J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
34 (1987), 127-157.

Cauchy problem for Fuchsian hyperbolic operators, II.

Dedicated to Prof. S. Itô on his 60th birthday

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In our previous paper [2], we have discussed the Cauchy problem for a class of Fuchsian hyperbolic operators in distribution spaces, and established the existence, uniqueness and propagation results of C^∞-singularities of distribution solutions, by constructing a right and a left parametrix (see also Uryu [8]).

The aim of this paper is to show that the discussion in [2] can be applied to a somewhat wider class of Fuchsian hyperbolic operators. The result here is a generalization of results in [2].

§ 1. Statement of main results.

Let us consider the Cauchy problem:

$$\begin{cases} P(t, x, D_t, D_x)u=f(t, x)\,, \\ D_t^j u|_{t=0}=g_j(x)\,, \quad j=0, 1, \cdots, m-k-1 \end{cases} \tag{1.1}$$

for a class of differential operators $P(t, x, D_t, D_x)(=P)$ with "regular singularities" on $\{t=0\}$ of the form

$$P=t^k D_t^m+\sum_{\substack{j+|\alpha|\leqq m\\ j<m}} t^{p(j,\alpha)}a_{j,\alpha}(t, x)D_t^j D_x^\alpha\,, \tag{1.2}$$

where $(t, x)=(t, x_1, \cdots, x_n)\in[0, T]\times\boldsymbol{R}^n$ $(T>0)$, $k\in\boldsymbol{Z}_+$ $(=\{0, 1, 2, \cdots\})$, $m\in\boldsymbol{N}$ $(=\{1, 2, 3, \cdots\})$, $\alpha=(\alpha_1, \cdots, \alpha_n)\in\boldsymbol{Z}_+^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $p(j, \alpha)\in\boldsymbol{Z}_+$ $(j+|\alpha|\leqq m$ and $j<m)$, $a_{j,\alpha}(t, x)\in C^\infty([0, T]\times\boldsymbol{R}^n)$ $(j+|\alpha|\leqq m$ and $j<m)$,

$$D_t=\frac{1}{\sqrt{-1}}\frac{\partial}{\partial t} \quad \text{and} \quad D_x^\alpha=\Big(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\cdots\Big(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_n}\Big)^{\alpha_n}.$$

In addition, we impose the following conditions on P:

(A-1) $0\leqq k\leqq m$.

(A-2) $p(j, \alpha)\in\boldsymbol{Z}_+$ $(j+|\alpha|\leqq m$ and $j<m)$ satisfy

$$\begin{cases} p(j,\alpha)=k+\nu|\alpha|\,, & \text{when } j+|\alpha|=m \text{ and } j<m, \\ p(j,\alpha)\geqq k-m+j+(\nu+1)|\alpha|\,, & \text{when } j+|\alpha|<m \end{cases}$$

for some $\nu\in \boldsymbol{Z}_+$.

(A-3) All the roots $\lambda_i(t,x,\xi)$ $(i=1,\cdots,m)$ of

$$\lambda^m+\sum_{\substack{j+|\alpha|=m\\ j<m}} a_{j,\alpha}(t,x)\lambda^j\xi^\alpha=0$$

are *real, simple* and *bounded* on $\{(t,x,\xi)\in[0,T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n\,;\,|\xi|=1\}$.

Then, P is a generalization of Fuchsian hyperbolic operators treated in [2] (in fact, the operators in [2] correspond to the case $\nu=0$). The indicial polynomial $C(x,\zeta)$ of P is defined by

$$C(x,\zeta)=\zeta(\zeta-1)\cdots(\zeta-m+1)+a_{m-1}(x)\zeta(\zeta-1)\cdots(\zeta-m+2)$$
$$+\cdots+a_{m-k}(x)\zeta(\zeta-1)\cdots(\zeta-m+k+1)\,,$$

where

$$a_j(x)=\begin{cases} \left(\dfrac{1}{\sqrt{-1}}\right)^j a_{j,(0,\cdots,0)}(0,x)\,, & \text{when } p(j,(0,\cdots,0))=k-m+j\,, \\ 0\,, & \text{when } p(j,(0,\cdots,0))>k-m+j\,. \end{cases}$$

To make (1.1) meaningful, at least at a formal power series level, we impose the following Fuchs condition on P:

(A-4) $C(x,\zeta)\neq 0$ for any $x\in\boldsymbol{R}^n$ and $\zeta\in\{\lambda\in\boldsymbol{Z}\,;\,\lambda\geqq m-k\}$.

Problem (1.1), even for more general operators P, has been solved by several authors. Baouendi-Goulaouic [1] solved (1.1) in analytic function spaces, Tahara [4] in hyperfunction spaces, Tahara [5] in C^∞ function spaces, Bove-Lewis-Parenti [2] in distribution spaces (when $\nu=0$), and Uryu [7], Tahara [6] in Gevrey function spaces. See also Uryu [8]. In particular, we should recall here the following:

THEOREM 0 (Tahara [5]). *Assume that* (A-1)$\sim$(A-4) *hold. Then, for any* $f(t,x)\in C^\infty([0,T]\times\boldsymbol{R}^n)$ *and any* $g_j(x)\in C^\infty(\boldsymbol{R}^n)$ $(j=0,1,\cdots,m-k-1)$ *there exists a unique* $u(t,x)\in C^\infty([0,T]\times\boldsymbol{R}^n)$ *which solves* (1.1). *Moreover, the domain* $D(t_0,x^0)$ *defined by*

$$D(t_0,x^0)=\{(t,x)\in[0,T]\times\boldsymbol{R}^n\,;\,|x^0-x|<\lambda_{\max}T^\nu(t_0-t)\} \tag{1.3}$$

(*where* $\lambda_{\max}=\sup\{|\lambda_i(t,x,\xi)|\,;\,i=1,\cdots,m,\,(t,x)\in[0,T]\times\boldsymbol{R}^n$ *and* $|\xi|=1\}$) *is a dependence domain of* $(t_0,x^0)\in(0,T]\times\boldsymbol{R}^n$. *In other words, if* $f(t,x)=0$ *on* $D(t_0,x^0)$ *and* $g_j(x)=0$ *on* $D(t_0,x^0)\cap\{t=0\}$ $(j=0,1,\cdots,m-k-1)$, *then the unique solution* $u(t,x)$ *also satisfies* $u(t,x)=0$ *on* $D(t_0,x^0)$.

In [2], we have constructed a right and a left parametrix for the case $\nu=0$, and obtained existence, uniqueness and propagation results of C^∞-singularities of distribution solutions of the Cauchy problem (1.1).

In this paper, we want to generalize some results in [2] to the general case $\nu\geqq 0$.

Now, let us give our results. The existence and uniqueness result is stated as follows. Let $\mathcal{D}'(\boldsymbol{R}^n)$ be the locally convex space of all distributions on $\boldsymbol{R}^n$ with strong topology, and let $C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ be the space of all infinitely differentiable functions on $[0, T]$ with values in $\mathcal{D}'(\boldsymbol{R}^n)$. Then, we have

THEOREM 1. *Assume that* (A-1)~(A-4) *hold. Then, for any* $f(t, x)\in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ *and any* $g_j(x)\in\mathcal{D}'(\boldsymbol{R}^n)$ $(j=0, 1, \cdots, m-k-1)$ *there exists a unique* $u(t, x)\in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ *which solves* (1.1). *Moreover, the domain* $D(t_0, x^0)$ *is a dependence domain of* $(t_0, x^0)\in(0, T]\times\boldsymbol{R}^n$.

The propagation result of C^∞-singularities is stated as follows. Following [2], we say that $f(t, x)\in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ is a regular distribution if

$$WF(f|_{t>0})\cap\{(t, x, \tau, \xi)\mid t>0, \xi=0\}=\emptyset .$$

For a regular distribution $f(t, x)$, we define the boundary wave front set $\partial WF(f)$ $(\subset T^*\boldsymbol{R}^n\setminus 0)$ over $\{t=0\}$ in the following way: we say that a point $(x, \xi)\in T^*\boldsymbol{R}^n\setminus 0$ does not belong to $\partial WF(f)$, if and only if there exists a classical pseudo-differential operator $B(x, D_x)$, elliptic near (x, ξ), such that $(Bf)(t, x)\in C^\infty([0, \varepsilon]\times\boldsymbol{R}^n)$ for some $\varepsilon>0$. Let $\nu\in\boldsymbol{Z}_+$ be as in (A-2), let $\lambda_i(t, x, \xi)$ $(i=1, \cdots, m)$ be as in (A-3), and let $(x^{(i)}(t, s, y, \eta),\ \xi^{(i)}(t, s, y, \eta))$ be the solution of the Hamiltonian equations:

$$\begin{aligned} &\frac{dx^{(i)}}{dt}=-t^\nu\nabla_\xi\lambda_i(t, x^{(i)}, \xi^{(i)}) , && \frac{d\xi^{(i)}}{dt}=t^\nu\nabla_x\lambda_i(t, x^{(i)}, \xi^{(i)}) , \\ &x^{(i)}|_{t=s}=y , && \xi^{(i)}|_{t=s}=\eta \end{aligned} \tag{1.4}$$

(where $t, s\in[0, T]$ and $(y, \eta)\in T^*\boldsymbol{R}^n\setminus 0$). Then, the following theorem holds.

THEOREM 2. *Assume that* (A-1)~(A-4) *hold and let* $f(t, x)$ *be a regular distribution. Then, the unique solution* $u(t, x)$ *in Theorem* 1 *is also a regular distribution and the following inclusions hold.*

(1) $\partial WF(u)\subset\partial WF(f)\cup\bigcup_{j=0}^{m-k-1} WF(g_j)$.

$$(2)\quad WF(u|_{t>0})\subset\{(t, x, \tau, \xi)\mid t>0, (t, x, \tau, \xi)\in WF(f)\}$$
$$\cup\bigcup_{i=1}^{m}\Big\{(t, x, t^{\nu}\lambda_i(t, x, \xi), \xi)\mid t>0, \exists s, \frac{s}{t}\in(0, 1), \exists(y, \eta)\in T^*\boldsymbol{R}^n\setminus 0,$$
$$x=x^{(i)}(t, s, y, \eta), \xi=\xi^{(i)}(t, s, y, \eta),$$
$$(s, y, s^{\nu}\lambda_i(s, y, \eta), \eta)\in WF(f)\Big\}$$
$$\cup\bigcup_{i=1}^{m}\Big\{(t, x, t^{\nu}\lambda_i(t, x, \xi), \xi)\mid t>0, \exists(y, \eta)\in T^*\boldsymbol{R}^n\setminus 0,$$
$$x=x^{(i)}(t, 0, y, \eta), \xi=\xi^{(i)}(t, 0, y, \eta),$$
$$(y, \eta)\in\partial WF(f)\cup\bigcup_{j=0}^{m-k-1}WF(g_j)\Big\}.$$

In view of Theorem 0 quoted above, we note that to obtain Theorems 1 and 2 it is sufficient to treat the Cauchy problem (1.1) in $C^{\infty}([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ modulo $C^{\infty}([0, T]\times\boldsymbol{R}^n)$.

The proof is done by constructing a right and a left parametrix for a reduced system modulo $C^{\infty}([0, T]\times\boldsymbol{R}^n)$. This construction and the preparations needed form the core of this paper.

For simplicity, we may assume from now on that

$$a_{j,\alpha}(t, x)\in B^{\infty}([0, T]\times\boldsymbol{R}^n)\quad(j+|\alpha|\leqq m\text{ and }j<m)\tag{1.5}$$

holds (in fact, to get rid of this condition we have only to apply a cut-off argument). Here, $B^{\infty}([0, T]\times\boldsymbol{R}^n)$ means the space of all functions $a(t, x)\in C^{\infty}([0, T]\times\boldsymbol{R}^n)$ such that every derivative $D_t^iD_x^{\alpha}a(t, x)$ is bounded on $[0, T]\times\boldsymbol{R}^n$.

§ 2. Reduction to a first-order system.

In this section, we shall reduce (1.1) to a suitable first-order $m\times m$ system of pseudo-differential equations. The method of reduction here is quite different from that used in [2]. The method of reduction proposed here has the advantage in that we need nothing in proving the equivalence between (1.1) and the reduced system, while in [2] a deep result of Hanges [3] was used.

