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A single point blow-up for solutions of semilinear parabolic systems

Dedicated to Professor Seizô Itô on his sixtieth birthday

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Introduction.

It is well known that solutions of the semi-linear heat equation

$$
\begin{aligned}
& u_t - u_{xx} = f(u) && (-a<x<a,\, t<0)\,, \\
& u(\pm a, t) = 0 && (t>0)\,, \\
& u(x,0) = \phi(x) && (-a<x<a)
\end{aligned} \tag{0.1}
$$

may blow-up in finite time; see [4] and the references given there. As for the precise nature of the blow-up, Weissler [6] proved (under some very restrictive assumptions on ϕ and f) that the solution blows up at the single point $x=0$. More recently Friedman and McLeod [3] established a single point blow-up under fairly general assumptions on ϕ, f. In particular, in the symmetric case where $\phi(x)=\phi(-x)$, it suffices to assume on ϕ that

$$
\begin{aligned}
& \phi'(x) \leqq 0 && \text{if } 0<x<a\,, \\
& \phi(0)>0\,, && \phi(a)=0\,.
\end{aligned} \tag{0.2}
$$

As for f it is required to satisfy some convexity type conditions; for instance, one may take

$$
\begin{aligned}
& f(u) = (u+\lambda)^p && \text{with } \lambda \geqq 0,\ p>1, \quad \text{or} \\
& f(u) = e^{\mu u}\,, && \mu>0\,.
\end{aligned} \tag{0.3}
$$

In this paper we consider a parabolic system

$$
u_t - u_{xx} = f(v) \qquad (-a<x<a,\ t>0)\,, \tag{0.4}
$$

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(0.5) $$v_t - v_{xx} = g(u) \qquad (-a<x<a,\ t>0)$$

with initial and boundary conditions

(0.6) $$\begin{aligned} &u=v=0 \quad \text{on } x=\pm a,\ t>0\,, \\ &u(x,0)=\phi(x)\,, \quad v(x,0)=\psi(x) \quad (-a<x<a) \end{aligned}$$

with f, g positive, increasing and superlinear and ϕ, ψ as in (0.2). We shall establish a single-point blow-up for both u and v. The method is based on extension of the method of [3] for one equation.

In § 1 we establish some general properties of the solution (u, v). In § 2 we prove a single point blow-up provided

(0.7) $$u \leqq C(v^{\gamma}+1)\,, \qquad v \leqq C(u^{1/\gamma}+1)$$

for some $C>0$, $\gamma>0$. The condition (0.7) is established, in § 3, for some specific examples, such as

$$f(u)=Ae^{\lambda u}\,, \qquad g(v)=Be^{\mu v}$$

and

$$f(u)=A(u+\lambda)^p\,, \qquad g(v)=B(v+\mu)^p\,.$$

Finally in § 4 we extend most of the results to systems

$$\begin{aligned} &u_t - u_{xx} = f(u, v)\,, \\ &v_t - v_{xx} = g(u, v)\,. \end{aligned}$$

Systems of nonlinear parabolic equations with blow-up are described in [1], [5] and in some of the references given in these papers.

§ 1. Preliminaries.

Consider the system

(1.1) $$u_t - \alpha u_{xx} = f(v) \qquad (-a<x<a,\ t>0)\,,$$

(1.2) $$v_t - \beta v_{xx} = g(u) \qquad (-a<x<a,\ t>0)$$

with

(1.3) $$\begin{aligned} &u(\pm a, t)=0 && (t>0)\,, \\ &u(x,0)=\phi(x) && (-a<x<a)\,, \end{aligned}$$

