J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
34 (1987), 111-125.

Volterra integro-differential equations of parabolic type of higher order in t

Dedicated to Professor Seizô Itô on his sixtieth birthday

By Hiroki TANABE

1. Introduction.

This paper is concerned with the initial-boundary value problem of the parabolic integro-differential equation of higher order in t:

$$\sum_{k=0}^{l} A_{l-k}(x, t, D_x) D_t^k u(x, t)$$

$$=\int_0^t B(x, t, s, D_x) u(x, s) ds + f(x, t) \qquad \Omega \times (0, T] \tag{1.1}$$

$$B_j(x, D_x) u(x, t) = 0\,, \qquad j=1, \cdots, m \qquad \partial\Omega \times (0, T] \tag{1.2}$$

$$(D_t^j u)(x, 0) = u_j\,, \qquad j=0, 1, \cdots, l-1 \qquad \Omega\,. \tag{1.3}$$

Here $A_j(x, t, D_x)$, $j=1, \cdots, l$, and $B(x, t, s, D_x)$ are linear differential operators in x with coefficients defined in $\bar{\Omega}\times[0, T]$ and $\bar{\Omega}\times\{(t, s): 0 \leqq s \leqq t \leqq T\}$ respectively, and $A_0(x, t, D_x)=1$. $\{B_j(x, D_x)\}_{j=1}^m$ is a system of linear differential operators with coefficients defined on $\partial\Omega$ which do not contain derivatives in t and are independent of t. The operator in the left side of (1.1) is assumed to be parabolic in the sense of Petrowsky.

In case $l=1$, when the boundary conditions depend on t, the problem (1.1)-(1.3) was solved by J. Prüss [4] in $L^p(\Omega)$, $1<p<\infty$, as an application of his general result on abstract equations.

We plan to solve the problem (1.1)-(1.3) in $L^p(\Omega)$, $1<p<\infty$, by constructing the fundamental solution $W(t, s)$ as in [4]:

$$\sum_{k=0}^{l} A_{l-k}(t) D_t^k W(t, s) = \int_s^t B(t, \sigma) W(\sigma, s) d\sigma\,, \tag{1.4}$$

$$D_t^j W(t, s) = 0 \quad \text{at} \quad t=s \qquad \text{for } j=0, \cdots, l-2 \tag{1.5}$$

$$D_t^{l-1} W(t, s) = I \quad \text{at} \quad t=s\,, \tag{1.6}$$

This work was supported by Grant-in-Aid for Scientific Research 61460003.

where $A_{l-k}(t)=A_{l-k}(x, t, D_x)$ for $k=1, \cdots, l-1$, $A_0(t)=I$, $A_l(t)$ is the realization of $A_l(x, t, D_x)$ in $L^p(\Omega)$ under the boundary conditions $B_j(x, D_x)u|_{\partial\Omega}=0$, $j=1, \cdots, m$, and $B(t, s)=B(x, t, s, D_x)$. $B(t, s)$ has the same order as $A_l(t)$, and the integral of the right side of (1.4) should be understood as an improper integral:

$$\int_s^t B(t, \sigma) W(\sigma, s) d\sigma = \lim_{\varepsilon \to +0} \int_{s+\varepsilon}^t B(t, \sigma) W(\sigma, s) d\sigma .$$

Once the fundamental solution $W(t, s)$ is constructed, the unique solution of the problem (1.1)-(1.3) can be represented as

$$u(t)=\sum_{j=0}^{l-1} u_j(t)+\int_0^t W(t, s) f(s) ds \tag{1.7}$$

$$u_j(t)=\frac{t^j}{j!} u_j+\int_0^t \int_s^t W(t, \tau) B(\tau, s) \frac{s^j}{j!} u_j d\tau ds \tag{1.8}$$

$$-\int_0^t W(t, s) \sum_{k=0}^{j} \frac{s^k}{k!} A_{l-j+k}(s) u_j ds \qquad \text{for } j=0, \cdots, l-2$$

$$u_{l-1}(t)=W(t, 0) u_{l-1} \tag{1.9}$$

provided that $u_j \in D(A_l) \equiv D(A_l(t))$ for $j=0, \cdots, l-2$, $u_{l-1} \in L^p(\Omega)$, and $f(t)$ is a Hölder continuous function with values in $L^p(\Omega)$.

We shall begin with the construction of the fundamental solution $U(t, s)$ to the equation without the integral term:

$$\sum_{k=0}^{l} A_{l-k}(t) D_t^k u(t)=f(t) . \tag{1.10}$$

In [3] J. E. Lagnese treated the equation (1.10) when A_{l-k} are independent of t. He reduced the equation to a system of first order in t, while we follow another method which is a direct extension of that used in the construction of the fundamental solution (or evolution operator) of parabolic evolution equations of first order in t (Section 5.2 of [7]). In this argument an essential role is played by the weighted elliptic estimates of S. Agmon and L. Nirenberg [1]. Finally following the method of J. Prüss [4] we construct the fundamental solution to the original integro-differential equation.