Put

$$L=(\sqrt{-1})^m t^{m-k}P,\tag{2.1}$$

and define a differential operator L_s with a parameter $s\in\boldsymbol{R}$ by

$$L_s v=t^{-s}L(t^s v),$$

i. e.:

$$L_s=(t\partial_t+s)(t\partial_t+s-1)\cdots(t\partial_t+s-m+1)$$
$$+\sum_{\substack{j+|\alpha|\leqq m\\ j<m}}(\sqrt{-1})^{m-j}t^{p(j,\alpha)+m-k-j}a_{j,\alpha}(t,x)D_x^\alpha \tag{2.2}$$
$$\times(t\partial_t+s)(t\partial_t+s-1)\cdots(t\partial_t+s-j+1)\,.$$

Recall that condition (A-4) guarantees the following fact: the Taylor coefficients $\{g_j(x)\}_{j=0}^\infty$ of the solution $u(t,x)$ $(\sim\sum_{j=0}^\infty g_j(x)t^j/j!)$ of (1.1) are uniquely determined by the Taylor coefficients (in t) of $f(t,x)$ and the Cauchy data $g_j(x)$ $(j=0,1,\cdots,m-k-1)$. Therefore, for any $s\in\boldsymbol{Z}_+$, $s\geqq m-k$, we can express $u(t,x)$ in the form

$$u(t,x)=\sum_{j=0}^{s-1}g_j(x)\frac{t^j}{j!}+t^su_s(t,x) \tag{2.3}$$

and therefore only $u_s(t,x)$ remains to be determined. Since within the space $C^\infty([0,T],\mathcal{D}'(\boldsymbol{R}^n))$ the equation $Pu=f$ is obviously equivalent to $Lu=(\sqrt{-1})^mt^{m-k}f$, we can rewrite (1.1) as an equation with respect to $u_s(t,x)$ and obtain

$$L_su_s=f_s \tag{2.4}$$

for some known $f_s(t,x)\in C^\infty([0,T],\mathcal{D}'(\boldsymbol{R}^n))$.

Hence, in order to transform (2.4) into a first-order system we introduce the unknown functions

$$\begin{cases}u_1=(1+t^{\nu+1}\Lambda)^{m-1}u_s\,,\\ u_2=(1+t^{\nu+1}\Lambda)^{m-2}(t\partial_t+s)u_s\,,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ u_m=(t\partial_t+s)(t\partial_t+s-1)\cdots(t\partial_t+s-(m-2))u_s\,,\end{cases} \tag{2.5}$$

where $\Lambda\in OPS_{\mathrm{cl}}^1(\boldsymbol{R}_x^n)$ is a pseudo-differential operator with symbol $\lambda(\xi)\in C^\infty(\boldsymbol{R}_\xi^n)$ such that $\lambda(\xi)\geqq 1/2$ on $\boldsymbol{R}_\xi^n$ and $\lambda(\xi)=|\xi|$ for $|\xi|\geqq 1$. Then, the relation

$$(t\partial_t+s-j+1)u_j=(\nu+1)(m-j)t^{\nu+1}\Lambda(1+t^{\nu+1}\Lambda)^{-1}u_j+(1+t^{\nu+1}\Lambda)u_{j+1}$$

holds for $j=1,\cdots,m-1$ and (2.4) is rewritten into the form

$$(t\partial_t+s-m+1)u_m=-\sum_{j=0}^{m-1}k_j(t)(1+t^{\nu+1}\Lambda)u_{j+1}+f_s(t)\,,$$

where

$$k_j(t)=(\sqrt{-1})^{m-j}\sum_{|\alpha|\leqq m-j}t^{p(j,\alpha)+m-k-j}a_{j,\alpha}(t,x)D_x^\alpha(1+t^{\nu+1}\Lambda)^{-m+j}\,. \tag{2.6}$$

Therefore, (2.4) is equivalent to the following first-order system

$$(t\partial_t+s)\vec{u}=K(t)(1+t^{\nu+1}\Lambda)\vec{u}+M(t)\vec{u}+\vec{f} \tag{2.7}$$

under the relations (2.5) (when (2.4)⇒(2.7)) and $u_s=(1+t^{\nu+1}\Lambda)^{-m+1}u_1$ (when (2.7)⇒(2.4)), where

$$K(t)=\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -k_0(t), & -k_1(t), & \cdots, & & -k_{m-1}(t) \end{pmatrix}, \quad \vec{u}=\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_m \end{pmatrix}, \quad \vec{f}=\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ f_s \end{pmatrix}$$

and

$$M(t)=\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & m-1 \end{pmatrix}+(\nu+1)\begin{pmatrix} m-1 & & & \\ & m-2 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}(t^{\nu+1}\Lambda)(1+t^{\nu+1}\Lambda)^{-1}.$$

Thus, we have reduced (1.1) to an equivalent first-order $m\times m$ system (2.7).

Now, let us make clear the structure of (2.7). Put

$$h_j(t, x, \xi)=\sum_{|\alpha|=m-j} a_{j,\alpha}(t, x)(\sqrt{-1}\,\xi)^{\alpha}\lambda(\xi)^{-m+j}, \tag{2.8}$$

$$\begin{aligned} s_j(t, x, \xi)=&\sum_{|\alpha|=m-j} a_{j,\alpha}(t, x)(\sqrt{-1}\,\xi)^{\alpha}\lambda(\xi)^{-m+j}\left\{\sum_{i=1}^{m-j}\left(\frac{t^{\nu+1}\lambda(\xi)}{1+t^{\nu+1}\lambda(\xi)}\right)^{i}\right\} \\ &+\sum_{|\alpha|<m-j}(\sqrt{-1})^{m-j}t^{p(j,\alpha)+m-k-j}a_{j,\alpha}(t, x)\xi^{\alpha} \\ &\qquad\times(t^{\nu+1}\lambda(\xi))(1+t^{\nu+1}\lambda(\xi))^{-m+j} \end{aligned} \tag{2.9}$$

for $j=0, 1, \cdots, m-1$, and denote by $h_j(t)$ $(=h_j(t, x, D_x))$, $s_j(t)$ $(=s_j(t, x, D_x))$ the corresponding pseudo-differential operators. Since $p(j, \alpha)=k+\nu|\alpha|$ $(=k+j-m+(\nu+1)|\alpha|)$ holds for $|\alpha|=m-j$ (by (A-2)), we have

$$k_j(t)t^{\nu+1}\Lambda=t^{\nu+1}h_j(t)\Lambda+s_j(t)$$

for $j=0, 1, \cdots, m-1$. Therefore, (2.7) can be expressed in the form

$$(t\partial_t+s)\vec{u}=t^{\nu+1}A(t)\vec{u}+B(t)\vec{u}+\vec{f}, \tag{2.10}$$

where

$$A(t)=\begin{pmatrix} 0 & \Lambda & & & \\ & 0 & \Lambda & & \\ & & \ddots & \ddots & \\ & & & 0 & \Lambda \\ -h_0(t)\Lambda, & -h_1(t)\Lambda, & \cdots, & & -h_{m-1}(t)\Lambda \end{pmatrix} \tag{2.11}$$

and

$$B(t)=M(t)+K(t)-\begin{pmatrix} & 0 & \\ s_0(t), & s_1(t), \cdots, & s_{m-1}(t) \end{pmatrix}. \tag{2.12}$$

The following lemma holds.

LEMMA 1. *$A(t)\in OPS^1_{cl}(\boldsymbol{R}^n_x; m\times m)$, that is, $A(t)$ $(=A(t, x, D_x))$ is an $m\times m$ matrix of classical first-order pseudo-differential operators on $\boldsymbol{R}^n_x$ (depending smoothly on $t\in[0, T]$). In addition, we have*

$$\det(\zeta I_m-\sigma_1(A)(t, x, \xi))=\zeta^m+\sum_{\substack{j+|\alpha|=m\\ j<m}} a_{j,\alpha}(t, x)\zeta^j(\sqrt{-1}\,\xi)^\alpha \tag{2.13}$$

for any $\zeta\in\boldsymbol{C}$, $t\in[0, T]$ and $(x, \xi)\in T^*\boldsymbol{R}^n\setminus 0$.

The proof is clear from (2.8) and (2.11). Hence, by (2.13) and (A-3) we can find a smooth invertible $m\times m$ matrix $U(t, x, \xi)$, $(t, x, \xi)\in[0, T]\times T^*\boldsymbol{R}^n\setminus 0$, positively homogeneous of degree zero in ξ, such that

$$U^{-1}(t, x, \xi)\sigma_1(A)(t, x, \xi)U(t, x, \xi)=\begin{pmatrix}\sqrt{-1}\,\lambda_1(t, x, \xi) & & 0\\ & \ddots & \\ 0 & & \sqrt{-1}\,\lambda_m(t, x, \xi)\end{pmatrix} \tag{2.14}$$

for any $(t, x, \xi)\in[0, T]\times T^*\boldsymbol{R}^n\setminus 0$, where λ_j are the roots in (A-3). Thus, the structure of $A(t)$ is clear.

To explain the structure of $B(t)$ $(=B(t, x, D_x))$, let us introduce some classes $S^{p,q}_\kappa$, $\Sigma^{p,q}_\kappa$ and $\hat{\Sigma}^{p,q}_\kappa$ of symbols and corresponding pseudo-differential operators.

Let $p, q\in\boldsymbol{R}$ and $\kappa\in\boldsymbol{N}$. By $S^{p,q}_\kappa$ we denote the space of all functions $a(t, x, \xi)\in C^\infty([0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$ such that for any $\Omega\Subset\boldsymbol{R}^n$, $j\in\boldsymbol{Z}_+$, multi-indices $\alpha, \beta\in\boldsymbol{Z}^n_+$ and $\delta>0$, there is a $C>0$ for which the inequality

$$|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(t, x, \xi)|\leqq C|\xi|^{p-|\beta|}\Big(t+\frac{1}{|\xi|^{1/\kappa}}\Big)^{q-j}$$

holds for any $(t, x)\in[0, T]\times\Omega$ and $|\xi|\geqq\delta$.

By $\$^q$ we denote the space of all functions $\varphi(x, \xi', z)\in C^\infty(\boldsymbol{R}^n\times S^{n-1}\times\overline{\boldsymbol{R}_+})$ for which there is a sequence $(\varphi_{-j})_{j\geqq 0}$, $\varphi_{-j}(x, \xi')\in C^\infty(\boldsymbol{R}^n\times S^{n-1})$, such that

$$\varphi(x, \xi', z)\sim\sum_{j\geqq 0}\varphi_{-j}(x, \xi')z^{q-j}\quad\text{as } z\to+\infty \tag{2.15}$$

holds in the following sense: for any $\Omega\Subset\boldsymbol{R}^n$, $M, k\in\boldsymbol{Z}_+$, $\alpha\in\boldsymbol{Z}^n_+$ and any family $\theta_1, \cdots, \theta_h$ of smooth vector fields on S^{n-1}, there is a $C>0$ such that

$$\Big|\theta_1\cdots\theta_h\partial_z^k\partial_x^\alpha\Big[\varphi-\sum_{j<M}\varphi_{-j}z^{q-j}\Big]\Big|\leqq C(1+|z|)^{q-M-k}$$

holds for any $x\in\Omega$, $\xi'\in S^{n-1}$ and $z\in\overline{\boldsymbol{R}_+}$.

By $\Sigma_\kappa^{p,q}$ we denote the space of all functions $a(t, x, \xi)\in C^\infty([0, T]\times \boldsymbol{R}^n\times \boldsymbol{R}^n)$ for which there exist $\hat{a}(x, \xi', z)\in \$^q$ and $\delta>0$ such that

$$a(t, x, \xi)=|\xi|^{p-q/\kappa}\hat{a}(x, \xi/|\xi|, t|\xi|^{1/\kappa})$$

holds for any $(t, x)\in[0, T]\times \boldsymbol{R}^n$ and $|\xi|\geqq\delta$.

By $\hat{\Sigma}_\kappa^{p,q}$ we denote the space of all functions $a(t, x, \xi)\in S_\kappa^{p,q}$ for which there exists a sequence $(a_j)_{j\geqq 0}$, $a_j\in\Sigma_\kappa^{p,q+j}$, such that

$$a\sim \sum_{j\geqq 0} a_j$$

holds in the following sense: for any $M\geqq 1$ we have

$$\left(a-\sum_{j<M} a_j\right)\in S_\kappa^{p,q+M} .$$

When $\kappa=1$, these classes $S_1^{p,q}$, $\Sigma_1^{p,q}$ and $\hat{\Sigma}_1^{p,q}$ coincide with $S^{p,q}$, $\Sigma^{p,q}$ and $\hat{\Sigma}^{p,q}$, respectively, introduced in [2]. Since all the properties stated in § 2 of [2] carry over (with slight modifications) to the general case $\kappa\geqq 1$, we omit the details of basic properties of $S_\kappa^{p,q}$, $\Sigma_\kappa^{p,q}$ and $\hat{\Sigma}_\kappa^{p,q}$. We may also omit the details of the corresponding classes $OPS_\kappa^{p,q}$, $OP\Sigma_\kappa^{p,q}$ and $OP\hat{\Sigma}_\kappa^{p,q}$ of pseudo-differential operators. However, for the reader's convenience, we recall from [2] the definition of partially regularizing operator as any operator of the form:

$$Rf(t, x)=\int r(t, x, y)f(t, y)dy\,, \quad f(t, x)\in C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))\,,$$

with a smooth kernel $r(t, x, y)\in C^\infty([0, T]\times \boldsymbol{R}^n\times \boldsymbol{R}^n)$.

