A|x|^\alpha$ for any $|x|>r$ for some $r>0$.

(Φ.2) $\Phi_2(x,t)\in C_b^\infty(\boldsymbol{R}^d\times(0,\infty))$ and for any multi indices $i,j\in(N_0)^d$ and $k\in N_0$ and any $T>0$, there exists a constant C such that for any $(x,t)\in\boldsymbol{R}^d\times(0,T)$

$$\left|(1+(x)^i)\left(\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial t}\right)^k\Phi(x,t)\right|\leqq C.$$

Let $q(t,y,x)$ be the transition probability density of the diffusion $Y(t)$ defined by (2.1). We set $L_0'=\frac{1}{2}\Delta-\frac{1}{2}(|\mathrm{grad}\,\Phi_1(x)|^2-\Delta\Phi_1(x))$, which is given by $L_0'=U^{-1}L_0U$, where $L_0=\frac{1}{2}\Delta-\mathrm{grad}\,\Phi_1(x)\cdot\mathrm{grad}$ is the generator of the diffusion (2.1), and $Uf(x)=Z_0^{1/2}\exp(-\Phi_1(x))f(x)$ is a unitary operator from $L^2(\boldsymbol{R}^d;dx)$ to $L^2(\boldsymbol{R}^d;Z_0\exp(-2\Phi_1(x))dx)$. Then, by the results of Davies and Simon [2] (Theorem 5.2 (b) and Theorem 3.2) we get

PROPOSITION A.1. *Let $\alpha>2$. Then the semigroup $\{e^{tL_0'};t\geqq0\}$ is intrinsically ultracontractive. Therefore, for any $0<t_0<T$, there exist*

positive constants C_1 *and* C_2 *such that for any* $x, y \in \boldsymbol{R}^d$ *and* $t \in [t_0, T]$

$$C_1 e^{-2\Phi_1(x)} \leqq q(t, y, x) \leqq C_2 e^{-2\Phi_1(x)}. \tag{A.9}$$

THEOREM A.2. *Under assumptions* (Φ.0), (Φ.1) *and* (Φ.2) *the following estimates hold.*

(I)(i) *Let* $0<\alpha<2$. *For any* $0<t_0<T$, *there exist positive constants* $C_1, C_2, C_3, C_3', C_4, C_4', C_5$ *and* C_5' *such that for any* $x, y \in \boldsymbol{R}^d$ *and any* $0 \leqq s<t \leqq T$, $t-s \geqq t_0$

$$\begin{aligned} C_1 \exp(-C_2(|x|^2+|y|^2)) &\leqq p(s, y, t, x) \\ &\leqq C_3 \exp(C_4|x|^{\alpha\vee 1}-C_5|x-y|^2) \wedge C_3' \exp(C_4'|y|^\alpha - C_5'|x-y|^2). \end{aligned}$$

(I)(ii) *Let* $\alpha=2$. *For any* $0<t_0<T$, *there exist positive constants* C_1, C_2, C_3 *and* C_4 *such that for any* $x, y \in \boldsymbol{R}^d$ *and any* $0 \leqq s<t \leqq T$, $t-s \geqq t_0$

$$C_1 \exp(-C_2(|x|^2+|y|^2)) \leqq p(s, y, t, x) \leqq C_3 \exp(-C_4|x-y|^2).$$

(I)(iii) *Let* $\alpha>2$. *For any* $0<t_0<T$, *there exist positive constants* C_1 *and* C_2 *such that for any* $x, y \in \boldsymbol{R}^d$ *and any* $0 \leqq s<t \leqq T$, $t-s \geqq t_0$

$$C_1 \exp(-2\Phi_1(x)) \leqq p(s, y, t, x) \leqq C_2 \exp(-2\Phi_1(x)).$$

(II) *Let* $\alpha>0$. *For any* $0<t_0<T$, *any multi-indices* i *and* $j \in (N_0)^d$ *and any* $a \in (1, \infty)$, *there exists a positive constant* C *such that for any* $0 \leqq s<t \leqq T$, $t-s \geqq t_0$ *and any* $x, y \in \boldsymbol{R}^d$

$$\begin{aligned} &|(\partial/\partial x)^i (\partial/\partial y)^j p(s, y, t, x)| \\ &\leqq (1+|x|^{|i|((\alpha-1)\vee 0)}+|y|^{|j|((\alpha-1)\vee 0)}+|x-y|^{|i|+|j|}) \\ &\quad \times (1+\exp(\Phi_1(y)/a) p(s, y, t, x)^{(1-1/a)}. \end{aligned} \tag{A.10}$$

