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Estimates for pseudo-differential operators with exotic symbols

Dedicated to Professor Seizô Itô on the occasion of his 60th birthday

By Akihiko MIYACHI*)

Abstract. We give some estimates of the operator norms of pseudo-differential operators regarded as operators between h^p, L^p, and bmo by means of certain Lipschitz norms of their symbols. We also give some negative results, which show that our norm estimates are sharp in a certain sense.

§ 1. Introduction.

The notations used in this paper will be explained in the latter half of this section.

In this paper, we shall consider the pseudo-differential operator of the following form:

$$a(X, D)f(x)=(2\pi)^{-n}\int_{R^n} e^{ix\xi}a(x, \xi)\mathcal{F}f(\xi)d\xi ,$$

where $\mathcal{F}$ denotes the Fourier transform. The function $a(x, \xi)$ is called the symbol of the pseudo-differential operator $a(X, D)$.

The following theorem is known.

THEOREM. *Let* $0\leqq\delta<1$, $0<p<\infty$, *and* $m=-n(1-\delta)|1/p-1/2|$. *If* k *and* k' *are sufficiently large integers and if the inequalities*

$$|\partial_x^\alpha\partial_\xi^\beta a(x, \xi)|\leqq C_{\alpha\beta}(1+|\xi|)^{m+\delta|\alpha|-\delta|\beta|} \tag{1.1}$$

hold for multi-indices α *and* β *with* $|\alpha|\leqq k$ *and* $|\beta|\leqq k'$, *then the pseudo-differential operator* $a(X, D)$, *originally defined on the Schwartz class* $\mathcal{S}(\boldsymbol{R}^n)$, *can be extended to a bounded operator in* h^p (*if* $p\leqq 1$) *or in* L^p (*if* $p>1$).

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This theorem is due to A. P. Calderón and R. Vaillancourt [3], [4] (the case $p=2$), C. Fefferman [9] (the case $1<p<\infty$; cf. also Wang-Li [26; p. 194]), and to L. Päivärinta and E. Somersalo [20] (the case $0<p\leqq 1$).

It is also known that the value of m mentioned in the theorem is the critical one, i.e., it is known that if δ and p are the same as in the theorem and if $m>-n(1-\delta)|1/p-1/2|$, then there are symbols $a(x, \xi)$ for which the inequalities (1.1) hold for all multi-indices α and β but the pseudo-differential operators $a(X, D)$ are not bounded in h^p (if $p\leqq 1$) or in L^p (if $p>1$). See [13; p. 163] and [16; § 5].

The purpose of the present paper is to refine the above theorem and, in particular, to give the critical values of k and k' in the theorem. More precisely, our results are as follows. We introduce classes $S_\delta^m(\kappa, \kappa')$, where $m\in \boldsymbol{R}$, $0\leqq\delta<1$, and κ and κ' are positive real numbers; a symbol $a(x, \xi)$ belongs to $S_\delta^m(\kappa, \kappa')$ if the inequalities (1.1) hold for multi-indices α and β with $|\alpha|<\kappa$ and $|\beta|<\kappa'$ and if the derivatives $\partial_x^\alpha\partial_\xi^\beta a(x, \xi)$ with $|\alpha|<\kappa\leqq|\alpha|+1$ and $|\beta|<\kappa'$ or with $|\alpha|<\kappa$ and $|\beta|<\kappa'\leqq|\beta|+1$ satisfy certain Lipschitz conditions. (This class is a modification of Hörmander's class $S_{\rho,\delta}^m$ with $\rho=\delta$; as for Hörmander's class, see [13] or [14; Chapt. 2].) Then we give the numbers κ_0 and κ_0' which are critical in the following sense: Let $1<p<\infty$ and $m=-n(1-\delta)|1/p-1/2|$; if $\kappa>\kappa_0$ and $\kappa'>\kappa_0'$, then the pseudo-differential operators with symbols in $S_\delta^m(\kappa, \kappa')$ are bounded in L^p, but if $\kappa<\kappa_0$ or $\kappa'<\kappa_0'$, then there are pseudo-differential operators with symbols in $S_\delta^m(\kappa, \kappa')$ which are not bounded in L^p. We also give the numbers κ_0 and κ_0' which are critical, in the sense similar to the above, for the $h^p\to L^p$ $(0<p<1)$ or $h^1\to h^1$ or $bmo\to bmo$ boundedness of pseudo-differential operators. We also give some results for the $h^p\to h^p$ $(0<p<1)$ boundedness. The detailed statements of our results are given in Sections 3 and 5 (cf. the paragraph below).

The contents of the following sections are as follows. In Section 2, we introduce certain Lipschitz classes on the product space $\boldsymbol{R}^n\times\boldsymbol{R}^n$ and give some properties of them; these are preliminaries to the subsequent sections. In Section 3, we introduce the classes $S_\delta^m(\kappa, \kappa')$ and state the positive parts of our results, i.e., we state such theorems as "if $\kappa>\kappa_0$ and $\kappa'>\kappa_0'$, then the pseudo-differential operators with symbols in $S_\delta^m(\kappa, \kappa')$ are bounded in $\cdots$". Section 4 is devoted to the proofs of the theorems of Section 3. In Section 5, we give the negative parts of our results, from which one can see that most of the results in Section 3 are sharp in a sense; in particular, we show such results as "if $\kappa<\kappa_0$ or $\kappa'<\kappa_0'$, then there are pseudo-differential operators with symbols in $S_\delta^m(\kappa, \kappa')$ which are not bounded in $\cdots$".

REMARK. (i) The results of the present paper are generalizations of those of the same author's paper [18], where only the case $\delta=0$ is treated. (ii) It is the most interesting problem to consider the case $\kappa=\kappa_0$ and $\kappa'=\kappa'_0$. In order to obtain positive result (boundedness of pseudo-differential operators) in this case, it is convenient to introduce some modification of our class $S^m_\delta(\kappa_0, \kappa'_0)$. Muramatu [19] and Sugimoto [23] have already obtained some such results.

Notations. The following notations are used throughout this paper.

We fix a Euclidean space $\boldsymbol{R}^n$; the letter n always denotes the dimension of this space.
If $x=(x_1, \cdots, x_n)$ and $\xi=(\xi_1, \cdots, \xi_n)$ are elements of $\boldsymbol{R}^n$, then $x\xi=\sum_{j=1}^n x_j\xi_j$, $|x|=(xx)^{1/2}$, and $\langle x\rangle=(1+|x|^2)^{1/2}$.
A multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers. If $\alpha=(\alpha_1, \cdots, \alpha_n)$ is a multi-index, then the length $|\alpha|$, the monomial z^α in $z=(z_1, \cdots, z_n)$, and the differential operator ∂^α are defined as follows:

$$|\alpha|=\alpha_1+\cdots+\alpha_n ,$$

$$z^\alpha=z_1^{\alpha_1}\cdots z_n^{\alpha_n} ,$$

$$\partial^\alpha f(x)=\partial_x^\alpha f(x)=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}f(x) ,$$

where $x=(x_1, \cdots, x_n)$.
If $u\in\boldsymbol{R}^n$, then the difference operators $\Delta(u)$ and $\Delta^2(u)$ are defined by

$$\Delta(u)f(x)=\Delta_x(u)f(x)=f(x+u)-f(x)$$

and

$$\Delta^2(u)f(x)=\Delta_x^2(u)f(x)=f(x+2u)-2f(x+u)+f(x) ,$$

where f denotes a function on $\boldsymbol{R}^n$.
The Fourier transform and the inverse Fourier transform are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively. We shall use these transforms only on $\boldsymbol{R}^n$ or $\boldsymbol{R}^n\times\boldsymbol{R}^n$. If f is a function on $\boldsymbol{R}^n$ and g is a function on $\boldsymbol{R}^n\times\boldsymbol{R}^n$, their transforms are defined as follows:

$$\mathcal{F}f(\xi)=\int_{R^n}e^{-ix\xi}f(x)dx , \quad \xi\in\boldsymbol{R}^n ,$$

$$\mathcal{F}^{-1}f(z)=(2\pi)^{-n}(\mathcal{F}f)(-z) , \quad z\in\boldsymbol{R}^n ,$$

$$\mathcal{F}g(\xi, \eta)=\iint_{R^n\times R^n}e^{-i(x\xi+y\eta)}g(x, y)dxdy , \quad (\xi, \eta)\in\boldsymbol{R}^n\times\boldsymbol{R}^n ,$$

$$\mathcal{F}^{-1}g(z, w)=(2\pi)^{-2n}(\mathcal{F}g)(-z, -w) , \quad (z, w)\in\boldsymbol{R}^n\times\boldsymbol{R}^n .$$

We shall explain the function (or distribution) spaces considered in this paper. For $0<p<\infty$, we denote by L^p the set of those measurable functions f on $\boldsymbol{R}^n$ for which

$$\|f\|_{L^p}=\left(\int|f(x)|^p dx\right)^{1/p}<\infty .$$

We denote by L^∞ and by $\|\ \|_{L^\infty}$ the set of all essentially bounded measurable functions on $\boldsymbol{R}^n$ and the essential supremum norm respectively. For $0<p\leqq 1$, we denote by h^p the set of those tempered distributions f on $\boldsymbol{R}^n$ for which $f^{*,1}\in L^p$, and we set $\|f\|_{h^p}=\|f^{*,1}\|_{L^p}$; here $f^{*,1}$ is defined by

$$f^{*,1}(x)=\sup_{0<t<1}|t^{-n}\phi(t^{-1}\cdot)*f(x)|\,, \quad x\in\boldsymbol{R}^n ,$$

where ϕ is a fixed function in $C_0^\infty(\boldsymbol{R}^n)$ such that $\int\phi(x)dx\neq 0$ and $*$ denotes the convolution (cf. Goldberg [11]). For $0<p\leqq 1$, we denote by H^p the set of those tempered distributions f on $\boldsymbol{R}^n$ for which $f^*\in L^p$, and we set $\|f\|_{H^p}=\|f^*\|_{L^p}$; here f^* is defined by

$$f^*(x)=\sup_{0<t<\infty}|t^{-n}\phi(t^{-1}\cdot)*f(x)|\,, \quad x\in\boldsymbol{R}^n ,$$

where ϕ is the same as above (cf. Fefferman-Stein [10]). We denote by bmo the set of those measurable functions f on $\boldsymbol{R}^n$ such that

$$\|f\|_{bmo}=\sup_{|Q|\leq 1}\frac{1}{|Q|}\int_Q|f(x)-f_Q|dx+\sup_{|Q|>1}\frac{1}{|Q|}\int_Q|f(x)|dx<\infty ,$$

where Q ranges over the cubes in $\boldsymbol{R}^n$, $|Q|$ denotes the Lebesgue measure of Q, and $f_Q=|Q|^{-1}\int_Q f(x)dx$ (cf. Goldberg [11]). We denote by $\mathcal{S}(\boldsymbol{R}^n)$ and by $\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ the spaces of rapidly decreasing smooth functions on $\boldsymbol{R}^n$ and on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ respectively. Other function spaces will be explained in due course.

If $a\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$, we define the operator $a(X,D)^*$, which is the dual of $a(X,D)$, by

$$a(X,D)^*g(y)=\int g(x)K(x,x-y)dx ,$$

where

$$K(x,z)=(2\pi)^{-n}\int e^{iz\xi}a(x,\xi)d\xi .$$

Observe that, if $a\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$, then the operators $a(X,D)$ and $a(X,D)^*$ are well defined on $\mathcal{S}(\boldsymbol{R}^n)$ and we have

$$\int g(x)a(X, D)f(x)dx=\int a(X, D)^*g(y)f(y)dy$$

for all f and g in $\mathcal{S}(\boldsymbol{R}^n)$.
If s is a real number, $[s]$ denotes the integer satisfying $[s]\leqq s<[s]+1$.
We fix a function θ in $C_0^\infty(\boldsymbol{R}^n)$ such that $\theta(\xi)=1$ if $|\xi|\leqq 1$, and $\theta(\xi)=0$ if $|\xi|\geqq 2$. For $j=0, 1, 2, \cdots$, we define θ_j and θ'_j by $\theta_0=\theta$, $\theta_j(\xi)=\theta(\xi/2^j)-\theta(\xi/2^{j-1})$ for $j=1, 2, 3, \cdots, \theta'_0=\theta_0+\theta_1$, and $\theta'_j=\theta_{j-1}+\theta_j+\theta_{j+1}$ for $j=1, 2, 3, \cdots$.
Finally, we shall use the letter C to denote a constant, which may be different in each occasion.