By using $\hat{\Sigma}_\kappa^{p,q}$, we can explain the structure of $B(t)$ $(=B(t, x, D_x))$ in (2.12).

LEMMA 2. *$B(t, x, D_x)\in OP\hat{\Sigma}_{\nu+1}^{0,0}(m\times m)$, that is, $B(t, x, D_x)$ is an $m\times m$ matrix of pseudo-differential operators belonging to $OP\hat{\Sigma}_{\nu+1}^{0,0}$.*

PROOF. Since $\lambda(\xi)=|\xi|$ for $|\xi|\geqq 1$, we can easily see the following:

$$(t^{\nu+1}\lambda(\xi))(1+t^{\nu+1}\lambda(\xi))^{-1}\in\Sigma_{\nu+1}^{0,0}\subset\hat{\Sigma}_{\nu+1}^{0,0}\,;$$

$$t^{p(j,\alpha)+m-k-j}\xi^\alpha(1+t^{\nu+1}\lambda(\xi))^{-m+j}\in\Sigma_{\nu+1}^{|\alpha|+j-m,\, p(j,\alpha)-k-\nu(m-j)}$$

$$\subset\Sigma_{\nu+1}^{0,\, p(j,\alpha)+m-k-j-(\nu+1)|\alpha|} \quad \text{for } j+|\alpha|\leqq m\,;$$

$$t^{p(j,\alpha)+m-k-j}\xi^\alpha(t^{\nu+1}\lambda(\xi))(1+t^{\nu+1}\lambda(\xi))^{-m+j}\in\Sigma_{\nu+1}^{|\alpha|+1+j-m,\, p(j,\alpha)-k-\nu(m-j)+\nu+1}$$

$$\subset\Sigma_{\nu+1}^{0,\, p(j,\alpha)+m-k-j-(\nu+1)|\alpha|} \quad \text{for } j+|\alpha|<m\,.$$

Since $p(j, \alpha) \geqq k-m+j+(\nu+1)|\alpha|$ holds for any (j, α) (by (A-2)), we have

$$\Sigma_{\nu+1}^{0, p(j,\alpha)+m-k-j-(\nu+1)|\alpha|} \subset \hat{\Sigma}_{\nu+1}^{0,0} .$$

Hence, by combining these with the facts: $\hat{\Sigma}_{\nu+1}^{0,0} \cdot \hat{\Sigma}_{\nu+1}^{0,0} \subset \hat{\Sigma}_{\nu+1}^{0,0}$ and $S_{\mathrm{cl}}^{0} \cdot \hat{\Sigma}_{\nu+1}^{0,0} \subset \hat{\Sigma}_{\nu+1}^{0,0}$ (if S_{cl}^{0} depends smoothly on $t \in [0, T]$), we can obtain $B(t, x, D_x) \in \hat{\Sigma}_{\nu+1}^{0,0}(m \times m)$.

Q. E. D.

§ 3. Decoupling of the reduced system.

By the reduction in § 2, we may discuss the following singular hyperbolic system instead of (1.1) from now on:

$$\mathscr{P}_s v = ((t\partial_t + s)I_m - t^{\nu+1}A(t, x, D_x) - B(t, x, D_x))v = g , \tag{3.1}$$

where $s \in \boldsymbol{Z}_+$,

$$A(t, x, D_x) = \begin{pmatrix} \sqrt{-1}\,\lambda_1(t, x, D_x) & & 0 \\ & \ddots & \\ 0 & & \sqrt{-1}\,\lambda_m(t, x, D_x) \end{pmatrix} \in OPS_{\mathrm{cl}}^{1}(\boldsymbol{R}^n ; m \times m) \tag{3.2}$$

(depending smoothly on $t \in [0, T]$), $B(t, x, D_x) \in OP\hat{\Sigma}_{\nu+1}^{0,0}(m \times m)$, and they may be assumed to be proper. Our hypothesis of (A-3) is stated as follows: $\lambda_i(t, x, \xi)$ $(i=1, \cdots, m)$ are real valued smooth functions on $[0, T] \times T^*\boldsymbol{R}^n \setminus 0$, positively homogeneous of degree 1 in ξ, such that $\lambda_i(t, x, \xi) \neq \lambda_j(t, x, \xi)$ for any $(t, x, \xi) \in [0, T] \times T^*\boldsymbol{R}^n \setminus 0$ and $1 \leqq i \neq j \leqq m$.

The purpose of this section is to prove the following result.

THEOREM 3. *Let $\mathscr{P}_s$ be as above. Then, there exist proper operators $Q, \tilde{B} \in OP\hat{\Sigma}_{\nu+1}^{0,0}(m \times m)$ which satisfy the following conditions.*

(1) *Q is invertible in $OP\hat{\Sigma}_{\nu+1}^{0,0}(m \times m)$ modulo a partially regularizing operator.*

(2) *For any $\omega \Subset \boldsymbol{R}^n$, there is a $\delta > 0$ such that the symbol $\tilde{b}(t, x, \xi)$ of $\tilde{B}$ is diagonal on $\{(t, x, \xi) ; x \in \omega$ and $t|\xi|^{1/(\nu+1)} \geqq \delta\}$.*

(3) *Put*

$$\tilde{\mathscr{P}}_s = (t\partial_t + s)I_m - t^{\nu+1}A(t, x, D_x) - \tilde{B}(t, x, D_x) . \tag{3.3}$$

Then, we have

$$\mathscr{P}_s Q - Q\tilde{\mathscr{P}}_s = \text{a partially regularizing operator.} \tag{3.4}$$

Let us first prove a weaker result.

PROPOSITION 1. *There exist proper operators* $Q, \tilde{B} \in \hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$ *which satisfy* (1), (2) *in Theorem* 3 *and*

$$\mathcal{P}_s Q - Q\tilde{\mathcal{P}}_s \in OPS^{0;\infty}_{\nu+1}(m\times m)\,, \tag{3.5}$$

where $S^{0;\infty}_{\nu+1} = \bigcap_{q=0}^{\infty} S^{0;q}_{\nu+1}$.

PROOF. To obtain this, it is sufficient to find Q and $\tilde{B}$ at the level of formal symbolic calculus in $\hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$.

Put

$$a(t, x, \xi) = A(t, x, \xi/|\xi|) \tag{3.6}$$

(where $A(t, x, \xi)$ is the symbol of $A(t, x, D_x)$ in (3.1)). Since $a(t, x, \xi) \in S^0_{\mathrm{cl}}(\boldsymbol{R}^n\,; m\times m)$ depends smoothly on $t \in [0, T]$, we can easily see that $a(t, x, \xi) \in \hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$ with the asymptotic expansion:

$$\begin{cases} a(t, x, \xi) \sim \sum_{j\geq 0} a_j(t, x, \xi)\,, \\ a_j(t, x, \xi) = |\xi|^{-j/(\nu+1)} \hat{a}_j(t, \xi/|\xi|, t|\xi|^{1/(\nu+1)}) \in \Sigma^{0;j}_{\nu+1}(m\times m)\,, \\ \hat{a}_j(x, \xi', z) = \dfrac{1}{j!}(\partial_t^j a)(0, x, \xi') z^j \in \$^j(m\times m)\,. \end{cases} \tag{3.7}$$

Let $b(t, x, \xi)$ be the symbol of $B(t, x, D_x)$. Since $B(t, x, D_x) \in \hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$, we have an asymptotic expansion of the form

$$\begin{cases} b \sim \sum_{j\geq 0} b_j\,, \quad b_j \in \Sigma^{0;j}_{\nu+1}(m\times m)\,, \\ b_j(t, x, \xi) = |\xi|^{-j/(\nu+1)} \hat{b}_j(x, \xi/|\xi|, t|\xi|^{1/(\nu+1)})\,, \\ \hat{b}_j(x, \xi', z) \in \$^j(m\times m)\,. \end{cases} \tag{3.8}$$

Denote by $q(t, x, \xi)$, $\tilde{b}(t, x, \xi) \in \hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$ the symbols of the unknown operators $Q(t, x, D_x)$, $\tilde{B}(t, x, D_x) \in \hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$, and let their asymptotic expansions (in $\hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$) be as follows:

$$\begin{cases} q \sim \sum_{j\geq 0} q_j\,, \quad q_j \in \Sigma^{0;j}_{\nu+1}(m\times m)\,, \\ q_j(t, x, \xi) = |\xi|^{-j/(\nu+1)} \hat{q}_j(x, \xi/|\xi|, t|\xi|^{1/(\nu+1)})\,, \\ \hat{q}_j(x, \xi', z) \in \$^j(m\times m)\,, \quad j \geqq 0\,; \end{cases}$$

$$\begin{cases} \tilde{b} \sim \sum_{j\geq 0} \tilde{b}_j\,, \quad \tilde{b}_j \in \Sigma^{0;j}_{\nu+1}(m\times m)\,, \\ \tilde{b}_j(t, x, \xi) = |\xi|^{-j/(\nu+1)} \hat{\tilde{b}}(x, \xi/|\xi|, t|\xi|^{1/(\nu+1)})\,, \\ \hat{\tilde{b}}_j(x, \xi', z) \in \$^j(m\times m)\,, \quad j \geqq 0\,. \end{cases}$$

To obtain Proposition 1 it is sufficient to find (matrix) functions $\hat{q}_j, \hat{\tilde{b}}_j \in \$^j$ $(j \geqq 0)$ which satisfy the following conditions:

(i) $\hat{q}_0(x, \xi', z)$ is invertible on $\boldsymbol{R}^n \times S^{n-1} \times \overline{\boldsymbol{R}_+}$.

(ii) For any $\omega \Subset \boldsymbol{R}^n$ there is a $\delta > 0$ such that $\hat{\tilde{b}}_j(x, \xi', z)$ $(j \geqq 0)$ are diagonal on $\{(x, \xi', z)\,;\, x \in \omega \text{ and } z \geqq \delta\}$.

(iii) By putting

$$\tilde{\mathcal{P}}_{s,l} = (t\partial_t + s)I_m - t^{\nu+1}A(t, x, D_x) - \Big(\sum_{j=0}^{l} \tilde{b}_j(t, x, D_x)\Big), \tag{3.9}$$

we have for any $M \geqq 0$

$$\mathcal{P}_s\Big(\sum_{l=0}^{M} q_l(t, x, D_x)\Big) - \Big(\sum_{l=0}^{M} q_l(t, x, D_x)\Big)\tilde{\mathcal{P}}_{s,M} \in OP\hat{\Sigma}_{\nu+1}^{0;M+1}(m \times m)\,. \tag{3.10}$$

Our next step will consist in obtaining (3.10) through a family of recursive differential equations involving the symbols q_j and $\tilde{b}_j$. To this purpose, it is essential that the following relations hold:

$$t^{\nu+1}[A(t, x, D_x), q_j(t, x, D_x)] \in OP\hat{\Sigma}_{\nu+1}^{0;j}(m \times m)\,, \quad j = 0, 1, \cdots. \tag{3.11}$$

It is not difficult to verify that (3.11) is satisfied if denoting by $\hat{q}_j(x, \xi', z) \sim \sum_{k \geq 0} \hat{q}_{j,-k}(x, \xi') z^{j-k}$ the asymptotic expansion of $\hat{q}_j \in \$^j$, we have

$$\begin{cases} \hat{q}_{0,0} = I_m\,, \quad \hat{q}_{0,-1} = \cdots = \hat{q}_{0,-\nu} = 0\,, \\ \hat{q}_{j,0} = \hat{q}_{j,-1} = \cdots = \hat{q}_{j,-\nu} = 0 \quad \text{for } j \geqq 1\,. \end{cases} \tag{3.12}$$

Under conditions (3.12), relations (3.10) $(M = 0, 1, 2, \cdots)$ can be expressed by the following recursive family of differential equations:

$$\begin{cases} t\partial_t q_0(t, x, \xi) - t^{\nu+1}|\xi|[a_0(x, \xi), q_0(t, x, \xi)] \\ \qquad - b_0(t, x, \xi) q_0(t, x, \xi) + q_0(t, x, \xi)\tilde{b}_0(t, x, \xi) = 0\,, \\ t\partial_t q_M(t, x, \xi) - t^{\nu+1}|\xi|[a_0(x, \xi), q_M(t, x, \xi)] \\ \qquad - b_0(t, x, \xi) q_M(t, x, \xi) + q_M(t, x, \xi)\tilde{b}_0(t, x, \xi) \\ \qquad\qquad + q_0(t, x, \xi)\tilde{b}_M(t, x, \xi) = \phi_M(t, x, \xi)\,, \\ \qquad\qquad M = 1, 2, 3, \cdots, \end{cases} \tag{3.13}$$

where $a_0(x, \xi) = a(0, x, \xi)$ (in (3.6)), and $\phi_M(t, x, \xi) \in \Sigma_{\nu+1}^{0;M}(m \times m)$ is a function determined by $q_0, \tilde{b}_0 \in \Sigma_{\nu+1}^{0;0}(m \times m), \cdots, q_{M-1}, \tilde{b}_{M-1} \in \Sigma_{\nu+1}^{0;M-1}(m \times m)$.

