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Generalized Kummer surfaces and their unirationality in characteristic p

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

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§ 0. Introduction.

Let X be an algebraic surface defined over an algebraically closed field of characteristic p. The surface X is called *a unirational surface* if there exists a generically surjective rational mapping φ from the projective plane $\boldsymbol{P}^2$ to X, and the surface X is called *a Zariski surface* if there exists φ as above which is a purely inseparable rational mapping of degree p. A non-singular projective surface X is called *supersingular* if the Picard number $\rho(X)$ is equal to the second Betti number $B_2(X)$. In characteristic zero, a unirational surface is nothing but a rational surface. In positive characteristic, however, the situation is different, and O. Zariski found in this case examples of irrational unirational surfaces in 1958 (cf. Zariski [34, p. 314]).

In this paper, we consider the following two questions in positive characteristic, and give partial answers to them.

QUESTION 1 (Artin and Shioda). For a $K3$ surface X, is X unirational if and only if X is supersingular?

QUESTION 2 (Shioda). Assume $p\neq 2$. Let E_1 and E_2 be two supersingular elliptic curves. Is the Kummer surface $\mathrm{Km}(E_1\times E_2)$ a Zariski surface?

As for Question 1, the "only if" part is proved in Shioda [27, Corollary 2]. The "if" part is still open, but two interesting partial answers are known. Namely, Question 1 is affirmative for Kummer $K3$ surfaces in characteristic $p\geqq 3$ (cf. Shioda [29, Theorem 1.1]), and for $K3$ surfaces in characteristic $p=2$ (cf. Rudakov and Shafarevich [24, Corollary on p. 151]). In this paper, we introduce the notion of a generalized Kummer $K3$ surface (cf. Definition 2.1), and we show that in case $p\geqq 7$, Question 1 is also affirmative for them. We will also give a new proof of Shioda's theorem for Kummer surfaces (cf. Section 4). The interesting point of our new proof is that it does not contain any explicit computation in birational geometry. To show the unirationality of supersingular generalized Kummer surfaces, we examine the structure of finite subgroups of automorphism groups of abelian surfaces. We give the list of finite subgroups which can give generalized Kummer surfaces (cf. Theorem 3.7). In characteristic 0, some of them cannot appear (see Fujiki [5], and see also Corollary 3.17). However, we show that every group G in the list can appear for some characteristic $p>0$ (cf. Section 7). As for Question 2, Professor T. Shioda communicated to the author that the question is affirmative if $p=3$ (cf. Remark 4.3). We show here that Conjecture 2 is affirmative if $p\not\equiv 1 \pmod{12}$. Moreover, even in case $p\equiv 1 \pmod{12}$, we show that after a separable extension of degree $(p-1)/2$, the surface becomes a Zariski surface. But the author does not know whether or not Question 2 is affirmative in case $p\equiv 1 \pmod{12}$.

Finally we give a brief outline of our paper. In Section 1, we prepare notations and lemmas which we use later. In Section 2, we give, in characteristic $p\neq 2$, a criterion for a minimal non-singular model of the quotient surface A/G of an abelian surface A by a finite group G to be a $K3$ surface (see also Katsura [13]). In Section 3, we give, in characteristic $p\neq 2, 3, 5$, the list of finite subgroups of automorphism groups of abelian surfaces which can give generalized Kummer surfaces (cf. Theorem 3.7). In Section 4, we give a new proof of above Shioda's theorem for Kummer surfaces. As a corollary, we give a weak answer to Question 2 (cf. Corollary 4.2 (i)). In Section 5, we calculate the discriminants of Néron-Severi groups of some generalized Kummer surfaces. Using the results, we give a partial affirmative answer to Question 2. In Section 6, we show that in case $p\geqq 7$, Qestion 1 is affirmative for generalized Kummer surfaces. In Section 7, we give examples of generalized Kummer surfaces, and show that any finite subgroups in the list in Theorem 3.7 can occur in the examples of generalized Kummer surfaces.

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§1. Notation and preliminaries.

Let k be an algebraically closed field of characteristic $p\geqq 0$. Let X be a projective variety of dimension n over k. Then, we use the following notations:

$k(X)$: the function field of X,
$\mathcal{O}_X$: the structure sheaf of X,
Ω_X^i: the sheaf of germs of regular i-forms of X,
$H^i(X, \mathcal{F})$: the i-th cohomology group of a coherent sheaf $\mathcal{F}$ on X,
$\chi(\mathcal{F}) = \sum_{i=0}^n (-1)^i \dim_k H^i(X, \mathcal{F})$,
$\boldsymbol{Z}$: the ring of rational integers,
$\boldsymbol{Q}$: the field of rational numbers,
$\boldsymbol{Q}_l$: the l-adic number field for a prime l different from p,
$H^i(X, \boldsymbol{Q}_l)$: the i-th l-adic étale cohomology group of X,
$B_i(X) = \dim_{\boldsymbol{Q}_l} H^i(X, \boldsymbol{Q}_l)$: the i-th Betti number of X,
id_X: the identity morphism of X.

Sometimes, a Cartier divisor and the associated invertible sheaf will be identified. For a morphism $f: X \to Y$ from a variety X to a variety Y and a subvariety Z of X, we denote by $f|_Z$ the restriction of f to Z. For a singular variety X, if a non-singular complete model $\tilde{X}$ of X exists, then we define the Kodaira dimension $\kappa(X)$ by $\kappa(\tilde{X})$. As is well-known, if $\dim X = 1$ or 2, then there exists a non-singular complete model $\tilde{X}$ of X even if the characteristic p is positive. Let G be a finite subgroup of the group of automorphisms of X. We denote by X/G the quotient variety of X by G. We denote by $|G|$ the order of G. For an element g of G we denote by $\langle g \rangle$ the subgroup of G generated by g. For a non-singular projective algebraic surface X, we use the following notations:

$\mathrm{NS}(X)$: the Néron-Severi group of X,
$\rho(X)$: the Picard number of X,
$p_a(D)$: the vertual genus of a divisor D, i.e.,

$$p_a(D) = (D^2 + K_X \cdot D)/2 + 1,$$

where K_X is a canonical divisor of X,
$D \sim D'$: linear equivalence for Cartier divisors D and D'.

For an abelian variety A, we use the following notations:

A^t: the dual of A,
T_x: the translation by an element x of A,
φ_L: the homomorphism from A to A^t defined by

$$x \longmapsto T_x^* L \otimes L^{-1} \quad (x \in A)$$

for an invertible sheaf L,

$K(L)=\mathrm{Ker}\,\varphi_L$,
$\mathrm{Aut}_v(A)$: the group of automorphisms of A as a variety,
$\mathrm{Aut}(A)$: the group of automorphisms of A as an abelian variety,
$\mathrm{End}(A)$: the endomorphism ring of A,
$\mathrm{End}^0(A)=\mathrm{End}(A)\otimes_Z \boldsymbol{Q}$.

Now, we assume char. $k=p>0$ until the end of this section. We use the notations:

$\boldsymbol{F}_p$: the finite field with p elements,
α_p: the local-local group scheme of rank p defined over k.

An abelian variety A of dimension n is called *supersingular* if A is isogenous to E^n with a supersingular elliptic curve E (cf. Oort [21, Theorem 4.2]). A curve C defined by some equations means the non-singular complete model of the curve defined by these equations, unless otherwise mentioned. A curve C is called *supersingular* if the Jacobian variety $J(C)$ of C is supersingular. As is well-known, an abelian surface A is supersingular if and only if A is supersingular as a projective surface.

LEMMA 1.1 (Shioda [27, Lemma on p. 234]). *Let X, Y be two non-singular complete algebraic varieties, and let $f: X\to Y$ be a generically surjective rational mapping. Then,*

$$B_2(X)-\rho(X)\geqq B_2(Y)-\rho(Y)\,.$$

LEMMA 1.2 (Deligne). *Let E_i $(i=1,2,\cdots,2n)$ be any supersingular elliptic curves. Assume $n\geqq 2$. Then, $E_1\times\cdots\times E_n$ is isomorphic to $E_{n+1}\times\cdots\times E_{2n}$.*

For the proof, see Shioda [30, Theorem 3.5].

LEMMA 1.3. *Let E be a supersingular elliptic curve, and let $\varepsilon:\alpha_p\hookrightarrow E\times E$ be an immersion such that $(E\times E)/\varepsilon(\alpha_p)$ is not isomorphic to a product of two supersingular elliptic curves. Let θ be an element of $\mathrm{Aut}((E\times E)/\varepsilon(\alpha_p))$. Then, we can find a unique automorphism $\tilde{\theta}$ of $E\times E$ such that the following diagram commutes:*

$$\begin{array}{ccc} E\times E & \xrightarrow{\quad\tilde{\theta}\quad} & E\times E \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ (E\times E)/\varepsilon(\alpha_p) & \xrightarrow{\quad\theta\quad} & (E\times E)/\varepsilon(\alpha_p)\,, \end{array} \tag{1.1}$$

where π is the canonical projection.

PROOF. Let F be the Frobenius morphism of $E\times E$. We have $\operatorname{Ker} F\cong\alpha_p\times\alpha_p$. By assumption, $\pi(\operatorname{Ker}(F))$ is the unique subgroup scheme of $(E\times E)/\varepsilon(\alpha_p)$ which is isomorphic to α_p. Therefore, θ preserves $\pi(\operatorname{Ker} F)$. Hence, we have an automorphism θ' of $(E\times E)/\operatorname{Ker} F$ such that the following diagram commutes:

$$\begin{array}{ccc}
(E\times E)/\varepsilon(\alpha_p) & \xrightarrow{\ \theta\ } & (E\times E)/\varepsilon(\alpha_p) \\
\Big\downarrow{\scriptstyle\pi'} & & \Big\downarrow{\scriptstyle\pi'} \\
(E\times E)/\operatorname{Ker} F & \xrightarrow{\ \theta'\ } & (E\times E)/\operatorname{Ker} F,
\end{array}$$

where π' is the canonical projection. Since $F=\pi'\circ\pi$, we set $\tilde{\theta}=(\theta')^{(1/p)}$, where $(\theta')^{(1/p)}$ is the rational mapping whose coefficients are the p-th roots of those of θ'. Then, $\tilde{\theta}$ is an element of $\operatorname{Aut}(E\times E)$ such that the diagram (1.1) commutes. q. e. d.

§2. Generalized Kummer surfaces.

Let A be an abelian surface defined over an algebraically closed field k of characteristic $p\geqq 0$. Let G be a finite subgroup of $\operatorname{Aut}_0(A)$. We denote by ω_A a non-zero regular two-form on A which is unique up to constant multiple. We denote the canonical projection from A to the quotient surface A/G by

$$\pi : A \longrightarrow A/G. \tag{2.1}$$

Since A/G is normal, the singular locus of A/G consists of isolated points.

DEFINITION 2.1. If A/G is birationally equivalent to a $K3$ surface, we call the minimal non-singular model of A/G *a generalized Kummer surface.* We denote it by $\operatorname{Km}(A, G)$.

DEFINITION 2.2. We call a reduced irreducible curve C on A *a fixed curve of a group G* (resp. *a fixed curve of an element g of G different from the identity*) if there exists an element g of G different from the identity which induces the identity mapping on C (resp. if g induces the identity mapping on C).

LEMMA 2.3. *A fixed curve by an element g of G different from the identity is a non-singular elliptic curve.*

PROOF. Let C be a fixed curve, and let g be an element of G different from the identity which induces the identity mapping on C. By a suitable choice of the origin of A, we may assume that g is an element of $\mathrm{Aut}(A)$. Then, the curve C is a reduced part of an irreducible component of the kernel of the homomorphism $g-\mathrm{id}_A$. Hence, C is a non-singular elliptic curve. q. e. d.

The main purpose of this section is to prove the following theorem.

THEOREM 2.4. *Assume that k is of characteristic $p\neq 2$. A relatively minimal model of the quotient surface A/G is a $K3$ surface if and only if G satisfies the following four conditions:*
(i) *G has no fixed curves,*
(ii) *there exists an element of G which has some isolated fixed points,*
(iii) *all the singular points of A/G are rational double points,*
(iv) *for any element g of G, $g^*\omega_A=\omega_A$ holds.*

If G satisfies these four conditions, we say that G satisfies Condition (K).

DEFINITION 2.5. Let X be an algebraic surface with isolated singular points P_i $(i=1,\cdots,n)$. Let $\varphi:\tilde{X}\to X$ be a resolution of X. The resolution is said to be *minimal* if $\varphi^{-1}(P_i)$'s $(i=1,\cdots,n)$ contain no exceptional curves of the first kind.

LEMMA 2.6. *Suppose that G has some fixed curves. Then, $\kappa(A/G)$ is equal to $-\infty$.*

PROOF. To begin with, we consider the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{A} & \xrightarrow{h} & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\
S & \xrightarrow{\varphi} & A/G,
\end{array}
\tag{2.2}
$$

where π is the canonical projection, $\varphi:S\to A/G$ is the minimal resolution of singularities, h is a birational morphism composed of suitable blowing-ups with center at the isolated points such that the induced mapping f becomes a morphism. Since $\kappa(A)=0$ and π is separable, we have $\kappa(A/G)\leqq 0$.