§ 2. Lipschitz classes.

In the previous paper [18], the author introduced certain Lipschitz classes on the product space $\boldsymbol{R}^n\times\boldsymbol{R}^n$. In this section, we shall give some generalizations of those classes.

Let w, ρ, and σ be positive functions on $\boldsymbol{R}^n\times\boldsymbol{R}^n$. We say the triplet (w, ρ, σ) is *slowly varying* if there exist positive constants a, b, and c such that the inequalities $1/a\leqq w(x', y')/w(x, y)\leqq a$, $1/b\leqq\rho(x', y')/\rho(x, y)\leqq b$, and $1/c\leqq\sigma(x', y')/\sigma(x, y)\leqq c$ hold for all those x, x', y, and y' satisfying $|x'-x|\leqq\rho(x, y)$ and $|y'-y|\leqq\sigma(x, y)$.

We now introduce the generalized Lipschitz classes on $\boldsymbol{R}^n\times\boldsymbol{R}^n$.

DEFINITION 2.1. Let λ and μ be positive numbers and let (w, ρ, σ) be a slowly varying triplet of positive functions. Let l and m be the nonnegative integers satisfying $l<\lambda\leqq l+1$ and $m<\mu\leqq m+1$. Then $\Lambda(\lambda, \mu; w, \rho, \sigma)$ denotes the set of those functions f on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ which have the following estimates:

(i) if $|\alpha|\leqq l$ and $|\beta|\leqq m$, then $\partial_x^\alpha\partial_y^\beta f(x, y)$ is a continuous function and

$$|\partial_x^\alpha\partial_y^\beta f(x, y)|\leqq Aw(x, y)\rho(x, y)^{-|\alpha|}\sigma(x, y)^{-|\beta|};$$

(ii) if $|\alpha|=l$, $|\beta|\leqq m$, $u\in\boldsymbol{R}^n$, and $|u|\leqq\rho(x, y)/2$, then

$$|\Delta_x^2(u)\partial_x^\alpha\partial_y^\beta f(x, y)|\leqq Aw(x, y)\rho(x, y)^{-\lambda}\sigma(x, y)^{-|\beta|}|u|^{\lambda-l};$$

(iii) if $|\alpha|\leqq l$, $|\beta|=m$, $v\in\boldsymbol{R}^n$, and $|v|\leqq\sigma(x, y)/2$, then

$$|\Delta_y^2(v)\partial_x^\alpha\partial_y^\beta f(x, y)|\leqq Aw(x, y)\rho(x, y)^{-|\alpha|}\sigma(x, y)^{-\mu}|v|^{\mu-m};$$

(iv) if $|\alpha|=l$, $|\beta|=m$, $u\in\boldsymbol{R}^n$, $|u|\leqq\rho(x, y)/2$, $v\in\boldsymbol{R}^n$, and $|v|\leqq\sigma(x, y)/2$, then

$$|\Delta_x^2(u)\Delta_y^2(v)\partial_x^\alpha\partial_y^\beta f(x, y)|\leqq Aw(x, y)\rho(x, y)^{-\lambda}\sigma(x, y)^{-\mu}|u|^{\lambda-l}|v|^{\mu-m}.$$

In the above, A denotes a constant which does not depend on α, β, x, y, u, and v. The smallest such constant A is denoted by $\|f\|_{\Lambda(\lambda,\mu;w,\rho,\sigma)}$. We set $\|f\|_{\Lambda(\lambda,\mu;w,\rho,\sigma)}=\infty$ if f does not belong to $\Lambda(\lambda,\mu;w,\rho,\sigma)$. If $w(x,y)=\rho(x,y)=\sigma(x,y)=1$ (constant functions), then $\Lambda(\lambda,\mu;w,\rho,\sigma)$ and $\|f\|_{\Lambda(\lambda,\mu;w,\rho,\sigma)}$ are denoted by $\Lambda(\lambda,\mu)$ and $\|f\|_{\Lambda(\lambda,\mu)}$ respectively.

We also recall the definition of the Lipschitz classes on $\boldsymbol{R}^n$.

DEFINITION 2.2. Let $\lambda>0$ and let l be the nonnegative integer satisfying $l<\lambda\leqq l+1$. Then $\Lambda(\lambda)$ denotes the set of those functions f on $\boldsymbol{R}^n$ which have the following estimates:
(i) if $|\alpha|\leqq l$, then $\partial_x^\alpha f(x)$ is a continuous function and $|\partial_x^\alpha f(x)|\leqq A$;
(ii) if $|\alpha|=l$ and $u\in\boldsymbol{R}^n$ with $|u|\leqq 1/2$, then $|\Delta_x^2(u)\partial_x^\alpha f(x)|\leqq A|u|^{\lambda-l}$.
Here A is a constant which does not depend on α, x, and u. The smallest such constant A is denoted by $\|f\|_{\Lambda(\lambda)}$.

It is easy to see that $\Lambda(\lambda,\mu;w,\rho,\sigma)$ with the norm $\|\ \|_{\Lambda(\lambda,\mu;w,\rho,\sigma)}$ and $\Lambda(\lambda)$ with the norm $\|\ \|_{\Lambda(\lambda)}$ are Banach spaces. We shall give some properties of these spaces in the propositions below. In those propositions, λ and μ will denote positive numbers, and (w,ρ,σ), (w_1,ρ_1,σ_1), etc. will denote slowly varying triplets of positive functions.

PROPOSITION 2.1. *The map $(f,g)\mapsto fg$ is continuous from $\Lambda(\lambda)\times\Lambda(\lambda)$ to $\Lambda(\lambda)$.*

PROPOSITION 2.2. *A tempered distribution f on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ belongs to $\Lambda(\lambda,\mu)$ if and only if there exists a constant A such that*

$$|\mathcal{F}^{-1}(\theta_j(\xi)\theta_k(\eta)\mathcal{F}f(\xi,\eta))(x,y)|\leqq A2^{-j\lambda}2^{-k\mu}$$

for all $(x,y)\in\boldsymbol{R}^n\times\boldsymbol{R}^n$ and all nonnegative integers j and k. If $\|f\|$ denotes the infimum of the above constant A, then $f\mapsto\|f\|$ is a norm in $\Lambda(\lambda,\mu)$ which is equivalent to the norm $\|\ \|_{\Lambda(\lambda,\mu)}$.

PROPOSITION 2.3. *For complex numbers z and w, and for $f\in\mathcal{S}'(\boldsymbol{R}^n\times\boldsymbol{R}^n)$, we define $\langle D_x\rangle^z\langle D_y\rangle^w f(x,y)\in\mathcal{S}'(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ by*

$$\langle D_x\rangle^z\langle D_y\rangle^w f(x,y)=\mathcal{F}^{-1}(\langle\xi\rangle^z\langle\eta\rangle^w\mathcal{F}f(\xi,\eta)).$$

If $\mathrm{Re}\,z<\lambda$ *and* $\mathrm{Re}\,w<\mu$, *then the operator $f\mapsto\langle D_x\rangle^z\langle D_y\rangle^w f(x,y)$ is bounded from $\Lambda(\lambda,\mu)$ to $\Lambda(\lambda-\mathrm{Re}\,z,\mu-\mathrm{Re}\,w)$ with the operator norm not exceeding $C(1+|\mathrm{Im}\,z|)^{n+1}(1+|\mathrm{Im}\,w|)^{n+1}$, where C is a constant depending only on n, λ, μ,* $\mathrm{Re}\,z$, *and* $\mathrm{Re}\,w$. *This constant C is bounded if* $\mathrm{Re}\,z$ *and* $\mathrm{Re}\,w$ *range on compact sets.*

PROPOSITION 2.4. *If α and β are multi-indices with $|\alpha|<\lambda$ and $|\beta|<\mu$, then the operator $f\mapsto\partial_x^\alpha\partial_y^\beta f(x, y)$ is bounded from $\Lambda(\lambda, \mu)$ to $\Lambda(\lambda-|\alpha|, \mu-|\beta|)$.*

PROPOSITION 2.5. *Let λ_0, λ_1, μ_0, and μ_1 be positive numbers. If $0<t<1$, $\lambda=(1-t)\lambda_0+t\lambda_1$, and $\mu=(1-t)\mu_0+t\mu_1$, then, for functions f on $\boldsymbol{R}^n\times\boldsymbol{R}^n$, we have*

$$\|f\|_{\Lambda(\lambda,\mu)}\leqq C(\|f\|_{\Lambda(\lambda_0,\mu_0)})^{1-t}(\|f\|_{\Lambda(\lambda_1,\mu_1)})^t,$$

where C is a constant depending only on n, λ_0, λ_1, μ_0, μ_1, and t.

PROPOSITION 2.6. *The map $(f, g)\mapsto fg$ is continuous from $\Lambda(\lambda, \mu\,; w_1, \rho_1, \sigma_1)\times\Lambda(\lambda, \mu\,; w_2, \rho_2, \sigma_2)$ to $\Lambda(\lambda, \mu\,; w, \rho, \sigma)$, where $w(x, y)=w_1(x, y)w_2(x, y)$, $\rho(x, y)=\min\{\rho_1(x, y), \rho_2(x, y)\}$, and $\sigma(x, y)=\min\{\sigma_1(x, y), \sigma_2(x, y)\}$.*

PROPOSITION 2.7. *Let $\{\phi_j\}$ be a partition of unity on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ which has the following properties*: (i) *As (x, y) ranges over $\boldsymbol{R}^n\times\boldsymbol{R}^n$, the cardinality of the set $\{j\mid \operatorname{supp}\phi_j\ni(x, y)\}$ is bounded*; (ii) *The set $\{\phi_j\}$ is bounded in the Banach space $\Lambda(\lambda, \mu\,; 1, \rho, \sigma)$. Then, a function f on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ belongs to $\Lambda(\lambda, \mu\,; w, \rho, \sigma)$ if and only if $\{f\phi_j\}$ is a bounded subset of $\Lambda(\lambda, \mu\,; w, \rho, \sigma)$, and the norm*

$$\|f\|=\sup_j\|f\phi_j\|_{\Lambda(\lambda,\mu;w,\rho,\sigma)}$$

is equivalent to the norm $\|f\|_{\Lambda(\lambda,\mu;w,\rho,\sigma)}$.

PROPOSITION 2.8. *Let w_0, ρ_0, σ_0, and B be positive numbers. Suppose f is a function on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ such that the inequalities $1/B\leqq w(x, y)/w_0\leqq B$, $1/B\leqq\rho(x, y)/\rho_0\leqq B$, and $1/B\leqq\sigma(x, y)/\sigma_0\leqq B$ hold for all $(x, y)\in\operatorname{supp}f$. Then it holds that*

$$\begin{aligned}C^{-1}\|f\|_{\Lambda(\lambda,\mu;w,\rho,\sigma)}&\leqq\|f\|_{\Lambda(\lambda,\mu;w_0,\rho_0,\sigma_0)}\\&\leqq C\|f\|_{\Lambda(\lambda,\mu;w,\rho,\sigma)},\end{aligned}$$

where C is a constant depending only on n, λ, μ, w, ρ, σ, and B.

PROPOSITION 2.9. *If w_0, ρ_0, and σ_0 are positive numbers, then*

$$\|f\|_{\Lambda(\lambda,\mu;w_0,\rho_0,\sigma_0)}=w_0^{-1}\|\tilde{f}\|_{\Lambda(\lambda,\mu)},$$

where f denotes a function on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ and the function $\tilde{f}$ is defined by $\tilde{f}(x, y)=f(\rho_0x, \sigma_0y)$.

We shall omit the proofs of these propositions. (See the remark below.)