2. Assumptions and Theorems.

Let Ω be a bounded domain in R^n, $n>1$, with boundary $\partial\Omega$. We put $D_x=(D_1, \cdots, D_n)$, $D_i=\partial/\partial x_i$, $D_t=\partial/\partial t$, $D_x^\alpha=D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ for a multi-integer $\alpha=$

$(\alpha_1, \cdots, \alpha_n)$, $\alpha_i \geqq 0$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We are interested in operators

$$\mathfrak{A}(x, t, D_x, D_t) = \sum_{k=0}^{l} A_{l-k}(x, t, D_x) D_t^k \qquad \text{and} \quad B(x, t, s, D_x) \tag{2.1}$$

where $A_j(x, t, D_x)$ is a linear differential operator in x with coefficients defined in $\bar{\Omega} \times [0, T]$, and $B(x, t, s, D_x)$ is a linear differential operator in x with coefficients defined in $\bar{\Omega} \times \bar{\Delta}$, where $\bar{\Delta} = \{(t, s) : 0 \leqq s \leqq t \leqq T\}$ is the closure of $\Delta = \{(t, s) : 0 \leqq s < t \leqq T\}$.

Let s_j be the order of A_j. It is assumed that

$$s_l = 2m , \quad s_j \leqq 2mj/l , \qquad j = 1, \cdots, l-1$$

for some integer $m \geqq 1$ and that m and l are related by the condition $2m/l = d$, an even integer. The order of B is assumed to be $2m$.

In addition to (2.1) there are m linear differential boundary operators $\{B_j(x, D_x)\}_{j=1}^m$ of respective orders $m_j \leqq 2m-1$ which do not contain D_t and are independent of t.

We denote by $A_j^{\#}(x, t, D_x)$ the sum of terms of $A_j(x, t, D_x)$ which are of order dj, and put

$$\mathfrak{A}^{\#}(x, t, D_x, D_t) = \sum_{k=0}^{l} A_{l-k}^{\#}(x, t, D_x) D_t^k .$$

Similarly $B_j^{\#}(x, D_x)$ is the sum of terms of $B_j(x, D_x)$ which are of order m_j.

We assume

(A.1) $\mathfrak{A}(x, t, D_x, D_t)$ is parabolic in the sense of Petrowsky, i. e. for all real n-vectors $\xi \neq 0$, all $(x, t) \in \bar{\Omega} \times [0, T]$ and all complex numbers λ with $\operatorname{Re} \lambda \geqq 0$, $\mathfrak{A}^{\#}(x, t, i\xi, \lambda) \neq 0$.

(A.2) At any point (x, t) of $\partial\Omega \times [0, T]$ let ν be the normal to $\partial\Omega$ at x and ξ be parallel to $\partial\Omega$ at x or $\xi = 0$. Let λ be any complex number with $\operatorname{Re} \lambda \geqq 0$. Then if $(\xi, \lambda) \neq 0$, the polynomials in s : $B_j^{\#}(x, \xi + s\nu)$, $1 \leqq j \leqq m$, are linearly independent modulo the polynomial $\Pi_{k=1}^m (s - s_k^+(\xi, \lambda))$ where $s_k^+(\xi, \lambda)$ are the roots of $\mathfrak{A}^{\#}(x, t, i(\xi + s\nu), \lambda)$ with positive imaginary part.

(A.3) Ω is a bounded domain of class C^{2m}. The coefficients of A_j, $j = 1, \cdots, l$, and their derivatives in t of order up to l are continuous in $\bar{\Omega} \times [0, T]$. The coefficients of B are continuous in $\bar{\Omega} \times \bar{\Delta}$ and uniformly Hölder continuous in (t, s) in $\bar{\Omega} \times \bar{\Delta}$. The coefficients of B_j are of class C^{2m-m_j} on $\partial\Omega$ for $j = 1, \cdots, m$.

Let $W^{j,p}(\Omega)$, $1 < p < \infty$, be the usual Sobolev space with the norm

$$\|u\|_{j,p}=\left(\sum_{|\alpha|\leq j}\int_{\Omega}|D^{\alpha}u|^{p}dx\right)^{1/p}.$$

The norm of $L^{p}(\Omega)$ is denoted by $\| \ \|_{p}$. We denote by $W^{2m,p}(\Omega, \{B_j\})$ the totality of functions in $W^{2m,p}(\Omega)$ which satisfy the boundary conditions $B_j u=0$, $1\leqq j\leqq m$. We use the notations $B(L^{p}, L^{p})$, $B(L^{p}, W^{j,p})$ to denote the set of all bounded linear operators from $L^{p}(\Omega)$ to $L^{p}(\Omega)$, $W^{j,p}(\Omega)$ respectively.

The operators $A_j(t)$, $j=0, \cdots, l$, are defined as follows: $A_0(t)=I$, $A_j(t)u=A_j(x, t, D_x)u$ for $u\in W^{d_j,p}(\Omega)$ if $j=1, \cdots, l-1$, and $A_l(t)u=A_l(x, t, D_x)u$ for $u\in W^{2m,p}(\Omega\,;\{B_j\})$. Similarly, the operator $B(t, s)$ is defined by $B(t, s)u=B(x, t, s, D_x)u$ for $u\in W^{2m,p}(\Omega)$.