Suppose $\kappa(A/G)=0$. Then, there exists a non-zero regular 12-ple two-form $\tilde{\omega}_S$ on S such that

$$h^*\omega_A^{\otimes 12}=f^*\tilde{\omega}_S\,. \tag{2.3}$$

Since G has a fixed curve, π has a ramification locus of codimension 1. Since the singular points of A/G are isolated, f has also the ramification locus of codimension 1. Therefore, the image $h(R)$ of the ramification locus R of f is of codimension 1. On the other hand, R is contained in the zero divisor of $f^*\tilde{\omega}_S$. Therefore, by (2.3), the support of R is contained in the exceptional divisor of the morphism h, that is, $h(R)$ consists of isolated points, a contradiction. Hence, we conclude $\kappa(A/G)=-\infty$. q. e. d.

LEMMA 2.7. *Suppose $\kappa(A/G)=0$. Then, the minimal resolution S of A/G is the minimal non-singular model of A/G.*

PROOF. By Lemma 2.6, we may assume that G has no fixed curves. Suppose that S is not a minimal model. Then, we have the following diagram:

$$\begin{tikzcd} S \arrow[d, "f"'] \arrow[drr, "\varphi"] & & A \arrow[d, "\pi"] \\ X & & A/G\,, \end{tikzcd} \tag{2.4}$$

where φ is the minimal resolution, X is the minimal model of S, and f is the contraction of exceptional curves. Since $\kappa(X)=0$, $K_X^{\otimes 12}$ is trivial (cf. Bombieri and Mumford [4, Theorem 1]). Hence, there exists a non-zero regular 12-ple two-form $\tilde{\omega}_X$ which never vanishes on X. We set

$$\tilde{\omega}_S=f^*\tilde{\omega}_X\,. \tag{2.5}$$

Then, the support of the zero divisor of $\tilde{\omega}_S$ consists of the exceptional curves of f, and contains all of them. On the other hand, since φ is birational, there exists a non-zero 12-ple rational two-form $\tilde{\omega}$ on A/G such that

$$\tilde{\omega}_S=\varphi^*\tilde{\omega}\,. \tag{2.6}$$

Since the singularities of A/G are isolated, $\tilde{\omega}$ is regular except at the singular points. Since π is a finite morphism, $\pi^*\tilde{\omega}$ has no zero divisors. Therefore, $\tilde{\omega}$ has no zero points except at the singular points. Hence by (2.6), we see that $\tilde{\omega}_S$ has no zero points except on the exceptional curves of the morphism φ. Thus, we conclude that the support of the exceptional divisor of f is contained in the support of the exceptional divisor of φ, a contradiction to the minimality of the resolution $\varphi: S \to A/G$. q. e. d.

LEMMA 2.8. *Suppose that A/G has some singular points which are not rational double points. Then, $\kappa(A/G)$ is equal to $-\infty$.*

PROOF. Suppose $\kappa(A/G)=0$. By Lemma 2.6, G has no fixed curves. Let P be a singular point of A/G which is not a rational double point. Let $f: S \to A/G$ be a minimal resolution, and let Z be a fundamental cycle corresponding to the singular point P (cf. Artin [1, p. 132]). Since S is the minimal model of A/G by Lemma 2.7, we see that $K_S^{\otimes 12}$ is trivial. Therefore, we have

$$2(p_a(Z)-1)=K_S \cdot Z+Z^2=Z^2 . \tag{2.7}$$

Again by Artin [1, Theorem 3], we have

$$p_a(Z) \geqq 0 , \tag{2.8}$$

where the equality holds if and only if P is a rational singular points.

First, suppose that P is not a rational singular point. Then, we have

$$p_a(Z) \geqq 1 . \tag{2.9}$$

Therefore, by (2.7), we have $Z^2 \geqq 0$, which contradicts the fact that the intersection matrix of the exceptional curves is negative definite. Hence, P is rational, that is, $p_a(Z)=0$. Therefore, we have $Z^2=-2$ by (2.7). Since $-Z^2$ is equal to the multiplicity of the rational singular point P (cf. Artin [1, Corollary 6]), P is a rational double point. q. e. d.

LEMMA 2.9. *Suppose that A/G has some singular points. Then, any non-singular model of A/G is neither abelian, hyperelliptic, nor quasi-hyperelliptic.*

PROOF. Suppose that A/G is birationally equivalent to an abelian surface, a hyperelliptic surface or a quasi-hyperelliptic surface. Then, we have $\kappa(A/G)=0$. Therefore, by Lemma 2.7, the minimal resolution S of A/G is a minimal model. Since A/G has at least one singular point, S contains a curve whose self-intersection number is negative. It is impossible if S is an abelian surface, a hyperelliptic surface or a quasi-hyperelliptic surface. q. e. d.

PROOF OF THEOREM 2.4. Suppose that the minimal non-singular model of A/G is a $K3$ surface. Then, Condition (i) holds by Lemma 2.6. Suppose that Condition (ii) does not hold. Then, the canonical projection

$$\pi : A \longrightarrow A/G \tag{2.10}$$

is an étale morphism. Therefore, we have

$$0=\chi(\mathcal{O}_A)=\deg(\pi)\cdot\chi(\mathcal{O}_{A/G})\,, \tag{2.11}$$

which contradicts the fact that A/G is a $K3$ surface. Condition (iii) holds by Lemma 2.8. Since ω_A is the pull-back by π of a non-zero regular two-form on A/G, we see that Condition (iv) also holds. Conversely, suppose that Conditions (i), (ii), (iii) and (iv) hold. We have the following diagram:

$$\begin{tikzcd} S \arrow[dr, "\varphi"'] & A \arrow[d, "\pi"] \\ & A/G\,, \end{tikzcd} \tag{2.12}$$

where $\varphi : S \to A/G$ is the minimal resolution of A/G. By Condition (iv), there exists a non-zero rational two-form ω on A/G such that $\pi^*\omega$ is a non-zero regular two-form ω_A. Since π is étale except at a finite number of points on A, and ω_A has neither zero nor pole, $\varphi^*\omega$ has neither zero nor pole except on the exceptional curves of φ. Let $\{E_i\}_{i=1,\dots,m}$ be all the exceptional curves of φ. Then, there exist integers n_i $(i=1,\cdots,m)$ such that

$$K_S \sim (\text{the divisor of } \varphi^*\omega) = \sum_{i=1}^{m} n_i E_i\,. \tag{2.13}$$

By Condition (iii), we have

$$p_a(E_i)=0 \quad \text{and} \quad E_i^2=-2 \quad (i=1,\cdots,m) \tag{2.14}$$

(cf. Artin [1, p. 135]). Hence, we have

$$K_S\cdot E_i=0 \quad (i=1,\cdots,m)\,. \tag{2.15}$$

By (2.13), we have

$$\sum_{i=1}^{m} n_i(E_iE_j)=0 \quad (j=1,\cdots,m)\,. \tag{2.16}$$

Since the intersection matrix of the exceptional curves is negative definite, we conclude

$$n_i=0 \quad (i=1,\cdots,n)\,, \tag{2.17}$$

that is, the canonical bundle of S is trivial. Hence, we have $\kappa(S)=0$. By Conditions (i) and (ii), A/G has some singular points. Hence, by Lemma 2.9, S is neither an abelian surface, a hyperelliptic surface nor a quasi-hyperelliptic surface. By the assumption $p\neq 2$, we conclude that S is a $K3$ surface. q. e. d.

REMARK 2.10. There exists an example of (A, G) with an abelian surface A and $G\subset \mathrm{Aut}_v(A)$ which satisfies Conditions (i), (ii) and (iv) such that a relatively minimal model of A/G is not a $K3$ surface. For example, in characteristic $p=3$, we consider the elliptic curve E defined by the equation

$$y^2=x^3-x\,.$$

We consider the automorphism defined by

$$\sigma\,:\,x\longmapsto x+1,\, y\longmapsto y\,.$$

We set $A=E\times E$ and $G=\langle\sigma\times\sigma\rangle$. Then, (A, G) satisfies Conditions (i), (ii) and (iv), but does not satisfy Condition (iii). In fact, it is not difficult to prove that A/G is a rational surface by the similar method to the proof of rationality in Katsura [12, p. 534]. However, we will show in Section 3 that in case $p\geqq 7$, Condition (iii) follows from the other conditions (cf. Theorem 3.12).

Finally, we give a criterion of rationality for the later use.

THEOREM 2.11. *Let A be an abelian surface defined over an algebraically closed field k of characteristic $p\geqq 0$. Let G be a finite subgroup of* $\mathrm{Aut}_v(A)$. *Suppose that G has no fixed curves. If A/G has at least one singular point other than rational double points, then A/G is a rational surface.*

PROOF. By Lemma 2.8, we have $\kappa(A/G)=-\infty$. Therefore, A/G is birationally equivalent to a ruled surface. We consider the following diagram:

$$\begin{tikzcd} A \arrow[r, dashed, "\psi"] \arrow[dr, "\pi"'] & S \arrow[r, "\theta"] \arrow[d, "\varphi"] & C \\ & A/G\,, & \end{tikzcd} \tag{2.18}$$

where φ is a resolution of singularities of A/G, ψ is the rational mapping such that $\psi=\varphi^{-1}\circ\pi$, and $\theta: S\to C$ gives the structure of ruled surface. Since $\theta\circ\psi$ gives a non-trivial morphism from an abelian surface to a curve, the genus of C is smaller than or equal to one. Suppose that the genus of C is equal to one. Then, by a suitable choice of the origin of C, we can assume that $\theta\circ\psi$ is a homomorphism. The reduced part $(\theta\circ\psi)^{-1}(P)_{\mathrm{red}}$ of $(\theta\circ\psi)^{-1}(P)$ consists of a finite number of elliptic curves for any point P of C. Since the fixed points of G are finite in number, $(\theta\circ\psi)^{-1}(P)_{\mathrm{red}}$ does not contain them for a general point P of C. Therefore, G acts freely on $(\theta\circ\psi)^{-1}(P)_{\mathrm{red}}$. Hence, the quotient scheme of $(\theta\circ\psi)^{-1}(P)_{\mathrm{red}}$ by G is an elliptic curve, and is isomorphic to $\theta^{-1}(P)$. This contradicts the fact that $\theta^{-1}(P)$ is isomorphic to a rational curve for a general point P of C. Therefore, C is a rational curve. Hence, S is a rational surface. q. e. d.

Let A be a supersingular abelian surface in characteristic two. Let ι be the inversion of A. Since the singularity of $A/\langle\iota\rangle$ is elliptic (cf. Katsura [12, Theorem A]), we have a new proof of the following fact.

COROLLARY 2.12 (Shioda [26] and Katsura [12]). *Let A be a supersingular abelian surface in characteristic two. Then, $A/\langle\iota\rangle$ is a rational surface.*

§ 3. The structure of groups.

Let k be an algebraically closed field of characteristic $p\geqq 0$. Let A be an abelian surface defined over k. We denote by ω_A a non-zero regular two-form on A. We investigate the structure of subgroups of the group $\mathrm{Aut}_v(A)$ of automorphisms of A. Let G be a finite subgroup of $\mathrm{Aut}_v(A)$. Let g be an element of G. We have the induced homomorphism

$$g^*: H^1(A, \boldsymbol{Q}_l) \longrightarrow H^1(A, \boldsymbol{Q}_l).$$

Let $f(X)$ be the characteristic polynomial of g^*. It is a polynomial of degree four which has rational integral coefficients. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be eigen-values of g^*. Since we have the canonical isomorphism

$$H^i(A, \boldsymbol{Q}_l)\cong \bigwedge^i H^1(A, \boldsymbol{Q}_l), \tag{3.1}$$

the eigen-values of g^* on $H^i(A, \boldsymbol{Q}_l)$ are given by

$$\omega_{\alpha_1}\cdot\omega_{\alpha_2}\cdot\cdots\cdot\omega_{\alpha_i} \quad (1\leqq\alpha_1,\cdots,\alpha_i\leqq 4\,;\ \alpha_m\neq\alpha_n \text{ if } m\neq n). \tag{3.2}$$

We denote by Φ_g (resp. Δ) the graph of g (resp. the diagonal) in $A\times A$.

Then, by the Lefschetz fixed point formula, we have

$$f(1)=(1-\omega_1)(1-\omega_2)(1-\omega_3)(1-\omega_4)=\varDelta\cdot\varPhi_g\,. \tag{3.3}$$

In case the order of g is not divisible by p, $\varDelta\cdot\varPhi_g$ is equal to the number of fixed points of g.

LEMMA 3.1 (Harder and Narasimhan). *Let G be a finite subgroup of the group of automorphisms of a projective variety X over k. Let l be a prime number which is prime to both p and the order of G. Then, $H^i(X/G, \boldsymbol{Q}_l)$ is isomorphic to the subspace $H^i(X, \boldsymbol{Q}_l)^G$ of G-invariants in $H^i(X, \boldsymbol{Q}_l)$:*

$$H^i(X/G, \boldsymbol{Q}_l)\cong H^i(X, \boldsymbol{Q}_l)^G\,. \tag{3.4}$$

For the proof, see Harder and Narasimhan [6, Proposition 3.2]. The following two lemmas are well-known.