REMARK 2.1. The space $\Lambda(\lambda)$ is sometimes denoted by $B^{\lambda}_{\infty,\infty}$, which is one of the Besov spaces $B^{\lambda}_{p,q}$. Our space $\Lambda(\lambda, \mu; w, \rho, \sigma)$ is a modification of the space $\Lambda(\lambda)=B^{\lambda}_{\infty,\infty}$. The theory of Besov spaces are well developped; see Bergh-Löfström [1; Chapter 6] and the references cited there. One can prove the above propositions by only slightly modifying the methods found in the theory of Besov spaces. Even if the reader is not familiar with the theory of Besov spaces, the remarks (I)$\sim$(IV) below will be sufficient to help him prove the propositions by himself.
(I) As for a direct proof of Proposition 2.2, consult Bergh-Löfström [1; § 6.2] or Grevholm [12; § 2].
(II) If $\lambda \geqq \lambda' > 0$ and $\mu \geqq \mu' > 0$, then $\Lambda(\lambda, \mu; w, \rho, \sigma) \subset \Lambda(\lambda', \mu'; w, \rho, \sigma)$ with continuous embedding; this is easy to check.
(III) If $0<\varepsilon \leqq 1/2$, then in Definition 2.1 we can replace the assumptions $|u| \leqq \rho(x, y)/2$ and $|v| \leqq \sigma(x, y)/2$ by $|u| \leqq \varepsilon\rho(x, y)$ and $|v| \leqq \varepsilon\sigma(x, y)$, respectively, without affecting the result (i. e., if we carry out this replacement, then the set $\Lambda(\lambda, \mu; w, \rho, \sigma)$ defined by the new definition is the same as the original set and the norm $\| \ \|_{\Lambda(\lambda, \mu; w, \rho, \sigma)}$ defined by the new definition is equivalent to the original norm); this is easy to check.
(IV) If λ is not a positive integer, then in Definition 2.1 we can replace the operators $\Delta^2_x(u)$ and $\Delta^2_x(u)\Delta^2_y(v)$ by $\Delta_x(u)$ and $\Delta_x(u)\Delta^2_y(v)$, respectively, without affecting the result; similar fact holds if μ is not a positive integer. As for a direct proof this fact, consult Zygmund [27; Proof of Theorem (3.4) of Chapter II and the accompanying footnote].

The following lemmas will be used in Section 4.

LEMMA 2.1. *Let $\mu>0$ and $R \geqq 1$. Suppose $f \in \Lambda(\mu)$ and* $\operatorname{supp} f \subset \{y \mid |y| \leqq R\}$. *Then the following hold.*
(i) *For every $\varepsilon>0$, we have*

$$\|\langle z\rangle^{\mu-\varepsilon}\mathcal{F}^{-1}f(z)\|_{L^2} \leqq C_\varepsilon R^{n/2}\|f\|_{\Lambda(\mu)}.$$

(ii) *For every $\varepsilon>0$ and every $v \in \boldsymbol{R}^n$ with $|v| \leqq 10/R$, we have*

$$\|\langle z\rangle^{\mu-\varepsilon}\Delta_z(v)\mathcal{F}^{-1}f(z)\|_{L^2_z} \leqq C_\varepsilon R^{n/2+1}|v|\,\|f\|_{\Lambda(\mu)}.$$

In the above, C_ε denotes a constant which depends only on n, μ, and ε.

PROOF. We set $A=\|f\|_{\Lambda(\mu)}$ and $k=\mathcal{F}^{-1}f$. First we shall prove (i). We have $|f(y)| \leqq A\chi_R(y)$ and $|\Delta^2(u)\partial^\alpha f(y)| \leqq A|u|^{\mu-|\alpha|}\chi_{2R}(y)$ if $|\alpha|<\mu \leqq |\alpha|+1$ and $|u| \leqq 1/2$, where χ_r denotes the defining function of the set $\{y \mid |y| \leqq r\}$. Hence, by Plancherel's theorem, we obtain

$$\|k\|_{L^2} \leqq CAR^{n/2} \tag{2.1}$$

and

$$\|(e^{-iuz}-1)^2 z^\alpha k(z)\|_{L_z^2} \leqq CA|u|^{\mu-|\alpha|}R^{n/2}$$

if $|\alpha|<\mu\leqq|\alpha|+1$ and $|u|\leqq 1/2$. From this latter estimate, we can easily deduce that

$$\|\langle z\rangle^{\mu-\varepsilon}k(z)\|_{L^2(2^{j-1}<|z|\leqq 2^j)} \leqq C_\varepsilon A 2^{-j\varepsilon}R^{n/2}$$

for all positive integers j. Summing up the above inequalities over j and taking (2.1) into account, we obtain the desired result. Next, we shall prove (ii). Let $v\in \boldsymbol{R}^n$ with $|v|\leqq 10/R$ and set $f_v(y)=(e^{ivy}-1)f(y)$. Then, by using Proposition 2.1, we can prove that $\|f_v\|_{\Lambda(\mu)}\leqq CA|v|R$. Since $\mathcal{F}^{-1}f_v(z)=\Delta_z(v)\mathcal{F}^{-1}f(z)$, the desired result now follows from (i). This completes the proof of Lemma 2.1.

LEMMA 2.2. *Let $0<\lambda<1$, $\mu>0$, and $R\geqq 1$. Suppose $f\in\Lambda(\lambda,\mu)$ and* $\operatorname{supp} f(x,\cdot)\subset\{y\in\boldsymbol{R}^n \mid |y|\leqq R\}$ *for all $x\in\boldsymbol{R}^n$. Define the function K on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ by $K(x,\cdot)=\mathcal{F}^{-1}(f(x,\cdot))$. Then, if $\varepsilon>0$, $u\in\boldsymbol{R}^n$, $|u|\leqq 10$, $v\in\boldsymbol{R}^n$, and $|v|\leqq 10/R$, we have*

$$\sup_x \|\langle z\rangle^{\mu-\varepsilon}(K(x+u,z+v)-K(x,z))\|_{L_z^2} \leqq C_\varepsilon R^{n/2}(|u|^\lambda+|v|R)\|f\|_{\Lambda(\lambda,\mu)},$$

where C_ε is a constant depending only on n, λ, μ, and ε.

PROOF. Let $A=\|f\|_{\Lambda(\lambda,\mu)}$, and let ε, u, and v satisfy the assumptions mentioned in the lemma. It is easy to see that $\|f(x+u,\cdot)\|_{\Lambda(\mu)}\leqq A$ and $\|f(x+u,\cdot)-f(x,\cdot)\|_{\Lambda(\mu)}\leqq C|u|^\lambda A$ with C depending only on n, λ, and μ (not on x nor u). (The latter inequality follows from the assertion in (IV) in Remark 2.1.) Hence, by Lemma 2.1, we have

$$\|\langle z\rangle^{\mu-\varepsilon}(K(x+u,z+v)-K(x+u,z))\|_{L_z^2} \leqq C_\varepsilon R^{n/2+1}|v|A$$

and

$$\|\langle z\rangle^{\mu-\varepsilon}(K(x+u,z)-K(x,z))\|_{L_z^2} \leqq C_\varepsilon R^{n/2}|u|^\lambda A\,.$$

Combining these inequalities, we obtain the desired inequality.

§ 3. Boundedness of pseudo-differential operators.

We define the class $S_\delta^m(\kappa,\kappa')$ as follows.

DEFINITION 3.1. Let $m\in\boldsymbol{R}$, $0\leqq\delta<1$, $\kappa>0$, and $\kappa'>0$. Then the class $S_\delta^m(\kappa,\kappa')$ is defined by $S_\delta^m(\kappa,\kappa')=\Lambda(\kappa,\kappa';w,\rho,\sigma)$ with $w(x,\xi)=\langle\xi\rangle^m$, $\rho(x,\xi)=\langle\xi\rangle^{-\delta}$, and $\sigma(x,\xi)=2^{-1}\langle\xi\rangle^\delta$. The norm in this space is denoted by $\|\ \|_{m,\delta,\kappa,\kappa'}$.

We shall use the following notation. Let (Y, Z) be a couple of function (or distribution) spaces over $\boldsymbol{R}^n$. Suppose nonnegative functions (norms, seminorms, etc.) $\| \|_Y$ and $\| \|_Z$ are defined on Y and Z respectively. Then we shall write as $\Psi_\delta^m(\kappa, \kappa') \subset \mathcal{L}(Y, Z)$ if there exists a constant C depending only on n, m, δ, κ, κ', Y, and Z for which the inequality

$$\|a(X, D)f\|_Z \leqq C\|a\|_{m,\delta,\kappa,\kappa'}\|f\|_Y$$

holds for all $a \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ and all $f \in \mathcal{S}(\boldsymbol{R}^n) \cap Y$. Similarly, we shall write as $(\Psi_\delta^m(\kappa, \kappa'))^* \subset \mathcal{L}(Y, Z)$ if the inequality

$$\|a(X, D)^* f\|_Z \leqq C\|a\|_{m,\delta,\kappa,\kappa'}\|f\|_Y$$

holds with C, a, and f being the same as above.

Our results on the boundedness of pseudo-differential operators in h^p and L^p are given in the following two theorems.

THEOREM 3.1. *Let* $0 \leqq \delta < 1$ *and* $m(n, \delta, p) = -n(1-\delta)|1/p - 1/2|$.
(1) *If* $0 < p \leqq 1$, $m = m(n, \delta, p)$, $\kappa > n/2 + n\delta(1/p - 1)/(1-\delta)$, *and* $\kappa' > n/p$, *then* $\Psi_\delta^m(\kappa, \kappa') \subset \mathcal{L}(h^p, h^p)$.
(1′) *If* $0 < p < 1$, $m = m(n, \delta, p)$, $\kappa > n/2$, *and* $\kappa' > n/p$, *then* $\Psi_\delta^m(\kappa, \kappa') \subset \mathcal{L}(h^p, L^p)$.
(2) *If* $1 < p \leqq 2$, $m = m(n, \delta, p)$, $\kappa > n/2$, *and* $\kappa' > n/p$, *then* $\Psi_\delta^m(\kappa, \kappa') \subset \mathcal{L}(L^p, L^p)$.
(3) *If* $2 < p < \infty$, $m = m(n, \delta, p)$, $\kappa > n/p$, *and* $\kappa' > n/2$, *then* $\Psi_\delta^m(\kappa, \kappa') \subset \mathcal{L}(L^p, L^p)$.
(4) *If* $m = -n(1-\delta)/2$, $\kappa > 0$, *and* $\kappa' > n/2$, *then* $\Psi_\delta^m(\kappa, \kappa') \subset \mathcal{L}(bmo, bmo)$.

THEOREM 3.2. *Let* δ *and* $m(n, \delta, p)$ *be the same as in Theorem* 3.1.
(1) *If* $0 < p \leqq 1$, $m = m(n, \delta, p)$, $\kappa > n/p - n$, *and* $\kappa' > n(1/p - 1/2)$, *then* $(\Psi_\delta^m(\kappa, \kappa'))^* \subset \mathcal{L}(h^p, h^p)$.
(2) *If* $1 < p \leqq 2$, $m = m(n, \delta, p)$, $\kappa > n - n/p$, *and* $\kappa' > n/2$, *then* $(\Psi_\delta^m(\kappa, \kappa'))^* \subset \mathcal{L}(L^p, L^p)$.
(3) *If* $2 < p < \infty$, $m = m(n, \delta, p)$, $\kappa > n/2$, *and* $\kappa' > n - n/p$, *then* $(\Psi_\delta^m(\kappa, \kappa'))^* \subset \mathcal{L}(L^p, L^p)$.
(4) *If* $m = -n(1-\delta)/2$, $\kappa > n/2$, *and* $\kappa' > n$, *then* $(\Psi_\delta^m(\kappa, \kappa'))^* \subset \mathcal{L}(bmo, bmo)$.

If $\delta = 0$, we can improve parts of these theorems. In order to give the improvement, we introduce a function space as follows.