By putting $z = t|\xi|^{1/(\nu+1)}$ relations (3.13) can be reexpressed in the form

$$\begin{cases} z\partial_z\hat{q}_0(x,\xi',z)-z^{\nu+1}[a_0(x,\xi'),\hat{q}_0(x,\xi',z)] \\ \qquad -\hat{b}_0(x,\xi',z)\hat{q}_0(x,\xi',z)+\hat{q}_0(x,\xi',z)\hat{\tilde{b}}(x,\xi',z)=0\,, \\ z\partial_z\hat{q}_M(x,\xi',z)-z^{\nu+1}[a_0(x,\xi'),\hat{q}_M(x,\xi',z)] \\ \qquad -\hat{b}_0(x,\xi',z)\hat{q}_M(x,\xi',z)+\hat{q}_M(x,\xi',z)\hat{\tilde{b}}_0(x,\xi',z) \\ \qquad\qquad +\hat{q}_0(x,\xi',z)\hat{\tilde{b}}_M(x,\xi',z)=\hat{\phi}_M(x,\xi',z)\,, \\ \qquad\qquad M=1,2,3,\cdots, \end{cases} \tag{3.14}$$

where $\hat{\phi}_M(x,\xi',z)\in\mathcal{S}^M(m\times m)$ is given by the relation:

$$\phi_M(t,x,\xi)=|\xi|^{-M/(\nu+1)}\hat{\phi}_M(x,\xi/|\xi|,t|\xi|^{1/(\nu+1)})\,.$$

To conclude, we have reduced the problem to finding $\hat{q}_j, \hat{\tilde{b}}_j\in\mathcal{S}^j(m\times m)$ $(j\geqq 0)$ which solve (3.14). As a consequence, Proposition 1 follows from Lemmas 3 and 4 given below.

LEMMA 3. *Let $a_0(x,\xi')$ and $b_0(x,\xi',z)$ be as above. Then, there exist $q(x,\xi',z)$, $\tilde{b}(x,\xi',z)\in\mathcal{S}^0(m\times m)$ with $q(x,\xi',z)\sim\sum_{j\geq 0}q_{-j}(x,\xi')z^{-j}$, $\tilde{b}(x,\xi',z)\sim\sum_{j\geq 0}\tilde{b}_{-j}(x,\xi')z^{-j}$ (as $z\to+\infty$) such that the following conditions are satisfied*:

(i) *$q(x,\xi',z)$ is invertible on $\boldsymbol{R}^n\times S^{n-1}\times\overline{\boldsymbol{R}_+}$.*

(ii) *$q_0(x,\xi')=I_m$, $q_{-1}(x,\xi')=\cdots=q_{-\nu}(x,\xi')=0$ and all the diagonal terms of $q_{-j}(x,\xi')$ $(j\geqq\nu+1)$ vanish.*

(iii) *For any $\omega\Subset\boldsymbol{R}^n$ there is a $\delta>0$ such that $\tilde{b}(x,\xi',z)$ is diagonal on $\{(x,\xi',z)\,;\,x\in\omega$ and $z\geqq\delta\}$.*

(iv) *$\tilde{b}_{-j}(x,\xi')$ $(j\geqq 0)$ are diagonal on $\boldsymbol{R}^n\times S^{n-1}$.*

(v) *The following equation is satisfied*:

$$\begin{aligned} &z\partial_z q(x,\xi',z)-z^{\nu+1}[a_0(x,\xi'),q(x,\xi',z)] \\ &\qquad -b_0(x,\xi',z)q(x,\xi',z)+q(x,\xi',z)\tilde{b}(x,\xi',z)=0\,. \end{aligned} \tag{3.15}$$

PROOF. It is easy to check that the formal power series $\sum_{j\geq 0}q_{-j}(x,\xi')z^{-j}$ and $\sum_{j\geq 0}\tilde{b}_{-j}(x,\xi')z^{-j}$ satisfying equation (3.15) are uniquely determined provided conditions (ii) and (iv) hold. Therefore, we can construct $q^*(x,\xi',z)$, $\hat{b}(x,\xi',z)\in\mathcal{S}^0(m\times m)$ such that $q^*\sim\sum_{j\geq 0}q_{-j}z^{-j}$, $\hat{b}\sim\sum_{j\geq 0}\tilde{b}_{-j}z^{-j}$, $\hat{b}(x,\xi',z)$ is diagonal, and

$$z\partial_z q^*-z^{\nu+1}[a_0,q^*]-b_0q^*+q^*\hat{b}\in\mathcal{S}^{-\infty}(m\times m)\,.$$

To prove Lemma 3, we must get rid of the $\mathcal{S}^{-\infty}(m\times m)$-part. Put $-g=z\partial_z q^*-z^{\nu+1}[a_0,q^*]-b_0q^*+q^*\hat{b}$, and let us consider the equation

$$z\partial_z\varphi-z^{\nu+1}[a_0,\varphi]-b_0\varphi+\varphi\hat{b}=g \tag{3.16}$$

on $\{(x, \xi', z); z>0\}$. Since (3.16) is a non-degenerate ordinary differential equation on $\{z>0\}$, by the same argument as in the proof of Lemma 3.1 in [2] we can obtain a solution $\varphi(x, \xi', z) \in C^{\infty}(\boldsymbol{R}^n \times S^{n-1} \times \{z>0\}; m \times m)$ of (3.16) such that $\varphi(x, \xi', z) \sim 0$ (as $z \to +\infty$) in the same sense as (2.15). Therefore, by putting $\hat{q}=q^*+\varphi$ we have $\hat{q}(x, \xi', z) \in C^{\infty}(\boldsymbol{R}^n \times S^{n-1} \times \{z>0\};$ $m \times m)$ with the same asymptotic expansion as q^* such that

$$z\partial_z \hat{q} - z^{\nu+1}[a_0, \hat{q}] - b_0 \hat{q} + \hat{q}\hat{b} = 0$$

on $\{(x, \xi', z); z>0\}$.

Since the asymptotic expansion of $\hat{q}$ satisfies the condition (ii), for any $\omega \Subset \boldsymbol{R}^n$ there is a $\delta>0$ such that

$$\sup_{\substack{x \in \omega, \xi' \in S^{n-1} \\ z \geqq \delta}} |I_m - \hat{q}(x, \xi', z)| < \frac{1}{2} .$$

Hence, we can choose a cut-off function $\chi(x, z)$ so that by defining

$$q(x, \xi', z) = \chi(x, z) I_m + (1-\chi(x, z)) \hat{q}(x, \xi', z)$$

we have $|I_m - q(x, \xi', z)| < 1/2$ on $\boldsymbol{R}^n \times S^{n-1} \times \overline{\boldsymbol{R}_+}$. The matrix $q(x, \xi', z) \in \$^0(m \times m)$ satisfies (i) and (ii). Moreover, by putting

$$\tilde{b} = \hat{b} - q^{-1}(z\partial_z q - z^{\nu+1}[a_0, q] - b_0 q + q\hat{b})$$

we can conclude that $\tilde{b}(x, \xi', z) \in \$^0(m \times m)$ and satisfies (iii), (iv) and (v). To prove (iii) and (iv) we use the remark that for the function $f = z\partial_z q - z^{\nu+1}[a_0, q] - b_0 q + q\hat{b}$ we have $\mathrm{supp}(f) \subset \{(x, \xi', z); \chi(x, z) \neq 0\}$. Q. E. D.

LEMMA 4. *Let* $a_0(x, \xi')$, $b_0(x, \xi', z)$, $q(x, \xi', z)$ *and* $\tilde{b}(x, \xi', z)$ *be as in Lemma* 3. *Let* $M \geqq 1$ *and let* $c(x, \xi', z) \in \$^M(m \times m)$ *with* $c(x, \xi', z) \sim \sum_{j \geq 0} c_{-j}(x, \xi') z^{M-j}$ *(as* $z \to +\infty$*). Then, there exist* $Q(x, \xi', z)$, $\tilde{B}(x, \xi', z) \in \$^M(m \times m)$ *with* $Q(x, \xi', z) \sim \sum_{j \geq 0} Q_{-j}(x, \xi') z^{M-j}$, $\tilde{B}(x, \xi', z) \sim \sum_{j \geq 0} \tilde{B}_{-j}(x, \xi') z^{M-j}$ *(as* $z \to +\infty$*) such that the following conditions are satisfied:*

(i) $Q_0(x, \xi') = Q_{-1}(x, \xi') = \cdots = Q_{-\nu}(x, \xi') = 0$ *and all the diagonal terms of* $Q_{-j}(x, \xi')$ $(j \geqq \nu+1)$ *vanish.*

(ii) $\tilde{B}(x, \xi', z)$ *is diagonal on* $\{(x, \xi', z); x \in \omega$ *and* $z \geqq \delta\}$ *for any* $\omega \Subset \boldsymbol{R}^n$ *with the same* $\delta>0$ *as in* (iii) *of Lemma* 3.

(iii) $\tilde{B}_{-j}(x, \xi')$ $(j \geqq 0)$ *are diagonal on* $\boldsymbol{R}^n \times S^{n-1}$.

(iv) *The following equation is satisfied:*

$$\begin{aligned} z\partial_z Q(x, \xi', z) - z^{\nu+1}[a_0(x, \xi'), Q(x, \xi', z)] \\ - b_0(x, \xi', z) Q(x, \xi', z) + Q(x, \xi', z) \tilde{b}(x, \xi', z) \\ + q(x, \xi', z) \tilde{B}(x, \xi', z) = c(x, \xi', z) . \end{aligned} \tag{3.17}$$

PROOF. Since the coefficients $Q_{-j}(x, \xi'), \tilde{B}_{-j}(x, \xi') \in C^\infty(\boldsymbol{R}^n \times S^{n-1}; m\times m)$ ($j \geqq 0$) are uniquely determined from (i), (iii) and (3.17) at a formal level, we can conclude in the same way as in Lemma 3. Q. E. D.

Having proved Proposition 1, to prove Theorem 3, we must get rid of the $S_{\nu+1}^{0;-\infty}(m\times m)$-part (note that operators with symbol in $S_{\nu+1}^{0;-\infty}(m\times m)$ are not partially regularizing).

PROOF OF THEOREM 3. By Proposition 1 and its proof, we have constructed two operators $\hat{Q}, \hat{B} \in OP\hat{\Sigma}_{\nu+1}^{0;0}(m\times m)$ with symbols $\hat{q}(t, x, \xi), \hat{b}(t, x, \xi)$ which satisfy the following conditions:

(i) $\hat{q}(t, x, \xi)$ can be splitted as the form

$$\hat{q}(t, x, \xi) = q_0(t, x, \xi) + q'(t, x, \xi) \tag{3.18}$$

for some invertible matrix $q_0(t, x, \xi) \in \Sigma_{\nu+1}^{0;0}(m\times m)$ and $q'(t, x, \xi) \in \hat{\Sigma}_{\nu+1}^{-1;-\nu}(m\times m) \subset \hat{\Sigma}_{\nu+1}^{-1/(\nu+1),0}(m\times m)$.

(ii) For any $\omega \Subset \boldsymbol{R}^n$ there is a $\delta > 0$ such that $\hat{b}(t, x, \xi)$ is diagonal on $\{(t, x, \xi); x \in \omega \text{ and } t|\xi|^{1/(\nu+1)} \geqq \delta\}$.

(iii) $\hat{Q}$ and $\hat{B}$ satisfy

$$\mathcal{P}_s\hat{Q} - \hat{Q}((t\partial_t + s)I_m - t^{\nu+1}A - \hat{B}) = R \in OPS_{\nu+1}^{0;-\infty}(m\times m). \tag{3.19}$$

(iv) There is a cut-off function $\chi(x, z)$ such that the diagonal terms of $q_0(t, x, \xi)$ do not vanish on the support of the function $(1-\chi(x, t|\xi|^{1/(\nu+1)})$.

To get rid of the term R in (3.19), it will be sufficient to construct two operators $S \in OPS_f^{-1}(m\times m)$, $V \in OPS_f^0(m\times m)$ with symbols $s(t, x, \xi)$ and $v(t, x, \xi)$ respectively such that the following conditions are satisfied:

(v) $v(t, x, \xi)$ is diagonal on $[0, T]\times \boldsymbol{R}^n \times \boldsymbol{R}^n$.

