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Knotted homology spheres defined by weighted homogeneous polynomials

Dedicated to Professor Itiro Tamura on his 60th birthday

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§1. Introduction.

Let $f(z)$ be a polynomial in $\boldsymbol{C}^{n+1}$ with $f(\mathbf{0})=0$. Suppose that f has an isolated critical point at the origin $\mathbf{0}$. Then the *algebraic knot* associated with f is the knot $K_f=S_\varepsilon^{2n+1}\cap f^{-1}(0)\subset S_\varepsilon^{2n+1}$ for $\varepsilon>0$ sufficiently small, where S_ε^{2n+1} is the $(2n+1)$-sphere of radius ε centered at the origin. (As a general reference for this and the following, see [9].) A polynomial $f(z)$ is called *weighted homogeneous* if there exist positive rational numbers $(w_1, \cdots, w_{n+1})$ such that for every monomial $cz_1^{a_1}\cdots z_{n+1}^{a_{n+1}}$ $(c\neq 0)$ of $f(z)$, $\sum_{i=1}^{n+1}\frac{a_i}{w_i}=1$. We call $(w_1, \cdots, w_{n+1})$ a *weight* of $f(z)$. For example, the Brieskorn type polynomial $f(z)=z_1^{a_1}+\cdots+z_{n+1}^{a_{n+1}}$ $(a_i\geqq 2)$ is weighted homogeneous of weight $(a_1, \cdots, a_{n+1})$.

Our main result of this paper is the following Theorem A, which concerns the case $n=2$.

THEOREM A. *Let $f(z_1, z_2, z_3)$ be a weighted homogeneous polynomial in $\boldsymbol{C}^3$ which has an isolated critical point at the origin. Suppose K_f is a homology 3-sphere. Then the knot (S^5, K_f) is of the same knot type as the algebraic knot associated with a Brieskorn type polynomial.*

Let $\Delta(t)=\Delta_f(t)$ be the Alexander polynomial of the knot (S^5, K_f). Then K_f is a homology 3-sphere if and only if $\Delta(1)=\pm 1$. Furthermore, by [18, 19] algebraic knots of Brieskorn type are classified by their Alexander polynomials. Thus Theorem A shows that algebraic knots in S^5 defined by weighted homogeneous polynomials with $\Delta(1)=\pm 1$ are classified by their Alexander polynomials.

In case $n=1$, Theorem A is also true (see [20, Lemma 2.1]). However, in case $n\geqq 3$, we shall show the following Example B.

Example B. Let $f_n(z_1, \cdots, z_{n+1}) = z_1^2 z_2 + z_1 z_2^6 + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_{n+1}^2$ and $g_n(z_1, \cdots, z_{n+1}) = z_1^3 z_2 + z_1 z_2^4 + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_{n+1}^2$ $(n \geqq 3)$, both of which are weighted homogeneous. Then $\Delta_{f_n}(t) = \Delta_{g_n}(t)$ and $\Delta_{f_n}(1) = \Delta_{g_n}(1) = 1$. However, K_{f_n} and K_{g_n} are of distinct knot types.

Since algebraic knots of Brieskorn type are classified by their Alexander polynomials, K_{f_n} or K_{g_n} is not of Brieskorn type. Thus Theorem A does not hold for $n \geqq 3$.

As an application, we show that for every $n \geqq 2$ there exists an algebraic knot in S^{2n+1} whose Seifert matrix cannot be realized as a Seifert matrix of any algebraic knot in S^3.

§ 2. Proof of Theorem A.

Our proof is based on a result of Orlik and Wagreich [13].