DEFINITION 3.2. For locally integrable functions f on $\boldsymbol{R}^n$, we define the function $N(f)$ on $\boldsymbol{R}^n$ by

$$N(f)(x) = \sup_{r \leq 1} \frac{1}{r^n} \int_{|x-y| \leq r} |f(y)| dy .$$

For $0<p\leqq 1$, we denote by X^p the set of those locally integrable functions f on $\boldsymbol{R}^n$ for which $N(f)$ belong to L^p. We define $\|f\|_{X^p}$ by $\|f\|_{X^p}=\|N(f)\|_{L^p}$.

The following two theorems improve parts of Theorems 3.1 and 3.2.

THEOREM 3.3. *If the assumptions of* (1) *of Theorem* 3.1 *are satisfied with* $\delta=0$, *then* $\Psi_0^m(\kappa, \kappa')\subset\mathcal{L}(h^p, X^p)$.

THEOREM 3.4. *If the assumptions of* (1) *of Theorem* 3.2 *are satisfied with* $\delta=0$, *then* $(\Psi_0^m(\kappa, \kappa'))^*\subset\mathcal{L}(h^p, X^p)$.

In Section 5, we shall give some negative results, which will show that Theorems 3.1～3.4 are sharp in a sense.

Proofs of the theorems will be given in the next section.

§ 4. Proofs of the theorems in Section 3.

Throughout this section, let $m(n, \delta, p)$ be the same as in Theorem 3.1.

In order to prove the theorems in Section 3, we use the following theorems and lemma.

THEOREM A. *If* $0\leqq\delta<1$, $\kappa>n/2$, *and* $\kappa'>n/2$, *then* $\Psi_\delta^0(\kappa, \kappa')\subset\mathcal{L}(L^2, L^2)$.

In the case $\delta=0$, this theorem is due to Cordes [8; Theorem D] (see also [18]). In the case $0<\delta<1$, it is a consequence of a stronger theorem of Muramatu [19].

THEOREM B. *If* $0\leqq\delta<1$, $m<-n(1-\delta)/2$, $\kappa>0$, *and* $\kappa'>n/2$, *then* $\Psi_\delta^m(\kappa, \kappa')\subset\mathcal{L}(L^2, L^2)$.

PROOF. Suppose δ, m, κ, and κ' satisfy the assumptions of the theorem. Let $a\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$. We use the following notations. Set $A=\|a\|_{m,\delta,\kappa,\kappa'}$,

$$a_j(x, \xi)=a(x, \xi)\theta_j(\xi)\ ,$$

$$\tilde{a}_j(x, \xi)=a_j(2^{-j\delta}x, 2^{j\delta}\xi)\ ,$$

$T=a(X, D)$, and $T_j=a_j(X, D)$. Let $K(x, \cdot)$ and $\tilde{K}_j(x, \cdot)$ be the inverse Fourier transforms of $a(x, \cdot)$ and $\tilde{a}_j(x, \cdot)$ respectively. Then it holds that $T=\sum_{j=0}^{\infty}T_j$,

$$T_jf(2^{-j\delta}x)=\int\tilde{K}_j(x, x-y)f(2^{-j\delta}y)dy$$

(the left hand side should be interpreted as $(T_jf)(2^{-j\delta}x)$),

$$\operatorname{supp} \tilde{a}_j(x, \cdot) \subset \{\xi \mid |\xi| \leqq 2^{1+j(1-\delta)}\},$$

and

$$\|\tilde{a}_j\|_{\Lambda(\kappa, \kappa')} \leqq CA2^{jm}.$$

(The last inequality can be obtained by the use of Propositions 2.7~2.9.)

Now take $\varepsilon>0$ such that $\kappa'-\varepsilon>n/2$. Schwarz's inequality gives

$$|T_j f(2^{-j\delta}x)| \leqq \|\langle x-y\rangle^{\kappa'-\varepsilon}\tilde{K}_j(x, x-y)\|_{L_y^2}\|\langle x-y\rangle^{-\kappa'+\varepsilon}f(2^{-j\delta}y)\|_{L_y^2}.$$

From this it follows that $\|(T_j f)(2^{-j\delta}\cdot)\|_{L^2}$ does not exceed the product of

$$\sup_x \|\langle x-y\rangle^{\kappa'-\varepsilon}\tilde{K}_j(x, x-y)\|_{L_y^2}$$

and

$$\left(\iint \langle x-y\rangle^{2(-\kappa'+\varepsilon)}|f(2^{-j\delta}y)|^2 dxdy\right)^{1/2}.$$

By Lemma 2.1, the first factor is majorized by $C_\varepsilon A2^{j(m+(1-\delta)n/2)}$. The second factor is equal to $C_\varepsilon\|f(2^{-j\delta}\cdot)\|_{L^2}$. Thus we have

$$\|T_j f\|_{L^2} \leqq CA2^{j(m+(1-\delta)n/2)}\|f\|_{L^2}.$$

Summing up these estimates over $j=0, 1, 2, \cdots$, we obtain $\|Tf\|_{L^2} \leqq CA\|f\|_{L^2}$ since $m+(1-\delta)n/2<0$. This completes the proof of Theorem B.

LEMMA 4.1. *Let* $0<p<2$ *and* $M>n(1/p-1/2)$.

(i) *For every measurable function* f *on* $\boldsymbol{R}^n$, *the inequality*

$$\|f\|_{L^p} \leqq C\|\langle x\rangle^M f(x)\|_{L^2}$$

holds with a constant C *depending only on* n, p, *and* M.

(ii) *Let* (E, μ) *be a measure space and* f *be a measurable function on* $E\times\boldsymbol{R}^n$. *Suppose there exist positive numbers* s *and* B *and a measurable map* $\phi: E\to\boldsymbol{R}^n$ *such that* $|\phi(u)|\leqq s$ *and*

$$\|\langle z-\phi(u)\rangle^M f(u, z)\|_{L_z^2(\boldsymbol{R}^n)} \leqq B$$

for all $u\in E$. *Then*

$$\left\|\int_E f(u, z)d\mu(u)\right\|_{L^p(|z|>2s)} \leqq CB\mu(E),$$

where C *is a constant depending only on* n, p, *and* M.

PROOF. Proof of (i). Set $q=(1/p-1/2)^{-1}$. Then Hölder's inequality gives

$$\|f\|_{L^p} \leqq \|\langle x\rangle^{-M}\|_{L^q}\|\langle x\rangle^M f(x)\|_{L^2} = C\|\langle x\rangle^M f(x)\|_{L^2}.$$

Proof of (ii). Minkowski's inequality gives

$$\left\|\int_E \langle z-\phi(u)\rangle^M |f(u,z)|d\mu(u)\right\|_{L^2_z} \leqq \int_E \|\langle z-\phi(u)\rangle^M f(u,z)\|_{L^2_z} d\mu(u) \leqq B\mu(E) .$$

If $|z|>2s$, then $\langle z-\phi(u)\rangle \geqq \langle z\rangle - |\phi(u)| \geqq \langle z\rangle/2$ (since $|\phi(u)|\leqq s$). Combining these inequalities, we have

$$\left\|(\langle z\rangle/2)^M \int_E |f(u,z)|d\mu(u)\right\|_{L^2(|z|>2s)} \leqq B\mu(E) .$$

This and the inequality in (i) give the desired result.

We shall give some propositions, which will contain the essential part of the proofs of the theorems in Section 3.

PROPOSITION 4.1. *If* $0<\delta<1$, $0<p<1$, $m=m(n,\delta,p)$, $\kappa>n/2$, *and* $\kappa'>n/p+1$, *then* $\Psi^m_\delta(\kappa,\kappa')\subset\mathcal{L}(h^p, L^p)$.

PROOF. Suppose δ, p, m, κ, and κ' satisfy the assumptions of the proposition. Let $a\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$. In this proof the constant C depends only on n, p, δ, κ, κ', and the parameters indicated as suffixes. We set $N=[n/p-n]+1$ and $M=[n(1/p-1/2)]+1$. We define A, a_j, $\tilde{a}_j$, T, T_j, K, and $\tilde{K}_j$ in the same way as in the proof of Theorem B.

We recall the atomic decomposition theorem for h^p, which is due to Goldberg [11; Lemma 5]. A function f on $\boldsymbol{R}^n$ is called an h^p atom if there exists a ball B of radius $r\leqq 1$ such that $\operatorname{supp} f\subset B$, $\|f\|_{L^\infty}\leqq r^{-n/p}$, and $\int f(x)x^\alpha dx=0$ for $|\alpha|\leqq[n/p-n]$, or if there exists a ball B of radius $r>1$ such that $\operatorname{supp} f\subset B$ and $\|f\|_{L^\infty}\leqq r^{-n/p}$. The atomic decomposition theorem for h^p asserts that every f in h^p can be decomposed as $f=\sum_{j=1}^\infty \lambda_j f_j$, where λ_j are complex numbers such that $\sum_j|\lambda_j|^p\leqq C\|f\|^p_{h^p}$ and f_j are h^p atoms. (This theorem also holds for $p=1$.)

By virtue of this atomic decomposition theorem, it is sufficient to show that $\|Tf\|_{L^p}\leqq CA$ for all h^p atoms f. In the present case, it is sufficient to consider the h^p atoms which are supported in the balls centered at the origin. Thus, we shall prove the estimate $\|Tf\|_{L^p}\leqq CA$ for all those f such that $\operatorname{supp} f\subset\{x\mid |x|\leqq r\}$ and $\|f\|_{L^\infty}\leqq r^{-n/p}$ with some $r>0$ and $\int f(x)x^\alpha dx=0$ for $|\alpha|\leqq[n/p-n]=N-1$ if $r\leqq 1$ (if $r>1$, there are no moment conditions).

By Theorem A or B, we see that the operator T is bounded in L^2 with operator norm not exceeding CA. Hence, using Hölder's inequality, we have

$$\begin{aligned}\|Tf(x)\|_{L^p(|x|\leqq 2r)} &\leqq Cr^{n(1/p-1/2)}\|Tf\|_{L^2} \\ &\leqq CAr^{n(1/p-1/2)}\|f\|_{L^2} \\ &\leqq CA\,.\end{aligned}$$

In the rest of the proof, we shall prove the estimate

$$\|Tf(x)\|_{L^p(|x|>2r)} \leqq CA\,.$$

In order to do this, we shall prove the following two estimates:

$$\|T_jf(x)\|_{L^p(|x|>2r)} \leqq CA(2^jr)^{-n/p+n}\,, \tag{4.1}$$

$$\|T_jf(x)\|_{L^p(|x|>2r)} \leqq CA(2^jr)^{N-n/p+n} \quad \text{if } r\leqq 1\,. \tag{4.2}$$

If we have proved these estimates, then we can deduce the desired estimate as follows:

$$\begin{aligned}\|Tf(x)\|^p_{L^p(|x|>2r)} &\leqq \sum_{j=0}^{\infty} \|T_jf(x)\|^p_{L^p(|x|>2r)} \\ &\leqq \sum_{2^jr\leqq 1} (CA(2^jr)^{N-n/p+n})^p + \sum_{2^jr>1} (CA(2^jr)^{-n/p+n})^p \\ &\leqq (CA)^p\,,\end{aligned}$$

where the first inequality is legitimate since $p<1$, the second inequality is due to (4.1) and (4.2), and the last inequality holds since $N-n/p+n>0$ and $-n/p+n<0$.