(1.4) $$\begin{aligned} &v(\pm a, t)=0 && (t>0)\,, \\ &v(x,0)=\psi(x) && (-a<x<a)\,, \end{aligned}$$

where $\alpha>0$, $\beta>0$, and assume:

$$\begin{aligned}&\phi(x)=\phi(-x)\,, \quad \phi(x)\geqq 0\,, \quad \phi\in C^1[-a,a]\,,\\ &\phi'(x)\leqq 0 \quad \text{if } 0<x<a\,, \quad \phi(a)=0\\ &\psi(x)=\psi(-x)\,, \quad \psi(x)\geqq 0\,, \quad \psi\in C^1[-a,a]\,,\\ &\psi'(x)\leqq 0 \quad \text{if } 0<x<a\,, \quad \psi(a)=0\,,\end{aligned} \tag{1.5}$$

$$\begin{aligned}&f, g\in C^1(\boldsymbol{R}^1), \quad f(s)>0, \quad g(s)>0 \quad \text{if } s>0\,;\\ &f'(s)>0, \quad g'(s)>0 \quad \text{if } s>0\,.\end{aligned} \tag{1.6}$$

Set

$$H_\alpha w=w_t-\alpha w_{xx}\,,$$

$$Q_\sigma=\{(x,t)\,;\ -a<x<a,\ 0<t<\sigma\}\,.$$

Then there exists a unique classical solution of (1.1)-(1.4) in some Q_{t_0}, and $u\geqq 0$, $v\geqq 0$ by the maximum principle. Let $T=\sup t_0$, for all t_0 as above. We claim

$$\sup_{Q_\sigma} u\longrightarrow\infty \quad \text{if } \sigma\to T\,. \tag{1.7}$$

Indeed, otherwise we deduce from (1.2), (1.4) that also v remain bounded in Q_T. Applying standard parabolic estimates to (1.1), (1.3) and to (1.2), (1.4) we can then continue the solution u, v into $Q_{T+\varepsilon}$ for some $\varepsilon>0$, which is a contradiction.

Similarly one can show that

$$\sup v\longrightarrow\infty \quad \text{if } \sigma\to T\,. \tag{1.8}$$

We call T the *blow-up time*.

LEMMA 1.1. *There holds*:

$$u_x<0,\quad v_x<0 \quad \textit{if } 0<x\leqq a,\quad 0<t<T\,. \tag{1.9}$$

PROOF. Differentiating (1.1), (1.2) in x and setting $U=u_x$, $V=v_x$, we get

$$\begin{aligned}&H_\alpha U=f'(v)V\,,\\ &H_\beta V=g'(u)U\end{aligned} \tag{1.10}$$

and $U(0,t)=0$ (since $u(x,t)=u(-x,t)$). Further, $U(a,t)=u_x(a,t)<0$ by the maximum principle, and $U(x,0)=\phi'(x)\leqq 0$ if $0<x<a$. V satisfies similar

initial and boundary conditions.

Consider first the case where

$$(1.11)\qquad \begin{aligned} &U(x,0)<0, \qquad V(x,0)<0 \qquad \text{if } 0<x<a\,, \\ &U_x(0,0)<0\,, \qquad V_x(0,0)<0\,. \end{aligned}$$

We claim that

$$(1.12)\qquad U\leqq 0,\quad V\leqq 0 \qquad \text{in } Q_T\cap\{x>0\}\,.$$

Indeed, otherwise there exists a largest σ such that

$$U\leqq 0,\quad V\leqq 0 \qquad \text{in } Q_\sigma$$

and $\sigma<T$; by (1.11) we also have that $\sigma>0$. From (1.10) we deduce that

$$U_t-\alpha U_{xx}\leqq 0 \qquad \text{in } Q_\sigma\cap\{x>0\}$$

and by the maximum principle it then follows that $U(x,\sigma)<0$ if $0<x<a$ and $U_x(0,\sigma)<0$. Noting that $U(a,\sigma)<0$, we now conclude by continuity that $U\leqq 0$ in $Q_{\sigma+\varepsilon}$ for some $\varepsilon>0$. Similarly $V\leqq 0$ in $Q_{\sigma+\varepsilon}$ and we therefore get a contradiction to the definition of σ.

To complete the proof of (1.12) we approximate $\phi(x)$, $\psi(x)$ by functions $\phi_n(x)$, $\psi_n(x)$ for which (1.11) holds and apply the above result. Finally, (1.9) follows from (1.12) and the maximum principle.