Let $f(z_1, z_2, z_3)$ be a weighted homogeneous polynomial in $\boldsymbol{C}^3$ which has an isolated critical point at the origin. Then by [13, 14], (S^5, K_f) is of the same knot type as the algebraic knot associated with one of the eight classes of polynomials as follows:

(I) $z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$,
(II) $z_1^{a_1} + z_2^{a_2} + z_2 z_3^{a_3}$ $(a_2 > 1)$,
(III) $z_1^{a_1} + z_2^{a_2} z_3 + z_3^{a_3} z_2$ $(a_2 > 1, a_3 > 1)$,
(IV) $z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3}$ $(a_1 > 1)$,
(V) $z_1^{a_1} z_2 + z_2^{a_2} z_3 + z_1 z_3^{a_3}$,
(VI) $z_1^{a_1} + z_2 z_3$,
(VII) $z_1^{a_1} + z_1 z_2^{a_2} + z_1 z_3^{a_3} + z_2^{b_2} z_3^{b_3}$ $((a_1 - 1)(a_2 b_3 + a_3 b_2)/a_1 a_2 a_3 = 1)$,
(VIII) $z_1^{a_1} z_2 + z_1 z_2^{a_2} + z_1 z_3^{a_3} + z_2^{b_2} z_3^{b_3}$ $((a_1 - 1)(a_2 b_3 + a_3 b_2)/a_3(a_1 a_2 - 1) = 1, a_2 > 1)$.

Note that there is an omission in the table of [13, Definition 3.1.1]. The complete table can be found in [14, p. 61].

Let $h(z)$ be one of the eight polynomials above. Let (w_1, w_2, w_3) be a weight of h. For integers $a_1, a_2, \cdots, a_k$ let $(a_1, a_2, \cdots, a_k)$ denote their greatest common divisor and $\langle a_1, a_2, \cdots, a_k \rangle$ their least common multiple. We set $w_i = u_i / v_i$ $((u_i, v_i) = 1)$, $d = \langle u_1, u_2, u_3 \rangle$, $q_i = d / w_i$, $c = (u_1, u_2, u_3)$, $c_1 = (u_2, u_3)/c$, $c_2 = (u_1, u_3)/c$, and $c_3 = (u_1, u_2)/c$. Furthermore, we define $\gamma_1, \gamma_2, \gamma_3 \in \boldsymbol{N}$ by $u_1 = c c_2 c_3 \gamma_1$, $u_2 = c c_1 c_3 \gamma_2$, and $u_3 = c c_1 c_2 \gamma_3$.

Now $K_h = h^{-1}(0) \cap S^5$ is a Seifert fibered 3-manifold. Orlik and Wagreich [13] have calculated the Seifert invariants of K_h, $\{-b; (g); n_1(\alpha_1, \beta_1), n_2(\alpha_2, \beta_2), n_3(\alpha_3, \beta_3), n_4(\alpha_4, \beta_4)\}$, which are described as follows.

	α_1	n_1	α_2	n_2
I	γ_1	cc_1	γ_2	cc_2
II	γ_1	$(cc_1-1)/v_3$	$v_3\gamma_1$	1
III	γ_1	$(cc_1-v_2-v_3)/v_2v_3$	$v_3\gamma_1$	1
IV	γ_3	$(c-1)/v_2$	v_3	1
V	v_1	1	v_2	1
VII	γ_2	$(c-1)/v_2$	γ_3	$(c-1)/v_3$
VIII	γ_3	$(c-v_1-v_2)/v_1v_2$	v_3	1

	α_3	n_3	α_4	n_4
I	γ_3	cc_3		0
II	γ_3	c		0
III	$v_2\gamma_1$	1		0
IV	$v_2\gamma_3$	1		0
V	v_3	1		0
VII	$v_2\gamma_3$	1	$v_3\gamma_2$	1
VIII	$v_2\gamma_3$	1	$v_1\gamma_3$	1

$$b=\frac{d}{q_1q_2q_3}+\sum_{i=1}^{4} n_i\frac{\beta_i}{\alpha_i}.$$

(g and β_i are determined by other formulas.)

Note that $z_1^{a_1}+z_2z_3$ is analytically equivalent to $z_1^{a_1}+z_2^2+z_3^2$. Thus the class VI is thought of as a subclass of I.

Now suppose K_h is a homology 3-sphere. Then there exist at least three exceptional orbits and their multiplicities are pairwise relatively prime (see [16, Satz 12]). Thus h must belong to the class I or the class V or the class VIII. (Note that K_h is not diffeomorphic to S^3 by [11].) Thus it suffices to show that h belongs to neither the class V nor the class VIII.