LEMMA 3.2. *The natural representation of* End(A) *in* $H^1(A, \boldsymbol{Q}_l)$ *is faithful.*

LEMMA 3.3. *An element g of* $\mathrm{Aut}_v(A)$ *induces the identity on* $H^1(A, \boldsymbol{Q}_l)$ *if and only if g is a translation of A.*

LEMMA 3.4. *Under the notations as above, an element g of G different from the identity either acts freely on A or has a fixed curve if and only if $f(1)=0$.*

PROOF. Suppose that g has a fixed curve E. By Lemma 2.3, E is an elliptic curve. Let $h: E\hookrightarrow A$ be the natural immersion. Then, we have a surjective homomorphism

$$h^*\ :\ H^1(A, \boldsymbol{Q}_l)\longrightarrow H^1(E, \boldsymbol{Q}_l)\,.$$

Since g^* acts on $H^1(E, \boldsymbol{Q}_l)$ as the identity, we see that at least one of eigen-values of g^* on $H^1(A, \boldsymbol{Q}_l)$ is equal to 1. Therefore, we have $f(1)=0$. The other parts of this lemma follow from (3.3). q. e. d.

LEMMA 3.5. *Let g be an element of order two of* $\mathrm{Aut}_v(A)$. *Assume that $\langle g\rangle$ satisfies Conditions* (i) *and* (ii) *in Theorem* 2.4. *Then, g is the inversion ι of A with a suitable choice of the origin of A.*

PROOF. Let P be a fixed point of g. We choose P as the origin of A. Then, g is an element of End(A). Since the order of g is equal to two, the eigen-values ω_1, ω_2, ω_3, ω_4 of g^* on $H^1(A, \boldsymbol{Q}_l)$ are equal to 1 or -1. By Conditions (i) and (ii) and Lemma 3.4, we have $(1-\omega_1)(1-\omega_2)(1-\omega_3)(1-\omega_4)\neq 0$. Hence, we have $\omega_i=-1$ for $i=1,2,3,4$. Since ι^* on $H^1(A, \boldsymbol{Q}_l)$ is given by the matrix

$$\iota^*=\begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix},$$

we conclude $g^*=\iota^*$. Hence, by Lemma 3.2, we have $g=\iota$. q. e. d.

LEMMA 3.6. *Assume* char. $k=p\neq 2,3$. *Let G be a finite subgroup of* $\mathrm{Aut}_v(A)$ *which satisfies Condition* (K). *Let g be an element of G which has no fixed points. Then, g is a translation of A.*

PROOF. Let ω_1, ω_2, ω_3, ω_4 be eigen-values of g^* on $H^1(A, \boldsymbol{Q}_l)$. Suppose that there exists an element g^n of $\langle g\rangle$ which has a fixed point, and which is not the identity. Since G satisfies Condition (K), $\langle g^n\rangle$ satisfies Condition (i) in Theorem 2.4. Therefore, by Lemma 3.4, we have

$$(1-\omega_1^n)(1-\omega_2^n)(1-\omega_3^n)(1-\omega_4^n)\neq 0\,.$$

Hence, we have

$$(1-\omega_1)(1-\omega_2)(1-\omega_3)(1-\omega_4)\neq 0\,,$$

which contradicts the assumption on g. Therefore, the group $\langle g\rangle$ acts freely on A. We set $S=A/\langle g\rangle$. We consider the canonical projection

$$\pi : A \longrightarrow A/\langle g\rangle = S\,,$$

which is an étale morphism. By Condition (K), we have $g^*\omega_A=\omega_A$. Therefore, there exists a rational two-form ω_S on S such that $\pi^*\omega_S=\omega_A$. Since π is étale, ω_S has neither zeros nor poles. Therefore, we see that K_S is trivial. Since $(\deg \pi)\chi(\mathcal{O}_S)=\chi(\mathcal{O}_A)=0$, S is an abelian surface by the assumption $p\neq 2,3$ (cf. Bombieri and Mumford [4, Section 3]). By Lemma 3.1, we have $H^1(A, \boldsymbol{Q}_l)^{\langle g\rangle}\cong H^1(S, \boldsymbol{Q}_l)$. Since $\dim_{\boldsymbol{Q}_l}\{H^1(S, \boldsymbol{Q}_l)\}=4$, the group $\langle g\rangle$ acts trivially on $H^1(A, \boldsymbol{Q}_l)$. Hence, by Lemma 3.3, g is a translation. q. e. d.

From here on, until the end of this section we assume that char. $k=p\neq 2,3,5$, unless otherwise mentioned. Let G be a finite subgroup of $\mathrm{Aut}_v(A)$ which satisfies Condition (K). We set

$$N=\{g\in G \mid g \text{ has no fixed points}\}\cup\{\mathrm{id}_A\}\,.$$

Then, by Lemma 3.6, N consists of translations of A, and it is easy to see that N is a normal subgroup of G. To examine the quotient surface A/G, we can consider $(A/N, G/N)$ instead of (A, G). Therefore, we add one more condition:

Condition (F). Any element of G has some fixed points on A.

We note that under Conditions (K) and (F), any subgroup H of G also satisfies Conditions (K) and (F). Now, we state the main theorem in this section.

THEOREM 3.7. *Let k be an algebraically closed field of characterisitc p. Assume $p \neq 2, 3, 5$. Let A be an abelian surface over k and let G be a finite subgroup of* $\mathrm{Aut}_v(A)$ *which satisfies Conditions* (K) *and* (F). *Then, G is isomorphic to one of the following groups*:

(i) *cyclic group of order* 2, 3, 4, 5, 6, 8, 10 *or* 12,
(ii) *binary dihedral group* $\langle 2, 2, n\rangle$ *with* $n=2, 3, 4, 5$ *or* 6,
(iii) *binary tetrahedral group* $\langle 2, 3, 3\rangle$,
(iv) *binary octahedral group* $\langle 2, 3, 4\rangle$,
(v) *binary icosahedral group* $\langle 2, 3, 5\rangle$.

REMARK 3.8. Let A be an abelian surface over k, and let G be a finite subgroup of $\mathrm{Aut}_v(A)$. Theorem 3.7 shows that if A/G is birationally equivalent to a $K3$ surface, then G is isomorphic to an extension of a group in Theorem 3.7 by a finite group consisted of translations of A. We will show in Section 7 that all groups in Theorem 3.7 can occur (in case $p=0$, see Corollary 3.17, and Fujiki [5]).

To prove Theorem 3.7, we need some lemmas.

LEMMA 3.9. *Let A be an abelian surface over k and let G be a finite subgroup of* $\mathrm{Aut}_v(A)$ *which has no fixed curves and satisfies Condition* (F). *Let $\pi: A \to A/G$ be the canonical projection and let P be a point of A. If the stabilizer of G at P is not trivial, then $\pi(P)$ is a singular point of A/G.*

PROOF. This follows from the fact that the branch locus of π through a non-singular point is of codimension one. q. e. d.

LEMMA 3.10. *Let A and G be as in Lemma* 3.9. *Let g be an element of order n $(n \geqq 2)$ of G. Then, any eigen-value of g^* on $H^1(A, \boldsymbol{Q}_l)$ is equal to one of primitive n-th roots of unity.*

PROOF. Suppose that one of eigen-values of g^* on $H^1(A, \boldsymbol{Q}_l)$, say ζ, is not equal to any primitive n-th root of unity. Since $(g^*)^n$ is the identity, there exists a divisor m of n such that $m<n$ and $\zeta^m=1$. We consider the subgroup $\langle g^m\rangle$ of G. By Lemma 3.4 and our choice of m, the subgroup $\langle g^m\rangle$ has a fixed curve or does not satisfy Condition (F). A contradiction. q. e. d.

The following lemma is essentially known.

LEMMA 3.11. *Let A and G be as in Lemma* 3.9. *Then, the order of any element g of G is equal to* 1, 2, 3, 4, 5, 6, 8, 10 *or* 12.

PROOF. Let n be the order of g. By Lemma 3.2, the order of g^* on $H^1(A, \boldsymbol{Q}_l)$ is also equal to n. By Lemma 3.10, any eigen-value of g^* is a zero of a cyclotomic polynomial $\Phi_n(X)$ whose zeros consist of primitive n-th roots of unity. Since the characteristic polynomial of g^* on $H^1(A, \boldsymbol{Q}_l)$ is of degree four, we have $\deg \Phi_n(X) \leqq 4$. Hence, we have $n=1, 2, 3, 4, 5, 6, 8, 10$ or 12. q. e. d.

THEOREM 3.12. *Assume that* $\mathrm{char.}\, k=p\neq 2, 3, 5$. *Let A be an abelian surface over k and let G be a finite subgroup of* $\mathrm{Aut}_0(A)$ *which satisfies Condition* (F). *Then, A/G is birationally equivalent to a $K3$ surface if and only if $g^*\omega_A=\omega_A$ for any element g of G.*

PROOF. The "only if" part follows from Theorem 2.4. Assume that $g^*\omega_A=\omega_A$ for any element g of G. Suppose that an element g of G different from the identity has a fixed curve E. We may assume that E is an abelian subvariety of A. Then, g induces an automorphism of the elliptic curve A/E which has at least one fixed point. Therefore, the order of g is equal to 2, 3, 4 or 6. Therefore, the order of g is prime to p by our assumption. Let P be a point on E. Since g is the identity on E, by a suitable choice of the local coordinate (s, t) of A at P the action of g at P is given by

$$g^* : \begin{cases} s \longrightarrow s \\ t \longrightarrow \varepsilon t \end{cases}$$

with an element ε of k which is different from 0 and 1, which contradicts the assumption $g^*\omega_A=\omega_A$. Hence, G has no fixed curves. By Lemma 3.11, the order of any element of the stabilizer of G at P is prime to p. Hence, by Pinkham [23], we conclude that Condition (iii) in Theorem 2.4 follows from the assumption $g^*\omega_A=\omega_A$ for any element g of G. Hence, the result follows from Theorem 2.4. q. e. d.

LEMMA 3.13. *Let A be an abelian surface and let G be a finite subgroup of* $\mathrm{Aut}_v(A)$ *which satisfies Conditions* (K) *and* (F). *Let g be an element of G, and let a_i $(i=1, \cdots, m)$ be fixed pionts of the automorphism g. Let $\pi : A \to A/\langle g \rangle$ be the canonical projection. Then, the local rings $\mathcal{O}_{\pi(a_i)}$ $(i=1, \cdots, m)$ are isomorphic to each other.*

PROOF. We may assume that the origin o of A is a fixed point of g. Then, the fixed points of the automorphism g are the points of $\mathrm{Ker}(g-\mathrm{id}_A)$. Let a be one of them. Then, we have $T_a \circ g = g \circ T_a$. Therefore, T_a induces an isomorphism from $\mathcal{O}_{\pi(o)}$ to $\mathcal{O}_{\pi(a)}$. q. e. d.

LEMMA 3.14. *Let A and G be as in Lemma* 3.13. *Let g be an element of order n $(n \geqq 2)$ of G, and let P be a fixed point of g. Let $\pi : A \to A/\langle g \rangle$ be the canonical projection. Then, the singularity at the point $\pi(P)$ is of type A_{n-1}.*

PROOF. By Katsura [15, Lemma 1.3], we can find a regular system of parameters (u, v) of the local ring $\mathcal{O}_P$ at P such that

$$g^* u = \zeta u\,, \quad g^* v = \zeta' v$$

with primitive roots ζ, ζ' of unity. Using Lemma 3.9 for subgroups of $\langle g \rangle$, we see that both ζ and ζ' are primitive n-th roots of unity. We denote by m_P the maximal ideal of $\mathcal{O}_P$. Since m_P/m_P^2 is naturally isomorphic to $\overset{2}{\wedge} H^0(A, \Omega_A^1)$, we have $\zeta' = \zeta^{-1}$ by (iv) in Condition (K). Hence, the singularity at the point $\pi(P)$ is of type A_{n-1}. q. e. d.

Let G be a finite subgroup of $\mathrm{Aut}_v(A)$ which satisfies Conditions (K) and (F). Let g be an element of order n of G, and let $f(X)$ be the characteristic polynomial of g^* on $H^1(A, \boldsymbol{Q}_l)$. Let $\pi : A \to A/\langle g \rangle$ be the canonical projection. Then, the singular points of $A/\langle g \rangle$ are given by the image of the set of points:

$$\{P \in A \mid g^m(P) = P \text{ for some integer } m\}\,.$$

Using these notations, by (3.3), Lemmas 3.10, 3.11, 3.13 and 3.14, we have the following table.

Table.

order g	$f(X)$	$H^2(A, \mathbf{Q}_l)^{\langle g\rangle}$	singularities of $A/\langle g\rangle$	$\rho(\mathrm{Km}(A, \langle g\rangle))$
2	$(X+1)^4$	6	$A_1(16)$	$\geqq 17$
3	$(X^2+X+1)^2$	4	$A_2(9)$	$\geqq 19$
4	$(X^2+1)^2$	4	$A_3(4), A_1(6)$	$\geqq 19$
5	$X^4+X^3+X^2+X+1$	2	$A_4(5)$	22
6	$(X^2-X+1)^2$	4	$A_5(1), A_2(4), A_1(5)$	$\geqq 19$
8	X^4+1	2	$A_7(1), A_3(1), A_1(3)$	22
10	$X^4-X^3+X^2-X+1$	2	$A_9(1), A_4(2), A_1(3)$	22
12	X^4-X^2+1	2	$A_{11}(1), A_3(1), A_2(2), A_1(2)$	22

In this table, by $A_m(\alpha)$ we mean that singularities of type A_m appear α times. The following lemma is also essentially known.