A point $x\in(-a,a)$ is called a *blow-up point* of u if there is a sequence (x_m,t_m) such that

$$t_m\uparrow T,\quad x_m\to x \quad\text{and}\quad u(x_m,t_m)\to\infty \qquad \text{as } m\to\infty\,,$$

where T is the blow-up time. The set of blow-up points of u are called the *blow-up set* for u.

THEOREM 1.2. *Suppose u and v solve* (1.1), (1.2) *with* (1.3)-(1.6). *Then the blow-up sets for u and v coincide with some interval* $[-\rho,\rho]$.

PROOF. From Lemma 1.1 it follows that the blow-up sets for u and v coincide with some intervals $[-a_1,a_1]$ and $[-b_1,b_1]$ respectively, i.e., if $-a_1\leqq\xi\leqq a_1$ then

$$\limsup_{\substack{x\to\xi\\ t\to T}} u(x,t)=\infty\,,$$

and if $|\xi|>a_1$ then

$$\limsup_{\substack{x\to\xi\\ t\to T}} u(x,t)<\infty\,,$$

and similarly for v.

Suppose now that

$$a_1 < b_1 . \tag{1.13}$$

Integrating (1.1) over $\{a_1+\lambda<x<a-\lambda, 0<t<T\}$ we get

$$\int_{a_1+\lambda}^{a} u(x, T)dx - \int_{a_1+\lambda}^{a} \phi(x, 0)dx + \alpha\int_0^T u_x(a_1+\lambda, t)dt$$
$$-\alpha\int_0^T u_x(a-\lambda, t)dt = \iint_{R_\lambda} f(u)dxdt$$

where

$$R_\lambda = \{a_1+\lambda<x<a-\lambda, 0<t<T\} .$$

Integrating the last relation with respect to $\lambda, \delta_0<\lambda<\delta_1$ and noting that

$$\int_{\delta_0}^{\delta_1} u_x(a_1+\lambda, t)d\lambda = u(a_1+\delta_1, t) - u(a_1+\delta_0, t) ,$$

$$\int_{\delta_0}^{\delta_1} u_x(a-\lambda, t)d\lambda = -u(a-\delta_1, t) + u(a-\delta_0, t) ,$$

we conclude that

$$\int_{\delta_0}^{\delta_1}\int_{R_\lambda} f(v)dxdtd\lambda \leqq C ,$$

which implies that

$$\int_{R_{\delta_1}} f(v)dxdt \leqq C .$$

Since $f(v) \geqq cv$ $(c>0)$ if $v>1$, and since $v_x \leqq 0$, we deduce that

$$\int_0^T\int_{a_1+\delta_1}^{a} v(x, t)dxdt \leqq C . \tag{1.14}$$

In view of (1.13) we may choose δ_1 such that $4\delta_1 < b_1 - a_1$. Let $a_1 + 4\delta_1 < \xi < b_1$. We represent $v(\xi, s)$ in $S_\lambda = \{a_1+\delta_1+\lambda<x<a, 0<t<s\}$ by means of Green's function G_λ (see [2]):

$$v(\xi, s) = \iint_{S_\lambda} G_\lambda g(u)dxdt + \int_{\partial S_\lambda \times (0<t<s)} \frac{\partial G_\lambda}{\partial \nu} vdSdt + \int_{a_1+\delta_1+\lambda}^{a} G_\lambda \phi dx$$

and integrate both sides with respect to λ, $\delta_1<\lambda<2\delta_1$. Recalling (1.14) and noting that $|g(u)| \leqq C$ and that $|G_\lambda| \leqq C$, $|\partial G_\lambda/\partial\nu| \leqq C$ on the domain of integration we deduce that

$$v(\xi, s) \leqq C$$

with C independent of s. Taking $s \to T$ we get a contradiction since $\xi < b_1$. Thus (1.13) cannot hold and similarly $b_1 < a_1$ cannot hold.

In the next section we shall establish, under some conditions, that $\rho = 0$, namely the blow-up set for u and v consists of a single point.

§ 2. A single point blow-up.