Now suppose h is of class V. Then v_1, v_2, and v_3 are greater than or equal to 2, and they are pairwise relatively prime. The weight (w_1, w_2, w_3) of h is given by

$$w_1=(a_1a_2a_3+1)/(a_2a_3-a_3+1)\,,$$
$$w_2=(a_1a_2a_3+1)/(a_1a_3-a_1+1)\,,\quad\text{and}$$
$$w_3=(a_1a_2a_3+1)/(a_1a_2-a_2+1)\,.$$

Since

$$\begin{aligned}a_1a_2a_3+1&=a_1(a_2a_3-a_3+1)+(a_1a_3-a_1+1)\\&=a_2(a_1a_3-a_1+1)+(a_1a_2-a_2+1)\\&=a_3(a_1a_2-a_2+1)+(a_2a_3-a_3+1)\,,\end{aligned}$$

we have

$$(a_1a_2a_3+1, a_2a_3-a_3+1)=(a_1a_2a_3+1, a_1a_3-a_1+1)=(a_1a_2a_3+1, a_1a_2-a_2+1)\,.$$

Thus $c=d=u_1=u_2=u_3$. This implies $v_i=q_i$ $(i=1, 2, 3)$ and $b=\dfrac{d}{v_1v_2v_3}+\sum_{i=1}^{3}\dfrac{\beta_i}{v_i}$.

Now consider the Brieskorn manifold $\Sigma(v_1, v_2, v_3)$, which is the algebraic knot associated with the polynomial $z_1^{v_1}+z_2^{v_2}+z_3^{v_3}$. Since v_1, v_2, and v_3 are pairwise relatively prime, $\Sigma(v_1, v_2, v_3)$ is a homology 3-sphere (see [2]). Its Seifert invariants are $\{-b'\,;(0)\,;(v_1, \beta_1'), (v_2, \beta_2'), (v_3, \beta_3')\}$, where $b'=\dfrac{1}{v_1v_2v_3}+\sum_{i=1}^{3}\dfrac{\beta_i'}{v_i}$. Recall that the homology 3-sphere K_h has Seifert invariants $\{-b\,;(g)\,;(v_1, \beta_1), (v_2, \beta_2), (v_3, \beta_3)\}$. Now by the uniqueness of Seifert fibered homology 3-sphere with given multiplicities v_1, v_2, and v_3 ([16, Satz 12]), we have

(1) $\beta_i=\beta_i'$ and $b=b'$ or
(2) $\beta_i=v_i-\beta_i'$ and $-b=-3+b'$.

In the case (1), $d=1$. Since $d=\langle u_1, u_2, u_3\rangle$, $u_1=u_2=u_3=1$. Now by the definition of weights,

$$a_1/w_1+1/w_2=1\,,$$
$$a_2/w_2+1/w_3=1\,,\quad\text{and}$$
$$1/w_1+a_3/w_3=1\,.$$

Since $1/w_i=v_i/u_i=v_i\geqq 2$, this is a contradiction.

In the case (2), $3=b'+b=\dfrac{d+1}{v_1v_2v_3}+3$. This implies $d=-1$. This is also a contradiction. Thus h cannot belong to the class V.

Next suppose that $h(z)$ is of class VIII. Since K_h is a homology

3-sphere, $\gamma_3=1$. Furthermore, v_1, v_2, and v_3 are greater than or equal to 2, and they are pairwise relatively prime. The weight of h is given by

$$w_1=(a_1a_2-1)/(a_2-1)\,,$$
$$w_2=(a_1a_2-1)/(a_1-1)\,, \quad \text{and}$$
$$w_3=a_3(a_1a_2-1)/a_2(a_1-1)\,.$$

Since $a_1a_2-1=a_1(a_2-1)+(a_1-1)=a_2(a_1-1)+(a_2-1)$, we have $(a_1a_2-1, a_1-1)=(a_1a_2-1, a_2-1)$. We write this value as r. Let $r'=(a_3(a_1a_2-1), a_2(a_1-1))$. Then

$$c=u_1=u_2=(a_1a_2-1)/r \quad \text{and}$$
$$d=u_3=c\gamma_3=a_3(a_1a_2-1)/r'\,.$$

Since $\gamma_3=1$, $u_1=u_2=u_3$. Thus $d=u_i$ and $q_i=d\cdot(v_i/u_i)=v_i$. Using the same argument as in the case that h is of class V, we see that $d=1$ or $d=-1$. This is a contradiction. Thus h cannot belong to the class VIII, either. This completes the proof.