(vi) S and V satisfy

$$\begin{aligned} t\frac{\partial S}{\partial t} - [t^{\nu+1}A, S] - BS + S\hat{B} + SV + \hat{Q}V \\ + (1-\chi(x, t|D_x|^{1/(\nu+1)}))R \in OPS_f^{-\infty}(m\times m). \end{aligned} \tag{3.20}$$

Here, we use the notation S_f^p to denote the space of all functions $\varphi(t, x, \xi) \in C^\infty([0, T]\times \boldsymbol{R}^n \times \boldsymbol{R}^n)$ such that for any $\Omega \Subset \boldsymbol{R}^n$, $M, j \in \boldsymbol{Z}_+$ and $\alpha, \beta \in \boldsymbol{Z}_+^n$, there is a $C>0$ for which the inequality

$$|\partial_t^j \partial_x^\alpha \partial_\xi^\beta \varphi(t, x, \xi)| \leqq Ct^M(1+|\xi|)^{p-|\beta|}$$

holds for any $(t, x) \in [0, T]\times \Omega$ and $\xi \in \boldsymbol{R}^n$.

If such operators S and V as above can be found, by defining

$$Q=\hat{Q}+(1-\chi(x, t|D_x|^{1/(\nu+1)}))S,$$
$$\tilde{B}=\hat{B}+(1-\chi(x, t|D_x|^{1/(\nu+1)}))V,$$

we obtain the desired operators Q, $\tilde{B}\in OP\hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$ as stated in Theorem 3. To verify (1)~(3) in Theorem 3, we have only to use (i)~(iii), (v), (vi) and the following inclusions (whose proof is left to the reader):

$$\chi(x, t|D_x|^{1/(\nu+1)})\cdot OPS^{0;\infty}_{\nu+1}\subset OPS^{-\infty}_{1,0},$$
$$(1-\chi(x, t|D_x|^{1/(\nu+1)}))\cdot OPS^{0;\infty}_{\nu+1}\subset OPS^{0}_{f},$$
$$\chi(x, t|D_x|^{1/(\nu+1)})\cdot OPS^{p}_{f}\subset OPS^{-\infty}_{1,0},$$
$$(1-\chi(x, t|D_x|^{1/(\nu+1)}))\cdot OPS^{p}_{f}\subset OPS^{p,\infty}_{\nu+1}.$$

The construction of S and V is done as follows. Put

$$\begin{cases} s(t, x, \xi)\sim \sum_{j\geq 0} s_{-1-j/(\nu+1)}(t, x, \xi), \\ s_{-1-j/(\nu+1)}(t, x, \xi)\in S_f^{-1-j/(\nu+1)}(m\times m), \end{cases} \tag{3.21}$$

$$\begin{cases} v(t, x, \xi)\sim \sum_{j\geq 0} v_{-j/(\nu+1)}(t, x, \xi), \\ v_{-j/(\nu+1)}(t, x, \xi)\in S_f^{-j/(\nu+1)}(m\times m), \end{cases} \tag{3.22}$$

and impose the following conditions:

(vii) All the diagonal terms of $s_{-1-j/(\nu+1)}(t, x, \xi)$ $(j\geqq 0)$ vanish.

(viii) $v_{-j/(\nu+1)}(t, x, \xi)$ $(j\geqq 0)$ are diagonal.

Under conditions (3.18), (3.21) and (3.22), the equation (3.20) modulo $OPS_f^{-1/(\nu+1)}(m\times m)$ is expressed in the form

$$\begin{aligned} -[t^{\nu+1}A(t, x, \xi), s_{-1}(t, x, \xi)]+q_0(t, x, \xi)v_0(t, x, \xi) \\ +(1-\chi(x, t|\xi|^{1/(\nu+1)}))r(t, x, \xi)=0 \quad \text{modulo } S_f^{-1/(\nu+1)}(m\times m). \end{aligned} \tag{3.23}$$

Now, by using (iv) we can uniquely find $s_{-1}(t, x, \xi)\in S_f^{-1}(m\times m)$ and $v_0(t, x, \xi)\in S_f^0(m\times m)$ satisfying (vii), (viii) and equation (3.23) (for details, see [2]).

Proceeding by induction on j, we can construct $s_{-1-j/(\nu+1)}\in S_f^{1-j/(\nu+1)}(m\times m)$, $v_{-j/(\nu+1)}\in S_f^{-j/(\nu+1)}(m\times m)$ for $j\geqq 0$. Thus, we can obtain $s\in S_f^{-1}(m\times m)$ and $v\in S_f^0(m\times m)$. Q. E. D.

§ 4. Construction of parametrices.

In this section, we construct a right and a left parametrix for the system $\tilde{\mathcal{P}}_s$ defined in Theorem 3, under the assumption that s is sufficiently large. To simplify notation, we drop the $\tilde{}$.

Let us state precisely our situation. The operator treated here is of the following type:

$$\mathscr{P}_s=(t\partial_t+s)I_m-t^{\nu+1}A(t, x, D_x)-B(t, x, D_x)\,, \tag{4.1}$$

where A is the matrix given by (3.2) with the condition (A-3), and $B\in OP\hat{\Sigma}^{0,0}_{\nu+1}(m\times m)$ with a symbol $b(t, x, \xi)\sim\sum_{j\geq 0}b_j(t, x, \xi)$, $b_j\in\Sigma^{0,j}_{\nu+1}(m\times m)$ satisfying the following condition: for any $\omega\Subset \boldsymbol{R}^n$ there is a $\delta>0$ such that $b(t, x, \xi)$ and $b_j(t, x, \xi)$ $(j\geqq 0)$ are diagonal on $\{(t, x, \xi)\,;\, x\in\omega$ and $t|\xi|^{1/(\nu+1)}\geqq\delta\}$. In addition, we may assume that A and B are proper, and that

$$b_0(t, x, \xi) \text{ is bounded on } [0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n\,. \tag{4.2}$$

In order to state our results, we need to define some symbol classes to which the amplitudes of the parametrices will belong.

Let $p, q\in\boldsymbol{R}$ and $\kappa\in\boldsymbol{N}$. By $HS^{p,q}_{\kappa}$ we denote the space of all functions $a(\rho, t, x, \xi)\in C^{\infty}((0, 1]\times[0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$ such that for any $\Omega\Subset\boldsymbol{R}^n$, $j, l\in\boldsymbol{Z}_+$, $\alpha, \beta\in\boldsymbol{Z}^n_+$ and $\varepsilon, \delta>0$, there is a $C>0$ for which

$$|\rho^{\varepsilon}(\rho\partial_\rho)^l\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(\rho, t, x, \xi)|\leqq C|\xi|^{p-|\beta|}\left(t+\frac{1}{|\xi|^{1/\kappa}}\right)^{q-j}$$

holds for any $(\rho, t, x)\in(0, 1]\times[0, T]\times\Omega$ and $|\xi|\geqq\delta$.

By $H\$^q$ we denote the space of all functions $\varphi(\rho, x, \xi', z)\in C^{\infty}((0, 1]\times\boldsymbol{R}^n\times S^{n-1}\times\overline{\boldsymbol{R}_+})$ such that for any $\Omega\Subset\boldsymbol{R}^n$, $l, k\in\boldsymbol{Z}_+$, $\alpha\in\boldsymbol{Z}^n_+$, $\varepsilon>0$ and any family $\theta_1, \cdots, \theta_h$ of smooth vector fields on S^{n-1}, there is a $C>0$ for which

$$|\theta_1\cdots\theta_h\rho^{\varepsilon}(\rho\partial_\rho)^l\partial_z^k\partial_x^\alpha\varphi(\rho, x, \xi', z)|\leqq C(1+|z|)^{q-k}$$

holds for any $\rho\in(0, 1]$, $x\in\Omega$, $\xi'\in S^{n-1}$ and $z\in\overline{\boldsymbol{R}_+}$.

By $H\Sigma^{p,q}_{\kappa}$ we denote the space of all functions $a(\rho, t, x, \xi)\in C^{\infty}((0, 1]\times[0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n)$ for which there exist $\hat{a}(\rho, x, \xi', z)\in H\$^q$ and $\delta>0$ such that

$$a(\rho, t, x, \xi)=|\xi|^{p-q/\kappa}\hat{a}(\rho, x, \xi/|\xi|, t|\xi|^{1/\kappa})$$

holds for any $(\rho, t, x)\in(0, 1]\times[0, T]\times\boldsymbol{R}^n$ and $|\xi|\geqq\delta$.

By $H\hat{\Sigma}^{p,q}_{\kappa}$ we denote the space of all functions $a(\rho, t, x, \xi)\in HS^{p,q}_{\kappa}$ for which there exists a sequence $(a_j)_{j\geq 0}$, $a_j\in H\Sigma^{p,q+j}_{\kappa}$, such that

$$a\sim\sum_{j\geq 0}a_j$$

holds in the following sense: for any $M\geqq 1$ we have

$$\left(a-\sum_{j<M}a_j\right)\in HS^{p,q+M}_{\kappa}\,.$$

When $\kappa=1$, the above classes were already defined in [2] to which we refer for some basic properties.

Now, let us state our construction of a right parametrix for $\mathcal{P}_s$ in (4.1). Let $\varphi_j(t, s, x, \xi)\in C^\infty([0, T]\times[0, T]\times \boldsymbol{R}^n\times \boldsymbol{R}^n\setminus 0)$ be the solution of

$$\begin{cases} \partial_t\varphi_j(t, s, x, \xi)=t^\nu\lambda_j(t, x, \nabla_x\varphi_j(t, s, x, \xi)) , \\ \varphi_j|_{t=s}=x\cdot\xi \end{cases} \tag{4.3}$$

$(j=1, \cdots, m)$. Then, $\varphi_j(t, s, x, \xi)$ is real valued and positively homogeneous of degree 1 in ξ. Put

$$\phi_j(\rho, t, x, \xi)=\varphi_j(t, \rho t, x, \xi)\,, \quad j=1, \cdots, m \tag{4.4}$$

for any $\rho\in[0, 1]$, put

$$e^{i\phi(\rho,t,x,\xi)}=\begin{pmatrix} e^{i\phi_1(\rho,t,x,\xi)} & & 0 \\ & \ddots & \\ 0 & & e^{i\phi_m(\rho,t,x,\xi)} \end{pmatrix}, \tag{4.5}$$

and for any $h(\rho, t, x, \xi)\in H\hat{\Sigma}^{0,0}_{\nu+1}(m\times m)$ define the operator $E(h)$: $C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m\to C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))^m$ by

$$E(h\,;f)=\int_0^1\int_{\boldsymbol{R}^n_\xi} e^{i\phi(\rho,t,x,\xi)}h(\rho, t, x, \xi)\hat{f}(\rho t, \xi)d\rho d\xi$$

for $f(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m$.

The following result holds.

THEOREM 4. *Let $\mathcal{P}_s$ be the operator in* (4.1) *and assume that s is sufficiently large. Then, there exists a matrix $h(\rho, t, x, \xi)\in H\hat{\Sigma}^{0,0}_{\nu+1}(m\times m)$ such that*

$$\mathcal{P}_sE(h)-id : C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m \longrightarrow C^\infty([0, T]\times \boldsymbol{R}^n)^m\,.$$

The following lemma will play an important role in the proof of this theorem.