We assume that, for some $M > 1$,

$$\begin{aligned} &pf(v) \leqq vf'(v) \quad \text{if } v > M, \quad \text{where } p > 1, \\ &qg(u) \leqq ug'(u) \quad \text{if } u > M, \quad \text{where } q > 1 \end{aligned} \tag{2.1}$$

and that the solution (u, v) satisfies the estimates:

$$\begin{aligned} &u \leqq C(v^{\gamma} + 1) \\ &v \leqq C(u^{1/\gamma} + 1) \quad \text{where } C > 0,\ \gamma > 0 \text{ and } p > \gamma,\ q > \frac{1}{\gamma}. \end{aligned} \tag{2.2}$$

Set

$$J = u_x + \varepsilon x^2 (A + u)^{1+\delta}, \tag{2.3}$$

$$K = v_x + \varepsilon x^2 (B + v)^{1+\tilde{\delta}}, \qquad \tilde{\delta} = \gamma \delta. \tag{2.4}$$

LEMMA 2.1. *Suppose that u and v solve* (1.1), (1.2) *with* (1.3)-(1.6) *and suppose that* (2.1), (2.2) *hold. Then for any large constants $A > 0$, $B > 0$ there exist δ, ε positive and small such that, for $0 < x < a$, $0 < t < T$,*

$$H_\alpha J - f'(v) K - bJ \leqq 0, \tag{2.5}$$

$$H_\beta K - g'(u) J - \tilde{b} K \leqq 0 \tag{2.6}$$

where b, $\tilde{b}$ are bounded functions in $Q_{T'}$, for any T' smaller than the blow-up time T.

PROOF. Set

$$G(u) = (A + u)^{1+\delta}, \qquad F(v) = (B + v)^{1+\tilde{\delta}}.$$

Then

$$H_\alpha J = f'(v) v_x + \varepsilon x^2 G'(u) f(v) - 4\alpha\varepsilon x G'(u) u_x - 4\alpha\varepsilon G(u) - \alpha\varepsilon x^2 G''(u) u_x^2.$$

Substituting v_x, u_x from (2.3), (2.4), we get

$$H_\alpha J - f'(v)K - bJ < -\varepsilon x^2 F(v) f'(v) + \varepsilon x^2 G'(u) f(v) \tag{2.7}$$
$$+ 4\alpha\varepsilon^2 x^3 G'(u) G(u) - 4\alpha\varepsilon G(u) \equiv R$$

since $G''(u) \geqq 0$.

On the set $\{v \leqq M\}$ we have, by (2.2),

$$u \leqq C(M^r + 1) . \tag{2.8}$$

Dropping the first term in R we then get

$$R \leqq \varepsilon[x^2 G'(u) f(M) - 2\alpha G(u)] + \varepsilon G(u)[4\alpha\varepsilon x^3 G'(u) - 2\alpha]$$
$$\leqq \varepsilon(A+u)^\delta [C_1 - 2\alpha(A+u)] + \varepsilon G(u)[\varepsilon C_2 - 2\alpha]$$

where $C_1 = C_1(M)$ and C_2 depends only on A and on the constants C, M in (2.8). Choosing $A = A(M)$ such that $C_1 - 2\alpha A < 0$ and then choosing $\varepsilon \leqq \varepsilon_0(M)$ where $\varepsilon_0(M) C_2 - 2\alpha < 0$, we get $R \leqq 0$. Thus (2.7) implies (2.5) on the set $\{v \leqq M\}$.