(III)(i) *Let* $0<\alpha<2$. *For any* $0<t_0<T$ *and any multi-indices* i *and* $j \in (N_0)^d$ *and any* $n \in N_0$, *there exist positive constants* C_1, C_2 *and* C_3 *such that for any* $0 \leqq s<t \leqq T$, $t-s \geqq t_0$ *and any* $x, y \in \boldsymbol{R}^d$

$$(1+|x|)^n (1+|y|)^n |(\partial/\partial x)^i (\partial/\partial y)^j p(s, y, t, x)| \leqq C_1 \exp(C_2|y|^\alpha - C_3|x-y|^2).$$

(III)(ii) *Let* $\alpha=2$. *For any* $0<t_0<T$ *and any multi-indices* i *and* $j \in (N_0)^d$ *and any* $n \in N_0$, *there exist positive constants* C_1, C_2, C_1' *and* C_2' *such that for any* $0 \leqq s<t \leqq T$, $t-s \geqq t_0$ *and any* $x, y \in \boldsymbol{R}^d$

$$\begin{aligned} &(1+|x|)^n (1+|y|)^n |(\partial/\partial x)^i (\partial/\partial y)^j p(s, y, t, x)| \\ &\leqq \{C_1 (1+|y|)^{2n+|i|+|j|} \exp(-C_2|x-e^{-2A(t-s)}y|^2)\} \\ &\quad \wedge \{C_1' (1+|x|)^{2n+|i|+|j|} \exp(-C_2'|x-e^{-2A(t-s)}y|^2)\}. \end{aligned}$$

(III)(iii) *Let* $\alpha>2$. *For any* $0<t_0<T$, *any multi-indices* i *and* $j\in(N_0)^d$, *any* $n\in N_0$ *and any* $a\in(1,\infty)$, *there exists a positive constant* C *such that for any* $0\leqq s<t\leqq T$, $t-s\geqq t_0$ *and any* $x, y\in \boldsymbol{R}^d$

$$\begin{aligned}&(1+|x|)^n(1+|y|)^n|(\partial/\partial x)^i(\partial/\partial y)^j p(s, y, t, x)|\\&\leqq C(1+\exp(\Phi_1(y)/a))\exp(-2(1-1/a)\Phi_1(x)).\end{aligned}$$

PROOF. By the assumptions on Φ, the condition (A.4) in Theorem A.1 is satisfied. Then, by (A.5),

$$p(s, y, t, x)=g_0(t-s, x-y)\exp(-\Phi(x, t)+\Phi(y, s))r(s, y, t, x) \tag{A.11}$$

where

$$\begin{aligned}r(s, y, t, x)&=E\Big[\exp\Big(-\int_s^t \Psi(B_{s,y}^{t;x}(u), u)du\Big)\Big],\\ \Psi(x, u)&=\frac{1}{2}|\mathrm{grad}\,\Phi_1(x)|^2-\frac{1}{2}\Delta\Phi_1(x)+\mathrm{grad}\,\Phi_1(x)\cdot\mathrm{grad}_x\Phi_2(x, u)\\&\quad+\frac{1}{2}|\mathrm{grad}_x\Phi_2(x, u)|^2-\frac{1}{2}\Delta_x\Phi_2(x, u)-\frac{\partial}{\partial u}\Phi_2(x, u),\end{aligned}$$

and

$B_{s,y}^{t;x}(u)$ is given in (A.7).