We try to solve the problem (1.1)-(1.3) in $L^{p}(\Omega)$, $1<p<\infty$, and formulate the problem as

$$\sum_{k=0}^{l}A_{l-k}(t)D_t^k u(t)=\int_0^t B(t, s)u(s)ds+f(t)\,, \qquad 0<t\leqq T\,, \tag{2.2}$$

$$(D_t^j u)(0)=u_j\,, \qquad j=0, \cdots, l-1\,. \tag{2.3}$$

For the sake of simplicity we put

$$A(t, D_t)=\sum_{k=0}^{l}A_{l-k}(t)D_t^k\,.$$

DEFINITION. An operator valued function $W(t, s)$, $(t, s)\in\Delta$, is called the fundamental solution of (2.2), (2.3) if it satisfies the following equation and initial conditions

$$A(t, D_t)W(t, s)=\int_s^t B(t, \sigma)W(\sigma, s)d\sigma \qquad (t, s)\in\Delta\,, \tag{2.4}$$

$$D_t^j W(t, s)=0 \qquad \text{at } t=s \text{ for } j=0, \cdots, l-2\,, \tag{2.5}$$

$$D_t^{l-1}W(t, s)=I \qquad \text{at } t=s\,. \tag{2.6}$$

We state the main results of this paper.

THEOREM 1. *Under the assumptions* (A.1)-(A.3) *the fundamental solution* $W(t, s)$ *of the problem* (2.2), (2.3) *exists and is uuique. To be precise the initial conditions* (2.5), (2.6) *are satisfied in the following sense*:

$$\lim_{t-s\to 0}D_t^j W(t, s)=0 \qquad j=0, \cdots, l-2 \tag{2.7}$$

in the strong operator topology of $B(L^{p}, W^{d(l-1-j)-1,p})$,

$$\lim_{t-s\to 0} D_t^{l-1} W(t, s) = I \tag{2.8}$$

in the strong operator topology of $B(L^p, L^p)$, *and*

$$\lim_{t-s\to 0} D_t^j W(t, s) = 0 \qquad j=0, \cdots, l-2 \tag{2.9}$$

in the weak operator topology of $B(L^p, W^{d(l-1-j), p})$.

Furthermore, the following estimates hold for $j+dk \leqq 2m$:

$$\|D_t^k W(t, s)\|_{B(L^p, W^{j,p})} \leqq C(t-s)^{l-1-k-j/d}, \tag{2.10}$$

where C *is a constant independent of* t, s.

THEOREM 2. *For any* $u_0, \cdots, u_{l-2} \in W^{2m,p}(\Omega, \{B_j\})$, $u_{l-1} \in L^p(\Omega)$, *and for any Hölder continuous function* $f(t)$ *with values in* $L^p(\Omega)$, *the unique solution of* (2.2), (2.3) *is given by* (1.7)-(1.9). *The integral in the right side of* (2.2) *exists in the improper sense*:

$$\int_0^t B(t, s)u(s)ds = \lim_{\varepsilon\to+0} \int_\varepsilon^t B(t, s)u(s)ds .$$

In what follows we denote by C constants which depend only on the assumptions (A.1), (A.2), (A.3) and p.

3. Fundamental solution of the equation without the integral term.

In this section we construct the fundamental solution of the equation without the integral term:

$$A(t, D_t)u(t) = f(t) \qquad 0 < t \leqq T \tag{3.1}$$

$$(D_t^j u)(0) = u_j \qquad j=0, \cdots, l-1 . \tag{3.2}$$

By definition the fundamental solution $U(t, s)$ to (3.1), (3.2) is the bounded operator valued function defined in $\bar{\Delta}$ satisfying

$$A(t, D_t)U(t, s) = 0 \qquad (t, s) \in \Delta \tag{3.3}$$

$$D_t^j U(t, s) = 0 \quad \text{at } t=s \text{ for } j=0, \cdots, l-2, \tag{3.4}$$

$$D_t^{l-1} U(t, s) = I \quad \text{at } t=s, \tag{3.5}$$

$$D_s^j U(t, s) = 0 \quad \text{at } t=s \text{ for } j=0, \cdots, l-2 \tag{3.6}$$

$$D_s^{l-1} U(t, s) = (-1)^{l-1} I \quad \text{at } t=s. \tag{3.7}$$

Moreover, we will show that for $j+dk\leqq 2m$

$$\|D_t^k U(t,s)\|_{B(L^p,W^{j,p})}\leqq C(t-s)^{l-1-k-j/d} \tag{3.8}$$

$$\|D_s^k U(t,s)\|_{B(L^p,W^{j,p})}\leqq C(t-s)^{l-1-k-j/d} \tag{3.9}$$

$$\operatorname*{w-lim}_{t-s\to 0} D_t^k U(t,s)=\operatorname*{w-lim}_{t-s\to 0} D_s^k U(t,s)=0\,, \qquad k=0,\cdots,l-2\,, \tag{3.10}$$

in $B(L^p, W^{d(l-1-k),p})$, where w-lim means the convergence in the weak operator topology.

For $t\in[0,T]$ and a complex number λ let $A(t,\lambda)$ be the operator defined by

$$D(A(t,\lambda))=W^{2m,p}(\Omega,\{B_j\})\,,$$

$$(A(t,\lambda)u)(x)=\mathfrak{A}(x,t,D_x,\lambda)u(x) \qquad \text{for } u\in W^{2m,p}(\Omega,\{B_j\})\,.$$

According to S. Agmon and L. Nirenberg [1] (see also J. E. Lagnese [3]) we have the following lemma.