REMARK. (1) In Theorem A, we cannot omit the condition that K_f be a homology 3-sphere. See Remark (1) in § 3.
(2) We do not know whether algebraic knots in S^5 with $\Delta(1)=\pm 1$ are classified by their Alexander polynomials. Note that there exists an algebraic knot in S^5 which is a homology 3-sphere but not of Brieskorn type. For example, consider the algebraic knot associated with the polynomial $f(z_1, z_2, z_3)=z_2^4-2z_1^3z_2^2-4z_1^5z_2+z_1^6-z_1^7+z_3^5$. We see easily that K_f is a homology 3-sphere. Furthermore, since the resolution diagram of $f^{-1}(0)$ is not star-shaped, K_f is not a Seifert fibered 3-manifold (see [12, Theorem 5]). Of course $f(z_1, z_2, z_3)$ is not weighted homogeneous.

§ 3. Higher dimensional cases.

Let $f(z)$ be a polynomial in $\boldsymbol{C}^{n+1}$ ($f(\boldsymbol{0})=0$) which has an isolated critical point at the origin. Then we denote by L_f the Seifert matrix of the algebraic knot (S^{2n+1}, K_f). (For the definition of Seifert matrices, see [15].) Note that the congruence class of L_f is a knot type invariant of (S^{2n+1}, K_f) (see [5, 8]).

In this section we show Example B in § 1.

Let $f(z_1, z_2)=z_1^2z_2+z_1z_2^6$ and $g(z_1, z_2)=z_1^3z_2+z_1z_2^4$, both of which are weighted homogeneous and have isolated critical points at the origin (see [20, § 3]). Furthermore we set

$$f_n(z_1, \cdots, z_{n+1}) = f(z_1, z_2) + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_{n+1}^2 \quad \text{and}$$
$$g_n(z_1, \cdots, z_{n+1}) = g(z_1, z_2) + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_{n+1}^2 \quad (n \geqq 3).$$

Using a result of Milnor and Orlik [10], we obtain

$$\Delta_{f_n}(t) = \Delta_{g_n}(t) = \begin{cases} \dfrac{(t^{286}+t^{143}+1)(t^{26}+t^{13}+1)}{(t^{22}+t^{11}+1)(t^2+t+1)} & (n : \text{odd}) \\ \dfrac{(t^{286}-t^{143}+1)(t^{26}-t^{13}+1)}{(t^{22}-t^{11}+1)(t^2-t+1)} & (n : \text{even}). \end{cases}$$

Thus $\Delta_{f_n}(1) = \Delta_{g_n}(1) = 1$. As for their Seifert matrices, we have $L_{f_n} = -\varepsilon_n \cdot L_f \otimes A_3 \otimes A_{13}$ and $L_{g_n} = -\varepsilon_n \cdot L_g \otimes A_3 \otimes A_{13}$ by Sakamoto [15], where $\varepsilon_n = (-1)^{(n+1)(n+2)/2}$ and A_m is the $(m-1) \times (m-1)$ matrix given by

$$A_m = \begin{pmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ 0 & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}.$$

Now let $\sigma(L_{f_n})$ and $\sigma(L_{g_n})$ be the signatures of the symmetrized matrices $L_{f_n} + {}^tL_{f_n}$ and $L_{g_n} + {}^tL_{g_n}$ respectively. Note that $-(L_{f_n} + {}^tL_{f_n})$ (or $-(L_{g_n} + {}^tL_{g_n})$) is an intersection matrix of $f_n^{-1}(1)$ (resp. $g_n^{-1}(1)$) for every even number n (for example see [15]). Thus one can apply a result of Steenbrink [17], obtaining $\sigma(L_{f_n}) = \varepsilon_n \cdot 144$ and $\sigma(L_{g_n}) = \varepsilon_n \cdot 136$. Hence L_{f_n} is not congruent to L_{g_n}. This implies that K_{f_n} and K_{g_n} are of distinct knot types.