Let $0<\alpha<2$. Then for any $b>0$, there exists a constant C such that for any $x, y\in \boldsymbol{R}^d$ $|y|^\alpha-b|x-y|^2\leqq 2^{\alpha\vee 1-1}|x|^{\alpha\vee 1}-\frac{b}{2}|x-y|^2+C$, which implies (I)(i). (I)(iii) follows from Proposition A.1. Let $\alpha=2$. Then, by Cameron-Martin-Maruyama-Girsanov's formula with Ornstein-Uhlenbeck process, we get

$$\begin{aligned}p(s, y, t, x)&=g_1(t-s, y, x)\exp(-\Phi'(x, t)+\Phi'(y, s))\\&\quad\times E\Big[\exp\Big(-\int_s^t \Psi'(B'^{t;x}_{s,y}(u), u)du\Big)\Big],\end{aligned} \tag{A.12}$$

where

$$g_1(t, y, x)=\Big(\frac{2A}{\pi(1-\exp(-4At))}\Big)^{d/2}\exp\Big(\frac{2A(x-y\exp(-2A))^2}{1-\exp(-4At)}\Big). \tag{A.13}$$

$$\Phi'(x, u)=\Phi(x, u)-A|x|^2,$$

$$\begin{aligned}\Psi'(x, u)&=\frac{1}{2}|\mathrm{grad}_x\Phi'(x, u)|^2+2A\,\mathrm{grad}_x\Phi'(x, u)\cdot x\\&\quad-\frac{1}{2}\Delta_x\Phi'(x, u)-\frac{\partial}{\partial u}\Phi'(x, u),\end{aligned} \tag{A.14}$$

(A.15) $$B'^{t;x}_{s,t}(u)=(\exp(-2A(u-s))-c(u;s,t)\exp(-2A(t-s)))y+c(u;s,t)x+B'(u-s)-c(u;s,t)B'(t-s)$$
(=the law of pinned Ornstein-Uhlenbeck process starting at y at time s), where
$$c(u;s,t)=\frac{1-\exp(-4A(u-s))}{1-\exp(-4A(t-s))}\exp(-2A(t-u)) \quad \text{and}$$
$B'(\cdot)$ is a Ornstein-Uhlenbeck process starting at 0 at time 0.

Using this expression, we get (I)(ii).

By (A.11), we get

$$\frac{\partial}{\partial y_i}r(s,y,t,x)=-E\left[\left(\int_s^t \partial_i\Psi(B^{t;x}_{s,y}(u),u)\frac{t-u}{t-s}du\right)\exp\left(-\int_s^t \Psi(B^{t;x}_{s,y}(u),u)du\right)\right].$$

Since $\Psi(x)\simeq|x|^{2(\alpha-1)}$ and $|\partial_i\Psi(x)|\simeq|x|^{2(\alpha-1)-1}$ for $|x|>r$, there exist positive constants C_1 and C_2 such that for any $x\in \boldsymbol{R}^d$, $|\partial_i\Psi(x)|\leqq C_1+C_2\Psi(x)$. Then by Hölder's inequality

$$\begin{aligned}&\left|\frac{\partial}{\partial y_i}r(s,y,t,x)\right|\\ &\leqq\left\{E\left[\left(C_1(t-s)+C_2\int_s^t\Psi(B^{t;x}_{s,y}(u),u)du\right)^a\exp\left(-\int_s^t\Psi(B^{t;x}_{s,y}(u),u)du\right)\right]\right\}^{1/a}\\ &\quad\times\left\{E\left[\exp\left(-\int_s^t\Psi(B^{t;x}_{s,y}(u),u)du\right)\right]\right\}^{1-1/a}\end{aligned}$$

for any $a\in(1,\infty)$. Then for any $a\in(1,\infty)$ and $T>0$ there exists a positive constant C such that for any $0\leqq s<t\leqq T$ and $x,y\in\boldsymbol{R}^d$

$$\left|\frac{\partial}{\partial y_i}r(s,y,t,x)\right|\leqq C\{r(s,y,t,x)\}^{1-1/a}.$$

In a similar way, we can show that for any multi-indices i and $j\in(N_0)^d$ and any $a\in(1,\infty)$ there exists a positive constant C such that

$$\left|\left(\frac{\partial}{\partial x}\right)^i\left(\frac{\partial}{\partial y}\right)^j r(s,y,t,x)\right|\leqq C\{r(s,y,t,x)\}^{1-1/a},$$

which implies (II). (III)(i) and (III)(iii) are immediate consequences of (I) and (II). (III)(ii) is given by differentiating (A.13). (Q.E.D.)