LEMMA 3.1. *There exists a constant λ_0 such that if* $\mathrm{Re}\,\lambda\geqq 0$ *and* $|\lambda|>\lambda_0$, *the operator $A(t,\lambda)$ is one-to-one from $W^{2m,p}(\Omega,\{B_j\})$ onto $L^p(\Omega)$ for each $t\in[0,T]$. The following estimate holds for $u\in W^{2m,p}(\Omega,\{B_j\})$:*

$$\sum_{j=0}^{2m}|\lambda|^{(2m-j)/d}\|u\|_{j,p}\leqq C\|A(t,\lambda)u\|_p\,. \tag{3.11}$$

The proof that $A(t,\lambda)$ is onto is not given in [1]; however, we can verify it by an analogous method to that of Section 3.8 of [7].

Replacing the unknown function u by $e^{-kt}u$ for some positive constant k if necessary we may and will assume that there exists an angle $\theta_0\in(\pi/2,\pi]$ such that the conclusion of Lemma 3.1 holds for $\lambda\in\Sigma=\{\lambda:|\arg\lambda|\leqq\theta_0\}\cup\{0\}$. Hence the bounded inverse $A(t,\lambda)^{-1}$ exists for $\lambda\in\Sigma$, and

$$\sum_{j=0}^{2m}|\lambda|^{(2m-j)/d}\|A(t,\lambda)^{-1}f\|_{j,p}\leqq C\|f\|_p \tag{3.12}$$

for any $f\in L^p(\Omega)$. Furthermore, it is not difficult to show that $A(t,\lambda)^{-1}$ is l times continuously differentiable in t for each fixed $\lambda\in\Sigma$ and

$$\sum_{j=0}^{2m}|\lambda|^{(2m-j)/d}\|D_t^k A(t,\lambda)^{-1}f\|_{j,p}\leqq C\|f\|_p\,, \quad k=1,\cdots,m\,. \tag{3.13}$$

For $\tau>0$, $s\in[0,T]$ we put

$$U_0(\tau,s)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}A(s,\lambda)^{-1}d\lambda\,, \tag{3.14}$$

where Γ is a smooth contour running in $\Sigma\setminus\{0\}$ from $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$. In view of (3.12), (3.13)

$$A(s, D_\tau)U_0(\tau, s)=0 \qquad \tau>0,\ s\in[0, T] \tag{3.15}$$

$$\|D_\tau^k D_s^i U_0(\tau, s)\|_{B(L^p, W^{j,p})}\leqq C\tau^{l-1-k-j/d} \tag{3.16}$$

for $i\leqq l$, $j+dk\leqq 2m$. In particular

$$\|D_\tau^k U_0(\tau, s)\|_{B(L^p, L^p)}\leqq C\tau^{l-1-k} \qquad k=0, \cdots, l \tag{3.17}$$

$$\|D_\tau^k U_0(\tau, s)\|_{B(L^p, W^{d(l-1-k),p})}\leqq C \qquad k=0, \cdots, l-1\,. \tag{3.18}$$

Hence, we get for $k=0, \cdots, l-2$

$$\lim_{\tau\to 0} D_\tau^k U_0(\tau, s)=0 \qquad \text{in } B(L^p, L^p)\,, \tag{3.19}$$

$$\text{w-}\lim_{\tau\to 0} D_\tau^k U_0(\tau, s)=0 \qquad \text{in } B(L^p, W^{d(l-1-k),p})\,. \tag{3.20}$$

Next we show

$$\lim_{\tau\to 0} D_\tau^{l-1} U_0(\tau, s)=I \tag{3.21}$$

in the strong operator topology of $B(L^p, L^p)$. Since $D_\tau^{l-1}U_0(\tau, s)$ is uniformly bounded in $B(L^p, L^p)$, it suffices to show that for each $u\in W^{2m,p}(\Omega\,; \{B_j\})$

$$\lim_{\tau\to 0} D_\tau^{l-1} U_0(\tau, s)u=u\,. \tag{3.22}$$

As is easily seen

$$\begin{aligned} D_\tau^{l-1}U_0(\tau, s)u &= \frac{1}{2\pi i}\int_\Gamma \lambda^{l-1}e^{\lambda\tau}A(s, \lambda)^{-1}u\,d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \lambda^{-1}e^{\lambda\tau}\{\lambda^l A(s, \lambda)^{-1}u-u\}d\lambda+u \\ &= \frac{1}{2\pi i}\int_\Gamma \lambda^{-1}e^{\lambda\tau}A(s, \lambda)^{-1}\{\lambda^l u-A(s, \lambda)u\}d\lambda+u \\ &= -\frac{1}{2\pi i}\int_\Gamma \lambda^{-1}e^{\lambda\tau}A(s, \lambda)^{-1}\sum_{k=0}^{l-1} A_{l-k}(s)\lambda^k u\,d\lambda+u\,. \end{aligned}$$

It is easy to show that the first term of the last member of the above equalities tends to 0 as $\tau\to 0$, and hence (3.22) follows.