LEMMA 3.15. *Assume* char. $k=p=0$. *Let G be as above. Then, the order of any element of G is equal to* 1, 2, 3, 4 *or* 6.

PROOF. In case char. $k=p=0$, we have $\rho(S)\leqq 20$ for any $K3$ surface S over k. Hence, this proposition follows from the above table. q.e.d.

LEMMA 3.16. *Let k be an algebraically closed field of characteristic p. Let G be as above. Assume that the order of G is prime to p if p is positive. Then, the group G acts faithfully on $H^0(A, \Omega_A^1)$.*

PROOF. Let g be an element of order n of G. We may assume that g fixes the origin o of A. Let $\mathcal{O}$ be the local ring of A at o, and let m be the maximal ideal of $\mathcal{O}$. Suppose that the action g^* of g on $H^0(A, \Omega_A^1)$ is trivial. Since $H^0(A, \Omega_A^1)$ is naturally isomorphic to m/m^2, g^* acts trivially on m/m^2. Let (x, y) be a regular system of parameters of $\mathcal{O}$. We set

$$\begin{cases} u=x+g^*x+\cdots+(g^*)^{n-1}(x)\,, \\ v=y+g^*y+\cdots+(g^*)^{n-1}(y)\,. \end{cases}$$

Then, we have $u\equiv nx \pmod{m^2}$ and $v\equiv ny \pmod{m^2}$. Since n is prime to p, we see that (u, v) is a regular system of parameters of $\mathcal{O}$ which is invariant under the action of $\langle g\rangle$. Hence, $\mathcal{O}^{\langle g\rangle}=\mathcal{O}$ is regular, which contradicts Lemma 3.9. q. e. d.

PROOF OF THEOREM 3.7. By Lemmas 3.11, 3.16 and (iv) in Condition (K), we have an injective homomorphism:

$$G \hookrightarrow SL(H^0(A, \Omega_A^1)) .$$

Since the order of every element of G is equal to 1, 2, 3, 4, 5, 6, 8, 10 or 12, by the same method as in Pinkham [23], we conclude that G is isomorphic to one of groups as in this theorem. q. e. d.

COROLLARY 3.17. *Under the same assumption as in Theorem* 3.7, *assume, moreover,* $p=0$. *Then, the group* G *is isomorphic to one of the following groups*:

(i) *cyclic group of order* 2, 3, 4 *or* 6,

(ii) *binary dihedral group* $\langle 2,2,2\rangle$ *or* $\langle 2,2,3\rangle$,

(iii) *binary tetrahedral group* $\langle 2,3,3\rangle$.

PROOF. This follows from Theorem 3.7 and Lemma 3.15. q. e. d.

REMARK 3.18. In char. $k=0$, all groups in Corollary 3.17 can occur (cf. Fujiki [5], and see also examples in a later section).

Here, we give some known examples of generalized Kummer surfaces which are used in later sections.

Example 1. $G\cong \boldsymbol{Z}/2$.

Assume $p\neq 2$. Let ι be the inversion of an abelian surface A. We set $G=\langle\iota\rangle$. $\mathrm{Km}(A, G)$ is called a Kummer surface, and is denoted by $\mathrm{Km}(A)$.

Example 2. $G\cong \boldsymbol{Z}/3$.

Assume $p\neq 2, 3$. We consider the non-singular complete models of elliptic curves E_j $(j=1, 2)$ defined by

$$E_j\ :\ y_j^2=x_j^3-1 \quad (j=1, 2) .$$

We set $A=E_1\times E_2$. We consider the automorphism defined by

$$\sigma\ :\ \begin{cases} x_1 \longmapsto \omega x_1 , & y_1 \longmapsto y_1 , \\ x_2 \longmapsto \omega^2 x_2 , & y_2 \longmapsto y_2 , \end{cases}$$

where ω is a primitive cube root of unity. We set $G=\langle\sigma\rangle$. Then, we get a generalized Kummer surface $\mathrm{Km}(A, G)$ (cf. Ueno [32, Example 16.13]). If

$k=\boldsymbol{C}$ (the complex number field), then $\mathrm{Km}(A, G)$ is not isomorphic to $\mathrm{Km}(A)$ (cf. Shioda and Inose [31, Lemma 5.1], and Inose [11, Theorem 0]).

Example 3. $G \cong \boldsymbol{Z}/4$.
Assume $p \neq 2$. We consider the non-singular complete elliptic curves E_j $(j=1, 2)$ defined by

$$E_j : y_j^2 = x_j^4 - 1 \quad (j=1, 2).$$

We set $A = E_1 \times E_2$. We consider the automorphism defined by

$$\tau : \begin{cases} x_1 \longmapsto -x_1, & y_1 \longmapsto iy_1, \\ x_2 \longmapsto -x_2, & y_2 \longmapsto -iy_2, \end{cases}$$

where i is a primitive fourth root of unity. We set $G=\langle\tau\rangle$. Then, we get a generalized Kummer surface $\mathrm{Km}(A, G)$ (cf. Ueno [32, Example 16.14]).

We give one more example of $G \cong \boldsymbol{Z}/3$.

Example 4. $G \cong \boldsymbol{Z}/3$.
Assume $p \neq 2, 3$. Let E be an arbitrary elliptic curve. Let ι (resp. id) be the inversion (resp. the identity) of E. We consider the automorphism g of $A = E \times E$ defined by

$$g = \begin{pmatrix} 0 & \iota \\ \mathrm{id} & \iota \end{pmatrix} : E \times E \longrightarrow E \times E.$$

We set $G = \langle g \rangle$. Then, $\mathrm{Km}(A, G)$ is a generalized Kummer surface with $G \cong \boldsymbol{Z}/3$.

We will give other examples in Section 7.

§ 4. A proof of Shioda's theorem.

Let k be an algebraically closed field of characteristic $p \geqq 3$. Let A be an abelian surface defined over k, and let ι be the inversion of A. We consider the Kummer surface $\mathrm{Km}(A)$. In [29], Shioda proved the following theorem.

THEOREM 4.1 (Shioda). $\mathrm{Km}(A)$ *is unirational if and only if* A *is a supersingular abelian surface.*

The "only if" part is an easier part. Shioda proved it by the calculation of the Picard number of $\mathrm{Km}(A)$. We will give a general lemma at the end of this section by which we can also prove this part. As for the "if" part, Shioda gave a proof which contains a tricky computation in birational geometry (for details, see Shioda [29, Theorem 1.1]). We give here a new proof of this part. As a corollary to our method, we get an estimate of the minimal degree of generically surjective rational mappings from $\boldsymbol{P}^2$ to Km(A). For this purpose, we recall first a construction of a family of supersingular curves of genus two due to Moret-Bailly [18].

Let E be a supersingular elliptic curve. We have a natural immersion

$$\alpha_p\times\alpha_p \hookrightarrow E\times E\,.$$

We denote by T the tangent space of $E\times E$ at the origin, and by S the projective line $\boldsymbol{P}(T)$ obtained from T. We set

$$K_S=\alpha_p\times\alpha_p\times S \text{ and } (E\times E)_S=E\times E\times S\,.$$

We consider the subgroup scheme H of $K_S=\operatorname{Spec}\mathcal{O}_S[\alpha,\beta]/(\alpha^p,\beta^p)$ defined by the equation $Y\alpha-X\beta=0$, where (X, Y) is a homogeneous coordinate of S. We set $\mathcal{X}=(E\times E)_S/H$. Then, we have the following diagram:

(4.1)
$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r] & (E\times E)_S \arrow[r, "\Pi"] \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \mathcal{X} \arrow[r] \arrow[d, "q"] & 0 \quad \text{(exact)},\\
& E\times E & & S\simeq\boldsymbol{P}^1 &
\end{tikzcd}$$

where Π is the canonical projection, where pr_1 and pr_2 are projections, and where q is the induced morphism. Moret-Bailly constructed a non-singular complete surface D in $\mathcal{X}$ such that $q|_D: D\to\boldsymbol{P}^1$ is a family of supersingular (not necessarily irreducible) curves of genus two. We set $D'=\Pi^{-1}(D)$. D' is a reduced irreducible algebraic surface. Then, by the construction, we have the following facts:

(4.2) $\Pi|_{D'}: D'\to D$ is a purely inseparable morphism of degree p,

(4.3) D (resp. D') is invariant under the action of the inversion of the abelian scheme $q:\mathcal{X}\to S$ (resp. $pr_2:(E\times E)_S\to S$),

(4.4) the morphism $\mathrm{pr}_1|_{D'}: D'\to E\times E$ is surjective and of degree $(p-1)/2$.

For the proofs of these facts, see Moret-Bailly [18, 2.3 on p. 131, and 2.5 on p. 133].

Now, we give a proof of the "if" part of Theorem 4.1. For any abelian surface A, we have a purely inseparable isogeny of degree p from $E\times E$ to A (cf. Oort [22, Corollary 7]). Therefore, as is shown in Shioda [28, Proposition 8], $\mathrm{Km}(A)$ is dominated by $\mathrm{Km}(E\times E)$ and it is sufficient to prove the result for $E\times E$. We have a commutative diagram:

(4.5)
$$\begin{tikzcd}
(E\times E)_S \arrow[r, "\Pi"] \arrow[dd, "\mathrm{pr}_1"'] & \mathcal{X} \arrow[r, "\Pi'"] & (E\times E)_S/K_S \simeq E\times E\times S \arrow[dd, "\mathrm{pr}_1"] \\
D' \arrow[u, phantom, "\cup"] \arrow[r, "f"] & D \arrow[u, phantom, "\cup"] & \\
E\times E \arrow[rr, "F"] & & E\times E,
\end{tikzcd}$$

where F is the Frobenius mapping, and Π' is the canonical projection. We set

(4.6)
$$f=\Pi|_{D'}\,,\quad g=(\mathrm{pr}_1\circ\Pi')|_D \quad \text{and} \quad h=\mathrm{pr}_1|_{D'}\,.$$

Since f, h and F are surjective, the morphism g is also surjective. By (4.3), the inversion of the abelian scheme $q:\mathcal{X}\to S$ induces the inversion $\bar{\iota}$ of the family $q|_D: D\to S$ of curves of genus two, and we have a commutative diagram:

$$\begin{tikzcd}
D \arrow[r, "\bar{\iota}"] \arrow[d, "g"'] & D \arrow[d, "g"] \\
E\times E \arrow[r, "\iota"] & E\times E,
\end{tikzcd}$$

where ι is the inversion of $E\times E$. Hence, we have a surjective morphism

(4.7)
$$\tilde{g}\ :\ D/\langle\bar{\iota}\rangle \longrightarrow (E\times E)/\langle\iota\rangle\,.$$

On the other hand, we have the morphism

(4.8)
$$\tilde{q}\ :\ D/\langle\bar{\iota}\rangle \longrightarrow S$$

which is induced by $q|_D$. Since a general fiber of $\tilde{q}$ is the non-singular rational curve $\boldsymbol{P}^1$, we see that by $S\cong\boldsymbol{P}^1$, $D/\langle\bar{\iota}\rangle$ is birationally equivalent to the projective space $\boldsymbol{P}^2$. Hence, by (4.7), $\mathrm{Km}(E\times E)$ is unirational. q. e. d.

COROLLARY 4.2. *Let A be a supersingular abelian surface defined over k.*

(i) *If A is isomorphic to a product of two supersingular elliptic curves, then there exists a separable covering X of* $\mathrm{Km}(A)$ *of degree $(p-1)/2$ such that X is a Zariski surface.*

(ii) *If A is not isomorphic to a product of two supersingular elliptic curves, then there exists a separable covering X of* Km(A) *of degree* $(p-1)/2$ *such that there exists a purely inseparable rational mapping of degree p^2 from $\boldsymbol{P}^2$ to X.*

PROOF. We use the notations in (4.4), (4.5) and (4.6). Since F is a purely inseparable morphism of degree p^2, by (4.1), (4.3) and (4.4) the purely inseparable degree (resp. the separable degree) of g is equal to p (resp. $(p-1)/2$). Therefore, the purely inseparable degree (resp. the separable degree) of g is also equal to p (resp. $(p-1)/2$). Hence, the results follows from Lemma 1.2, the proof of Theorem 4.1 and Oort [22, Corollary 7]. q. e. d.

REMARK 4.3. By Corollary 4.2 (i), we see that $\mathrm{Km}(E\times E)$ with a supersingular elliptic curve E is a Zariski surface if $p=3$. Professor T. Shioda communicated to the author that he had also a proof of this result for $p=3$. His proof is based on the fact that $\mathrm{Km}(E\times E)$ is isomorphic to the Fermat surface of degree four if $p\equiv 3 \pmod 4$ (cf. Shioda [25, Theorem 3]). For further results, see Theorem 5.10.

The "only if" part of Theorem 4.1 also follows from the following lemma.