LEMMA 5. *Let $\phi(\rho, t, x, \xi)$ denote any of the ϕ_j's in* (4.4). *Then, we have the following*:

(1) $\phi(\rho, t, x, \xi)\in H\hat{\Sigma}^{1,0}_{\nu+1}$ *with the asymptotic expansion* $\phi(\rho, t, x, \xi)\sim \sum_{k\geq 0}\phi^{(k)}(\rho, t, x, \xi)$, $\phi^{(k)}\in H\Sigma^{1,k}_{\nu+1}$ $(k\geq 0)$ *such that*

$$\begin{cases} \phi^{(0)}=x\cdot\xi\,, \\ \phi^{(1)}=\cdots=\phi^{(\nu)}=0 \quad (if\ \nu\geqq 1)\,, \\ \phi^{(\nu+1)}=\dfrac{(1-\rho^{\nu+1})}{(\nu+1)}\lambda(0,x,\xi/|\xi|)(t|\xi|^{1/(\nu+1)})^{\nu+1}\,, \\ \phi^{(k)}=\dfrac{1}{k!}(\partial_t^k\phi)(\rho,0,x,\xi/|\xi|)|\xi|^{1-k/(\nu+1)}(t|\xi|^{1/(\nu+1)})^k \quad (k\geqq\nu+2)\,. \end{cases}$$

(2) *For any cut-off function* $\chi(x,z)$, *we have*

$$e^{-ix\cdot\xi}\chi(x,t|\xi|^{1/(\nu+1)})e^{i\phi(\rho,t,x,\xi)}\in H\hat{\Sigma}^{0,0}_{\nu+1}\,.$$

PROOF. To obtain (1), it is sufficient to show that $\phi(\rho,t,x,\xi)$ has the form

$$\phi(\rho,t,x,\xi)=x\cdot\xi+\frac{(1-\rho^{\nu+1})}{(\nu+1)}t^{\nu+1}\lambda(0,x,\xi)+t^{\nu+2}\Phi(\rho,t,x,\xi) \tag{4.6}$$

for some $\Phi(\rho,t,x,\xi)\in C^\infty([0,1]\times[0,T]\times \boldsymbol{R}^n\times\boldsymbol{R}^n\setminus 0)$. This can be verified as follows. Let $\varphi(t,s,x,\xi)\sim\sum_{i,j\geq 0}a_{i,j}(x,\xi)t^is^j$ be the Taylor expansion in (t,s) of $\varphi(t,s,x,\xi)$. Then, by (4.3) we have

$$\begin{cases} t^\nu\lambda(t,x,\nabla_x\varphi(t,s,x,\xi))\sim\sum_{i,j\geq 0}ia_{i,j}t^{i-1}s^j\,, \\ x\cdot\xi\sim\sum_{i,j\geq 0}a_{i,j}t^{i+j}\,. \end{cases} \tag{4.7}$$

Moreover, by putting $t=s$ in the first relation in (4.7) we have

$$t^\nu\lambda(t,x,\xi)\sim\sum_{i,j\geq 0}ia_{i,j}t^{i-1+j}\,. \tag{4.8}$$

Therefore, by comparing the coefficients in (4.7) and (4.8) we have

$$\begin{cases} a_{i,j}=0 \qquad \text{for } 1\leqq i\leqq\nu \text{ and } j\geqq 0\,, \\ a_{0,0}=x\cdot\xi\,, \\ \sum_{i+j=l}a_{i,j}=0 \qquad \text{for } l\geqq 1\,, \\ \sum_{i+j=\nu+1}ia_{i,j}=\lambda(0,x,\xi)\,. \end{cases}$$

Hence, we obtain

$$\begin{cases} a_{0,0}=x\cdot\xi\,, \\ a_{i,j}=0 \quad \text{for } 1\leqq i+j\leqq\nu\,, \\ a_{\nu+1,0}=\dfrac{\lambda(0,x,\xi)}{(\nu+1)},\ a_{\nu,1}=\cdots=a_{1,\nu}=0,\ a_{0,\nu+1}=-\dfrac{\lambda(0,x,\xi)}{(\nu+1)}. \end{cases}$$

This implies (4.6), because $\phi(\rho, t, x, \xi)=\varphi(t, \rho t, x, \xi)$.

By (1), we have

$$\begin{aligned} &\chi(x, t|\xi|^{1/(\nu+1)})e^{i\phi^{(\nu+1)}(\rho,t,x,\xi)}\in H\Sigma^{0,0}_{\nu+1}\,, \\ &\theta(\rho, t, x, \xi)=(\phi-x\cdot\xi-\phi^{(\nu+1)})(\rho, t, x, \xi)\in H\hat{\Sigma}^{1,\nu+2}_{\nu+1}\,. \end{aligned} \tag{4.9}$$

Therefore, $(i\theta)^k/k!\in H\hat{\Sigma}^{k,(\nu+2)k}_{\nu+1}$ with an asymptotic expansion $(i\theta)^k/k!\sim\sum_{j\geq 0}\theta^{(k)}_j$, $\theta^{(k)}_j\in H\Sigma^{k,(\nu+2)k+j}_{\nu+1}$ $(j\geqq 0)$. Since $\chi\theta^{(k)}_j\in H\Sigma^{0,k+j}_{\nu+1}$ for any $k, j\geqq 0$, it follows from (4.9) that $\chi e^{i\phi^{(\nu+1)}}\theta^{(k)}_j\in H\Sigma^{0,k+j}_{\nu+1}$. Hence, (2) is a consequence of the following relations:

$$\begin{aligned} e^{-ix\cdot\xi}\chi e^{i\phi} &=\chi e^{i\phi^{(\nu+1)}}e^{i\theta} \\ &=\chi e^{i\phi^{(\nu+1)}}\sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!} \\ &\sim\sum_{r=0}^{\infty}\Big(\sum_{k+j=r}\chi e^{i\phi^{(\nu+1)}}\theta^{(k)}_j\Big). \end{aligned}$$

Q. E. D.

To prove Theorem 4, let us first show the following weaker result.

PROPOSITION 2. *Let $\mathcal{P}_s$ be as in* (4.1) *and assume that s is sufficiently large. Then, there exists a matrix $h(\rho, t, x, \xi)\in H\hat{\Sigma}^{0,0}_{\nu+1}(m\times m)$ such that*:

$$\mathcal{P}_s E(h)-(id+E(q))\;:\;C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m\longrightarrow C^\infty([0, T]\times\boldsymbol{R}^n)^m$$

for a suitable matrix $q(\rho, t, x, \xi)\in HS^{0,\infty}_{\nu+1}(m\times m)=\bigcap_{q>0}HS^{0,q}_{\nu+1}(m\times m)$.

PROOF. Let $h(\rho, t, x, \xi)\in H\hat{\Sigma}^{0,0}_{\nu+1}(m\times m)$ be such that $h(1, t, x, \xi)=I_m$. Then, using Lemma 5 and proceeding as in the proof of Theorem 4.1 in [2], we obtain

$$\mathcal{P}_s E(h)-id=R_1+E((t\partial_t+s-\rho\partial_\rho-1)h-p(h)) \tag{4.10}$$

for some partially regularizing operator R_1 and a matrix $p(h)\in H\hat{\Sigma}^{0,0}_{\nu+1}(m\times m)$ (depending on h) satisfying the following condition: if $h(\rho, t, x, \xi)\in H\hat{\Sigma}^{0,l}_{\nu+1}(m\times m)$ for some $l\in\boldsymbol{Z}_+$ with an asymptotic expansion $h(\rho, t, x, \xi)\sim\sum_{j\geq 0}h_{l+j}(\rho, t, x, \xi)$, $h_{l+j}\in H\Sigma^{0,l+j}_{\nu+1}(m\times m)$ $(j\geqq 0)$, then we have $p(h)\in H\hat{\Sigma}^{0,l}_{\nu+1}(m\times m)$ and $p(h)$ is expressed in the form

$$\begin{aligned} p(h)(\rho, t, x, \xi)=&\,b'_0(t, x, \xi)h_l(\rho, t, x, \xi) \\ &+\Lambda^-(\rho, x, \xi/|\xi|, t|\xi|^{1/(\nu+1)})\chi(x, t|\xi|^{1/(\nu+1)})b''_0(t, x, \xi) \\ &\qquad\times\Lambda^+(\rho, x, \xi/|\xi|, t|\xi|^{1/(\nu+1)})h_l(\rho, t, x, \xi) \\ &+p'(h)(\rho, t, x, \xi) \end{aligned}$$

for some $p'(h)\in H\hat{\Sigma}^{0,l+1}_{\nu+1}(m\times m)$, where $b_0'=b_0'(t,x,\xi)$, $b_0''=b_0''(t,x,\xi)$, $\chi(x,z)$ and $\Lambda^{\pm}(\rho,x,\xi',z)$ are as follows:

$$b_0'=\begin{pmatrix} b_0^{(1,1)} & & 0 \\ & \ddots & \\ 0 & & b_0^{(m,m)} \end{pmatrix},\quad b_0''=\begin{pmatrix} 0 & & b_0^{(i,j)} \\ & \ddots & \\ b_0^{(i,j)} & & 0 \end{pmatrix}$$

(where $b_0^{(i,j)}$ is the (i,j)-component of $b_0=b_0(t,x,\xi)$), $\chi(x,z)$ is a cut-off function satisfying $\chi(x,t|\xi|^{1/(\nu+1)})b_0''(t,x,\xi)=b_0''(t,x,\xi)$, and

$$\Lambda^{\pm}(\rho,x,\xi',z)=\begin{pmatrix} e^{\pm iz^{\nu+1}(1-\rho^{\nu+1})\lambda_1(0,x,\xi')/(\nu+1)} & & 0 \\ & \ddots & \\ 0 & & e^{\pm iz^{\nu+1}(1-\rho^{\nu+1})\lambda_m(0,x,\xi')/(\nu+1)} \end{pmatrix}.$$

Hence, by putting $h(\rho,t,x,\xi)\sim\sum_{j\ge 0}h_j(\rho,t,x,\xi)$, $h_j\in H\Sigma^{0,j}_{\nu+1}(m\times m)$ $(j\geqq 0)$ we are reduced to finding $h_j\in H\Sigma^{0,j}_{\nu+1}(m\times m)$ $(j\geqq 0)$ which solve the following transport equations:

$$\begin{cases} (t\partial_t+s-\rho\partial_\rho-1)h_j(\rho,t,x,\xi)-b_0'(t,x,\xi)h_j(\rho,t,x,\xi) \\ \qquad -\Lambda^-(\rho,x,\xi/|\xi|,t|\xi|^{1/(\nu+1)})\chi(x,t|\xi|^{1/(\nu+1)})b_0''(t,x,\xi) \\ \qquad\quad \times\Lambda^+(\rho,x,\xi/|\xi|,t|\xi|^{1/(\nu+1)})h_j(\rho,t,x,\xi) \\ \quad =f_j(\rho,t,x,\xi)\,,\quad j=0,1,2,\cdots, \\ h_j|_{\rho=1}=\begin{cases} I_m\,, & \text{when } j=0\,, \\ 0\,, & \text{when } j>0\,, \end{cases} \end{cases} \tag{4.11}$$

where $f_0=0$ and f_j is a matrix in $H\Sigma^{0,j}_{\nu+1}(m\times m)$ determined by $h_0\in H\Sigma^{0,0}_{\nu+1}(m\times m),\cdots,h_{j-1}\in H\Sigma^{0,j-1}_{\nu+1}(m\times m)$.

Put

$$\begin{cases} h_j(\rho,t,x,\xi)=|\xi|^{-j/(\nu+1)}\hat{h}_j(\rho,x,\xi/|\xi|,t|\xi|^{1/(\nu+1)})\,, \\ \hat{h}_j(\rho,x,\xi',z)\in H\$^j(m\times m)\quad (j\geqq 0)\,, \end{cases}$$

and put $z=t|\xi|^{1/(\nu+1)}$. Then, (4.11) can be rewritten as

$$\begin{cases} (z\partial_z+s-\rho\partial_\rho-1)\hat{h}_j(\rho,x,\xi',z)-\hat{b}_0'(x,\xi',z)\hat{h}_j(\rho,x,\xi',z) \\ \qquad -\Lambda^-(\rho,x,\xi',z)\chi(x,z)\hat{b}_0''(x,\xi',z)\Lambda^+(\rho,x,\xi',z)\hat{h}_j(\rho,x,\xi',z) \\ \quad =\hat{f}_j(\rho,x,\xi',z)\,,\quad j=0,1,2,\cdots, \\ \hat{h}_j|_{\rho=1}=\begin{cases} I_m\,, & \text{when } j=0\,, \\ 0\,, & \text{when } j>0\,, \end{cases} \end{cases} \tag{4.12}$$

where $\hat{b}_0'(x,\xi',z)$ [resp. $\hat{b}_0''(x,\xi',z)]\in\$^0(m\times m)$ is such that $b_0'(t,x,\xi)=\hat{b}_0'(x,\xi/|\xi|,t|\xi|^{1/(\nu+1)})$ [resp. $b_0''(t,x,\xi)=\hat{b}_0''(x,\xi/|\xi|,t|\xi|^{1/(\nu+1)})$], $\hat{f}_0=0$ and $\hat{f}_j$ is a matrix in $H\$^j(m\times m)$ determined by $\hat{h}_0\in H\$^0(m\times m),\cdots,\hat{h}_{j-1}\in H\$^{j-1}(m\times m)$. Hence, our problem is reduced to finding $\hat{h}_j\in H\$^j(m\times m)$ $(j\geqq 0)$ which solve (4.12). Thus, Proposition 2 is reduced to proving the following lemma.