The fundamental solution $U(t, s)$ of (3.1), (3.2) is constructed in the following manner:

$$U(t, s)=U_0(t-s, s)+Z(t, s) \tag{3.23}$$

$$Z(t, s)=\int_s^t U_0(t-\tau, \tau)R(\tau, s)d\tau \tag{3.24}$$

$$R(t, s)-\int_s^t R_1(t, \tau)R(\tau, s)d\tau=R_1(t, s) \tag{3.25}$$

$$\begin{aligned} R_1(t, s) &= -A(t, D_t)U_0(t-s, s) \\ &= \sum_{k=0}^{l-1}(A_{l-k}(s)-A_{l-k}(t))D_t^k U_0(t-s, s). \end{aligned} \tag{3.26}$$

If $l=1$, $U_0(t-s, s)=\exp(-(t-s)A(s))$, and hence the above construction of the fundamental solution is nothing but a direct extension of the argument of Section 5.2 of [7]. Therefore we only sketch the proof.

In view of (3.16), (3.17), (3.10), (3.20), (3.21) the conclusions (3.3)-(3.10) follow from the following estimates:

$$\|D_t^k Z(t, s)\|_{B(L^p, W^{j,p})}\leqq C(t-s)^{l-k-j/d} \tag{3.27}$$

$$\|D_s^k Z(t, s)\|_{B(L^p, W^{j,p})}\leqq C(t-s)^{l-k-j/d} \tag{3.28}$$

for $j+dk\leqq 2m$.

The following inequalities are easily seen:

$$\|R(t, s)\|_{B(L^p, L^p)}\leqq C \qquad (t, s)\in\Delta \tag{3.29}$$

$$\|R(t, s)-R(\tau, s)\|_{B(L^p, L^p)}\leqq C\left\{\frac{t-\tau}{t-s}+(t-\tau)\log\frac{t-s}{t-\tau}\right\} \tag{3.30}$$

$$\text{for } 0\leqq s<\tau<t\leqq T$$

$$\|A_l(t)(U_0(t-s, s)-U_0(t-s, t))\|_{B(L^p, L^p)}\leqq C \qquad (t, s)\in\Delta \tag{3.31}$$

$$\|(D_t+D_s)^i D_t^k U_0(t-s, s)\|_{B(L^p, L^p)}\leqq C(t-s)^{l-1-k} \tag{3.32}$$

$$(t, s)\in\Delta, \quad i\leqq l, \quad k\leqq l.$$

In the proof of (3.30) we use

$$\|D_t^{k+1}U_0(t-s, s)\|_{B(L^p, W^{d(l-k), p})}\leqq C(t-s)^{-2}.$$

Expressing as

$$D_t^k Z(t, s)=\int_s^t D_t^k U_0(t-\tau, \tau)(R(\tau, s)-R(t, s))d\tau$$

$$+\int_s^t (D_t+D_\tau)D_t^{k-1}U_0(t-\tau, \tau)d\tau R(t, s)+D_t^{k-1}U_0(t-s, s)R(t, s)$$

and making use of (3.17), (3.29), (3.30), (3.32), we can easily establish (3.27) for $1\leqq j\leqq l$. In order to show (3.27) with $j=0$ we first note

$$\int_s^t A_l(t)U_0(t-\tau, t)d\tau=I-\sum_{k=1}^{l} A_{l-k}(t)(-D_s)^{k-1}U_0(t-s, t) \tag{3.33}$$

which follows from $A(t, -D_\tau)U_0(t-\tau, t)=0$. The desired estimate is a consequence of

$$\begin{aligned} A_l(t)Z(t, s)=&\int_s^t A_l(t)(U_0(t-\tau, \tau)-U_0(t-\tau, t))R(\tau, s)d\tau \\ &+\int_s^t A_l(t)U_0(t-\tau, t)(R(\tau, s)-R(t, s))d\tau \\ &+\int_s^t A_l(t)U_0(t-\tau, t)d\tau R(t, s)\,, \end{aligned} \tag{3.34}$$

and (3.29), (3.30), (3.31), (3.33), (3.18) as well as the well-known elliptic estimates.

Following the argument of [6 ; p. 529] we can show

$$\|(D_t+D_s)^i R(t, s)\|_{B(L^p, L^p)}\leqq C\,, \qquad (t, s)\in \Delta \quad \text{for } i\leqq l\,. \tag{3.35}$$

The inequality (3.28) with $k=0$ is nothing other than (3.27) with $k=0$. For $0<k\leqq l$ (3.28) is a consequence of

$$\begin{aligned} D_s^k Z(t, s)=&\sum_{i=1}^{k}\binom{k}{i}\int_s^t D_\tau^{k-i}U_0(t-\tau, \tau)(D_\tau+D_s)^i R(\tau, s)d\tau \\ &+\int_s^t D_\tau^k U_0(t-\tau, \tau)(R(\tau, s)-R(t, s))d\tau \\ &-D_s^{k-1}U_0(t-s, s)R(t, s)\,, \end{aligned} \tag{3.36}$$

and (3.18), (3.29), (3.30), (3.35).