LEMMA 4.4. *Assume* char. $k=p\geqq 3$. *Let A be an abelian surface defined over k, and let G be a finite subgroup of* $\mathrm{Aut}_v(A)$ *which satisfies Conditions* (K) *and* (F). *Assume that the order $|G|$ of G is prime to p. Then,* $\mathrm{Km}(A, G)$ *is supersingular if and only if A is supersingular.*

PROOF. If A is supersingular, then by the general theory in Lemma 1.1 we see that $\mathrm{Km}(A, G)$ is supersingular. We prove the "only if" part. We have the following commutative diagram:

(4.9)
$$\begin{tikzcd} A & \tilde{A} \arrow[l, "h"'] \arrow[d, "\psi"] \\ A/G \arrow[from=u, "\pi"'] & \mathrm{Km}(A, G), \arrow[l, "\varphi"'] \end{tikzcd}$$

where φ is the minimal resolution of singularities, h is a composition of blowing-ups such that the mapping ψ induced by π is a morphism. We use the theory of formal Brauer group (cf. Artin and Mazur [3]). We have morphisms $\tilde{h}$ and $\tilde{\psi}$ induced by h and ψ, respectively:

$$\hat{\mathrm{Br}}(A) \xrightarrow{\tilde{h}} \hat{\mathrm{Br}}(\tilde{A}) \xleftarrow{\tilde{\psi}} \hat{\mathrm{Br}}(\mathrm{Km}(A, G))\,.$$

Since h is a composition of blowing-ups, we see that $\tilde{h}$ is an isomorphism (cf. Artin and Mazur [3, p. 122]). Corresponding to the diagram (4.9), we have a commutative diagram:

$$\begin{array}{ccc} H^2(A, \mathcal{O}_A) & \xrightarrow{\quad h^* \quad} & H^2(\tilde{A}, \mathcal{O}_{\tilde{A}}) \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \psi^*} \\ H^2(A/G, \mathcal{O}_{A/G}) & \xrightarrow{\quad \varphi^* \quad} & H^2(\mathrm{Km}(A, G), \mathcal{O}_{\mathrm{Km}(A,G)})\,. \end{array}$$

We denote by Tr the trace map from $H^2(A, \mathcal{O}_A)$ to $H^2(A/G, \mathcal{O}_{A/G})$. Since $|G|$ is prime to p, we see that $(1/|G|)\mathrm{Tr}\circ\pi^*$ is the identity mapping on $H^2(A/G, \mathcal{O}_{A/G})$. Therefore, π^* is injective. Since A/G has no other singularities than rational double points, we have

$$\varphi_*(\mathcal{O}_{\mathrm{Km}(A,G)}) \cong \mathcal{O}_{A/G} \quad \text{and} \quad R^1\varphi_*(\mathcal{O}_{\mathrm{Km}(A,G)}) = 0\,.$$

Therefore, we see that φ^* is an isomorphism. Since h is a birational morphism, h^* is an isomorphism. Hence, ψ^* is an isomorphism. Since $H^2(\tilde{A}, \mathcal{O}_{\tilde{A}})$ (resp. $H^2(\mathrm{Km}(A, G), \mathcal{O}_{\mathrm{Km}(A,G)})$) is the tangent space of $\hat{\mathrm{Br}}(\tilde{A})$ (resp. $\hat{\mathrm{Br}}(\mathrm{Km}(A, G))$ (cf. Artin and Mazur [3, p. 109]), we see that $\tilde{\psi}$ is a non-trivial homomorphism. Since $\hat{\mathrm{Br}}(\tilde{A})$ and $\hat{\mathrm{Br}}(\mathrm{Km}(A, G))$ are formal groups of dimension one, we conclude that the height of $\hat{\mathrm{Br}}(A)$ is equal to the height of $\hat{\mathrm{Br}}(\mathrm{Km}(A, G))$. Hence, $\hat{\mathrm{Br}}(A)$ is isomorphic to $\hat{\mathrm{Br}}(\mathrm{Km}(A, G))$. Now, suppose that $\mathrm{Km}(A, G)$ is supersingular. Then, by Artin [2, p. 544], the height of $\hat{\mathrm{Br}}(\mathrm{Km}(A, G))$ is equal to the infinity. Therefore, the height of $\hat{\mathrm{Br}}(A)$ is equal to the infinity. Hence, A is a supersingular abelian surface (cf. Illusie [10, p. 652]). q. e. d.

§ 5. The discriminant of Néron-Severi groups and Zariski Kummer surfaces.

In this section, we investigate the discriminants of Néron-Severi groups of generalized Kummer surfaces, and we will show that a certain Kummer surface is a Zariski surface (cf. Theorem 5.10).

Let E be a supersingular elliptic curve defined over an algebraically closed field k of characteristic $p>0$. We set $B=\mathrm{End}^0(E)$ and $\mathcal{O}=\mathrm{End}(E)$. B is a quaternion division algebra over the field $\boldsymbol{Q}$ of rational numbers with discriminant p, and $\mathcal{O}$ is a maximal order of B. We denote by $^-$ the canonical involution of B. We set $A=E\times E$. We consider a divisor

$$X=E\times\{o\}+\{o\}\times E.$$

This gives a principal polarization on A. We have a homomorphism

$$\varphi_X : A \longrightarrow A^t$$

defined by

$$x \longmapsto T_x^* X - X \quad \text{for } x\in A.$$

Then, we have an injective homomorphism as abelian groups:

$$\begin{array}{cccc} j : & \mathrm{NS}(A) & \longrightarrow & \mathrm{End}^0(A)\cong M_2(B). \\ & \Cup & & \Cup \\ & L & \longmapsto & \varphi_X^{-1}\circ\varphi_L \end{array}$$

By the homomorphism j, we often identify $\mathrm{NS}(A)$ with $j(\mathrm{NS}(A))$. An element g of $\mathrm{Aut}(A)$ induces an automorphism g^* of $\mathrm{NS}(A)$. The following two lemmas are easily proved (cf. Ibukiyama, Katsura and Oort [8, Section 2], for instance).

LEMMA 5.1. *The image of j is given by*

$$\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \,\middle|\, \alpha,\delta\in \boldsymbol{Z}\,;\,\gamma,\beta\in\mathcal{O}\,;\,\gamma=\bar{\beta}\right\}.$$

For two elements L_1 and L_2 of $\mathrm{NS}(A)$, *set* $j(L_1)=\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix}$ *and* $j(L_2)=\begin{pmatrix}\alpha_2 & \beta_2\\ \gamma_2 & \delta_2\end{pmatrix}$. *Then,*

$$(L_1\cdot L_2)=\alpha_2\delta_1+\alpha_1\delta_2-\gamma_1\beta_2-\gamma_2\beta_1. \tag{5.1}$$

LEMMA 5.2. *Let g be an element of* $\mathrm{Aut}(A)$. *Let g be given by* $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ *in $M_2(\mathcal{O})$. Then, the action g^* on* $\mathrm{NS}(A)$ *is given in $M_2(B)$ by*

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \longmapsto \begin{pmatrix}\bar{a} & \bar{c}\\ \bar{b} & \bar{d}\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

We denote by $\left(\frac{l}{p}\right)$ the Legendre symbol. The following lemma is due to Ibukiyama [7].

LEMMA 5.3 (Ibukiyama). *Assume $p\neq 2$. Let q be a prime number such that $-q\equiv 5$ (mod 8) and $\left(\frac{-q}{p}\right)=-1$. We set*

$$B=\boldsymbol{Q}+\boldsymbol{Q}\alpha+\boldsymbol{Q}\beta+\boldsymbol{Q}\alpha\beta \tag{5.2}$$

$$\textit{with}\quad \alpha^2=-p\,,\quad \beta^2=-q\,,\quad \alpha\beta=-\beta\alpha\,.$$

Let a be an integer such that $a^2 \equiv -p \pmod q$. We set

$$\mathcal{O} = \boldsymbol{Z}\omega_1 + \boldsymbol{Z}\omega_2 + \boldsymbol{Z}\omega_3 + \boldsymbol{Z}\omega_4 \tag{5.3}$$

$$\text{with} \quad \omega_1 = 1\,,\ \omega_2 = (1+\beta)/2\,,\ \omega_3 = \alpha(1+\beta)/2\,,\ \omega_4 = (a+\alpha)\beta/q\,.$$

Then, B is a quaternion division algebra over $\boldsymbol{Q}$ with discriminant p, and $\mathcal{O}$ is a maximal order of B. Moreover, in case $p \equiv 3 \pmod 4$, B and $\mathcal{O}$ are also given by the following form:

$$\begin{cases} B = \boldsymbol{Q} + \boldsymbol{Q}\alpha + \boldsymbol{Q}\beta + \boldsymbol{Q}\alpha\beta\,, \\ \alpha^2 = -p\,,\ \beta^2 = -1\,,\ \alpha\beta = -\beta\alpha\,, \\ \mathcal{O} = \boldsymbol{Z} + \boldsymbol{Z}\beta + \boldsymbol{Z}(1+\alpha)/2 + \boldsymbol{Z}\beta(1+\alpha)/2\,. \end{cases} \tag{5.4}$$

LEMMA 5.4. (i) *Under the notations in Example 2 in Section 3, assume $p \equiv 2 \pmod 3$. Then, $B = \mathrm{End}^0(E)$ and $\mathcal{O} = \mathrm{End}(E)$ are given by*

$$B = \boldsymbol{Q} + \boldsymbol{Q}\alpha + \boldsymbol{Q}\beta + \boldsymbol{Q}\alpha\beta \tag{5.5}$$

$$\text{with} \quad \alpha^2 = -p\,,\ \beta^2 = -3\,,\ \alpha\beta = -\beta\alpha\,,$$

$$\mathcal{O} = \boldsymbol{Z}\omega_1 + \boldsymbol{Z}\omega_2 + \boldsymbol{Z}\omega_3 + \boldsymbol{Z}\omega_4 \tag{5.6}$$

$$\text{with} \quad \omega_1 = 1\,,\ \omega_2 = \alpha\,,\ \omega_3 = (-1+\beta)/2\,,\ \omega_4 = (1+\alpha)(3+\beta)/6\,.$$

A basis of the invariant subspace $\mathrm{NS}(A)^G$ of $\mathrm{NS}(A)$ by G is given by

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & \bar{\alpha} \\ \alpha & 0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & 2+\bar{\omega}_3 - 3\bar{\omega}_4 \\ 2+\omega_3 - 3\omega_4 & 0 \end{pmatrix}.$$

(ii) *Under the notations in Example 3 in Section 3, assume $p \equiv 3 \pmod 4$. Then, $B = \mathrm{End}^0(E)$ and $\mathcal{O} = \mathrm{End}(E)$ are given by the form in* (5.4). *A basis of the invariant subspace $\mathrm{NS}(A)^G$ of $\mathrm{NS}(A)$ by G is given by*

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & -\alpha\beta \\ \alpha\beta & 0 \end{pmatrix}.$$

PROOF. In both cases, elliptic curves E are supersingular and are defined over $\boldsymbol{F}_p$. Therefore, the Frobenius morphism satisfies $F^2 = -p$. In case (i), we set $\alpha = F$ and $\beta = 2\sigma + 1$. Then, we get (5.2) and (5.3) with $q = 3$ and $a = 1$. Since $\mathrm{End}(E)$ is a maximal order of $\mathrm{End}^0(E)$, we have $\mathcal{O} = \mathrm{End}(E)$ by Lemma 5.3. Since $\sigma = (-1+\beta)/2$ is a unit of $\mathrm{End}(E)$, we see that

$$\{\sigma,\ -(1+\beta)\sigma/2,\ -\alpha(1+\beta)\sigma/2,\ -(1+\alpha)\beta\sigma/3\}$$

is also a basis of $\mathrm{End}(E)$. This basis gives the basis in (5.6). In case (ii), we set $\alpha = F$ and $\beta = \tau$. Then, we get (5.4). Since $\mathrm{End}(E)$ is a maximal order of $\mathrm{End}^0(E)$, we have $\mathcal{O}=\mathrm{End}(E)$ by Lemma 5.3. By Lemma 5.2, the action of σ^* (resp. τ^*) on $\mathrm{NS}(A)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$$

$$\left(\text{resp. } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in j(\mathrm{NS}(A))$. By direct calculation, we can easily obtain a basis of the invariant space $\mathrm{NS}(A)^G$. q. e. d.

For a quaternion division algebra B in (5.2) and its maximal order $\mathcal{O}$ in (5.3), there exists a supersingular elliptic curve E such that $\mathrm{End}^0(E)\cong B$ and $\mathrm{End}(E)\cong\mathcal{O}$ (cf. Waterhouse [33, Theorem 3.13]). In Example 4 in Section 3, we take this elliptic curve E. Then, we have the following.

LEMMA 5.5. *Under the notations in Example* 4 *in Section* 3, *let E be an elliptic curve as above. Then, a basis of the invariant subspace* $\mathrm{NS}(A)^G$ *is given by*

$$e_1=\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},\quad e_2=\begin{pmatrix} -1 & \bar{\omega}_2 \\ \omega_2 & -1 \end{pmatrix},\quad e_3=\begin{pmatrix} 0 & \bar{\omega}_3 \\ \omega_3 & 0 \end{pmatrix},\quad e_4=\begin{pmatrix} 0 & \bar{\omega}_4 \\ \omega_4 & 0 \end{pmatrix}.$$

PROOF. The action g^* on $\mathrm{NS}(A)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in j(\mathrm{NS}(A))$. By direct calculation, we get easily a basis of the invariant subspace $\mathrm{NS}(A)^G$. q. e. d.

LEMMA 5.6. *In Example* 2 (*resp. Erample* 3, *resp. Example* 4 *with an elliptic curve E as above*) *in Section* 3, *the discriminant of* $\mathrm{NS}(A)^G$ *is equal to* $-3p^2$ (*resp.* $-4p^2$, *resp.* $-3p^2$).