Before we prove (4.1) and (4.2), we shall prove that

$$\|\langle x-y\rangle^M D_2^\beta \tilde{K}_j(x, x-y)\|_{L^2_x} \leqq C_\beta A 2^{j((1-\delta)(|\beta|+n/2)+m)} \tag{4.3}$$

for any multi-index β, where the function $D_2^\beta \tilde{K}_j$ is defined by $D_2^\beta\tilde{K}_j(x, z)=\partial_z^\beta\tilde{K}_j(x, z)$. (Note that C_β is independent of y and j.) In order to prove this, observe, by integration by parts, that

$$(-iz)^\alpha\partial_z^\beta\tilde{K}_j(x, z)=(2\pi)^{-n}\int e^{i\xi z}\partial_\xi^\alpha[(i\xi)^\beta\tilde{a}_j(x, \xi)]\theta_j'(2^{j\delta}\xi)d\xi\,.$$

(Note that $\theta_j'(2^{j\delta}\xi)=1$ on the support of the integrand.) Setting $z=x-y$ in the above equality, we obtain

$$(-i(x-y))^\alpha D_2^\beta\tilde{K}_j(x, x-y)=T_{\alpha,\beta,j}g_{y,j}(x)\,,$$

where $T_{\alpha,\beta,j}$ is the pseudo-differential operator with the symbol $a_{\alpha,\beta,j}(x, \xi)=\partial_\xi^\alpha[(i\xi)^\beta\tilde{a}_j(x, \xi)]$ and $g_{y,j}$ is the inverse Fourier transform of the function $\xi\mapsto e^{-i\xi y}\theta_j'(2^{j\delta}\xi)$. Using Plancherel's theorem, we have $\|g_{y,j}\|_{L^2}\leqq C2^{j(1-\delta)n/2}$. On the other hand, by Propositions 2.6~2.9 and 2.4, we have $\|a_{\alpha,\beta,j}\|_{0,0,\kappa,\kappa'-M}$

$\leqq C_\beta A 2^{j((1-\delta)|\beta|+m)}$ if $|\alpha|\leqq M$; hence, since $\kappa'-M>n/2$, using Theorem A, we see that the operators $T_{\alpha,\beta,j}$ with $|\alpha|\leqq M$ are bounded in L^2 with operator norms not exceeding $C_\beta A 2^{j((1-\delta)|\beta|+m)}$. Thus we have

$$\|(x-y)^\alpha D_2^\beta \tilde{K}_j(x, x-y)\|_{L_x^2} \leqq C_\beta A 2^{j((1-\delta)(|\beta|+n/2)+m)}$$

if $|\alpha|\leqq M$, which implies (4.3).

Now we shall prove (4.1) and (4.2). It holds that

$$T_j f(2^{-j\delta}x)=\int_{|y|\leq 2^{j\delta}r} \tilde{K}_j(x, x-y) f(2^{-j\delta}y)dy\,. \tag{4.4}$$

If $r\leqq 1$, then we also have

$$T_j f(2^{-j\delta}x)=\int (\tilde{K}_j(x, x-y)-P(y)) f(2^{-j\delta}y)dy\,,$$

where $P(y)$ denotes any polynomial in y of degree not exceeding $N-1$; if we take as $P(y)$ Maclaurin's series of the function $y\mapsto \tilde{K}_j(x, x-y)$ up to the terms of degree $N-1$, then we have

$$T_j f(2^{-j\delta}x)=\iint_{\substack{0<t<1\\ |y|\leq 2^{j\delta}r}} N(1-t)^{N-1} \sum_{|\beta|=N} D_2^\beta \tilde{K}_j(x, x-ty)\frac{(-y)^\beta}{\beta!} f(2^{-j\delta}y)dtdy\,. \tag{4.5}$$

Now we can easily derive (4.1) and (4.2) from (4.3), (4.4), (4.5), and the estimate $\|f\|_{L^\infty}\leqq r^{-n/p}$ with the aid of Lemma 4.1. This completes the proof of Proposition 4.1.

PROPOSITION 4.2. *If δ, p, and m satisfy the same assumptions as in Proposition 4.1 and if $\kappa>(n/p+n+4)\delta/(1-\delta)+3n/2+3$ and $\kappa'>n/p+2$, then $\Psi_\delta^m(\kappa, \kappa')\subset \mathcal{L}(h^p, h^p)$.*

Before we go to the proof of this proposition, we recall some properties of h^p and H^p. In this paragraph, we assume $0<p\leqq 1$. First, if $\operatorname{supp}\mathcal{F}f \subset\{\xi \mid |\xi|\leqq 2\}$ and $f\in L^p$, then $f\in h^p$ and $\|f\|_{h^p}\leqq C\|f\|_{L^p}$. This can be shown as follows: if $\operatorname{supp}\mathcal{F}f\subset\{\xi \mid |\xi|\leqq 2\}$, then

$$\sup_{|y|\leq 1} |f(x-y)|\leqq C(M(|f|^r)(x))^{1/r} \tag{4.6}$$

for all $r>0$ and all $x\in \boldsymbol{R}^n$, where M denotes the Hardy-Littlewood maximal function and C is a constant depending only on n and r (as for this inequality, see Triebel [24; § 1.3.1]); if further $f\in L^p$ and if we take $r<p$, then the right hand side of (4.6) belongs to L^p and so is the left hand side and hence *a fortiori* f belongs to h^p. Secondly, there exist a finite number of functions m_j $(j=1,\cdots,H)$ on $\boldsymbol{R}^n$ which are homogeneous of

degree zero and smooth away from the origin and have the following property: f belongs to $L^2 \cap H^p$ if and only if all $m_j(D)f$, where $j=1, \cdots, H$, belong to $L^2 \cap L^p$, and there exists a constant C such that

$$C^{-1}\|f\|_{H^p} \leqq \sum_{j=1}^{H} \|m_j(D)f\|_{L^p} \leqq C\|f\|_{H^p}$$

for all those f. This fact is due to Fefferman and Stein [10] (cf. also Coifman-Dahlberg [5]). Thirdly, if b is a smooth function on $\boldsymbol{R}^n$ such that $|\partial^\alpha b(\xi)| \leqq C_\alpha \langle\xi\rangle^{-|\alpha|}$ for all α, then the operator $b(D)$ is bounded in h^p. This fact is due to Goldberg [11; Theorem 4]. Combining the above facts, we obtain the following characterization of h^p: Define functions b_j ($j=0, 1, \cdots, H$) on $\boldsymbol{R}^n$ by $b_0=\theta$ and $b_j=(1-\theta)m_j$ for $j=1, \cdots, H$; then f belongs to $L^2 \cap h^p$ if and only if all $b_j(D)f$, where $j=0, 1, \cdots, H$, belong to $L^2 \cap L^p$, and there exists a constant C such that

$$C^{-1}\|f\|_{h^p} \leqq \sum_{j=0}^{H} \|b_j(D)f\|_{L^p} \leqq C\|f\|_{h^p}$$

for all those f.

Now we go to the proof of Proposition 4.2.

PROOF OF PROPOSITION 4.2. Suppose δ, p, m, κ, and κ' satisfy the assumptions mentioned in the proposition. Let $a \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$. By virtue of the characterization of h^p as given above, it is sufficient to estimate the $h^p \to L^p$ operator norms of the operators $b_j(D)a(X, D)$ for $j=0, 1, \cdots, H$. It holds that $b_j(D)a(X, D)$ is equal to the pseudo-differential operator $c_j(X, D)$ whose symbol $c_j(x, \xi)$ satisfies

$$c_j(x, \xi) = (2\pi)^{-n} \iint e^{-iy\eta} b_j(\xi+\eta) a(x+y, \xi) dy d\eta$$

and

$$\|c_j\|_{m, \delta, \kappa, \kappa'} \leqq C\|a\|_{m, \delta, \kappa+s, \kappa'+1},$$

where $s=(\kappa'+n+3)\delta/(1-\delta)+n+3$ (cf. [14; Chapt. 2, § 2]). Hence Proposition 4.1 gives the desired estimate for $c_j(X, D)=b_j(D)a(X, D)$. This completes the proof.

PROPOSITION 4.3. *If p, κ, and κ' satisfy the same assumptions as in Proposition* 4.1 *and if $m=m(n, 0, p)$, then $\Psi_0^m(\kappa, \kappa') \subset \mathcal{L}(h^p, X^p)$.*

We can prove this proposition by slightly modifying the proof of Proposition 4.1. In fact, we can easily prove that, if $\delta=0$, the estimates in the proof of Proposition 4.1 remain true if we replace Tf and $T_j f$ by

$N(Tf)$ and $N(T_jf)$ respectively. In order to see this, we need only the following two properties of the maximal function $N(f)$. First, there exists a constant C depending only on n such that $\|N(f)\|_{L^2}\leqq C\|f\|_{L^2}$; this follows from the Hardy-Littlewood maximal theorem (see e. g. [22; Chapt. II, § 3]). Secondly, for $M>0$, $y\in\boldsymbol{R}^n$, and $f\in L^1_{\mathrm{loc}}$, we have

$$\langle x-y\rangle^M N(f)(x)\leqq 2^M N(\langle\cdot-y\rangle^M f(\cdot))(x)\,;$$

this follows from the fact that $|x-x'|\leqq 1$ implies $\langle x\rangle\leqq 2\langle x'\rangle$. We shall omit the details of the proof of Proposition 4.3.

PROPOSITION 4.4. *If* $0<\delta<1$, $1>p>n/(n+1)$, $m=m(n,\delta,p)$, $\kappa>n/p-n$, *and* $\kappa'>n(1/p-1/2)$, *then* $(\Psi^m_\delta(\kappa,\kappa'))^*\subset\mathcal{L}(h^p,h^p)$.

PROOF. We shall prove a modified form of this proposition; i. e., we shall prove that the proposition holds if $\mathcal{L}(h^p,h^p)$ is replaced by $\mathcal{L}(h^p,L^p)$. This is sufficient to prove the original proposition as we shall see now. Let $b_j(D)$ be the operators as given before the proof of Proposition 4.2. In order to show the $h^p\to h^p$ estimate for $a(X,D)^*$, it is sufficient to show the $h^p\to L^p$ estimate for $b_j(D)a(X,D)^*$. It holds that $b_j(D)a(X,D)^*=c_j(X,D)^*$ with $c_j(x,\xi)=a(x,\xi)b_j(-\xi)$ and that this symbol satisfies $\|c_j\|_{m,\delta,\kappa,\kappa'}\leqq C\|a\|_{m,\delta,\kappa,\kappa'}$ (this latter fact follows from Proposition 2.6). Combining these facts, we see that, if the modified Proposition 4.4 (i. e., the one with $\mathcal{L}(h^p,h^p)$ replaced by $\mathcal{L}(h^p,L^p)$) holds, then the original Proposition 4.4 holds as well.

Now, we shall prove the modified Proposition 4.4. Suppose δ, p, m, κ, and κ' satisfy the assumptions of the proposition; without loss of generality, we may and shall assume $\kappa<1$. Let $a\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$. We use the same notations as in the proof of Theorem B. We also set $T^*=a(X,D)^*$ and $T_j^*=a_j(X,D)^*$. It holds that $T^*=\sum_{j=0}^\infty T_j^*$ and

$$T_j^*g(2^{-j\delta}y)=\int g(2^{-j\delta}x)\tilde{K}_j(x,x-y)dx \tag{4.7}$$

(the left hand side should be interpreted as $(T_j^*g)(2^{-j\delta}y)$). In this proof, the constant C depends only on n, p, δ, κ, κ', and the parameters indicated as suffixes.

By the same reasoning as in the proof of Proposition 4.1, it is sufficient to show the estimate $\|T^*g\|_{L^p}\leqq CA$ for all those g such that $\mathrm{supp}\,g\subset\{x\mid|x|\leqq r\}$ and $\|g\|_{L^\infty}\leqq r^{-n/p}$ with some $r>0$ and $\int g(x)dx=0$ if $r\leqq 1$ (if $r>1$, the integral of g need not vanish).

By Theorem B, we see that the operator T is bounded in L^2 with operator norm not exceeding CA. Hence, the same is true for T^*. Thus, using Hölder's inequality in the same way as in the proof of Proposition 4.1, we obtain

$$\|T^*g(y)\|_{L^p(|y|\leq 2r)} \leqq CA .$$

In the rest of the proof, we shall prove the estimate

$$\|T^*g(y)\|_{L^p(|y|>2r)} \leqq CA .$$

We can deduce this estimate from the following two estimates:

$$\|T_j^*g(y)\|_{L^p(|y|>2r)} \leqq CA(2^j r)^{-n/p+n} , \tag{4.8}$$

$$\|T_j^*g(y)\|_{L^p(|y|>2r)} \leqq CA(2^j r)^{-n/p+n+\kappa} \quad \text{if } 2^j r \leqq 1 . \tag{4.9}$$

(The deduction is the same as in the proof of Proposition 4.1.)

