J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
34 (1987), 51-55.

Generic homeomorphisms of S^1 have the pseudo orbit tracing property

Dedicated to Professor Itiro Tamura on his sixtieth birthday

By Koichi YANO

Introduction.

In this paper, we study the pseudo orbit tracing property for homeomorphisms of S^1. We give a characterization of such homeomorphisms and obtain

THEOREM. *The pseudo orbit tracing property is generic in the space* Homeo(S^1) *of all the homeomorphisms of* S^1.

The notion of the pseudo orbit tracing property (or p.o.t.p. for short) was first introduced by Bowen [1], and it is related to various concepts of stability: one is Zeeman's tolerance stability, for which we refer to [4]. In [4], Takens showed that Zeeman's tolerance stability conjecture is reduced to so called Takens' conjecture, which asserts for a generic f, that the extended f-orbits are really f-orbits. It is not so hard to see that this is satisfied by homeomorphisms with p.o.t.p. Therefore the theorem above answers in the affirmative a version of Takens' conjecture for S^1 and implies

COROLLARY. *Generic elements of* Homeo(S^1) *are tolerance stable.*

Another related notion is the topological stability (see [6]). Bowen [1] essentially proved that the p.o.t.p. with expansiveness implies the topological stability (see also Walters [5]). Conversely the topological stability implies the p.o.t.p.: For dimensions greater than one, Walters [5] and Morimoto [2] gave a direct proof and the one dimensional case is seen via characterization of topologically stable homeomorphisms of S^1 in [6]. We show in Section 2, however, that the topologically stable homeomorphisms constitute only a meagre subset of Homeo(S^1). This contrasts with the space of C^1-diffeomorphisms of S^1, in which Morse-Smale diffeomorphisms form an open and dense subset by Peixoto's theorem (see [3]). While every homeomorphism of S^1 is topologically conjugate to a C^1-diffeomorphism, generic ones are then sent to a nowhere dense subset.

Finally we refer to [7] for the wildness of generic homeomorphisms of higher dimensional manifolds.

The author would like to thank the referee for pointing out a gap in the first version of this paper.

1. Pseudo orbit tracing property.

In this section, we give a characterization of homeomorphisms of S^1 that have the p.o.t.p. We begin with notation and definitions. The standard metric on S^1 is denoted by d. Let f be a homeomorphism of S^1. A sequence $\{x_n\}_{n\in Z}$ of points of S^1 is a *δ-pseudo orbit* of f if $d(f(x_n), x_{n+1})<\delta$ for all n, and is *ε-traced* by a point x if $d(f^n(x), x_n)<\varepsilon$ for all n. We say that f has the *pseudo orbit tracing property* (or *p.o.t.p.*) if for any ε, there is a δ such that every δ-pseudo orbit of f is ε-traced by some point.

A periodic point p of f with the period n is called *topologically hyperbolic* if the map f^n is locally conjugate to a linear map $x\mapsto\lambda x$ with $\lambda\neq\pm 1$ near p. The set of fixed points and that of periodic points of f are denoted by $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$, respectively. When f has a periodic point of period n, the complement of $\mathrm{Per}(f)$ is a union of $U_+=\{x\in S^1 ; f^n(x)>x\}$ and $U_-=\{x\in S^1 ; f^n(x)<x\}$. More precisely, let $\mathrm{pr}: \boldsymbol{R}\to S^1$ be the covering and $F: \boldsymbol{R}\to\boldsymbol{R}$ the covering map of f^n which has fixed points. Then $U_+=\mathrm{pr}\{x\in\boldsymbol{R} ; F(x)>x\}$ and $U_-=\mathrm{pr}\{x\in\boldsymbol{R} ; F(x)<x\}$. We say that U_+ and U_- *separate each other* if U_+ and U_- are non-empty and any two connected components of U_+ (resp. of U_-) are separated by components of U_- (resp. of U_+).