PROOF. By Lemmas 5.1, 5.4 and 5.5, the intersection matrix of $\mathrm{NS}(A)^G$ is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2p & 3p \\ 0 & 0 & 3p & -6p \end{pmatrix}$$

$$\left(\text{resp. } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2p & 0 \\ 0 & 0 & 0 & -2p \end{pmatrix},\ \text{resp. } \begin{pmatrix} 6 & 3 & 0 & 0 \\ 3 & (q+3)/2 & 0 & a \\ 0 & 0 & p(1-q)/2 & -p \\ 0 & a & -p & 2(a^2-p)/q \end{pmatrix}\right).$$

Computing the determinant, we get the result. q. e. d.

The following theorem is due to Ogus [20, Corollary 7.14] (see also Shioda [30, Theorem 4.2]).

THEOREM 5.7 (Ogus). *Assume* char. $k=p\geqq 3$. *Let* X *be a supersingular* $K3$ *surface.* X *is isomorphic to a Kummer surface* $\mathrm{Km}(A)$ *with an abelian surface* A *which is isomorphic to a product of two supersingular elliptic curves if and only if the discriminant of the Néron-Severi group* $\mathrm{NS}(X)$ *is equal to* $-p^2$.

Now, we examine the structure of generalized Kummer surfaces in Examples 2, 3 and 4 in Section 3. Let σ (resp. g) be the automorphism of order three in Example 2 (resp. Example 4) in Section 3. Then, by Table in Section 3 (or by direct calculation), σ (resp. g) has nine fixed points. We consider the blowing-up of A at these nine points:

$$\psi_1 : A_1 \longrightarrow A .$$

Then, the automorphism g of A lifts to an automorphism σ_1 (resp. g_1) of order three of A_1. The automorphism σ_1 (resp. g_1) has two fixed points on each exceptional curve. We again blow-up these fixed points:

$$\psi_2 : A_2 \longrightarrow A_1 .$$

Then, the automorphism σ_1 (resp. g_1) of A_1 lifts to an automorphism σ_2 (resp. g_2) of order three of A_2. We consider the quotient surface $A_3=A_2/\langle\sigma_2\rangle$ (resp. $A_3=A_2/\langle g_2\rangle$). Then, by a local calculation, we see that $A_3=A_2/\langle\sigma_2\rangle$ (resp. $A_3=A_2/\langle g_2\rangle$) is non-singular and contains nine exceptional curves. We blow-down these nine exceptional curves. Then, we obtain $\mathrm{Km}(A, G)$ which is the minimal resolution of A/G. Thus, we get the following commutative diagram:

(5.7)
$$\begin{array}{ccccc} A_2 & & \xrightarrow{\quad\quad\psi\quad\quad} & & A \\ \big\downarrow{\scriptstyle\tilde{\pi}} & & & & \big\downarrow{\scriptstyle\pi} \\ A_3 & \xrightarrow{\ h\ } & \mathrm{Km}(A, G) & \xrightarrow{\ \varphi\ } & A/G , \end{array}$$

where $\psi=\psi_1\circ\psi_2$, π and $\tilde{\pi}$ are the canonical projections, φ is the minimal resolution of singularities, and h is the blowing-down as above.

Now, let τ be the automorphism in Example 2 in Section 3. We have $\tau^2=\iota$, where ι is the inversion of A. The automorphism ι has sixteen fixed points, and τ acts on these sixteen fixed points. We blow-up these sixteen fixed points:

$$\psi_1 : A_1 \longrightarrow A .$$

Then, the automorphism τ lifts to an automorphism τ_1 of A_1. We set $\tau_1^2 = \iota_1$. Then ι_1 is a lifting of ι. We set $\mathrm{Km}(A) = A_1/\langle \iota_1 \rangle$. This surface is the Kummer surface of A, and the automorphism τ_1 descends to an automorphism τ_1' of order two of $\mathrm{Km}(A)$. It is easy to see that τ_1' has eight fixed points on $\mathrm{Km}(A)$. We blow-up these eight points:

$$\psi_2 : X_2 \longrightarrow \mathrm{Km}(A) .$$

The automorphism τ_1' lifts to an automorphism τ_2 of X_2. By a local calculation, we see that the quotient surface $X_2/\langle \tau_2 \rangle$ is a non-singular minimal surface and is isomorphic to $\mathrm{Km}(A, G)$. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
 & A_1 \arrow[r, "\psi_1"] \arrow[d, "\pi_1"] & A \arrow[dd, "\pi"] \\
X_2 \arrow[r, "\psi_2"] \arrow[d, "\pi_2"'] & \mathrm{Km}(A) & \\
\mathrm{Km}(A, G) \arrow[rr, "\varphi"] & & A/G
\end{tikzcd} \tag{5.8}$$

where π, π_1 and π_2 are the canonical projections, and φ is the minimal resolution of singularities.

The essential idea of the proof of the following lemma is due to Shioda [30, Proposition 3.1].

LEMMA 5.8. *In Example* 2 (*resp. Example* 3, *resp. Example* 4) *in Section* 3, *assume that* $p \equiv 2$ (*mod* 3) (*resp.* $p \equiv 3$ (*mod* 4), *resp.* E *is a supersingular elliptic curve as in Lemma* 5.5). *Then, the discriminant of the Néron-Severi group* $\mathrm{NS}(\mathrm{Km}(A, G))$ *is equal to* $-p^2$.

PROOF. First, we consider Example 2 (resp. Example 4) in Section 3. Let $\tilde{G}$ be a group generated by σ_2 (resp. g_2). By (5.7), we have the following diagram:

(5.9)
$$\begin{array}{l} NS(A_2)\cong NS(A)\oplus L_2 \\ \quad\cup \\ NS(A_2)^{\tilde{G}}\cong NS(A)^{G}\oplus L_2 \\ \tilde{\pi}^*\uparrow\!\cup \\ NS(A_3)\cong NS(Km(A,G))\oplus L_3\,, \end{array}$$

where L_2 and L_3 are subgroups generated by exceptional curves obtained from blowing-ups. Since $\operatorname{rank}(NS(A)^G\oplus L_2)=\operatorname{rank}(NS(Km(A,G))\oplus L_3)=31$, $\tilde{\pi}^*(NS(A_3))$ has a finite index in $NS(A_2)^{\tilde{G}}$. Since $|\tilde{G}|=3$, we have

$$3NS(A_2)^{\tilde{G}}\subset\tilde{\pi}^*(NS(A_3))\subset NS(A_2)^{\tilde{G}}\,. \tag{5.10}$$

It is easy to see that the discriminant of L_2 (resp. L_3) is equal to $(-1)^{27}=-1$ (resp. $(-1)^9=-1$). By Artin [2, (4.6) on p. 556], the discriminant of $NS(Km(A,G))$ is of the form $-p^{2a}$ with a positive integer a. Therefore, by (5.10) and Lemma 5.6, we have

$$(3^2)^{31}(-1)^{27}(-3p^2)=3^{31}(-1)^9(-p^{2a})n^2$$

with a positive integer n. Since p is a prime number such that $p\neq 2,3$, we have $n=3^{16}$ and $a=1$. Hence, the discriminant of $NS(Km(A,G))$ is equal to $-p^2$.

Now, we consider Example 3. Let G_1 (resp. G_1', resp. G_2) be a group generated by τ_1 (resp. τ_1', resp. τ_2). Since $NS(A_1)$ is $\langle\iota_1\rangle$-invariant, we have by (5.8)

$$\psi_1^*(NS(A)^G)\oplus L_1^{G_1}\cong NS(A_1)^{G_1}\supset\pi_1^*(NS(Km(A))^{G_1'}\supset 2NS(A_1)^{G_1}\,, \tag{5.11}$$

and

$$\psi_2^*(NS(Km(A))^{G_1'})\oplus L_2^{G_2}\cong NS(X_2)^{G_2}\supset\pi_2^*(NS(Km(A,G)))\supset 2NS(X_2)^{G_2}\,, \tag{5.12}$$

where L_1 and L_2 are subgroups generated by exceptional curves obtained from blowing-ups. Since G_1 preserves four exceptional curves and interchanges other twelve exceptional curves, the discriminant of $L_1^{G_1}$ is equal to $(-1)^4(-2)^6=2^6$. We see easily $\operatorname{rank} NS(A_1)^{G_1}=14$. We denote by $\operatorname{disc}(NS(Km(A))^{G_1'})$ the discriminant of $NS(Km(A))^{G_1'}$. Then, by (5.11), Lemma 5.6 and the same argument above, we have

$$2^{28}(-4p^2\cdot 2^6)=2^{14}\operatorname{disc}(NS(Km(A))^{G_1'})n^2$$

with a positive integer n, that is,

$$\operatorname{disc}(NS(Km(A))^{G_1'})=-2^{22}p^2/n^2\,. \tag{5.13}$$

Since $L_2^{G_2}=L_2$ and rank $L_2=8$, we see that the discriminant of $L_2^{G_2}$ is equal to $(-1)^8$. Therefore, by (5.12), (5.13) and the same argument as above, we have

$$2^{44}(-2^{22}\cdot p^2/n^2)(-1)^8=2^{22}(-p^{2a})m^2 \tag{5.14}$$

with a positive integer m. Since p is a prime number such that $p\neq 2$, we have $mn=2^{22}$ and $a=1$. Hence, the discriminant of NS(Km(A, G)) is equal to $-p^2$. q. e. d.

THEOREM 5.9. *In Example* 2 (*resp. Example* 3, *resp. Example* 4), *the generalized Kummer surface* Km(A, G) *is isomorphic to the Kummer surface* Km(A) *if* $p\equiv 2$ (*mod* 3) (*resp. if* $p\equiv 3$ (*mod* 4), *resp. if* E *is a supersingular elliptic curve as in Lemma* 5.5).

PROOF. This follows from Theorem 5.7 and Lemma 5.8. q. e. d.

THEOREM 5.10. *Let* k *be an algebraically closed field of characteristic* $p\geqq 3$. *Let* X *be a supersingular abelian surface defined over* k *which is isomorphic to a product of two supersingular elliptic curves. If* $p\not\equiv 1$ (*mod* 12), *then the Kummer surface* Km(X) *is a Zariski surface.*

PROOF. We assume $p\equiv 2$ (mod 3) (resp. $p\equiv 3$ (mod 4)). Let E, A and G be as in Example 2 (resp. Example 3) in Section 3. Then, the elliptic curve E is supersingular, and by Lemma 1.2, we have $A\cong X$. Therefore, by Theorem 5.9, the generalized Kummer surface Km(A, G) is isomorphic to the Kummer surface Km(X). By Katsura [14, Proposition 5.1], Km(A, G) is birationally equivalent to the elliptic surface defined by

$$y^2=4x^3-t^4(t-1)^4 \quad (\text{resp. } y^2=4x^3-t^3(t-1)^3x)\,.$$

By the base change by a purely inseparable morphism of degree p, this surface is transformed into a rational surface (cf. Katsura [14, Proposition 5.2]). Hence, Km(X) is a Zariski surface. q. e. d.

§ 6. The unirationality.

Let k be an algebraically closed field of characteristic p. In this section, we prove the following theorem.

THEOREM 6.1. *Assume* char. $k=p\geqq 7$. *Let* Km(A, G) *be a generalized Kummer surface. Then, the following three conditions are equivalent:*

(i) $\mathrm{Km}(A, G)$ *is unirational,*
(ii) $\mathrm{Km}(A, G)$ *is supersingular,*
(iii) A *is supersingular.*

LEMMA 6.2. *Assume* char. $k=p\geqq 7$. *Let* G *be a finite subgroup of* $\mathrm{Aut}_v(A)$ *which satisfies Conditions* (K) *and* (F). *Assume that* $|G|$ *is divisible by two, then* G *has a unique element of order two. Moreover, the element of order two is the inversion of* A *with a suitable choice of the origin of* A.

PROOF. Since $|G|$ is divisible by two, G has an element of order two. The uniqueness follows from the structure of groups in Theorem 3.7. By Lemma 3.5, this element is the inversion of A with a suitable choice of the origin of A. q. e. d.

The following lemma is known. We give a proof for reader's convenience.

LEMMA 6.3. *Assume* $p\equiv 1$ (*mod* 5). *Let* A *be a supersingular abelian surface. Then, there exists no automorphism of order five which has some fixed points on* A.