PROPOSITION 3.1. *For any* $u_0, \cdots, u_{l-2}\in W^{2m,p}(\Omega, \{B_j\})$, $u_{l-1}\in L^p(\Omega)$, *and any Hölder continuous function* $f(t)$ *with values in* $L^p(\Omega)$, *the unique solution of the initial value problem* (3.1), (3.2) *is given by*

$$u(t)=\sum_{j=0}^{l-1} u_j(t)+\int_0^t U(t, s)f(s)ds \tag{3.37}$$

$$u_j(t)=\frac{t^j}{j!}u_j-\int_0^t U(t, s)\sum_{k=0}^{j}\frac{s^k}{k!}A_{l-j+k}(s)u_j ds\,, \qquad j=0, \cdots, l-2 \tag{3.38}$$

$$u_{l-1}(t)=U(t, 0)u_{l-1}\,. \tag{3.39}$$

PROOF. It can be shown with the aid of a direct calculation and the argument of Section 5.2 of [7] that the function $u(t)$ given by (3.37)-(3.39) is a solution of (3.1), (3.2). If $p=2$, the uniqueness follows from M. S. Agranovič and M. I. Višik [2] or H. Tanabe [5].

For any $v_0 \in W^{2m,2}(\Omega\,;\{B_j\})$ the functions $U(t, s)v_0$ and

$$v(t\,;s)=\frac{(t-s)^{l-1}}{(l-1)!}v_0-\int_s^t U(t, \sigma)\sum_{k=0}^{l-1}\frac{(\sigma-s)^k}{k!}A_{k+1}(\sigma)v_0\,d\sigma$$

are both solutions of the initial value problem in $L^2(\Omega)$:

$$A(t, D_t)u(t)=0 \qquad s<t\leqq T,$$

$$(D_t^j u)(s)=0 \qquad \text{for } j=0, \cdots, l-2, \qquad (D_t^{l-1}u)(s)=v_0.$$

Hence, owing to the uniqueness we get

$$U(t, s)v_0=v(t\,;s). \tag{3.40}$$

When v_0 is an arbitrary element of $W^{2m,p}(\Omega\,;\{B_j\})$, we see that (3.40) holds by approximating v_0 by a sequence in $W^{2m,p}(\Omega\,;\{B_j\})\cap W^{2m,2}(\Omega\,;\{B_j\})$ in the strong topology of $W^{2m,p}(\Omega)$. Differentiating both sides of (3.40) l times in s, we get

$$\sum_{k=0}^{l}(-D_s)^k(U(t, s)A_{l-k}(s)v_0)=0. \tag{3.41}$$

Let $u(t)$ be the solution of (3.1), (3.2) with $u_0=\cdots=u_{l-1}=0$, $f(t)\equiv 0$. With the aid of (3.41) and integration by parts we get

$$0=\int_0^t\sum_{k=0}^{l}(-D_s)^k(U(t, s)A_{l-k}(s))u(s)ds$$

$$=-u(t)+\int_0^t U(t, s)A(s, D_s)u(s)ds=-u(t).$$

Thus a solution of (3.1), (3.2) is unique, and the proof of Proposition 3.1 is complete.

4. Proofs of Theorems.

In this section following the method of J. Prüss [4] we construct the fundamental solution $W(t, s)$ to

$$A(t, D_t)u(t)=\int_0^t B(t, s)u(s)ds+f(t) \qquad 0<t\leqq T \tag{4.1}$$

$$(D_t^j u)(0) = u_j \qquad j=0, \cdots, l-1 . \tag{4.2}$$

In what follows we simply write $\| \ \|$ instead of $\| \ \|_{B(L^p, L^p)}$.

Let $K(t, s)$ be the operator defined by

$$B(t, s) = K(t, s) A_l(s) . \tag{4.3}$$

By the assumption (A.3) $K(t, s)$ is a bounded operator valued function defined in $\bar{\Delta}$ which is uniformly Hölder continuous:

$$\|K(t', s') - K(t, s)\| \leqq C(|t'-t|^\rho + |s'-s|^\rho) , \qquad \rho > 0 . \tag{4.4}$$

It would be natural to expect that $W(t, s)$ is the solution of the integral equation

$$W(t, s) = U(t, s) + \int_s^t U(t, \tau) \int_s^\tau B(\tau, \sigma) W(\sigma, s) d\sigma d\tau . \tag{4.5}$$

Putting $V(t, s) = A_l(t)(W(t, s) - U(t, s))$ and calculating formally we get

$$V(t, s) = \int_s^t A_l(t) \int_\sigma^t U(t, \tau) K(\tau, \sigma) d\tau A_l(\sigma) W(\sigma, s) d\sigma$$

$$= \int_s^t A_l(t) \int_\sigma^t U(t, \tau) K(\tau, \sigma) d\tau \, V(\sigma, s) d\sigma + V_0(t, s)$$

where

$$V_0(t, s) = \int_s^t A_l(t) \int_\sigma^t U(t, \tau) K(\tau, \sigma) d\tau A_l(\sigma) U(\sigma, s) d\sigma . \tag{4.6}$$

Hence letting $V(t, s)$ be the solution of the integral equation

$$V(t, s) = V_0(t, s) + \int_s^t P(t, \sigma) V(\sigma, s) d\sigma \tag{4.7}$$

where

$$P(t, \sigma) = A_l(t) \int_\sigma^t U(t, \tau) K(\tau, \sigma) d\tau , \tag{4.8}$$

we define $W(t, s)$ by

$$W(t, s) = U(t, s) + A_l(t)^{-1} V(t, s) . \tag{4.9}$$

Rigorously $V_0(t, s)$ is expressed as follows:

$$V_0(t, s) = \int_s^t (P(t, \sigma) - P(t, s)) A_l(\sigma) U(\sigma, s) d\sigma + P(t, s) \int_s^t A_l(\sigma) U(\sigma, s) d\sigma. \tag{4.10}$$