THEOREM 1.1. *A homeomorphism f of S^1 has the p.o.t.p. if and only if* $\mathrm{Per}(f)$ *is non-empty,* $\mathrm{int}\,\mathrm{Per}(f)$ *is empty and U_+ and U_- separate each other.*

PROOF. First we show the necessity. If a homeomorphism f is periodic point free, then it is semi-conjugate to an irrational rotation and thus does not have the p.o.t.p. Since any positive iterate f^n of f has the p.o.t.p. if and only if so does f, we may assume that f has a fixed point. Suppose either $\mathrm{int}\,\mathrm{Fix}(f)\neq\emptyset$ or U_+ and U_- are non-empty but do not separate each other. Then there exists an invariant interval $[a, b]$ in S^1 such that $f(x)\geqq x$ (or $f(x)\leqq x$) for any x in $[a, b]$ and some interior point c is fixed by f. In this case, for any δ, there is a δ-pseudo orbit that moves from a to b (or from b to a), but the fixed point c obstructs the existence of tracing points. A similar argument applies to the case that U_+ or U_- is empty. Thus the necessity is proved.

The proof of the sufficiency is essentially the same as that of Theorem 2 in [6]. Suppose that f satisfies the required conditions and let ε be a positive number. Then there are a finite number of components I_i and J_j of U_+ and U_- respectively such that

(1) I_i's and J_j's are alternatingly arranged on S^1,

(2) the union $\cup I_i \cup J_j$ is invariant under f, and

(3) each connected component of $S^1 - \cup I_i \cup J_j$ has the length smaller than $\varepsilon/2$.

Let T be a space obtained by collapsing each connected component of $S^1 - \cup I_i \cup J_j$ to a point. Then T is homeomorphic to S^1 and is equipped with a metric induced from d. Let $h: S^1 \to T$ be the quotient map. By the condition (2), there is a homeomorphism g of T that is semi-conjugate to f by h. All the periodic points of g are topologically hyperbolic by (1) and thus g has the p.o.t.p. Let ε' be a positive number smaller than $\varepsilon/2$ and also than the length of any of I_i's and J_j's. Take a δ such that any δ-pseudo orbit of g is ε'-traced and let $\{x_i\}$ be a δ-pseudo orbit of f. Then so is $\{h(x_i)\}$ with respect to g and is ε'-traced by some point. Let y be in the inverse image under h of such a tracing point. By the choice of ε', there is at most one or a part of one connected component of $S^1 - \cup I_i \cup J_j$ between $f^i(y)$ and x_i, and thus by the condition (3) their distance is less than $\varepsilon' + \varepsilon/2 < \varepsilon$. Therefore $\{x_i\}$ is ε-traced by y and this completes the proof.

2. Genericity.

We prove the main result in this section. Let Homeo(S^1) be the set of all the homeomorphisms of S^1, and give it a complete metric $\tilde{d}$ by

$$\tilde{d}(f, g) = \sup_x d(f(x), g(x)) + \sup_x d(f^{-1}(x), g^{-1}(x)).$$

Then Homeo(S^1) is a Baire space. A property for homeomorphisms of S^1 is called *generic* if it is satisfied by all the elements that belong to a countable intersection of open and dense subsets of Homeo(S^1).

Now we prove the theorem in the introduction. By the density of Morse-Smale diffeomorphisms and the stability of topologically hyperbolic periodic points (see [6]), the set of homeomorphisms with periodic points contains an open and dense subset of Homeo(S^1). Let l denote the length of an interval, and consider the condition:

(C_n) For each connected component K_λ of $S^1 - U_-$ with $l(K_\lambda) \geqq 1/n$, there is a component I_λ of U_+ contained in K_λ such that $l(I_\lambda)>((n-1)/n)l(K_\lambda)$.

If a homeomorphism f satisfies all the (C_n) and all the corresponding conditions (C'_n) for U_-, then $\operatorname{int}\operatorname{Per}(f)\neq\emptyset$ and U_+ and U_- separate each other. Therefore the proof of the theorem is reduced to

PROPOSITION 2.1. *The condition (C_n) is assumed by an open and dense subset of* $\operatorname{Homeo}(S^1)$.