Appendix B.

In this appendix we consider the case of linear equations in one dimension (i.e., $d=1, \Phi_2=0$).

PROPOSITION B.1. *Let $\alpha=2$ and $A=1/2$ and q_0 be the invariant probability measure of Ornstein-Uhlenbeck process. Then, there exists a positive constant C such that for any $p_0 \in \mathcal{P}^{(2)}(\boldsymbol{R})$,*

$$\|T_t p_0 - q_0\|_{L^1(\boldsymbol{R};dx)} \leqq C\left(1+\int_{\boldsymbol{R}} |x|^2 p_0(dx)\right)^{1/2} e^{-(1/2)t}.$$

PROOF. Let $H_{q_0}(p_0)<\infty$. Then, our proof is simple in a linear case as follows: By Theorem 4.2, we have

$$H_{q_0}(T_t p_0) - H_{q_0}(T_s p_0) \leqq -4\int_s^t I_{q_0}(T_u p_0)du.$$

By Gross' logarithmic Sobolev inequality for Ornstein-Uhlenbeck process [7], we have $H_{q_0}(p) \leqq 4I_{q_0}(p)$ for any $p \in \mathcal{P}(\boldsymbol{R}^d)$. Then, $H_{q_0}(T_t p_0) \leqq H_{q_0}(p_0)e^{-t}$. By Lemma 6.1, we have

$$\|T_t p_0 - q_0\|_{L^1(\boldsymbol{R}^1;dx)} \leqq (2H_{q_0}(T_t p_0))^{1/2} \leqq (2H_{q_0}(p_0))^{1/2} e^{-(1/2)t}.$$

By (A.13), there exist positive constants C_1, C_2 and C_3 such that

$$C_1 e^{-C_2(|x|^2+|y|^2)} \leqq g_1(1, x, y) \leqq C_3.$$

Then by Jensen's inequality,

$$\begin{aligned}
&\int_{\boldsymbol{R}} |\log T_1 p_0(x)| T_1 p_0(x)dx \\
&\leqq (|\log C_1| \vee |\log C_3| \vee C_2)\left(1+\int_{\boldsymbol{R}} |y|^2 p_0(dy) + \int_{\boldsymbol{R}} |x|^2 T_1 p_0(x)dx\right) \\
&\leqq C_4\left(1+\int_{\boldsymbol{R}} |y|^2 p_0(dy)\right)
\end{aligned}$$

for some positive constant C_4. Then there exists a positive constant C_5 such that

$$H_{q_0}(T_1 p_0) \leqq C_5\left(1+\int_{\boldsymbol{R}} |y|^2 p_0(dy)\right)$$

for any $p_0 \in \mathcal{P}^{(2)}(\boldsymbol{R})$, which completes the proof. (Q.E.D.)

Let p_R ($R \in \boldsymbol{R}$) be the probability measure on $\boldsymbol{R}$ with the density $p_R(x)=1$ if $R+1 \leqq x \leqq R+2$ and $p_R(x)=0$, otherwise.

Let m be a given positive constant and let $\{R_n; n \in N\}$ be a sequence of positive numbers that satisfies the following conditions: (i) $2 \leqq R_1 < R_2$

$<\cdots\to\infty$, (ii) $\sum_{n=1}^{\infty} R_n^{-1}<\infty$ and (iii) there exists a positive constant $a\geqq 1$ such that $R_n\leqq(R_{n-1})^a$ for any $n\in N$. (e.g., $R_n=(n+1)^2$, $R_n=b^n$ $(b>1)$, $R_n=b^{a^n}$ $(b>1, a>1)$.) We set $Z=\sum_{n=1}^{\infty}(1+R_n)^{-(m+1)}$ ($<\infty$ by (ii)).