Consider next the case where $v > M$. Then, by (2.2),

$$G'(u) = (1+\delta)(A+u)^\delta \leqq (1+\delta)\left(\frac{A}{C} + v^r + 1\right)^\delta C^\delta \leqq (1+\delta)(3v^r)^\delta C^\delta \tag{2.9}$$

since we may always increase the constant C in (2.2), if necessary. Dropping the last term in R, we have

$$R \leqq -\varepsilon x^2 (S_1 + S_2) \tag{2.10}$$

where

$$S_1 = F(v) f'(v) - G'(u) f(v) ,$$
$$S_2 = -4\alpha\varepsilon x G'(u) G(u) .$$

By (2.1), (2.9),

$$S_1 \geqq \left[p \frac{F(v)}{v} - G'(u)\right] f(v)$$
$$\geqq f(v)[p(B+v)^{\tilde{\delta}} - (1+\delta)(3C)^\delta v^{r\delta}]$$
$$= f(v)\left[p - (1+\delta)(3C)^\delta \left(\frac{v}{B+v}\right)^{r\delta}\right](B+v)^{\tilde{\delta}}$$
$$\geqq f(v)\theta(B+v)^{\tilde{\delta}}$$

for any $0 < \theta < p-1$ provided δ is sufficiently small (independently of B). Since, by (2.1), $f(v) \geqq cv^p$ if $v > M$, where $c > 0$, we conclude that

$$S_1 \geqq c\theta v^p (B+v)^{\tilde{\delta}}. \tag{2.11}$$

Next, as in (2.9),

$$\begin{aligned} -S_2 &= 4\alpha\varepsilon x(1+\delta)(A+u)^{1+2\delta} \\ &\leqq 4\alpha\varepsilon x(1+\delta)(3v^{\gamma})^{1+2\delta} C^{1+2\delta} \\ &= [4\alpha x(3C)^{1+2\delta}]\varepsilon v^{\tilde{\delta}} v^{\gamma(1+\delta)}. \end{aligned}$$

Comparing with (2.11) and recalling, by (2.2), that $p>\gamma(1+\delta)$ if δ is small enough, we obtain

$$S_1+S_2>0$$

provided ε is small enough, depending on M. In view of (2.10), we again conclude that $R \leqq 0$ and thus (2.5) holds also on the set $\{v \geqq M\}$.

The proof of (2.6) is similar.

COROLLARY 2.2. *Under the assumptions of Lemma* 2.1,

$$J<0, \quad K<0 \qquad \text{if } 0<x<a,\ 0<t<T. \tag{2.12}$$

PROOF. By Lemma 2.1, for any $\eta>0$,

$$u_x(x, \eta)<0 \qquad \text{if } 0<x\leqq a\,;$$

further, from the proof of that lemma, also

$$u_{xx}(0, \eta)>0, \quad u_x(a, t) \leqq -c<0 \qquad \text{if } \eta \leqq t<T.$$

Hence, if ε is very small (depending on A) then

$$\begin{aligned} &J(x, \eta)<0 \qquad \text{if } 0<x<a, \\ &J(a, t)<0 \qquad \text{if } \eta \leqq t<T. \end{aligned}$$

Clearly also $J(0, t)=0$ if $\eta \leqq t<T$. The same holds for K. Using (2.5), (2.6) we can now proceed to argue as in Lemma 1.1 (with U, V replaced by J, K) in order to establish the assertion (2.12).

THEOREM 2.3. *Suppose that u and v solves* (1.1), (1.2) *with* (1.3)-(1.6). *If the conditions* (2.1), (2.2) *are satisfied, then there is a single blow-up point.*

PROOF. We proceed as in [3]. From (2.12) we have

$$-\frac{u_x}{(A+u)^{1+\delta}} \geqq \varepsilon x^2 .$$

Integrating with respect to x, $0<x<\xi$, we get

$$(A+u)^{-\delta}(\xi, t) \geqq (A+u)^{-\delta}(0, t)+\frac{\delta\varepsilon}{3}\xi^3 \geqq \frac{\delta\varepsilon}{3}\xi^3 .$$

It follows that

$$\limsup_{\substack{\xi\to\xi_0\\ t\to T}}(A+u(\xi, t))^{\delta}<\infty \qquad \text{if } \xi_0>0 .$$

The same holds for v.

§ 3. Sufficient conditions for (2.2).

The conditions in (2.1) hold for a large class of functions f, g, including

(3.1) $$f(v)=Ae^{\lambda v}, \qquad g(u)=Be^{\mu u}$$

with A, λ, B, μ positive constants,

(with any $p>1$, $q>1$) and

(3.2) $$f(v)=A(v+\lambda)^p, \qquad g(u)=B(u+\mu)^q$$

with $A>0$, $B>0$, $\lambda\geqq 0$, $\mu\geqq 0$, $p\geqq q>1$.