PROOF*). Let E be a supersingular elliptic curve. Then, A is isogenous to $E\times E$ (cf. Oort [21, Theorem 4.2]). We set $\mathrm{End}^0(E)=B$. The algebra B is a division algebra over $\boldsymbol{Q}$ which is ramified at p and ∞. $\mathrm{End}^0(A)$ is isomorphic to $M_2(B)$. Suppose that there exists an element of order five which has some fixed points. By a suitable choice of the origin of A, we may assume that the element is contained in $\mathrm{Aut}(A)$. Therefore, we have an element g of $\mathrm{End}^0(A)$ such that $\mathrm{End}^0(A)\supset \boldsymbol{Q}(g)$ and $g^5=1$. Since $\dim_{\boldsymbol{Q}}\mathrm{End}^0(A)=16$ and $\dim_{\boldsymbol{Q}}\boldsymbol{Q}(g)=4$, we see that $\boldsymbol{Q}(g)$ is a splitting field of $\mathrm{End}^0(A)$. We have a commutative diagram of Brauer groups:

$$\begin{array}{ccc} \mathrm{Br}(\boldsymbol{Q}_p) & \xrightarrow{\ \tilde{\varphi}\ } & \mathrm{Br}(\boldsymbol{Q}_p(g)) \\ \mathrm{Inv}_p \uparrow & & \uparrow \psi \\ \mathrm{Br}(\boldsymbol{Q}) & \xrightarrow[\ \varphi\ ]{} & \mathrm{Br}(\boldsymbol{Q}(g)) . \end{array}$$

We have $\mathrm{Br}(\boldsymbol{Q}_p)\cong \boldsymbol{Q}/\boldsymbol{Z}$ and $\mathrm{Br}(\boldsymbol{Q}_p(g))\cong \boldsymbol{Q}/\boldsymbol{Z}$, and the homomorphism $\tilde{\varphi}$ is given by the multiplication by the degree $[\boldsymbol{Q}_p(g):\boldsymbol{Q}_p]$ of the algebraic extension $\boldsymbol{Q}_p(g)$ of $\boldsymbol{Q}_p$ (for instance, see Mumford [19, p. 196]). Since $p\equiv 1$ mod 5, we see $\boldsymbol{Q}_p(g)=\boldsymbol{Q}_p$. Therefore, $\tilde{\varphi}$ is an isomorphism. Since B is

*) The author thanks Professor Y. Morita for communicating the idea of this proof.

ramified at p, we have $\mathrm{Inv}_p(B)\neq 0$. Therefore, $\tilde{\varphi}\circ\mathrm{Inv}_p(B)\neq 0$. On the other hand, since $\boldsymbol{Q}(g)$ is a splitting field of B, we have $\varphi(B)=0$. Therefore, we have $\psi\circ\varphi(B)=0$, a contradiction. q. e. d.

LEMMA 6.4. *Assume $p=0$ or $p\equiv 1 \pmod 5$. Let A be an abelian surface over k. Then, there exists no automorphism g of order five of A such that $A/\langle g\rangle$ is birationally equivalent to a $K3$ surface.*

PROOF. In case $p=0$, this follows from Lemma 3.15. Assume $p\equiv 1$ (mod 5). Suppose that there exists an automorphism g of order five of A such that $A/\langle g\rangle$ is birationally equivalent to a $K3$ surface. Then, by Table in Section 3 and Lemma 4.4, A is a supersingular abelian surface. Hence, this lemma follows from Lemma 6.3. q. e. d.

The following lemma is well-known.

LEMMA 6.5 (Skolem-Noether). *Let A be a central simple algebra over a field K, and let B a simple subalgebra of A. For any algebra-homomorphisms σ and τ from B to A, there exists an element t of A such that*

$$\tau(b)=t\sigma(b)t^{-1} \text{ for any element } b \text{ of } B.$$

LEMMA 6.6. *Let E be a supersingular elliptic curve. Let f be an element of order three (resp. order five) of $\mathrm{Aut}(E\times E)$. If there exists an element g of order three (resp. order five) of $\mathrm{Aut}_v(E\times E)$ such that g has only isolated fixed points and that $(E\times E)/\langle g\rangle$ is unirational, then $(E\times E)/\langle f\rangle$ is also unirational.*

PROOF. By our assumption, we can find an element a of $E\times E$ such that $T_a\circ g\circ T_a^{-1}$ is contained in $\mathrm{Aut}(E\times E)$. Therefore, replacing g by $T_a\circ g\circ T_a^{-1}$, we may assume that g is also contained in $\mathrm{End}(E\times E)$. We set $B=\boldsymbol{Q}(g)$. This is a simple algebra over $\boldsymbol{Q}$. We have two homomorphisms σ, τ from $\boldsymbol{Q}(g)$ to $\mathrm{End}^0(E\times E)$ defined by

$$\begin{cases} \sigma : g \longrightarrow g\,, \\ \tau : g \longrightarrow f\,. \end{cases}$$

By Lemma 6.5, we can find an element t of $\mathrm{End}^0(E\times E)$ such that

$$\tau(b)=t\sigma(b)t^{-1} \text{ for any element } b \text{ of } \boldsymbol{Q}(g)\,. \tag{6.1}$$

By a suitable choice of t, we may assume that t is contained in $\mathrm{End}(E\times E)$. We set $b=g$. Then, by (6.1), we have

$$f\circ t = t\circ g\,.$$

Hence, we have the morphism

$$t\,:\,(E\times E)/\langle g\rangle \longrightarrow (E\times E)/\langle f\rangle$$

induced by t. Since $(E\times E)/\langle g\rangle$ is unirational, we conclude that $(E\times E)/\langle f\rangle$ is unirational. q. e. d.

LEMMA 6.7. *Let A be a supersingular abelian surface, and let f be an element of order three (resp. order five) of* $\mathrm{Aut}(A)$. *Let E be any supersingular elliptic curve. If there exists an element g of order three (resp. order five) of* $\mathrm{Aut}_0(E\times E)$ *such that g has only isolated fixed points and that $(E\times E)/\langle g\rangle$ is unirational, then $A/\langle f\rangle$ is also unirational.*

PROOF. We may assume that g is contained in $\mathrm{End}(E\times E)$. If A is isomorphic to a product of two supersingular elliptic curves, then we have $A\cong E\times E$ by Lemma 1.2. Therefore, in this case, this lemma follows from Lemma 6.6. If A is not isomorphic to a product of two supersingular elliptic curves, then there exists an immersion, $\varepsilon:\alpha_p\hookrightarrow E\times E$ such that $A\cong(E\times E)/\varepsilon(\alpha_p)$ (cf. Oort [22, Corollary 7]). We identity A with $(E\times E)/\varepsilon(\alpha_p)$ by this isomorphism. Then, by Lemma 1.3, we can find an automorphism $\tilde f$ of $E\times E$ such that the following diagram commutes:

$$\begin{array}{ccc} E\times E & \xrightarrow{\ \tilde f\ } & E\times E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\ f\ } & A\,, \end{array}$$

where π is the canonical projection. Therefore, we have a morphism $\tilde\pi:(E\times E)/\langle\tilde f\rangle\to A/\langle f\rangle$ induced by π. By assumption and Lemma 6.6, the surface $(E\times E)/\langle\tilde f\rangle$ is unirational. Hence, $A/\langle f\rangle$ is also unirational. q. e. d.

LEMMA 6.8. *Assume $p\neq 5$ and $p\not\equiv 1$ (mod 5). Let C be the non-singular complete model of the curve defined by the equation*:

$$y^2 = x^5 - 1\,.$$

Let $J(C)$ be the Jacobian variety of C, and let g be the automorphism of $J(C)$ induced by the automorphism g' of order five of the curve defined by

$$\begin{cases} x \longmapsto \zeta x\,, \\ y \longmapsto y\,, \end{cases}$$

where ζ is a primitive fifth root of unity. Then, g is of order five and $J(C)/\langle g\rangle$ is rational.

PROOF. We have a morphism

$$\varphi\,:\,C \longrightarrow J(C)$$

which commutes with the actions of our automorphisms g', g of order five. This morphism φ induces an isomorphism

$$\varphi^*\,:\,H^0(J(C),\,\Omega^1_{J(C)}) \xrightarrow{\sim} H^0(C,\,\Omega^1_C)\,.$$

A basis of $H^0(C, \Omega^1_C)$ is given by $\{dx/y, xdx/y\}$. Therefore, the action of g' is given with respect to this basis by $\begin{pmatrix}\zeta & 0\\ 0 & \zeta^2\end{pmatrix}$. Therefore, the action of g at a fixed point is given by $\begin{pmatrix}\zeta & 0\\ 0 & \zeta^2\end{pmatrix}$ with respect to a suitable regular system of parameters. Hence, $J(C)/\langle g\rangle$ has singularities which are not rational double points. Hence, by Theorem 2.11, $J(C)/\langle g\rangle$ is rational. q. e. d.

PROOF OF THEOREM 6.1. By Shioda [27, Corollary 2], (i) implies (ii). By Lemma 4.4, (ii) is equivalent to (iii). Now, suppose that (iii) holds. By the argument of Section 3, we may assume that G satisfies Conditions (K) and (F). Then, the structure of the group G is given as in Theorem 3.7. Suppose that $|G|$ is divisible by two. Then, there exists an element g of order two of G. We have a morphism

$$A/\langle g\rangle \longrightarrow A/G\,.$$

By Lemma 6.2 and Theorem 4.1, $A/\langle g\rangle$ is unirational. Therefore, A/G is unirational. Suppose that $|G|$ is divisible by three. Then, there exists an element g of order three of G. By Theorems 4.1, 5.9 and Lemma 6.7, $A/\langle g\rangle$ is unirational. Therefore, A/G is unirational. Finally, suppose that $|G|$ is divisible by five. Then, by Lemmas 6.7 and 6.8, we see that A/G is unirational by the same argument as above. Hence, in any case in Theorem 3.7, we conclude that $\mathrm{Km}(A, G)$ is unirational. Hence, (iii) implies (i). q. e. d.

§ 7. Examples.

Let k be an algebraically closed field of characteristic p. In Section 3, we gave some examples of generalized Kummer surfaces. In this section, we give other examples and show that every group in Theorem 3.7 can occur. In characteristic 0, every group in Corollary 3.17 can occur (see also Fujiki [5]). In this section, we denote by E, E_1 and E_2 elliptic curves, and by A an abelian surface. We denote by G a finite subgroup of $\mathrm{Aut}_0(A)$ which satisfies Conditions (K) and (F). Then, the minimal non-singular model of A/G gives a generalized Kummer surface $\mathrm{Km}(A, G)$.

Example 5. $G\cong \boldsymbol{Z}/6$.

We have the following well-known example. Assume $p\neq 2, 3$. Let E_j $(j=1, 2)$, A and ω be as in Example 2 in Section 3. We consider the automorphism defined by

$$\rho : \begin{cases} x_1 \longmapsto -\omega x_1, & y_1 \longmapsto y_1, \\ x_2 \longmapsto -\omega^2 x_2, & y_2 \longmapsto y_2. \end{cases}$$

We set $G=\langle\rho\rangle$. Then, we have $G\cong \boldsymbol{Z}/6$, and G satisfies Conditions (K) and (F). Hence, we have a generalized Kummer surface $\mathrm{Km}(A, G)$.

Let E be a supersingular elliptic curve, and let $\gamma : \alpha_p=\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^p) \hookrightarrow E\times E$ be an immersion. We set $A=(E\times E)/\gamma(\alpha_p)$, and denote by $\pi : E\times E \to A$ the canonical projection. Let g (resp. $\tilde{g}$) be an element of $\mathrm{Aut}(A)$ (resp. $\mathrm{Aut}(E\times E)$) such that $g\circ\pi=\pi\circ\tilde{g}$. Assume that p is prime to the orders of g and $\tilde{g}$. Let α and β be eigen-values of $\tilde{g}^*$ on the cotangent space of $E\times E$ at the origin. Then, it is obvious that the eigen-values of $\tilde{g}_*$ on the tangent space of $E\times E$ are also given by α and β. Since $g\circ\pi=\pi\circ\tilde{g}$, the subgroup scheme $\gamma(\alpha_p)$ is fixed by $\tilde{g}$. Hence, a non-zero tangent vector to $\gamma(\alpha_p)$ is an eigen-vector of $\tilde{g}_*$. We may assume that the tangent vector corresponds to the eigen-value α.

LEMMA 7.1. *Under the above notations and assumptions, the eigen-values of g^* on the cotangent space of A at the origin are given by α^p and β.*

PROOF. We denote by $\mathcal{O}$ the local ring of $E\times E$ at the origin, and by m the maximal ideal of $\mathcal{O}$. We have homomorphisms

$$\alpha_p \overset{\gamma}{\hookrightarrow} \alpha_p\times\alpha_p \hookrightarrow E\times E. \tag{7.1}$$

By a suitable choice of a regular system of parameters (x, y) of $\mathcal{O}$, by (7.1) we have homomorphisms

$$k[\varepsilon]/(\varepsilon^p) \xleftarrow{\gamma^*} k[x]/(x^p)\otimes k[y]/(y^p) \longleftarrow k[[x]]\otimes k[[y]]$$

$$\begin{cases} a\varepsilon \longleftarrow\!\shortmid x \\ \varepsilon \longleftarrow\!\shortmid y \end{cases} \qquad \begin{cases} x\otimes 1 \longleftarrow\!\shortmid x \\ 1\otimes y \longleftarrow\!\shortmid y \end{cases}$$

with an element a of k, and the action of $\gamma(\alpha_p)$ on $k[[x, y]]$ is given by

$$x \longmapsto x+a\varepsilon\,, \quad y \longmapsto y+\varepsilon$$

(cf. Katsura and Ueno [16, Appendix 2]). Then $x-ay$ is invariant under the action of $\gamma(\alpha_p)$. Since $\gamma(\alpha_p)$ is fixed by $\tilde{g}$, $x-ay$ gives an eigen-vector of $\tilde{g}^*$ on the cotangent space m/m^2 which corresponds to the eigen-value β. Since $(x-ay, y)$ gives a regular system of parameters of $\mathcal{O}$, we see that $(x-ay, y^p)$ gives a regular system of parameters of A at the origin. Since $g\circ\pi=\pi\circ\tilde{g}$, $x-ay$ gives an eigen-vector of g^* on the cotangent space of A at the origin which corresponds to the eigen-value β. Since $\{x-ay, y^p\}$ gives a basis of the cotangent space of A at the origin, we conclude that the eigen-values of g^* are given by α^p and β. q. e. d.