First we shall prove (4.8). Since $\operatorname{supp} \tilde{a}_j(x, \cdot) \subset \{\xi \mid |\xi| \leqq 2^{j(1-\delta)+1}\}$ and $\|\tilde{a}_j\|_{\Lambda(\kappa,\kappa')} \leqq CA2^{jm}$, we can use Lemma 2.1 to obtain

$$\|\langle x-y\rangle^{\kappa'-\varepsilon}\tilde{K}_j(x, x-y)\|_{L_y^2} \leqq C_\varepsilon A 2^{j(m+(1-\delta)n/2)} \tag{4.10}$$

(C_ε independent of x and j). We take ε so small that $\kappa'-\varepsilon > n(1/p-1/2)$. Then we can derive (4.8) from (4.7), (4.10), and the estimate $\|g\|_{L^\infty} \leqq r^{-n/p}$ with the aid of Lemma 4.1.

Next we shall prove (4.9). We assume $2^j r \leqq 1$. Then, since $r \leqq 2^j r \leqq 1$, we have $\int g(x)dx = 0$ and hence

$$T_j^*g(2^{-j\delta}y) = \int_{|x|\leq 2^{j\delta}r} g(2^{-j\delta}x)(\tilde{K}_j(x, x-y) - \tilde{K}_j(0, -y))dx . \tag{4.11}$$

Using Lemma 2.2, we obtain, for x with $|x| \leqq 2^{j\delta}r$,

$$\|\langle y\rangle^{\kappa'-\varepsilon}(\tilde{K}_j(x, x-y) - \tilde{K}_j(0, -y))\|_{L_y^2} \leqq C_\varepsilon A 2^{j(m+(1-\delta)n/2)}((2^{j\delta}r)^\kappa + 2^j r) \tag{4.12}$$

(C_ε independent of x and j). We take ε so small that $\kappa'-\varepsilon > n(1/p-1/2)$. Then we can derive the estimate

$$\|T_j^*g(y)\|_{L^p(|y|>2r)} \leqq CA(2^j r)^{-n/p+n}((2^{j\delta}r)^\kappa + 2^j r)$$

from (4.11), (4.12), and the estimate $\|g\|_{L^\infty} \leqq r^{-n/p}$ with the aid of Lemma 4.1. The above estimate implies (4.9) since $\kappa<1$, $\delta<1$, and $2^j r \leqq 1$. This completes the proof of Proposition 4.4.

PROPOSITION 4.5. *If p, κ, and κ' satisfy the same assumptions as in Proposition* 4.4 *and if $m=m(n, 0, p)$, then $(\Psi_0^m(\kappa, \kappa'))^* \subset \mathcal{L}(h^p, X^p)$.*

We can prove this proposition by slightly modifying the proof of Proposition 4.4. We shall omit the details. (Cf. the paragraph just below Proposition 4.3.)

PROPOSITION 4.6. *If* $0<\delta<1$, $0<p<1$, $m=m(n,\delta,p)$, $\kappa>[n/p-n]+1$, *and* $\kappa'>n(1/p-1/2)$, *then* $(\Psi_\delta^m(\kappa,\kappa'))^*\subset\mathcal{L}(h^p,h^p)$.

PROOF. The proof of this proposition is very similar to that of Proposition 4.4. We shall only point out the necessary modifications. Suppose δ, p, m, κ, and κ' satisfy the assumptions in the proposition. Let $a\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$. We shall use the same notations as in the proofs of Theorem B and Proposition 4.4. By the same reasoning as in the proofs of Propositions 4.1 and 4.4, it is sufficient to show the estimate $\|T^*g\|_{L^p}\leqq CA$ for all those g such that $\operatorname{supp} g\subset\{x \mid |x|\leqq r\}$ and $\|g\|_{L^\infty}\leqq r^{-n/p}$ with some $r>0$ and $\int g(x)x^\alpha dx=0$ for $|\alpha|\leqq[n/p-n]$ if $r\leqq 1$ (if $r>1$, there are no moment conditions). We shall deduce the above estimate from the following three estimates:

$$\|T^*g(y)\|_{L^p(|y|\leqq 2r)}\leqq CA\,,$$

$$\|T_j^*g(y)\|_{L^p(|y|>2r)}\leqq CA(2^jr)^{-n/p+n}\,,$$

$$\|T_j^*g(y)\|_{L^p(|y|>2r)}\leqq CA(2^jr)^{N-n/p+n} \quad \text{if } r\leqq 1\,, \tag{4.13}$$

where $N=[n/p-n]+1$. Among the above three estimates, the first and the second ones can be proved by just the same reasoning as in the proof of Proposition 4.4. The last one, (4.13), can be proved as follows. If $r\leqq 1$, then we have

$$T_j^*g(2^{-j\delta}y)=\int g(2^{-j\delta}x)(\tilde{K}_j(x,x-y)-P(x))dx\,,$$

where $P(x)$ is any polynomial in x of degree not exceeding $N-1$. If we take as $P(x)$ Maclaurin's series of the function $x\mapsto\tilde{K}_j(x,x-y)$ up to the terms of degree $N-1$, then we have

$$T_j^*g(2^{-j\delta}y) \tag{4.14}$$
$$=\iint_{\substack{0<t<1\\ |x|\leqq 2^{j\delta}r}} g(2^{-j\delta}x)\sum_{|\alpha+\beta|=N}C_{\alpha,\beta}D_1^\alpha D_2^\beta\tilde{K}_j(tx,tx-y)x^{\alpha+\beta}(1-t)^{N-1}dtdx\,,$$

where the function $D_1^\alpha D_2^\beta\tilde{K}_j$ is defined by $D_1^\alpha D_2^\beta\tilde{K}_j(x,z)=\partial_x^\alpha\partial_z^\beta\tilde{K}_j(x,z)$. Now, by Propositions 2.6$\sim$2.9 and 2.4, we have

$$\|(i\xi)^\beta\partial_x^\alpha\tilde{a}_j(x,\xi)\|_{\Lambda(\kappa-|\alpha|,\kappa')}\leqq C_\beta A2^{j(m+(1-\delta)|\beta|)}$$

if $|\alpha|<\kappa$. From this and the fact that $\operatorname{supp}\tilde{a}_j(x,\cdot)\subset\{\xi|\,|\xi|\leqq 2^{j(1-\delta)+1}\}$, we obtain the estimate

$$\|\langle z\rangle^{\kappa'-\varepsilon}\partial_x^\alpha\partial_z^\beta\tilde{K}_j(x,z)\|_{L_z^2}\leqq C_\varepsilon A2^{j(m+(1-\delta)(N+n/2))} \tag{4.15}$$

for α, β with $|\alpha+\beta|=N$ (we used Lemma 2.1). Now (4.13) follows from (4.14), (4.15), and the estimate $\|g\|_{L^\infty}\leqq r^{-n/p}$ with the aid of Lemma 4.1. This completes the proof of Proposition 4.6.

PROPOSITION 4.7. *If p, κ, and κ' satisfy the same assumptions as in Proposition* 4.6 *and if $m=m(n,0,p)$, then $(\Psi_0^m(\kappa,\kappa'))^*\subset\mathcal{L}(h^p,X^p)$.*

We shall omit the proof of this proposition; it is a slight modification of the proof of Proposition 4.6. (Cf. also the paragraph just below Proposition 4.3.)

Now we go to the proofs of the theorems in Section 3. First, observe that (3)'s and (4)'s of Theorems 3.1 and 3.2 can be derived from (1)'s and (2)'s of the theorems by the use of the duality between L^p and L^q, where $1/p+1/q=1$, or the duality between h^1 and bmo. (As for the latter duality, see Goldberg [11].) Hence we omit the direct proofs of (3)'s and (4)'s of Theorems 3.1 and 3.2.

PROOF OF (1′) AND (2) OF THEOREM 3.1. The claim in (2) for the case $p=2$ is the same as Theorem A. We shall prove the claims in (1′) and (2) for $0<p<2$. Let $0<p<2$ and let m, κ, and κ' be as mentioned in (1′) or (2) of Theorem 3.1. Let $a\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ and set $A=\|a\|_{m,\delta,\kappa,\kappa'}$. Take real numbers r, t, and B such that $0<r<p$, $0<t<1$, $1/p=(1-t)/r+t/2$, $\kappa'+Bt>n/r+1$, and $\kappa'-B(1-t)>n/2$. (Observe that this is possible if we choose r sufficiently small.) For complex numbers z, set

$$a_z(x,\xi)=e^{(z-t)^2}\langle\xi\rangle^{-m'(z-t)}\sum_{j=0}^{\infty}\theta_j'(\xi)a_{z,j}(x,\xi)\,,$$

where $m'=m(n,\delta,r)$ and

$$a_{z,j}(x,\xi)=[\langle D_\eta\rangle^{B(z-t)}\tilde{a}_j(2^{j\delta}x,\eta)]_{\eta=2^{-j\delta}\xi}\,.$$

(As for the operator $\langle D_\eta\rangle^{B(z-t)}$, see Proposition 2.3.) Then, using the propositions in Section 2, we see that

$$\|a_z\|_{m',\delta,\kappa,\kappa'+Bt}\leqq CA \quad \text{if } \operatorname{Re}z=0$$

and

$$\|a_z\|_{0,\delta,\kappa,\kappa'-B(1-t)}\leqq CA \quad \text{if } \operatorname{Re}z=1\,.$$

Hence, by Proposition 4.1 (the case $0<\delta<1$) or 4.3 (the case $\delta=0$) and by Theorem A, we have

$$\|a_z(X, D)\|_{h^r\to L^r}\leqq CA \quad \text{if } \operatorname{Re} z=0$$

and

$$\|a_z(X, D)\|_{L^2\to L^2}\leqq CA \quad \text{if } \operatorname{Re} z=1 .$$

We now apply to the family $\{a_z(X, D)\}$ the interpolation theorem for analytic families of operators (see [21] and [2 ; Section 3]) and conclude that the operator norm of $a_t(X, D)$ from h^p (if $p\leqq 1$) or L^p (if $1<p<2$) to L^p does not exceed CA. This proves the desired result since $a_t(X, D)=a(X, D)$.

PROOF OF (1) OF THEOREM 3.1. Suppose p, m, κ, and κ' satisfy the assumptions of (1) of Theorem 3.1. Let $a\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$. Take q, r, t, m', B, and B' such that $0<q<p$, $1<r<2$, $0<t<1$, $1/p=(1-t)/q+t/r$, $m'=n(1-\delta)(1/q-1/r)$, $\kappa+Bt>(n/q+n+4)\delta/(1-\delta)+3n/2+3$, $\kappa-B(1-t)>n/2$, $\kappa'+B't>n/q+2$, and $\kappa'-B'(1-t)>n/r$. (Observe that this is possible if we take q sufficiently small and r sufficiently near to 1.) Set

$$a_z(x, \xi)=e^{(z-t)^2}\langle\xi\rangle^{m'(z-t)}\sum_{j=0}^{\infty}\theta'_j(\xi)a_{z,j}(x, \xi) ,$$

where

$$a_{z,j}(x, \xi)=[\langle D_y\rangle^{B(z-t)}\langle D_\eta\rangle^{B'(z-t)}\tilde{a}_j(y, \eta)]_{y=2^{j\delta}x,\, \eta=2^{-j\delta}\xi} .$$

Then, using the propositions in Section 2, Proposition 4.2, and (2) of Theorem 3.1, we see that

$$\|a_z(X, D)\|_{h^q\to h^q}\leqq CA \quad \text{if } \operatorname{Re} z=0$$

and

$$\|a_z(X, D)\|_{L^r\to L^r}\leqq CA \quad \text{if } \operatorname{Re} z=1 ,$$

where $A=\|a\|_{m,\delta,\kappa,\kappa'}$. We now apply to the family $\{a_z(X, D)\}$ the interpolation theorem for analytic families of operators and conclude that the operator norm of $a_t(X, D)=a(X, D)$ from h^p to h^p does not exceed CA. This completes the proof.

The assertions (1) and (2) of Theorem 3.2 can be derived from Propositions 4.4, 4.5, 4.6, and 4.7 and from Theorem A by means of the interpolation. Since this argument is very similar to the proofs of (1), (1′), and (2) of Theorem 3.1, we shall omit the details.