Thus in order to apply Theorem 2.3 we only need to find effective sufficient conditions for (2.2) to hold. We shall consider here the two examples (3.1) and (3.2) (with $p=q$), restricting ourselves to

(3.3) $$\alpha=\beta .$$

THEOREM 3.1. *In case* (3.1), (3.3), *the condition* (2.2) *is satisfied and, consequently, there is a single point blow-up for the initial-boundary valve problem* (1.1)-(1.5).

PROOF. Without loss of generality we may take $\lambda=\mu=1$; otherwise we can work with λu and μv. Let $J=Me^u-e^v$. Then

$$H_\alpha J=(MA-B)e^{u+v}-\alpha Me^u u_x^2+\alpha e^v v_x^2 .$$

Since

$$Me^u u_x=e^v v_x+J ,$$

we have

$$u_x^2=\frac{e^{2v}}{M^2e^{2u}}v_x^2+bJ_x$$

for some function b and, therefore,

$$H_\alpha J-\tilde{b}J_x=(MA-B)e^{u+v}+\alpha\Big(e^v-Me^u\frac{e^{2v}}{M^2e^{2u}}\Big)v_x^2$$

$$=(MA-B)e^{u+v}+\alpha e^v\frac{J}{Me^u}v_x^2\ ;$$

thus

$$H_\alpha J-\tilde{b}J_x-cJ=(MA-B)e^{u+v}>0$$

if $M>B/A$, where $\tilde{b}$, c are suitable functions, bounded in $Q_{T'}$ for any $T'<T$. Applying the maximum principle we easily deduce that $J>0$ if M is large enough so that $J(x, 0)>0$. Consequently $v\leqq C(u+1)$ for some constant C. Similarly $u\leqq C(v+1)$, and (2.2) follows with $\gamma=1$.

We now turn to the case (3.2).

LEMMA 3.2. *In case* (3.2), (3.3), *the second inequality of* (2.2) *holds with* $\gamma=(p+1)/(q+1)$.

PROOF. Introduce the functions

$$h(v)=\frac{(v+\lambda)^{p+1}}{p+1}\ , \qquad k(u)=M\frac{(u+\mu)^{q+1}}{q+1}$$

and set

(3.4) $$J=k(u)-h(v)\ .$$

Then

$$H_\alpha J=Ak'(u)(v+\lambda)^p-\alpha k''(u)u_x^2-Bh'(v)(u+\mu)^q+\alpha h''(v)v_x^2\ .$$

Since

$$k'(u)u_x=h'(v)v_x+J_x\ ,$$

we have

$$u_x^2=\frac{h'(v)^2}{k'(u)^2}v_x^2+bJ_x$$

with some coefficient b. Hence

$$H_\alpha J-\tilde{b}J_x=(MA-B)(u+\mu)^q(v+\lambda)^p$$
$$+\alpha\left[p(v+\lambda)^{p-1}-\frac{(v+\lambda)^{2p}}{M^2(u+\mu)^{2q}}Mq(u+\mu)^{q-1}\right]v_x^2 .$$

Since, by (3.4),

$$\frac{(v+\lambda)^{p+1}}{p+1}=M\frac{(u+\mu)^{q+1}}{q+1}+J ,$$

the last expression in brackets is equal to

$$(v+\lambda)^{p-1}\left[p-\frac{p+1}{q+1}q\right].$$

Thus

$$H_\alpha J-\tilde{b}J_x-cJ=(MA-B)(u+\mu)^q(v+\lambda)^p+\alpha q(v+\lambda)^{p-1}\left(\frac{p}{q}-\frac{p+1}{q+1}\right)v_x^2 \tag{3.5}$$

for some coefficient c. Observing that $p/q>(p+1)/(q+1)$ and choosing $M>B/A$, we obtain

$$H_\alpha J-\tilde{b}J_x-cJ\geqq 0 .$$

We now fix any small $\eta>0$ and choose M such that $Mu(x,\eta)\geqq v(x,\eta)$. Then, by the maximum principle, $J\geqq 0$ in $Q_T\backslash Q_\eta$, i.e., $k(u)\geqq h(v)$, and the second part of (2.2) follows.