PROPOSITION B.2. *Let* $1<\alpha<2$ *and let* q_0 *be the invariant probability measure of diffusion process defined by* SDE (2.1). *Set*

$$p_0=\sum_{n=1}^{\infty} a_n p_{R_n}$$

where

$$a_n=Z^{-1}(1+R_n)^{-(m+1)}.$$

Then, $p_0\in\mathcal{P}^{(m)}(\boldsymbol{R})$ *and*

$$\overline{\lim_{t\to\infty}}\left(-\frac{1}{t}\log\|T_t p_0-q_0\|_{L^1(\boldsymbol{R};dx)}\right)=0.$$

PROOF. First we have

$$\int_{\boldsymbol{R}}|x|^m p_0(dx)\leqq\sum_{n=1}^{\infty}a_n(R_n+2)^m=Z^{-1}\sum_{n=1}^{\infty}(2+R_n)^m(1+R_n)^{-(m+1)}$$
$$\leqq Z^{-1}2^m\sum_{n=1}^{\infty}R_n^{-1}<\infty.$$

Thus, $p_0\in\mathcal{P}^{(m)}(\boldsymbol{R})$.

Let f_R be a bounded function in $C^\infty(\boldsymbol{R})$ such that (i) $f_R(x)=0$ if $x\leqq 0$ and $f_R(x)=R$ if $x\geqq R+1$, (ii) $0\leqq f_R(x)\leqq x$ for any $x\geqq 0$ and $x\leqq f_R(x)+1$ for $0\leqq x\leqq R+1$ and (iii) $0\leqq f_R'(x)\leqq 1$, $|f_R''(x)|\leqq 2$ for any $x\in\boldsymbol{R}$, here we use the following notation: $f'(x)=(d/dx)f(x)$ and $f''(x)=(d/dx)^2 f(x)$. We set $h_R(x)=(1+f_R(x))^{m+2}$.

By the assumption on Φ_1, there exists a positive constant K such that $\Phi_1'(x)\leqq K(1+x)^{\alpha-1}$ for any $x\geqq 0$.

Let $\{Y(t); t\geqq 0\}$ be the diffusion process defined by (2.1) with the initial distribution p_0. Then by Ito's formula,

$$\frac{d}{dt}E[h_R(Y(t))]=\frac{1}{2}E[h_R''(Y(t))]-E[h_R(Y(t))\Phi_1'(Y(t))]$$
$$=\frac{1}{2}(m+2)(m+1)E[(1+f_R(Y(t)))^m|f_R'(Y(t))|^2; 0\leqq Y(t)\leqq R+1]$$
$$+\frac{1}{2}(m+2)E[(1+f_R(Y(t)))^{m+1}f_R''(Y(t)); 0\leqq Y(t)\leqq R+1]$$

$$
\begin{aligned}
&\quad -(m+2)E[(1+f_R(Y(t)))^{m+1}f'_R(Y(t))\Phi'_1(Y(t));\ 0\leqq Y(t)\leqq R+1]\\
&\geqq -(m+2)E[(1+f_R(Y(t)))^{m+1};\ 0\leqq Y(t)\leqq R+1]\\
&\quad -2K(m+2)E[(1+f_R(Y(t)))^{m+\alpha};\ 0\leqq Y(t)\leqq R+1]\\
&\geqq -(2K+1)(m+2)E[h_R(Y(t))^{1-(2-\alpha)/(m+2)}].
\end{aligned}
$$

By Jensen's inequality we have

$$
\frac{d}{dt}E[h_R(Y(t))]\geqq -(2K+1)(m+2)E[h_R(Y(t))]^{1-(2-\alpha)/(m+2)}.
$$

By this and $E[h_{R_n}(Y(0))]\geqq a_n(1+R_n)^{m+2}$, we have

(B.1) $$E[h_{R_n}(Y(t))]\geqq\{(a_n(1+R_n)^{m+2})^{(2-\alpha)/(m+2)}-(2K+1)(2-\alpha)t\}^{(m+2)/(2-\alpha)}$$

for $t\leqq(2K+1)^{-1}(2-\alpha)^{-1}(a_n(1+R_n)^{m+2})^{(2-\alpha)/(m+2)}$.