LEMMA 6. *Let $k\geqq 0$. Assume that s satisfies the following condition:*

$$\mathrm{Re}[\hat{b}_0'(x,\xi',z)+\Lambda^-(\rho,x,\xi',z)\chi(x,z)\hat{b}_0''(x,\xi',z)\Lambda^+(\rho,x,\xi',z)]\leqq(s-1)I_m$$

for any $\rho\in(0,1]$, $x\in\boldsymbol{R}^n$, $\xi'\in S^{n-1}$ and $z\in\overline{\boldsymbol{R}_+}$. Then, for any $\phi(x,\xi',z)\in \$^k(m\times m)$ and $g(\rho,x,\xi',z)\in H\$^k(m\times m)$ there exists a unique matrix $f(\rho,x,\xi',z)\in H\$^k(m\times m)$ such that

$$\begin{cases}(z\partial_z+s-\rho\partial_\rho-1)f(\rho,x,\xi',z)-\hat{b}_0'(x,\xi',z)f(\rho,x,\xi',z)\\ \qquad -\Lambda^-(\rho,x,\xi',z)\chi(x,z)\hat{b}_0''(x,\xi',z)\Lambda^+(\rho,x,\xi',z)f(\rho,x,\xi',z)\\ \quad =g(\rho,x,\xi',z)\,,\\ f|_{\rho=1}=\phi(x,\xi',z)\,.\end{cases}$$

By putting $C(x,\xi',z)=\hat{b}_0'(x,\xi',z)-(s-1)I_m$ and $C'(\rho,x,\xi',z)=\Lambda^-(\rho,x,\xi',z)\chi(x,z)\hat{b}_0''(x,\xi',z)\Lambda^+(\rho,x,\xi',z)$, we can obtain this lemma directly from Lemma 4.3 in [2]. Thus, the proof of Proposition 2 is completed.

Q. E. D.

PROOF OF THEOREM 4. Let $h(\rho,t,x,\xi)\in H\hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$ and $q(\rho,t,x,\xi)\in HS^{0;-\infty}_{\nu+1}(m\times m)$ be as in Proposition 2. Let $\chi(x,z)$ be a cut-off function and put

$$\begin{aligned}&q_0(\rho,t,x,\xi)=\chi(x,t|\xi|^{1/(\nu+1)})q(\rho,t,x,\xi)\,,\\ &p_0(\rho,t,x,\xi)=(1-\chi(x,t|\xi|^{1/(\nu+1)}))q(\rho,t,x,\xi)\,.\end{aligned}\tag{4.13}$$

It is easy to check that $E(q_0)$ is a partially regularizing operator and that $p_0(\rho,t,x,\xi)\in HS^0_f(m\times m)$ (the definition of HS^r_f is analogous to the definition of S^r_f given in the proof of Theorem 2, the only modification being the usual ρ-behavior of the symbols). As a consequence, to obtain Theorem 4 it is sufficient to find $r(\rho,t,x,\xi)\in HS^0_f(m\times m)$ such that

$$\mathcal{P}_sE(r)-E(p_0)\ :\ C^\infty([0,T],\mathcal{E}'(\boldsymbol{R}^n))^m\longrightarrow C^\infty([0,T]\times\boldsymbol{R}^n)^m\,.\tag{4.14}$$

In fact, if such an $r\in HS^0_f(m\times m)$ is found, then we have $(h-r)\in H\hat{\Sigma}^{0;0}_{\nu+1}(m\times m)$ (since $HS^0_f\subset HS^{0;-\infty}_{\nu+1}$) and therefore

$$\mathcal{P}_sE(h-r)-id\ :\ C^\infty([0,T],\mathcal{E}'(\boldsymbol{R}^n))^m\longrightarrow C^\infty([0,T]\times\boldsymbol{R}^n)^m\,.$$

Now, let us find the matrix r in (4.14). Put $L_i=L_i(\rho,t,x,\xi,\partial_x)$ $(i=1,\cdots,m)$, $\tilde{b}_{i,j}=\tilde{b}_{i,j}(\rho,t,x,\xi)$ $(i,j=1,\cdots,m)$, $\tilde{b}'=\tilde{b}'(\rho,t,x,\xi)$ and $\tilde{b}''=\tilde{b}''(\rho,t,x,\xi)$ as follows:

$$L_i=\langle\nabla_\xi\lambda_i(t, x, \nabla_x\phi_i(\rho, t, x, \xi)), \partial_x\rangle$$
$$+\sum_{|\alpha|=2}\frac{1}{\alpha!}(\partial_\xi^\alpha\lambda_i)(t, x, \nabla_x\phi_i(\rho, t, x, \xi))\partial_x^\alpha\phi_i(\rho, t, x, \xi)\,,$$

$$\tilde{b}_{i,j}=b_{i,j}(t, x, \nabla_x\phi_j(\rho, t, x, \xi))$$

(where $b_{i,j}(t, x, \xi)$ is the (i, j)-component of $b(t, x, \xi)$),

$$\tilde{b}'=\begin{pmatrix}\tilde{b}_{1,1} & & 0\\ & \ddots & \\ 0 & & \tilde{b}_{m,m}\end{pmatrix} \quad\text{and}\quad \tilde{b}''=\begin{pmatrix}0 & & \tilde{b}_{i,j}\\ & \ddots & \\ \tilde{b}_{i,j} & & 0\end{pmatrix}.$$

Let $\chi(x, z)$ be a cut-off function such that $\chi(x, t|\xi|^{1/(\nu+1)})b_{i,j}(t, x, \xi)=b_{i,j}(t, x, \xi)$ holds for any $1\leqq i\neq j\leqq m$. Let $r(\rho, t, x, \xi)\in HS_f^0(m\times m)$ such that $r(1, t, x, \xi)=0$. Then, by a formal symbolic calculus as in the proof of (4.10) we obtain:

$$\mathcal{P}_sE(r)=E((t\partial_t+s-\rho\partial_\rho-1)r)-E(M(\rho, t, x, \xi, \partial_x)r)+E(l(r))\,,$$

modulo partially regularizing operators, where $M=M(\rho, t, x, \xi, \partial_x)$ is a matrix of differential operators given by

$$M=t^{\nu+1}\begin{pmatrix}L_1(\rho, t, x, \xi, \partial_x) & & 0\\ & \ddots & \\ 0 & & L_m(\rho, t, x, \xi, \partial_x)\end{pmatrix}$$
$$+\tilde{b}'(\rho, t, x, \xi)+e^{-i\phi(\rho,t,x,\xi)}\chi(x, t|\xi|^{1/(\nu+1)})\tilde{b}''(\rho, t, x, \xi)e^{i\phi(\rho,t,x,\xi)}\,, \tag{4.15}$$

and $l(r)$ is a matrix having the following property: if $r\in HS_f^{-k}(m\times m)$ for some $k\in \mathbf{Z}_+$, then $l(r)\in HS_f^{-k-1}(m\times m)$. Hence, by putting $r(\rho, t, x, \xi)\sim\sum_{j\geq 0}r_{-j}(\rho, t, x, \xi)$, $r_{-j}(\rho, t, x, \xi)\in HS_f^{-j}(m\times m)$ $(j\geqq 0)$, we reduce our problem to finding $r_{-j}\in HS_f^{-j}(m\times m)$ $(j\geqq 0)$ which solve the following equations:

$$\begin{cases}(t\partial_t+s-\rho\partial_\rho-1)r_{-j}(\rho, t, x, \xi)-M(\rho, t, x, \xi, \partial_x)r_{-j}(\rho, t, x, \xi)=p_{-j}(\rho, t, x, \xi)\,,\\ r_{-j}|_{\rho=1}=0\,, \quad j=0, 1, 2, \cdots,\end{cases}$$

where $p_0\in HS_f^0(m\times m)$ is the same as in (4.13) and $p_{-j}\in HS_f^{-j}(m\times m)$ is a matrix determined by $r_0\in HS_f^0(m\times m), \cdots, r_{-j+1}\in HS_f^{-j+1}(m\times m)$. The following lemma shows how the preceding equations can be solved.

LEMMA 7. *Let M be as in* (4.15) *and let $k\geqq 0$. Then, for any $g(\rho, t, x, \xi)\in HS_f^{-k}(m\times m)$ there exists a unique matrix $\varphi(\rho, t, x, \xi)\in HS_f^{-k}(m\times m)$ such that*

$$\begin{cases} (t\partial_t+s-\rho\partial_\rho-1)\varphi(\rho, t, x, \xi)-M(\rho, t, x, \xi, \partial_x)\varphi(\rho, t, x, \xi)=g(\rho, t, x, \xi)\,, \\ \varphi|_{\rho=1}=0\,. \end{cases} \tag{4.16}$$

PROOF. Put $\rho=e^{-z}$, $t=t_0e^z$, $z\geqq 0$ and $t_0\in[0, T]$. Then (4.16) is rewritten into the form

$$\begin{cases} (\partial_z+s-1)\Phi(z, t_0, x, \xi)-M(e^{-z}, t_0e^z, x, \xi, \partial_x)\Phi(z, t_0, x, \xi)=g(e^{-z}, t_0e^z, x, \xi)\,, \\ \Phi|_{z=0}=0 \end{cases} \tag{4.17}$$

under the relation $\varphi(\rho, t, x, \xi)=\Phi(-\log\rho, \rho t, x, \xi)$. Since (4.17) is nothing but the Cauchy problem for a symmetric hyperbolic system (in (z, x)) in the direction dz, we can solve (4.17) and obtain a unique solution $\Phi(z, t_0, x, \xi)$, that is, we can obtain a unique solution $\varphi(\rho, t, x, \xi)\in C^\infty((0, 1]\times[0, T]\times \boldsymbol{R}^n \times\boldsymbol{R}^n\,; m\times m)$ of (4.16). In addition, by the energy inequality for the symmetric hyperbolic system we can obtain the following: if $\varphi(\rho, t, x, \xi)$ and $g(\rho, t, x, \xi)$ belong to $L^2(\boldsymbol{R}^n_x\,; m\times m)$ in x, then we have

$$|||\varphi(\rho, t, \xi)|||^2\leqq\frac{1}{\varepsilon}\int_\rho^1\left(\frac{\rho}{\mu}\right)^{-C-\varepsilon}\left|\left|\left|g\left(\mu, \frac{\rho t}{\mu}, \xi\right)\right|\right|\right|^2\frac{d\mu}{\mu} \tag{4.18}$$

for any $\varepsilon>0$ and $(\rho, t, \xi)\in(0, 1]\times[0, T]\times\boldsymbol{R}^n$, where C is a suitable positive constant and

$$|||\varphi(\rho, t, \xi)|||^2=\int_{\boldsymbol{R}^n_x}\|\varphi(\rho, t, x, \xi)\|^2dx\,.$$

Hence, by combining a cut-off argument with the energy inequality (4.18) we can easily see that $\varphi(\rho, t, x, \xi)\in HS_f^{-k}(m\times m)$. Q. E. D.

Thus, the proof of Theorem 4 is completed and a right parametrix for $\mathcal{P}_s$ is constructed.

Next, let us construct a left parametrix for $\mathcal{P}_s$. Let $\varphi_j(t, s, x, \xi)$ be the same as in (4.3) and define now:

$$\psi_j(\rho, t, y, \eta)=-\varphi_j(\rho t, 0, y, \eta)\,,\quad j=1, \cdots, m\,.$$

Put:

$$e^{i[\psi(\rho, t, y, \eta)+x\cdot\eta]}=\begin{pmatrix} e^{i(\psi_1(\rho, t, y, \eta)+x\cdot\eta)} & & 0 \\ & \ddots & \\ 0 & & e^{i(\psi_m(\rho, t, y, \eta)+x\cdot\eta)} \end{pmatrix},$$

and for any $h(\rho, t, y, \eta)\in H\hat{\Sigma}^{0,0}_{\nu+1}(m\times m)$ define the operator $F(h)$: $C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m\rightarrow C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))^m$ by the following oscillatory integral:

$$F(h:f)=\int_0^1\int_{R_y^n}\int_{R_\eta^n}h(\rho, t, y, \eta)e^{i[\varphi(\rho,t,y,\eta)+x\cdot\eta]}f(\rho t, y)d\rho dy d\eta$$

for $f(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m$. Let I denote the Fourier integral operator defined by

$$I(f)(t, x)=\int_{R_y^n}\int_{R_\eta^n}e^{i[\varphi(1,t,y,\eta)+x\cdot\eta]}f(t, y)dy d\eta\,. \tag{4.19}$$

Then we have the following result.

THEOREM 5. *Let $\mathcal{P}_s$ be the operator in* (4.1) *and assume that s is sufficiently large. Then, there exists a matrix $h(\rho, t, x, \xi)\in H\hat{\Sigma}_{\nu+1}^{0;0}(m\times m)$ such that*

$$F(h)\mathcal{P}_s-I\,:\,C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m\longrightarrow C^\infty([0, T]\times\boldsymbol{R}^n)^m\,.$$

Since the Fourier integral operator I defined by (4.19) is invertible modulo partially regularizing operators, by Theorem 5 we can obtain a left parametrix $I^{-1}F(h)$ for $\mathcal{P}_s$ such that $I^{-1}F(h)\mathcal{P}_s-id$ is a partially regularizing operator.