It appears difficult to establish the first part of (2.2) in case $p>q$.

From Lemma 3.2 and Theorem 2.3 we get:

THEOREM 3.3. *In case* (3.2), (3.3) *with* $p=q$, *there is a single point blow-up for the initial-boundary problem* (1.1)-(1.5).

§ 4. Generalizations.

In this section we extend most of the results of the previous sections to the system

$$u_t-\alpha u_{xx}=f(u,v) \qquad (-a<x<a,\ t>0) , \tag{4.1}$$

$$v_t-\beta v_{xx}=g(u,v) \qquad (-a<x<a,\ t>0) , \tag{4.2}$$

with the same initial-boundary conditions (1.3), (1.4), and with ϕ, ψ satisfying (1.5). We assume that

$$
\begin{aligned}
&f, g \in C^1(\boldsymbol{R}^2)\,, \\
&f(u,v)>0, \quad g(u,v)>0 \qquad \text{if } u>0, \quad v>0\,, \qquad (4.3)\\
&f_u \geqq 0, \quad f_v \geqq 0, \quad g_u \geqq 0, \quad g_v \geqq 0\,.
\end{aligned}
$$

Then the assertions $u_x<0$, $v_x<0$ in Lemma 1.1 remain valid with minor changes in the proof.

In order to extend the results of § 2 we assume that

$$u \leqq C(v+1)\,, \qquad v \leqq C(u+1) \tag{4.4}$$

for the solution, and that, for some $M>1$,

$$
\begin{aligned}
&pf \leqq uf_u + vf_v \qquad \text{if } v>M, \quad u>\frac{v}{C}-1, \qquad \text{where } p>1\,,\\
&pg \leqq ug_u + vg_v \qquad \text{if } u>M, \quad v>\frac{u}{C}-1\,.
\end{aligned} \tag{4.5}
$$

LEMMA 4.1. *Let J, K be defined by* (2.3), (2.4) *with $\tilde{\delta}=\delta$. Then for any large constants $A>0$, $B>0$ there exist δ, ε positive and small such that*

$$H_\alpha J - f_v K - bJ \leqq 0\,,$$

$$H_\beta K - g_u J - \tilde{b} K \leqq 0\,,$$

where b, $\tilde{b}$ are bounded functions in $Q_{T'}$, for any $T'<T$.

PROOF. Proceeding as in Lemma 2.1, we have

$$
\begin{aligned}
H_\alpha J - f_v K - bJ \leqq & -\varepsilon x^2(f_u G(u) + f_v F(v)) + \varepsilon x^2 G'(u) f \\
& + 4\alpha\varepsilon^2 x^3 G'(u) G(u) - 4\alpha\varepsilon G(u) \equiv R
\end{aligned}
$$

where $f_u G(u)$ is a new term. On the set $\{v \leqq M\}$ we can establish that $R \leqq 0$ precisely as before, provided we replace $f(M)$ by $f(C(M+1), M)$.

Consider next the case where $v>M$. Then (2.10) remains valid with

$$S_1 = f_u G(u) + f_v F(v) - G'(u) f(u, v)$$

and with the same S_2 as before; notice that $f_u G(u)$ is a new term.