The following lemma is easy and the proof is omitted.

LEMMA 2.2. *Let $h: \boldsymbol{R}\to\boldsymbol{R}$ be an orientation preserving homeomorphism which fixes an interval $I=[a,b]$ and moves interior points x as $h(x)>x$. Then for any positive ε, there is a δ such that any homeomorphism $h': \boldsymbol{R}\to\boldsymbol{R}$ with $\sup_t d(h(t),h'(t))<\delta$ moves any point x in the interval $[a+\varepsilon, b-\varepsilon]$ as $h'(x)>x$.*

PROOF OF PROPOSITION 2.1. It suffices to show the openness. If f satisfies (C_n), then there is a finite number of components J_j's of U_- such that each component K of $S^1-\cup J_j$ either coincides with some K_λ in the condition (C_n) or satisfies $l(K)<1/n$. Apply Lemma 2.2 to the covering map of some positive iterate of f with respect to components I_λ's and J_j's. Then if a homeomorphism f' is sufficiently close to f, there are corresponding I'_λ and J'_j so that any component K' of $S^1-\cup J'_j$ either contains some I'_λ with $l(I'_\lambda)>((n-1)/n)l(K')$ or satisfies $l(K')<1/n$. This completes the proof of Proposition 2.1 and thus that of the theorem.

Using this, we have another genericity result.

THEOREM 2.3. *The periodic point set of generic homeomorphisms of S^1 is a Cantor set with Lebesgue measure zero.*

PROOF. Let X be the subspace of $\operatorname{Homeo}(S^1)$ consisting of homeomorphisms which satisfy all the (C_n) and the (C_n'), and consider the subset X_m of X each element of which satisfies

(D_m) Let $I_i=[(i-1)/m, i/m]\subset S^1\cong[0,1]/\sim$, $i=1,\cdots,m$. Then either $\operatorname{Per}(f)\cap I_i=\emptyset$ or $\#(\operatorname{Per}(f)\cap\operatorname{int} I_i)\geqq 2$ for each i.

By the stability of connected components of U_+ and U_- due to Lemma 2.2, X_m is open in X and it is easy to see the density of X_m. Thus for generic f, each periodic point is not isolated in $\operatorname{Per}(f)$. Since $\operatorname{int}\operatorname{Per}(f)=\emptyset$ is also a generic condition, $\operatorname{Per}(f)$ is a Cantor set generically. The proof for the statement about Lebesgue measure is easy and is omitted.

It is proved in [6] that the topologically stable homeomorphisms of S^1 are topologically conjugate to Morse-Smale diffeomorphisms. Thus Theorem 2.3 implies the following, which seems to be true in any dimension (see [7] for some evidence).

COROLLARY 2.4. *Topological instability is generic in* Homeo(S^1).

References

[1] Bowen, R., ω-limit sets for Axiom A diffeomorphisms, J. Differential Equations **18** (1975), 333-339.
[2] Morimoto, A., Stochastic stable diffeomorphisms and Takens' conjecture, Sûrikaisekikenkyûsho Kôkyûroku **303** (1977), 8-24.
[3] Nitecki, Z., Differentiable Dynamics, M. I. T. Press, Cambridge, 1971.
[4] Takens, F., Tolerance stability, Lecture Notes in Math. vol. 468, Springer-Verlag, Berlin-Heidelberg-New York, 1975, 293-304.
[5] Walters, P., On the pseudo orbit tracing property and its relationship to stability, Lecture Notes in Math. vol. 668, Springer-Verlag, Berlin-Heidelberg-New York, 1979, 231-244.
[6] Yano, K., Topologically stable homeomorphisms of the circle, Nagoya Math. J. **79** (1980), 145-149.
[7] Yano, K., A remark on the topological entropy of homeomorphisms, Invent. Math. **59** (1980), 215-220.

(Received August 22, 1986)
(Revised September 20, 1986)

Department of Mathematics
Kyushu University 33
Hakozaki, Fukuoka
812 Japan