PROOF OF THEOREM 3.3. This theorem can be derived from Proposition 4.3 and Theorem A by means of the interpolation. The argument is similar to the proof of (1′) and (2) of Theorem 3.1. The only difference is that we must deal with the nonlinear operator $f \mapsto N(f)$; this point can be got around in the following way. Set $\delta=0$ and construct the family $\{a_z\}$ in the same way as in the proof of (1′) and (2) of Theorem 3.1. Let $\{\phi_x \mid x \in \boldsymbol{R}^n\}$ be a family of functions in $C_0^\infty(\boldsymbol{R}^n)$ such that $\operatorname{supp}\phi_x \subset \{y \mid |y| \leqq r_x\}$ and $\|\phi_x\|_{L^\infty} \leqq r_x^{-n}$ for some r_x satisfying $0<r_x\leqq 1$. Define the operator T_z by

$$T_z f(x) = \int \phi_x(x-y) a_z(X, D) f(y) dy .$$

Then $N(a_z(X, D)f)(x) = \sup |T_z f(x)|$, where the supremum is taken over all those families $\{\phi_x\}$ satisfying the above conditions. Applying the interpolation theorem for analytic families of operators to the family $\{T_z\}$, which is a family of linear operators, we can derive Theorem 3.3 from Proposition 4.3 and Theorem A. This completes the proof.

PROOF OF THEOREM 3.4. If p, m, κ, and κ' satisfy the assumptions of (2) of Theorem 3.2 with $\delta=0$, then we have

$$\|N(a(X, D)^* g)\|_{L^p} \leqq C \|a\|_{m, 0, \kappa, \kappa'} \|g\|_{L^p}$$

for all $a \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ and all $g \in \mathcal{S}(\boldsymbol{R}^n)$; this follows from (2) of Theorem 3.2 and the Hardy-Littlewood maximal theorem. From this fact and Proposition 4.7, we can derive Theorem 3.4 by means of the interpolation. The way to use the interpolation is similar to that in the proofs of (1), (1′), and (2) of Theorem 3.1 and in the proof of Theorem 3.3; we shall omit the details.

§ 5. Negative results.

In this section, we give some negative results, which will show that the results in Section 3 are sharp in a sense.

We define $m(n; \delta, p)$ in the same way as in Theorem 3.1. We extend the definition of H^p as follows: if $1<p<\infty$, we set $H^p=L^p$ and $\| \ \|_{H^p} = \| \ \|_{L^p}$. For $a \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$, we define $\|a(X, D)\|_{H^p \to L^p}$ as the supremum of $\|a(X, D)f\|_{L^p} / \|f\|_{H^p}$ for all those $f \in \mathcal{S}(\boldsymbol{R}^n) \cap H^p$, $f \neq 0$.

In order to prove the results in this section, we shall frequently use the lemmas given below. The idea of the first lemma is due to Coifman and Meyer [6; p. 31].

LEMMA 5.1. *Suppose that* $0<\delta<1$, $0<p<\infty$, $m \in \boldsymbol{R}$, $\kappa>0$, $\kappa'>0$, $B>0$, *and that the inequality*

$$\|a(X, D)\|_{H^p \to L^p} \leqq B\|a\|_{m, \delta, \kappa, \kappa'}$$

holds for all $a \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$. *Then the following hold.*

(i) *If* $b \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ *and there exists* $s>1$ *for which* $\operatorname{supp} b(x, \cdot) \subset \{\xi \mid s/10 \leqq |\xi| \leqq 10s\}$ *for all* $x \in \boldsymbol{R}^n$, *or if* $b \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ *and* $\operatorname{supp} b(x, \cdot) \subset \{\xi \mid |\xi| \leqq 10\}$ *for all* $x \in \boldsymbol{R}^n$, *then*

$$\|b(X, D)\|_{H^p \to L^p} \leqq CB\|b\|_{m/(1-\delta), 0, \kappa, \kappa'},$$

where C is a constant depending only on n, m, δ, κ, *and* κ'.

(ii) *The inequality*

$$\|a(X, D)\|_{H^p \to L^p} \leqq C_\varepsilon B\|a\|_{m/(1-\delta)-\varepsilon, 0, \kappa, \kappa'}$$

holds for all $\varepsilon>0$ *and all* $a \in \mathcal{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$, *where* C_ε *is a constant depending only on* ε, *n, m,* δ, *p,* κ, *and* κ'.

PROOF. Proof of (i). Suppose that $s>1$ and $\operatorname{supp} b(x, \cdot) \subset \{\xi \mid s/10 \leqq |\xi| \leqq 10s\}$ for all $x \in \boldsymbol{R}^n$ or that $s=1$ and $\operatorname{supp} b(x, \cdot) \subset \{\xi \mid |\xi| \leqq 10s\}$. Set $t=s^{1/(1-\delta)}$. By Propositions 2.8 and 2.9, we have

$$\begin{aligned}\|b(x, \xi)\|_{m/(1-\delta), 0, \kappa, \kappa'} &\approx \|b(x, \xi)\|_{\Lambda(\kappa, \kappa'; t^m, 1, 1)} \\ &= \|b(t^\delta x, t^{-\delta}\xi)\|_{\Lambda(\kappa, \kappa'; t^m, t^{-\delta}, t^\delta)} \\ &\approx \|b(t^\delta x, t^{-\delta}\xi)\|_{m, \delta, \kappa, \kappa'}.\end{aligned}$$

Hence, from the assumption of the lemma, we have

$$\|b(t^\delta X, t^{-\delta}D)\|_{H^p \to L^p} \leqq CB\|b(x, \xi)\|_{m/(1-\delta), 0, \kappa, \kappa'}.$$

On the other hand, the $H^p \to L^p$ operator norm of $b(t^\delta X, t^{-\delta}D)$ is equal to that of $b(X, D)$. (This can be seen easily from the fact that the operation $f \mapsto u^{n/p}f(u\cdot)$, where $u>0$, does not change the norms in H^p and L^p.) Hence the above inequality implies the desired result.

Proof of (ii). Set $a_j(x, \xi)=a(x, \xi)\theta_j(\xi)$. By (i), it holds that

$$\|a_j(X, D)\|_{H^p \to L^p} \leqq CB\|a_j\|_{m/(1-\delta), 0, \kappa, \kappa'}.$$

On the other hand, by Propositions 2.8, 2.9, and 2.7, we have

$$\begin{aligned}\|a_j\|_{m/(1-\delta), 0, \kappa, \kappa'} &\approx \|a_j\|_{\Lambda(\kappa, \kappa'; 2^{jm/(1-\delta)}, 1, 1)} \\ &= 2^{-j\varepsilon}\|a_j\|_{\Lambda(\kappa, \kappa', 2^{j(m/(1-\delta)-\varepsilon)}, 1, 1)} \\ &\approx 2^{-j\varepsilon}\|a_j\|_{m/(1-\delta)-\varepsilon, 0, \kappa, \kappa'} \\ &\leqq C_\varepsilon 2^{-j\varepsilon}\|a\|_{m/(1-\delta)-\varepsilon, 0, \kappa, \kappa'}.\end{aligned}$$

Hence

$$\|a_j(X, D)\|_{H^p\to L^p}\leqq C_\varepsilon B2^{-j\varepsilon}\|a\|_{m/(1-\delta)-\varepsilon, 0, \kappa, \kappa'} .$$

From this, follows the desired inequality. This completes the proof of Lemma 5.1.

LEMMA 5.2. *Let $0<a<1$ and let b and c be real numbers. Let K be the inverse Fourier transform of the function*

$$(1-\theta(\xi/2))|\xi|^{-b}(\log|\xi|)^c\exp(i|\xi|^a) .$$

Then K is smooth in $\boldsymbol{R}^n\backslash\{0\}$ and, if $b<n-na/2$, we have

$$|K(x)|\sim A|x|^{(b-n+na/2)/(1-a)}(\log|x|^{-1})^c \quad as\ x\to 0 ,$$

where A is a positive constant depending only on n, a, b, and c.

As for a proof of this lemma, cf. Wainger [25; Part II] or [16; § 5] or [17; § 3].

In the assertions (1)$\sim$(7) below, δ denotes a real number such that $0\leqq\delta<1$. We shall write as $\Psi_\delta^m(\kappa, \infty)\not\subset\mathcal{L}(Y, Z)$ if the inclusion $\Psi_\delta^m(\kappa, \kappa')\subset\mathcal{L}(Y, Z)$ (in the sense defined in Section 3) does not hold for any $\kappa'>0$. We shall also use the notations $\Psi_\delta^m(\infty, \kappa')$, $(\Psi_\delta^m(\kappa, \infty))^*$, and $(\Psi_\delta^m(\infty, \kappa'))^*$, together with the notation $\not\subset\mathcal{L}(Y, Z)$, in the similar meaning.

(1) *If $0<p\leqq 2$, $m=m(n, \delta, p)$, and $\kappa<n/2$, then $\Psi_\delta^m(\kappa, \infty)\not\subset\mathcal{L}(H^p, L^p)$.*

PROOF. Suppose p, m, and κ satisfy the above assumptions and $\Psi_\delta^m(\kappa, \kappa')\subset\mathcal{L}(H^p, L^p)$ with some $\kappa'>0$. If $0<\delta<1$, then we use Lemma 5.1 to see that the inequality

$$\|b(X, D)\|_{H^p\to L^p}\leqq C\|b\|_{m(n, 0, p), 0, \kappa, \kappa'} \tag{5.1}$$

holds for all those $b\in\mathcal{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ satisfying $\operatorname{supp} b(x, \cdot)\subset\{\xi \mid s/10\leqq|\xi|\leqq 10s\}$ with some $s>1$. If $\delta=0$, the same is *a fortiori* true. For $s>1$, set

$$b_s(x, \xi)=\theta_1(\xi/s)\langle\xi\rangle^{m(n, 0, p)-\kappa}\exp(-|x|^2-ix\xi)$$

and

$$f_s(x)=s^{n/p}\mathcal{F}^{-1}\theta_1(sx) .$$

It holds that $\{b_s\}_{s>1}$ is bounded in $S_0^{m(n, 0, p)}(\kappa, \kappa')$ and that the H^p-norm of f_s does not depend on s. Hence (5.1) implies that the L^p-norms of $b_s(X, D)f_s$ are bounded for $s>1$. On the other hand, we have

$$b_s(X, D)f_s(x) = \exp(-|x|^2)s^{n/p-n}(2\pi)^{-n}\int\langle\xi\rangle^{m(n,0,p)-\kappa}(\theta_1(\xi/s))^2 d\xi$$
$$\geqq Cs^{n/2-\kappa}\exp(-|x|^2)$$

and hence $\|b_s(X, D)f_s\|_{L^p}\to\infty$ as $s\to\infty$ since $\kappa<n/2$. This contradiction proves (1).

(2) *Let* $0<p\leqq 2$. *Then* $\Psi_\delta^m(\infty, n/p)\not\subset\mathcal{L}(H^p, L^p)$ *for any* $m\in\boldsymbol{R}$.

PROOF. This can be seen from the following example:

$$a(x, \xi) = \langle x\rangle^{-n/p}\exp(-ix\xi-|\xi|^2)\,.$$

This belongs to $S_\delta^m(\kappa, n/p)$ for all $m\in\boldsymbol{R}$ and all $\kappa>0$. On the other hand, $a(X, D)$ is not bounded from H^p to L^p since we have

$$a(X, D)f(x) = \langle x\rangle^{-n/p}(2\pi)^{-n/2}\int\exp(-|\xi|^2)\mathcal{F}f(\xi)d\xi$$

and hence $a(X, D)f$ does not belong to L^p for generic f.

(3) *If* $0<p<1$, $m\in\boldsymbol{R}$, *and* $\kappa<n/p-n$, *then* $(\Psi_\delta^m(\kappa, \infty))^*\not\subset\mathcal{L}(H^p, L^p)$.