We easily estimate

$$S_1 \geqq f_u u(A+u)^\delta + f_v v(B+v)^\delta - (1+\delta)(3C)^\delta v^\delta f$$

and

$$(A+u)^{\delta}\geqq\left(A+\frac{v-1}{C}\right)^{\delta}\geqq\frac{\sigma}{C^{\delta}}(v+1), \quad \sigma=\sigma(M)\,.$$

We may assume that $B\geqq 1$ and $\sigma/C^{\delta}<1$. Hence

$$S_1\geqq(f_u u+f_v v)(v+1)^{\delta}\frac{\sigma}{C^{\delta}}-(1+\delta)(3C)^{\delta}v^{\delta}f$$

$$\geqq f\left[\frac{p\sigma}{C^{\delta}}-(1+\delta)(3C)^{\delta}\left(\frac{v}{1+v}\right)^{\delta}\right](1+v)^{\delta}, \quad \text{by (4.5)}\,.$$

From (4.5) we also infer that

$$f(u, v)\geqq\theta(u^2+v^2)^{p/2}\geqq\theta' v^p \qquad \text{if } v>M\,,$$

where θ, θ' are positive constants. Hence, if δ is chosen small enough, then

$$S_1\geqq\bar{\theta}v^p(1+v)^{\delta}\,, \qquad \bar{\theta}>0\,.$$

Since the estimate of S_2 is precisely as in Lemma 2.1, we conclude that $R\leqq 0$. The rest of the proof now proceeds as before.

THEOREM 4.2. *Suppose that u and v solve* (4.1), (4.2) *with* (1.3)-(1.5). *If the conditions* (4.3)-(4.5) *are satisfied, then there is a single blow-up point.*

This follows from Lemma 4.1 by the same arguments as in Theorem 2.3.

In order to establish the condition (4.4) we need a comparison lemma.

LEMMA 4.3. *Let $\alpha=\beta=1$ and assume that for some sufficiently small $\varepsilon>0$,*

(4.6) $$g(y+1, \varepsilon y)\geqq\varepsilon f(y+1, \varepsilon y) \qquad \text{if } y\geqq 0$$

and

(4.7) $$\varepsilon\phi\leqq\psi+\varepsilon\,.$$

Then

(4.8) $$\varepsilon u\leqq v+\varepsilon \qquad \text{in } Q_T\,.$$

PROOF. Set $w=v-\varepsilon(u-1)$. Then

$$Hw = Hv - \varepsilon Hu = g(u, \varepsilon(u-1)+w) - \varepsilon f(u, \varepsilon(u-1)+w)$$

$$= g(u, \varepsilon(u-1)) - \varepsilon f(u, \varepsilon(u-1)) + \tilde{c}w$$

where $\tilde{c}$ is a function of (x, t). Denote by $\tilde{Q}$ the open set $\{u>1\}$ and by $\partial\tilde{Q}$ the parabolic boundary of $\tilde{Q}$. Then

$$Hw - \tilde{c}w \geqq 0 \qquad \text{in } \tilde{Q},$$

by (4.6), and by (4.7) $w \geqq 0$ at any point of $\partial\tilde{Q} \cap \partial Q_T$. On $\partial\tilde{Q} \cap Q_T$ we clearly have $u=1$ and thus $w=v>0$. By the maximum principle it then follows that $w \geqq 0$ in $\tilde{Q}$, i.e., $\varepsilon u \leqq v+\varepsilon$. Outside $\tilde{Q}$ we also have $\varepsilon u \leqq \varepsilon \leqq v+\varepsilon$.

The condition (4.6) is satisfied for a large class of functions, such as,

$$f(u, v) = (v+\alpha_1)^p (A_1 u + A_2 v)^m,$$

(4.9)

$$g(u, v) = (u+\beta_1)^q (B_1 u + B_2 v)^m, \qquad m \geqq 1,\ q \geqq p \geqq 1,$$

with

(4.10) $$\alpha_1 \geqq 0,\quad \beta_1 \geqq 0,\quad A_1 \geqq 0,\quad A_2 \geqq 0,\quad B_1 \geqq 0,\quad B_2 \geqq 0$$

provided $B_1>0$. Since (4.7) is always satisfied by taking ε small, we can assert:

THEOREM 4.4. *Consider* (4.1), (4.2), (1.3), (1.4) *with* f, g *given by* (4.9), (4.10) *with* $m \geqq 1$, $p=q \geqq 1$, $A_2>0$, $B_1>0$. *If* ϕ, ψ *satisfy* (1.5) *then the solution has a single point blow-up.*

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