By virtue of (3.31), (3.16), (4.5), (3.33), (3.27) $P(t, s)$ is expressed as follows:

$$P(t, s)=\int_s^t A_l(t)(U_0(t-\tau, \tau)-U_0(t-\tau, t))K(\tau, s)d\tau$$

$$+\int_s^t A_l(t)U_0(t-\tau, t)(K(\tau, s)-K(t, s))d\tau \tag{4.11}$$

$$+\int_s^t A_l(t)U_0(t-\tau, t)d\tau K(t, s)+\int_s^t A_l(t)Z(t, \tau)K(\tau, s)d\tau .$$

It is easy to verify that each term of the right side of (4.11) is strongly continuous and uniformly bounded in Δ.

The following inequality is a simple consequence of (4.4):

$$\|K(\tau, \sigma)-K(t, \sigma)-K(\tau, s)+K(t, s)\| \leqq C(t-\tau)^{\rho/2}(\sigma-s)^{\rho/2}$$

$$s \leqq \sigma \leqq \tau \leqq t. \tag{4.12}$$

With the aid of (3.33) and (3.16) we can easily show

$$\left\|\int_\sigma^t A_l(t)U_0(t-\tau, t)d\tau-\int_s^t A_l(t)U_0(t-\tau, t)d\tau\right\| \leqq C\log\frac{t-s}{t-\tau} . \tag{4.13}$$

Hence making use of (3.31), (4.4), (3.16), (4.12), (4.13), (3.27) we get

$$\|P(t, \sigma)-P(t, s)\|$$

$$\leqq C\left\{(\sigma-s)^\rho+(t-s)^{\rho-1}(\sigma-s)+(t-\sigma)^{\rho/2}(\sigma-s)^{\rho/2}+\log\frac{t-s}{t-\sigma}\right\} . \tag{4.14}$$

With the aid of (3.3) and integration by parts we get

$$\int_s^t A_l(\sigma)U(\sigma, s)d\sigma$$

$$=I-\sum_{k=1}^{l} A_{l-k}(t)D_t^{k-1}U(t, s)+\int_s^t \sum_{k=1}^{l-1} \dot{A}_{l-k}(\sigma)D_\sigma^{k-1}U(\sigma, s)d\sigma . \tag{4.15}$$

In view of (3.8) the right side of (4.15) is uniformly bounded, and so

$$\left\|\int_s^t A_l(\sigma)U(\sigma, s)d\sigma\right\| \leqq C . \tag{4.16}$$

The inequality (3.8) with $k=0$, $j=d(l-k)$ implies

$$\|A_l(\sigma)U(\sigma, s)\| \leqq C/(\sigma-s) . \tag{4.17}$$

From (4.10), (4.14), (4.16), (4.17) it follows that $V_0(t, s)$ is strongly continuous and uniformly bounded in Δ. Thus the integral equation (4.7) can be solved by successive approximation, and the solution $V(t, s)$ is strongly continuous and uniformly bounded in Δ.

Let $W(t, s)$ be the operator valued function defined by (4.9). We intend to show that $W(t, s)$ is the desired fundamental solution.

LEMMA 4.1. *The integral* $\int_s^t B(t, \sigma) U(\sigma, s) d\sigma$ *exists in the improper sense, and strongly continuous and uniformly bounded in* Δ. *Moreover the following inequality holds for* $0 \leqq s < \tau \leqq T$:

$$\left\| \int_s^t B(t, \sigma) U(\sigma, s) d\sigma - \int_s^\tau B(\tau, \sigma) U(\sigma, s) d\sigma \right\| \leqq C \left\{ (t-\tau)(t-s)^{\rho-1} + (t-\tau)^{\rho/2} (\tau-s)^{\rho/2} + (t-\tau)^\rho + \log \frac{t-s}{\tau-s} \right\}. \tag{4.18}$$

PROOF. The assertion of the lemma follows from

$$\int_s^t B(t, \sigma) U(\sigma, s) d\sigma = \int_s^t (K(t, \sigma) - K(t, s)) A_l(\sigma) U(\sigma, s) d\sigma + K(t, s) \int_s^t A_l(\sigma) U(\sigma, s) d\sigma ,$$

and (4.4), (4.12), (4.16), (4.17).