Example 6. $G\cong \boldsymbol{Z}/5$, $\boldsymbol{Z}/10$.

Assume $p\neq 2$ and $p\equiv 2 \pmod 5$. Let C be the non-singular complete model of the curve of genus two defined by the equation

$$y^2=x^5-1\,. \tag{7.2}$$

Let σ be the automorphism of order five of C defined by

$$\sigma\ :\ \begin{cases} x \longmapsto \zeta x\,, \\ y \longmapsto y\,, \end{cases}$$

where ζ is a primitive fifth root of unity. We denote again by σ the automorphism of the Jacobian variety $J(C)$ which is induced by σ. We denote by ι the inversion of $J(C)$. We may regard C as a divisor on $J(C)$ such that $\iota^*(C)=C$. We may assume that σ fixes the origin of $J(C)$. Since the action of σ on $H^0(C, \Omega_C^1)$ is given by the matrix

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}$$

with respect to a basis $\{dx/y, xdx/y\}$ the eigen-values of σ^* on the cotangent space of $J(C)$ at the origin are given by ζ and ζ^2. By our assumption, $J(C)$ is supersingular and is not isomorphic to a product of two supersingular elliptic curves (cf. Ibukiyama, Katsura and Oort [8, Proposition 1.13]). Let E be a supersingular elliptic curve. Then, by Oort [22, Corollary 7], there exists an immersion $\gamma : \alpha_p \hookrightarrow E\times E$ such that $J(C)\cong(E\times E)/\gamma(\alpha_p)$. We identify $J(C)$ with $(E\times E)/\gamma(\alpha_p)$ by this isomorphism. Let $\pi : E\times E\to J(C)$ be the canonical projection. Then, by Lemma 1.3, we have an automorphism $\tilde{\sigma}$ of order five of $E\times E$ such that $\pi\circ\tilde{\sigma}=\sigma\circ\pi$. By Lemma 7.1, the eigen-values of $\tilde{\sigma}^*$ on the cotangent space of $E\times E$ at the origin are given by either $\zeta^{1/p}=\zeta^3$ and ζ^2, or ζ and $\zeta^{2/p}=\zeta$ by the assumption $p\equiv 2 \pmod 5$.

Suppose that the eigen-values of $\tilde{\sigma}^*$ are given by ζ and ζ. Since $\pi^*(C)$ satisfies Conditions (a) and (b) in Moret-Bailly [18, p. 126], by $\pi^*(C)$ we can construct a family $q:\mathcal{X}\to S$ of principally polarized supersingular abelian surfaces as in (4.1). By our construction, the divisor $\pi^*(C)$ is preserved by $\tilde{\sigma}$. Therefore, by Katsura and Oort [17, Theorem 4.1], the automorphism $\tilde{\sigma}$ induces an automorphism of order five of the family $q:\mathcal{X}\to S$. Since the eigen-values of $\tilde{\sigma}^*$ on the cotangent space at the origin of $E\times E$ are given by ζ and ζ, $\tilde{\sigma}$ preserves all directions at the origin of $E\times E$ (cf. Katsura and Oort [17, Section 3]). Therefore, the action of the automorphism on S is trivial. This means that general fibres of $q:\mathcal{X}\to S$ have automorphisms of order five which preserve the polarizations. Therefore, general fibres are isomorphic to $(J(C), C)$ with C defined by (7.2) (cf. Igusa [9, p. 645]). Therefore, all fibres of $q:\mathcal{X}\to S$ are isomorphic to each other, which contradicts the fact that this family with polarization D is not a constant family (cf. Moret-Bailly [18, p. 131]). Hence, the eigen-values of $\tilde{\sigma}^*$ are given by ζ^3 and ζ^2.

Since the cotangent space of $E\times E$ at the origin is naturally isomorphic to $H^0(E\times E, \Omega^1_{E\times E})$, the eigen-values of $\tilde{\sigma}^*$ on $H^0(E\times E, \Omega^1_{E\times E})$ are given by ζ^3 and ζ^2. Hence, $\tilde{\sigma}^*$ preserves a non-zero regular two-form on $E\times E$. Now, it is easy to see that the group $\langle\tilde{\sigma}\rangle$ satisfies Conditions (K) and (F). Hence, we have a generalized Kummer surfaces $\mathrm{Km}(E\times E, G)$ with $G=\langle\tilde{\sigma}\rangle\cong \boldsymbol{Z}/5$ by Theorem 2.4 if $p\neq 2$ and $p\equiv 2 \pmod 5$. By the similar method, if $p\equiv 3$ (mod 5), we have also a generalized Kummer surface $\mathrm{Km}(E\times E, G)$ with $G\cong \boldsymbol{Z}/5$.

Now, assume either $p\neq 2$ and $p\equiv 2 \pmod 5$, or $p\equiv 3 \pmod 5$. Let $E\times E$ and $\tilde{\sigma}$ be as above. Let $\tilde{\iota}$ be the inversion of $E\times E$. We set $G=\langle\tilde{\sigma}, \tilde{\iota}\rangle$. Then, we have $G\cong \boldsymbol{Z}/10$, and G satisfies Conditions (K) and (F). Hence, if either $p\neq 2$ and $p\equiv 2 \pmod 5$, or $p\equiv 3 \pmod 5$, we have a generalized Kummer surface $\mathrm{Km}(A, G)$ with $G\cong \boldsymbol{Z}/10$.

Example 7. $G\cong \boldsymbol{Z}/8$.

Assume $p\equiv 3 \pmod 8$. Let E_j $(j=1, 2)$ and τ be as in Example 3 in Section 3. Then, E_j's are supersingular by the assumption $p\equiv 3 \pmod 8$. We consider the automorphism defined by

$$\tilde{g}=\begin{pmatrix} 0 & \tau \\ \mathrm{id} & 0 \end{pmatrix} : E_1\times E_2 \longrightarrow E_1\times E_2 .$$

The eigen-values of $\tilde{g}^*$ on the cotangent space at the origin of $E_1\times E_2$ are given by ξ and ξ^5, where ξ is a primitive eighth root of unity such that $\xi^2=i$. Let t be an eigen-vector of $\tilde{g}_*$ on the tangent space of $E_1\times E_2$ at the origin which corresponds to the eigen-value ξ. Let γ be the immersion of α_p into $E_1\times E_2$ which corresponds to the direction t. We set $A=(E_1\times E_2)/\gamma(\varepsilon_p)$. Let $\pi: E_1\times E_2\to A$ be the canonical projection. Then, by our choice of t, we have an automorphism g of A such that $\pi\circ\tilde{g}=g\circ\pi$. By Lemma 7.1, the eigen-values of g^* on the cotangent space of A are given by $\xi^p=\xi^3$ and ξ^5. Therefore, g^* preserves a non-zero regular two-form on A. We set $G=\langle g\rangle$. Then, G satisfies Conditions (K) and (F). Hence, we have a generalized Kummer surface $\mathrm{Km}(A, G)$ with $G\cong \boldsymbol{Z}/8$ if $p\equiv 3$ (mod 8).

Example 8. $G\cong \boldsymbol{Z}/12$.

Assume $p\equiv 5$ (mod 12). Let E_j and ρ be as in Example 5. We consider the following automorphism of order twelve:

$$\tilde{g}=\begin{pmatrix} 0 & \rho \\ \mathrm{id} & 0 \end{pmatrix} : E_1\times E_2 \longrightarrow E_1\times E_2 .$$

Then, by the similar method to the one in Example 7, we have a generalized Kummer surface $\mathrm{Km}(A, G)$ with $G\cong \boldsymbol{Z}/12$ if $p\equiv 5$ (mod 12).

Example 9. $G\cong\langle 2, 2, n\rangle$ with $2\leqq n\leqq 6$, $\langle 2, 3, 3\rangle$, $\langle 2, 3, 4\rangle$ and $\langle 2, 3, 5\rangle$.

Assume char. $k=p\geqq 7$. We consider a family $q:\mathcal{X}\to S$ with relative polarization D of principally polarized supersingular abelian surfaces as in Section 4. Under the notations in (4.1), we set $A=E\times E$. Then, by Katsura and Oort [17, Theorem 4.1], the group of automorphisms of the family $q:\mathcal{X}\to S$ which preserves the relative polarization D is isomorphic to a subgroup $\tilde{G}$ of $\mathrm{Aut}_v(E\times E)$. Since $\tilde{G}$ contains the inversion $\tilde{\imath}$ of $E\times E$, we set $G=\tilde{G}/\langle\tilde{\imath}\rangle$. Then, by Katsura and Oort [17, Lemma 7.2], G is isomorphic to one of the following groups:

(7.3) (1) $\boldsymbol{Z}/d$ $(1\leqq d\leqq 6)$, (2) D_{2e} $(2\leqq e\leqq 6)$, (3) A_4, (4) S_4, (5) A_5.

By Katsura and Oort [17, Sections 6 and 7], we have methods to construct concretely elements of $\tilde{G}$. The methods are similar to the methods in Examples 6 and 7. Using Table III in Katsura and Oort [17, Section 7], we can calculate eigen-values of each element of $\tilde{G}$ on $H^0(E\times E, \Omega^1_{E\times E})$. Hence, we can prove the following Lemma.

LEMMA 7.2. *A non-zero regular two-form on $E\times E$ is invariant under $\tilde{G}$.*

Since G is isomorphic to one of the groups in (7.3), we see that $\tilde{G}$ is isomorphic to one of groups in Theorem 3.7. Using Lemma 7.2 and the construction of $\tilde{G}$, we see that $\tilde{G}$ satisfies Conditions (K) and (F). Hence, we have a generalized Kummer surface $\mathrm{Km}(E\times E, \tilde{G})$. For example, by Katsura and Oort [17, Section 8, Table IV], we have the following:

(1) $\tilde{G}\cong\langle 2,3,5\rangle$ appears if $p=7,13,17$,
(2) $\tilde{G}\cong\langle 2,3,4\rangle$ appears if $p=11,13,19,29$,
(3) $\tilde{G}\cong\langle 2,3,3\rangle$ appears if $p=19,31$,
(4) $\tilde{G}\cong\langle 2,2,6\rangle$ appears if and only if $p\neq 5$ and $p\equiv 5 \pmod{12}$.

The other groups in Theorem 3.7 can be obtained as subgroups of these four groups.

Example 10. $G\cong\langle 2,2,2\rangle$, $\langle 2,2,3\rangle$.

We give here a concrete example of a generalized Kummer surface $\mathrm{Km}(A,G)$ with $G\cong\langle 2,2,2\rangle$ (resp. $G\cong\langle 2,2,3\rangle$). This example was pointed out by Professor Tadao Oda (see also Fujiki [5]). Assume $p\neq 2,3$ (resp. $p\neq 2$). Let E_j ($j=1,2$), A and σ (resp. τ) be as in Example 2 (resp. Example 3) in Section 3. Let σ' (resp. τ') be the automorphism of A defined by

$$\sigma' \text{ (resp. } \tau') : \begin{cases} x_1 \longmapsto x_2\,, & y_1 \longmapsto -y_2\,, \\ x_2 \longmapsto x_1\,, & y_2 \longmapsto y_1\,. \end{cases}$$

We set $G=\langle\sigma,\sigma'\rangle$ (resp. $G=\langle\tau,\tau'\rangle$). Then, the group G is isomorphic to $\langle 2,2,2\rangle$ (resp. $\langle 2,2,3\rangle$), and satisfies Conditions (K) and (F). Hence, we have a generalized Kummer surface $\mathrm{Km}(A,G)$ with $G\cong\langle 2,2,2\rangle$ (resp. $G\cong\langle 2,2,3\rangle$).

Example 11. $G\cong\langle 2,3,3\rangle$.

The author learned the existence of the following concrete example by Professor A. Fujiki (see also Fujiki [5]). Assume $p\neq 5$. We consider a non-singular complete model C of the curve defined by

$$y^2=x(x^2-1)(x^2+1)\,.$$

The reduced group $RA(C)$ of automorphisms of C is isomorphic to the symmetric group S_4 of degree four (cf. Igusa [9, p. 645]). We consider two automorphisms defined by

$$\begin{cases} g : x \longmapsto -x, & y \longmapsto iy \\ g' : x \longmapsto (x+i)/(x-i), & y \longmapsto 2\sqrt{2}\zeta y/(x-i)^3. \end{cases}$$

Then, the images of g and g' in $RA(C)$ generate the alternating group A_4 of degree four. We again denote by g (resp. g') the automorphism of the Jacobian variety $J(C)$ of C which is induced by g (resp. g'). We set $G=\langle g, g'\rangle$. We may regard C as a divisor of $J(C)$ such that $\iota^*C=C$ with the inversion ι of $J(C)$. Then, G contains ι and $G/\langle\iota\rangle$ is isomorphic to A_4. Computing the actions of g and g' on $H^0(C, \Omega_C^1)$, we see that a non-zero regular two-form on $J(C)$ is invariant under G. Hence, as before, we obtain a generalized Kummer surface $\mathrm{Km}(A, G)$ with $G\cong\langle 2, 3, 3\rangle$.

REMARK 7.3. Assume $p=0$. Using Examples 1, 2, 3, 4, 5, 10 and 11, we get all examples of groups in Corollary 3.17 (see also Fujiki [5]).

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