PROOF. This can be seen from the following example. Let p and κ satisfy the above assumptions. Take a function ϕ on $\boldsymbol{R}^n$ such that $\phi\in\Lambda(\kappa)$, $\phi\notin\Lambda(n/p-n)$, and $\operatorname{supp}\phi\subset\{x \mid |x|\leqq 1\}$. Set

$$a(x, \xi) = \phi(x)\exp(-ix\xi-|\xi|^2)\,.$$

This belongs to $S_\delta^m(\kappa, \kappa')$ for all $m\in\boldsymbol{R}$ and all $\kappa'>0$. On the other hand, the operator $a(X, D)^*$, which is given by

$$a(X, D)^*g(y) = \int g(x)\phi(x)dx(4\pi)^{-n/2}\exp(-|y|^2/4)\,,$$

is not bounded from H^p to L^p since the linear functional $g\mapsto\int g(x)\phi(x)dx$ is not bounded in H^p. (This last fact can be seen from the duality between H^p and the Lipschitz space; see [7; Theorem B, p. 593], [15], or [24; Section 2.11].)

(4) *The condition*

$$|\partial_\xi^\alpha a(x, \xi)|\leqq C_\alpha\langle\xi\rangle^{-n(1-\delta)/2-\delta|\alpha|} \quad \textit{for all } \alpha$$

does not imply that the operator $a(X, D)^*$ *is bounded from* H^1 *to* L^1.

PROOF. As for the case $\delta=0$, see [18; Section 5]. We assume $0<\delta<1$. We shall give an example of $a(x, \xi)$ which satisfies the above condition and for which $a(X, D)^*$ is not bounded from H^1 to L^1. The example is

$$a(x, \xi)=\chi\{x_1>0\}(1-\theta(\xi))|\xi|^{-n(1-\delta)/2}\exp(i|\xi|^{1-\delta}),$$

where $\chi\{x_1>0\}$ denotes the defining function of the set $\{x\in \boldsymbol{R}^n \mid x_1>0\}$. It is easy to check that this symbol satisfies the condition mentioned in (4). The operator $a(X, D)^*$ is given by

$$a(X, D)^*g(y)=\int_{x_1>0} g(x)K(x-y)dx,$$

where K is the inverse Fourier transform of the function $(1-\theta(\xi))|\xi|^{-n(1-\delta)/2}\times\exp(i|\xi|^{1-\delta})$. Take a function $f\in C_0^\infty(\boldsymbol{R}^n)$ such that $\mathrm{supp} f\subset\{x| x_1>0\}$ and $\int f(x)dx=1$, and set $g_r(x)=r^{-n}(f(x/r)-f(-x/r))$ for $r>0$. Then $\{g_r\}$ is bounded in H^1. It holds that, as r tends to 0, the function

$$a(x, D)^*g_r(y)=r^{-n}\int f(x/r)K(x-y)dx$$

tends to $K(-y)$ in the sense of distribution. Hence, if $a(X, D)^*$ is bounded from H^1 to L^1, then it follows that $\{a(X, D)^*g_r\}$ is bounded in L^1 and hence that the distribution K is a bounded complex measure. But this last fact does not hold since $|K(x)|\sim A|x|^{-n}$ as $x\to 0$ (see Lemma 5.2). Hence $a(X, D)^*$ is not bounded from H^1 to L^1. This proves (4).

(5) *If* $1<p\leqq 2$, $m=m(n, \delta, p)$, *and* $\kappa<n-n/p$, *then* $(\Psi_\delta^m(\kappa, \infty))^*\not\subset \mathcal{L}(L^p, L^p)$.

PROOF. Suppose p and m satisfy the above assumptions, $\kappa>0$, $\kappa'>0$, and $(\Psi_\delta^m(\kappa, \kappa'))^*\subset\mathcal{L}(L^p, L^p)$. We shall deduce that $\kappa\geqq n-n/p$. In order to do this, we shall show the following two assertions. First, $(\Psi_0^{m'}(\kappa, \kappa'))^*\subset \mathcal{L}(L^p, L^p)$ for all $m'<-n(1/p-1/2)$. This is clear if $\delta=0$; if $0<\delta<1$, we can deduce it by using Lemma 5.1. Secondly, if $(\Psi_0^{m'}(\kappa, \kappa'))^*\subset\mathcal{L}(L^p, L^p)$, then $m'+n/2\leqq\kappa$. In order to show this, let $0<c<1$ and consider the symbol

$$a(x, \xi)=(1-\theta(\xi))|\xi|^{m'-\kappa}\exp(-|x|^2-ix\xi+i|\xi|^c).$$

The operator $a(X, D)^*$ is given by

$$a(X, D)^*g(y)=\int g(x)\exp(-|x|^2)dxK(-y),$$

where K denotes the inverse Fourier transform of the function $(1-\theta(\xi))|\xi|^{m'-\kappa}\exp(i|\xi|^c)$. If $a(X, D)^*$ is bounded from L^p to L^p, then K belongs to L^p and hence

$$-m'+\kappa-n+nc/2>-n(1-c)/p \tag{5.2}$$

(we used Lemma 5.2). On the other hand, for all c satisfying $0<c<1$, the symbol $a(x,\xi)$ belongs to $\Psi_0^{m'}(\kappa,\kappa')$. Hence, if $(\Psi_0^{m'}(\kappa,\kappa'))^*\subset\mathcal{L}(L^p,L^p)$, then (5.2) holds for all c satisfying $0<c<1$ and hence $-m'+\kappa-n/2\geqq 0$. This shows the second assertion. Combining the two assertions, we have $\kappa\geqq n-n/p$. This proves (5).

(6) *If* $0<p\leqq 1$, $m\in\boldsymbol{R}$, *and* $\kappa'<n(1/p-1/2)$, *then* $(\Psi_\delta^m(\infty,\kappa'))^*\not\subset\mathcal{L}(H^p,L^p)$.

PROOF. This can be seen from the following counter example. Suppose p, m, and κ' satisfy the above assumptions. Take positive numbers a and b such that $-b\leqq m$, $-b+\kappa'(a-1)\leqq m-\kappa'\delta$, and $b<na(1/p-1/2)$, and set

$$a(\xi)=(1-\theta(\xi))|\xi|^{-b}\exp(i|\xi|^a)\,.$$

This belongs to $S_\delta^m(\kappa,\kappa')$ for all $\kappa>0$ and yet $a(D)^*=a(-D)$ is not bounded from H^p to L^p; as for the latter fact, see [16].

(7) *If* $1<p\leqq 2$, $m\in\boldsymbol{R}$, *and* $\kappa'<n/2$, *then* $(\Psi_\delta^m(\infty,\kappa'))^*\not\subset\mathcal{L}(L^p,L^p)$.

PROOF. Suppose p and m satisfy the above assumptions, $\kappa>0$, $\kappa'>0$, and $(\Psi_\delta^m(\kappa,\kappa'))^*\subset\mathcal{L}(L^p,L^p)$. We shall show $\kappa'\geqq n/2$. By duality and Lemma 5.1, it holds that

$$\Psi_0^{m'}(\kappa,\kappa')\subset\mathcal{L}(L^r,L^r) \tag{5.3}$$

for all $m'<m/(1-\delta)$ and for $r=p/(p-1)$. Consider now the following functions:

$$f_t(x)=(1+it)^{-n/2}\exp(-|x|^2/2(1+it))\,,$$
$$a_t(x,\xi)=\theta(x/t)\exp(-ix\xi-(1-it)|\xi|^2/2)\,,$$

where t is a large positive number. We have

$$\begin{aligned}a_t(X,D)f_t(x)&=\theta(x/t)(2\pi)^{-n/2}\int\exp(-|\xi|^2)d\xi\\&=2^{-n/2}\theta(x/t)\,.\end{aligned}$$

It is easy to see that

$$\|f_t\|_{L^r}\sim At^{-n/2+n/r}\qquad\text{as } t\to\infty$$

and

$$\|a_t(X,D)f_t\|_{L^r}=Bt^{n/r}\,,$$

where A and B are positive constants depending only on n and r. As for a_t, it holds that

$$\|a_t\|_{m',0,\kappa,\kappa'}\leqq Ct^{\kappa'}\qquad\text{if } t>1\,,$$

where C is a positive constant depending only on n, m', κ, and κ' (this can be shown by elementary calculations if κ and κ' are integers; the general case can be obtained with the aid of Proposition 2.5). From these estimates and (5.3), we obtain $t^{n/r}=O(t^{\kappa'-n/2+n/r})$ as $t\to\infty$, which implies $\kappa'\geqq n/2$. This proves (7).

The following two examples show that Theorems 3.3 and 3.4 cannot be directly generalized to the case $0<\delta<1$.

(8) *Let* $0<\delta<1$, $0<p<1$, $m=m(n,\delta,p)$, *and* $0<\varepsilon<n\delta(1/p-1)$. *Set*

$$a(x,\xi)=a(\xi)=(1-\theta(\xi))|\xi|^m\exp(i|\xi|^{1-\delta})$$

and

$$f=\mathcal{F}^{-1}((1-\theta(\xi))|\xi|^{n/p-n-\varepsilon}).$$

Then $a\in S_\delta^m(\kappa,\kappa')$ *for all* $\kappa,\kappa'>0$ *and* $f\in H^p$, *and yet* $a(D)f$ *is not locally integrable.*

PROOF. The fact that $a\in S_\delta^m(\kappa,\kappa')$ for all $\kappa,\kappa'>0$ is easy to check. Set $f_j=\mathcal{F}^{-1}(\theta_j(\xi)|\xi|^{n/p-n-\varepsilon})$ for $j\geqq 1$. Then $f_j(x)=2^{(j-1)(n/p-\varepsilon)}f_1(2^{j-1}x)$ and $\|f_j\|_{H^p}=2^{-\varepsilon(j-1)}\|f_1\|_{H^p}$. Hence $f=\sum_{j=1}^\infty f_j$ belongs to H^p. On the other hand, Lemma 5.2 shows that

$$|a(D)f(x)|\sim A|x|^{\varepsilon/\delta-n/p}\quad\text{as } x\to 0.$$

Hence, since $\varepsilon/\delta-n/p<-n$, $a(D)f$ is not locally integrable. This completes the proof.

(9) *Let* $0<\delta<1$, $m=-n(1-\delta)/2$, *and* $1<b\leqq 2$. *Set*

$$a(x,\xi)=a(\xi)=(1-\theta(\xi))|\xi|^m\exp(i|\xi|^{1-\delta})$$

and

$$f=\mathcal{F}^{-1}((1-\theta(\xi/4))(\log|\xi|)^{-b}).$$

Then $a\in S_\delta^m(\kappa,\kappa')$ *for all* $\kappa,\kappa'>0$ *and* $f\in H^1$, *and yet* $N(a(D)f)$ *is not locally integrable.*

PROOF. The fact that $a\in S_\delta^m(\kappa,\kappa')$ for all $\kappa,\kappa'>0$ is easy to check. We set $f_j=\mathcal{F}^{-1}(\theta_j(\xi)(\log|\xi|)^{-b})$ for $j\geqq 3$. Then

$$\begin{aligned}\|f_j\|_{H^1}&=\|\mathcal{F}^{-1}(\theta_j(2^{j-2}\xi)(\log(2^{j-2}|\xi|))^{-b})\|_{H^1}\\&=(j-2)^{-b}\|\mathcal{F}^{-1}(\theta_2(\xi)(\log 2+(j-2)^{-1}\log|\xi|)^{-b})\|_{H^1}\\&\leqq Cj^{-b}.\end{aligned}$$

Hence $f=\sum_{j=3}^\infty f_j$ belongs to H^1 since $b>1$. On the other hand, Lemma 5.2 shows that

$$|a(D)f(x)| \sim A|x|^{-n}(\log|x|^{-1})^{-b} \qquad \text{as } x \to 0\,.$$

Hence, if $|x|$ is small, we have

$$\begin{aligned} N(a(D)f)(x) &\geqq 2^{-1}A(2|x|)^{-n}\int_{|y|\leq|x|}|y|^{-n}(\log|y|^{-1})^{-b}dy \\ &= cA(b-1)^{-1}|x|^{-n}(\log|x|^{-1})^{-b+1}\,, \end{aligned}$$

where c is a positive constant depending only on n. From this, we see that $N(a(D)f)$ is not locally integrable since $b \leqq 2$. This completes the proof.

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Department of Mathematics
Hitotsubashi University
Kunitachi, Tokyo
186 Japan