We put

$$Q(t, s) = \int_s^t B(t, \sigma) W(\sigma, s) d\sigma . \tag{4.19}$$

In view of Lemma 4.1 the integral on the right of (4.19) exists in the improper sense, and

$$Q(t, s) = \int_s^t B(t, \sigma) U(\sigma, s) d\sigma + \int_s^t K(t, \sigma) V(\sigma, s) d\sigma .$$

Furthermore, by virtue of (4.18)

$$\| Q(t, s) - Q(\tau, s) \| \leqq C \left\{ (t-\tau)(t-s)^{\rho-1} + (t-\tau)^{\rho/2} (\tau-s)^{\rho/2} + (t-\tau)^\rho + \log \frac{t-s}{\tau-s} \right\}. \tag{4.20}$$

Noting that

$$A_l(t)^{-1} V_0(t, s) = \lim_{\varepsilon \to +0} \int_{s+\varepsilon}^t \int_\sigma^t U(t, \tau) K(\tau, \sigma) d\tau A_l(\sigma) U(\sigma, s) d\sigma = \lim_{\varepsilon \to +0} \int_{s+\varepsilon}^t U(t, \tau) \int_{s+\varepsilon}^\tau B(\tau, \sigma) U(\sigma, s) d\sigma d\tau ,$$

we see that

$$A_l(t)^{-1}V(t,s)=\int_s^t U(t,\tau)\int_s^\tau B(\tau,\sigma)W(\sigma,s)d\sigma d\tau = \int_s^t U(t,\tau)Q(\tau,s)d\tau\,. \tag{4.21}$$

According to (3.16), (3.27), (3.31), (3.33), (4.20) the right members of the following equalities exist:

$$\begin{aligned}&A_{l-k}(t)D_t^k\int_s^t U(t,\tau)Q(\tau,s)d\tau\\ &=\int_s^t A_{l-k}(t)(D_t+D_\tau)D_t^{k-1}U_0(t-\tau,\tau)Q(\tau,s)d\tau\\ &-\int_s^t A_{l-k}(t)D_\tau D_t^{k-1}U_0(t-\tau,\tau)(Q(\tau,s)-Q(t,s))d\tau\\ &+A_{l-k}(t)D_t^{k-1}U_0(t-s,s)Q(t,s)\\ &+\int_s^t A_{l-k}(t)D_t^k Z(t,\tau)Q(\tau,s)d\tau\,,\qquad k=1,\cdots,l\,,\end{aligned} \tag{4.22}$$

$$\begin{aligned}&A_l(t)\int_s^t U(t,\tau)Q(\tau,s)d\tau\\ &=\int_s^t A_l(t)U(t,\tau)(Q(\tau,s)-Q(t,s))d\tau+\int_s^t A_l(t)U(t,\tau)d\tau Q(t,s)\,,\end{aligned} \tag{4.23}$$

$$\begin{aligned}\int_s^t A_l(t)U(t,\tau)d\tau&=\int_s^t A_l(t)(U_0(t-\tau,\tau)-U_0(t-\tau,t))d\tau\\ &+\int_s^t A_l(t)U_0(t-\tau,t)d\tau+\int_s^t A_l(t)Z(t,\tau)d\tau\,.\end{aligned} \tag{4.24}$$

Hence, recalling (4.21) we see that

$$A(t,D_t)(A_l(t)^{-1}V(t,s))=Q(t,s)\qquad (t,s)\in\Delta\,.$$

$$D_t^j(A_l(t)^{-1}V(t,s))=0\qquad \text{at } t=s \text{ for } j=0,\cdots,l-1\,.$$

Thus we conclude that $W(t,s)$ is the fundamental solution of (2.2), (2.3), and the proof of Theorem 1 is complete if the uniqueness of the solution is shown.

It is not difficult to show that the function $u(t)$ defined by (1.7)-(1.9) is the solution of (2.2), (2.3) if the hypothesis of Theorem 2 is satisfied. If u is the solution with $u_0=\cdots=u_{l-1}=0$, $f(t)\equiv 0$, then in view of Proposition 3.1 and (4.3)

$$A_l(t)u(t)=\int_s^t A_l(t)\int_s^t U(t,\tau)K(\tau,s)d\tau A_l(s)u(s)ds\,. \tag{4.25}$$

It follows from (4.25) that $A_l(t)u(t)\equiv 0$, which implies $u(t)\equiv 0$. Thus the solution of (2.2), (2.3) is unique, and the proofs of Theorems 1, 2 are complete.

Note added. After submitting the manuscript the author noticed E. Obrecht's paper [8]. With the aid of his result it can be shown that (2.9) and (3.10) of the present paper hold in the strong operator topology.

Bibliography

[1] Agmon, S. and L. Nirenberg, Properties of solutions of ordinary differential equations in Banach space, Comm. Pure Appl. Math. **16** (1963), 121-239.

[2] Agranovič, M. S. and M. I. Višik, Elliptic problems with a parameter and parabolic problems of general type, Uspekhi Mat. Nauk **19** (1964), 53-161; Translated in Russian Math. Surveys **19** (1964), 53-157.

[3] Lagnese, J. E., On equations of evolution and parabolic equations of higher order in t, J. Math. Anal. Appl. **32** (1970), 15-37.

[4] Prüss, J., On resolvent operators for linear integrodifferential equations of Volterra type, J. Integral Equations **5** (1983), 211-236.

[5] Tanabe, H., Note on uniqueness of solutions of differential inequalities of parabolic type, Osaka J. Math. **2** (1965), 191-204.

[6] Tanabe, H., On regularity of solutions of abstract differential equations of parabolic type in Banach space, J. Math. Soc. Japan **19** (1967), 521-542.

[7] Tanabe, H., Equations of Evolution (English translation), Pitman, London, 1979.

[8] Obrecht, E., Evolution operators for higher order abstract parabolic equations, Czechoslovak Math. J. **36** (111) (1986), 210-222.

(Received July 2, 1986)

Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka
560 Japan