REMARK. (1) As we have seen, $L_f \otimes A_3 \otimes A_{13}$ and $L_g \otimes A_3 \otimes A_{13}$ are not congruent. This implies that L_f and L_g are not congruent. Thus, if we let $f'(z_1, z_2, z_3) = f(z_1, z_2) + z_3^2$ and $g'(z_1, z_2, z_3) = g(z_1, z_2) + z_3^2$, then $K_{f'}$ and $K_{g'}$ are of distinct knot types. Since $\Delta_{f'}(t) = \Delta_{g'}(t)$, $K_{f'}$ or $K_{g'}$ is not of Brieskorn type. (See the argument in § 1 subsequent to Example B.) Note that $\Delta_{f'}(1) = \Delta_{g'}(1) = 4$. Thus in Theorem A, the condition that K_f be a homology 3-sphere (i. e. $\Delta_f(1) = \pm 1$) cannot be omitted.
(2) By [9, Remark 8.7], the diffeomorphism type of K_{f_n} (or K_{g_n}) is determined by $\Delta_{f_n}(-1)$ (resp. $\Delta_{g_n}(-1)$) if n is odd and by $\sigma(L_{f_n})$ (resp. $\sigma(L_{g_n})$) if n is even. Thus if n is odd, both K_{f_n} and K_{g_n} are diffeomorphic to the standard sphere S^{2n-1}. However, if n is even, the homotopy spheres K_{f_n} and K_{g_n} are not diffeomorphic to each other.

§ 4. Application to the problem of realizing Seifert matrices.

In this section, we consider what kind of matrices cannot be realized as Seifert matrices of algebraic knots in S^3. Refer to [6, Problem 1.5].

Example C. Let $F_n(z_1, \cdots, z_{n+1})=f(z_1, z_2)+z_3^3+z_4^2+\cdots+z_{n+1}^2$ and $G_n(z_1, \cdots, z_{n+1})=g(z_1, z_2)+z_3^3+z_4^2+\cdots+z_{n+1}^2$ $(n\geqq 2)$, where f and g are the polynomials as in § 3. Then L_{F_n} or L_{G_n} cannot be realized as a Seifert matrix of any algebraic knot in S^3.

PROOF. First note that $L_{F_n}=-\varepsilon_n\cdot L_f\otimes A_3$ and $L_{G_n}=-\varepsilon_n\cdot L_g\otimes A_3$ are not congruent. This is because $L_f\otimes A_3\otimes A_{13}$ and $L_g\otimes A_3\otimes A_{13}$ are not congruent.

Suppose that K_1 and K_2 are algebraic knots in S^3 with Seifert matrices L_{F_n} and L_{G_n} respectively. Let $\Delta_i(t)$ be the Alexander polynomial of K_i $(i=1, 2)$. Since $\Delta_1(t)=\det(tL_{F_n}-{}^tL_{F_n})=\Delta_{F_3}(t)$ and $\Delta_2(t)=\det(tL_{G_n}-{}^tL_{G_n})=\Delta_{G_3}(t)$, we have $\Delta_1(t)=\Delta_2(t)=(t^{22}-t^{11}+1)(t^2-t+1)$. Furthermore, $\Delta_1(1)=\Delta_2(1)=1$. Thus K_1 and K_2 are connected. Since algebraic knots of one component in S^3 are classified by their Alexander polynomials ([4, 7]), K_1 and K_2 are of the same knot type. Thus L_{F_n} and L_{G_n} must be congruent. This is a contradiction.

REMARK. Let L be a Seifert matrix of an algebraic knot in S^{2n+1}. It is a well-known fact that L has the following properties.

(1) $\det(tL+(-1)^n\cdot{}^tL)$ is a product of cyclotomic polynomials ([3]).

(2) The trace of ${}^tL^{-1}\cdot L$ is equal to 1 ([1]).

(3) L is congruent to a lower triangular matrix with each diagonal entry equal to ε_n ([5]).

In Example C, since $L_f\otimes A_3=L_{F_5}$ and $L_g\otimes A_3=L_{G_5}$, they have all the above properties (1), (2), and (3) for $n=5$, so that also for $n=1$. Nevertheless $L_f\otimes A_3$ or $L_g\otimes A_3$ cannot be realized as a Seifert matrix of any algebraic knot in S^3.

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