Suppose that $\overline{\lim}_{t\to\infty}\left(-\frac{1}{t}\log\|T_tp_0-q_0\|_{L^1(R;dx)}\right)=2\lambda>0$. Then there exists a sequence $\{t_k;\ k\in N\}$ such that $t_1<t_2<\cdots\to\infty$ and $\|T_{t_k}p_0-q_0\|_{L^1(R;dx)}\leqq e^{-\lambda t_k}$. Choose $n=n(k)$ such that

$$
(a_{n-1}(1+R_{n-1})^{m+2})^{(2-\alpha)/(m+2)}\leqq 2(2K+1)(2-\alpha)t_k\leqq(a_n(1+R_n)^{m+2})^{(2-\alpha)/(m+2)}.
$$

Set

$$
M_{m+2}=\int_R(1+|x|^{m+2})q_0(x)dx.
$$

Then,

$$
\begin{aligned}
&E[h_{R_n}(Y(t_k))]\leqq M_{m+2}+\int_R h_{R_n}(x)(T_{t_k}p_0(x)-q_0(x))dx\\
&\leqq M_{m+2}+\|h_{R_n}\|_{L^\infty(R)}\|T_{t_k}p_0-q_0\|_{L^1(R;dx)}\\
&\leqq M_{m+2}+(1+R_n)^{m+2}\exp\left\{-\frac{\lambda}{2(2K+1)(2-\alpha)}(a_{n-1}(1+R_{n-1})^{m+2})^{(2-\alpha)/(m+2)}\right\}\\
&\leqq M_{m+2}+(1+R_{n-1})^{a(m+2)}\exp\left\{-\frac{\lambda Z^{-(2-\alpha)/(m+2)}}{2(2K+1)(2-\alpha)}(1+R_{n-1})^{(2-\alpha)/(m+2)}\right\}\\
&\longrightarrow M_{m+2}<\infty \qquad \text{as } k\to\infty.
\end{aligned}
$$

On the other hand, by (B.1) we have

$$
\begin{aligned}
E[h_{R_n}(Y(t_k))]&\geqq 2^{-(m+2)/(2-\alpha)}a_n(1+R_n)^{m+2}\\
&\geqq 2^{-(m+2)/(2-\alpha)}Z^{-1}(1+R_n)\longrightarrow\infty \qquad \text{as } k\to\infty,
\end{aligned}
$$

which is the contradiction. (Q.E.D.)

References

[1] Csiszár, I., Information-type measures of difference of probability distributions and indirect observations, Studia Sci. Math. Hungar. **2** (1967), 299-318.

[2] Davies, E. B. and B. Simon, Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians, J. Funct. Anal. **59** (1984), 335-395.

[3] Davies, E. B. and B. Simon, L^1-properties of Schrödinger semigroups, J. Funct. Anal. **65** (1986), 126-146.

[4] Donsker, M. D. and S. R. S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time I, Comm. Pure Appl. Math. **28** (1975), 1-47; III, Comm. Pure Appl. Math. **29** (1976), 389-461.

[5] Ershov, M. P., On the absolute continuity of measures corresponding to diffusion type processes, Theory Probab. Appl. **17** (1972), 169-174.

[6] Funaki, T., A certain class of diffusion processes with nonlinear parabolic equation, Z. Wahrsch. verw. Gebiete **67** (1984), 331-348.

[7] Gross, L., Logarithmic Sobolev inequalities, Amer. J. Math. **97** (1976), 1061-1083.

[8] Kusuoka, S. and D. Stroock, Some boundedness properties of certain stationary diffusion semigroups, J. Funct. Anal. **60** (1985), 243-264.

[9] McKean, H. P., Propagation of chaos for a class of non-linear parabolic equations, Stochastic Differential Equations, Lecture Series in Differential Equations 7, Catholic Univ., 1967, pp. 41-57.

[10] Shiga, T. and H. Tanaka, Central limit theorem for a system of Markovian particles with mean field interactions, Z. Wahrsch. verw. Gebiete **69** (1985), 439-459.

[11] Tamura, Y., On asymptotic behaviors of the solution of a non-linear diffusion equation, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 195-221.

[12] Tanemura, H., Propagation of Chaos and Asymptotic Behaviors for Diffusion Processes with Mean Field Interactions, Master thesis, Keio Univ., Yokohama, 1984.

(Received November 10, 1986)

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