The proof of Theorem 5 is quite parallel to that of Theorem 4. So, we may omit the details (compare also with the proof of Theorem 4.2 in [2]).

COROLLARY. *Let $\mathcal{P}_s$ be the operator in* (4.1) *and assume that s is sufficiently large. Then, we have the following results.*

(1) *For any $f(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m$ there exists a $u(t, x)\in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))^m$ such that*

$$\mathcal{P}_s u-f\in C^\infty([0, T]\times\boldsymbol{R}^n)^m\,.$$

(2) *If $u(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m$ satisfies $\mathcal{P}_s u\in C^\infty([0, T]\times\boldsymbol{R}^n)^m$, then we have $u(t, x)\in C^\infty([0, T]\times\boldsymbol{R}^n)^m$.*

PROOF. Let E and $I^{-1}F$ be the right and the left parametrices constructed in Theorems 4 and 5. Then, (1) is obtained by putting $u=Ef$, and (2) follows from the relation $u-I^{-1}F\mathcal{P}_s u\in C^\infty([0, T]\times\boldsymbol{R}^n)^m$. Q. E. D.

§ 5. Proof of Theorem 1.

By the reduction in (2.1)$\sim$(2.4), to prove Theorem 1 it is sufficient to show the following result.

THEOREM 6. *Let L_s be the operator in* (2.2) *and assume that s is sufficiently large. Then, for any $f(t, x)\in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ there exists a unique solution $u(t, x)\in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ of $L_s u=f$. Moreover, if $f(t, x)=0$ on $D(t_0, x^0)$, then $u(t, x)$ also satisfies $u(t, x)=0$ on $D(t_0, x^0)$ (where $D(t_0, x^0)$ is defined in* (1.3)).

Let us recall a result in C^∞ theory. For a compact subset K of $\boldsymbol{R}^n$ and a positive constant λ, we write

$$C(K, \lambda)=\{(t, x)\in[0, T]\times \boldsymbol{R}^n\ ;\ \min_{y\in K}|x-y|\leqq\lambda|t|\}\ . \tag{5.1}$$

Let $\lambda_{\max}$ be the same as in (1.3). Then, we have

PROPOSITION 3 (Tahara [5]). *Let L_s be the operator in* (2.2) *and assume that s is sufficiently large. Then, for any $f(t, x)\in C^\infty([0, T]\times \boldsymbol{R}^n)$ satisfying* $\mathrm{supp}(f)\subset C(K, \lambda)$ *for some $\lambda\geqq\lambda_{\max}T^\nu$ and some compact subset K of $\boldsymbol{R}^n$, there exists a unique solution $u(t, x)\in C^\infty([0, T]\times \boldsymbol{R}^n)$ of $L_s u=f$ with* $\mathrm{supp}(u)\subset C(K, \lambda)$.

The following holds:

PROPOSITION 4. *Let L_s be the operator in* (2.2) *and assume that s is sufficiently large. Then, there is a positive constant λ_0 such that:*

(1) *For any $f(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ satisfying* $\mathrm{sing.supp}(f)\subset C(K, \lambda)$ *for some $\lambda\geqq\lambda_0$ and some compact subset K of $\boldsymbol{R}^n$, there exists a $u(t, x)\in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ with* $\mathrm{sing.supp}(u)\subset C(K, \lambda)$ *and $L_s u-f\in C^\infty([0, T]\times \boldsymbol{R}^n)$.*

(2) *If $u(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ satisfies $L_s u\in C^\infty([0, T]\times \boldsymbol{R}^n)$, then we have $u(t, x)\in C^\infty([0, T]\times \boldsymbol{R}^n)$.*

The proof is a direct consequence of the reduction in § 2, Corollary in § 4 and the following lemma.

LEMMA 8. *Let $E(h)$ be the right parametrix for $\mathcal{P}_s$ constructed in Theorem 4. Then, there is a positive constant λ_0 such that: if $f(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m$ and* $\mathrm{sing.supp}(f)\subset C(K, \lambda)$ *for some $\lambda\geqq\lambda_0$ and some compact subset K of $\boldsymbol{R}^n$, we have* $\mathrm{sing.supp}(E(h\,;f))\subset C(K, \lambda)$.

PROOF OF LEMMA 8. Let $\phi(\rho, t, x, \xi)$ $(=\varphi(t, \rho t, x, \xi))$ denote any of the ϕ_j's in (4.4), let $h(\rho, t, x, \xi)\in HS^{0,0}_{\nu+1}$, and define the operator K by

$$Kf(t, x)=\int_0^1\int_{\boldsymbol{R}_y^n}\int_{\boldsymbol{R}_\xi^n} e^{i(\phi(\rho,t,x,\xi)-y\cdot\xi)}h(\rho, t, x, \xi)f(\rho t, y)d\rho dy d\xi$$

for $f(t, x)\in C^{\infty}([0, T], \mathcal{E}'(\boldsymbol{R}^n))$. Since $\varphi(t, s, x, \xi)$ is the solution of (4.3), we have $|\nabla_{\xi}\varphi(t, s, x, \xi)-x|\leqq\lambda_0|t-s|$ for some $\lambda_0>0$. If we choose such a λ_0, we can see that $\nabla_{\xi}\varphi(t, s, x, \xi)\neq y$ holds on $\{(t, s, x, \xi, y)\,;\, s\leqq t, (t, x)\notin C(K, \lambda)$ and $(s, y)\in C(K, \lambda)\}$ (where $\lambda\geqq\lambda_0$). Therefore, on $\{(\rho, t, x, \xi, y)\,;\,(t, x)\notin C(K, \lambda)$ and $(\rho t, y)\in C(K, \lambda)\}$ we can define the operator

$$L=|\nabla_{\xi}\phi(\rho, t, x, \xi)-y|^{-2}\langle\nabla_{\xi}\phi(\rho, t, x, \xi)-y, \partial_{\xi}\rangle$$

and obtain the relation

$$L(e^{i(\varphi(\rho,t,x,\xi)-y\cdot\xi)})=e^{i(\phi(\rho,t,x,\xi)-y\cdot\xi)}\,.$$

Hence, by using the standard stationary-phase-method we can obtain the following: sing. supp$(Kf)\subset C(K, \lambda)$. This proves Lemma 8. Q. E. D.

As a corollary of Propositions 3 and 4 we have

COROLLARY. *Let L_s be the operator in* (2.2) *and assume that s is sufficiently large. Then, for any $f(t, x)\in C^{\infty}([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ satisfying* supp$(f)\subset C(K, \lambda)$ *for some $\lambda\geqq\max\{\lambda_{\max}T^{\nu}, \lambda_0\}$ and some compact subset K of $\boldsymbol{R}^n$, there exists a unique solution $u(t, x)\in C^{\infty}([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ of $L_s u=f$ with* supp$(u)\subset C(K, \lambda)$.

PROOF. Let $f(t, x)\in C^{\infty}([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ be such that supp$(f)\subset C(K, \lambda)$. Then, by Proposition 4 we have a $v(t, x)\in C^{\infty}([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ which satisfies sing. supp$(v)\subset C(K, \lambda)$ and $L_s v-f\in C^{\infty}([0, T]\times\boldsymbol{R}^n)$. Let U be an open neighbourhood of K in $\boldsymbol{R}^n$, let $\varphi(t, x)\in C^{\infty}([0, T]\times\boldsymbol{R}^n)$ such that $\varphi(t, x)=1$ in a neighbourhood of $C(K, \lambda)$ and that supp$(\varphi)\subset C(\bar{U}, \lambda)$, and put $g=f-L_s(\varphi v)$. Then, $g(t, x)\in C^{\infty}([0, T]\times\boldsymbol{R}^n)$ and supp$(g)\subset C(\bar{U}, \lambda)$. Therefore, by applying Proposition 3 to $L_s w=g$ we obtain a solution $w(t, x)\in C^{\infty}([0, T]\times\boldsymbol{R}^n)$ of $L_s w=g$ which satisfies supp$(w)\subset C(\bar{U}, \lambda)$. Hence, by putting $u=\varphi v+w$ we obtain a solution $u(t, x)\in C^{\infty}([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ of $L_s u=f$ such that supp$(u)\subset C(\bar{U}, \lambda)$. Since the uniqueness of solution is clear (from Propositions 3 and 4) and since U $(\supset K)$ is chosen arbitrarily, we can conclude that the unique solution $u(t, x)$ satisfies supp$(u)\subset C(K, \lambda)$. Q. E. D.

PROOF OF THEOREM 6. First, we prove the existence part. Let $f(t, x)\in C^{\infty}([0, T], \mathcal{D}'(\boldsymbol{R}^n))$. Let $\{\varphi_i(x)\}_{i=1}^{\infty}$ be a partition of unity on $\boldsymbol{R}^n$, and put $f_i(t, x)=\varphi_i(x)f(t, x)$. Then, by applying the Corollary to $L_s u_i=f_i$ we can find a solution $u_i(t, x)\in C^{\infty}([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ of $L_s u_i=f_i$. Since $\sum_{i=1}^{\infty}u_i(t, x)$ is a locally finite sum, by putting $u(t, x)=\sum_{i=1}^{\infty}u_i(t, x)$ we obtain a solution $u(t, x)\in C^{\infty}([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ of $L_s u=f$.

Next let us prove the uniqueness part. Let $u(t, x)\in C^{\infty}([0, T], \mathcal{D}'(\boldsymbol{R}^n))$

such that $L_s u=0$ in a neighbourhood of $\overline{D(t_0, x^0)}$. Our aim is to show that $u(t, x)=0$ holds in a neighbourhood of $\overline{D(t_0, x^0)}$. To see this, it is sufficient to prove that $u(t, x)=0$ holds on $[0, \varepsilon]\times(D(t_0, x^0)\cap\{t=0\})$ for some $\varepsilon>0$, because L_s is a strictly hyperbolic operator on $[\varepsilon, T]\times\boldsymbol{R}^n$. Put $K=\overline{D(t_0, x^0)}\cap\{t=0\}$. Choose a $\delta>0$ and an open subset U of $\boldsymbol{R}^n$ such that $K\Subset U$ and $L_s u=0$ on $[0, \delta]\times U$. Let $\varphi(x)\in C_0^\infty(U)$ such that $\varphi(x)=1$ in a neighbourhood of K, and put $g=L_s(\varphi u)$. Then, $g(t, x)\in C^\infty([0, \delta], \mathcal{E}'(\boldsymbol{R}^n))$ and $g(t, x)=0$ in a neighbourhood of $\{0\}\times K$. Therefore, by applying the Corollary to $L_s v=g$ we obtain a solution $v(t, x)\in C^\infty([0, \delta], \mathcal{E}'(\boldsymbol{R}^n))$ of $L_s v=g$ on $[0, \delta]\times\boldsymbol{R}^n$ such that $v(t, x)=0$ in a neighbourhood of $\{0\}\times K$. Put $w=\varphi u-v$; then we have $w(t, x)\in C^\infty([0, \delta], \mathcal{E}'(\boldsymbol{R}^n))$ and $L_s w=0$. Therefore, by the uniqueness part of the Corollary we obtain $w(t, x)=0$ on $[0, \delta]\times\boldsymbol{R}^n$. This immediately leads us to the fact that $u(t, x)=0$ holds on $[0, \varepsilon]\times K$ for sufficiently small $\varepsilon>0$, because $u(t, x)=w(t, x)$ holds in a neighbourhood of $\{0\}\times K$. Q. E. D.

§6. Proof of Theorem 2.

We first note the following: since the boundedness of the dependence domain is already established in Theorem 1, in the proof of Theorem 2 we may assume that $u(t, x)$, $f(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ and $g_j(x)\in\mathcal{E}'(\boldsymbol{R}^n)$ $(j=0, 1, \cdots, m-k-1)$.

Let $\chi(t)\in C_0^\infty(\boldsymbol{R})$ be such that $\chi(t)=1$ in a neighbourhood of $t=0$, and define the operator $R:\mathcal{E}'(\boldsymbol{R}^n)\to C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ by

$$Rv(t, x)=\int_{\boldsymbol{R}_\xi^n} e^{ix\cdot\xi}\chi(t(1+|\xi|^2)^{1/2(\nu+1)})\hat{v}(\xi)d\xi$$

for $v(x)\in\mathcal{E}'(\boldsymbol{R}^n)$. Then, we have $R\in S_{\nu+1}^{0,0}$, $Rv|_{t=0}=v$, $\partial_t^i(Rv)|_{t=0}=0$ for $i\geqq 1$, $\partial WF(Rv)=WF(v)$ and $WF(Rv|_{t>0})=\emptyset$.

Let $u(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ be the unique solution of (1.1) with data $f(t, x)\in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))$ and $g_j(x)\in\mathcal{E}'(\boldsymbol{R}^n)$ $(j=0, 1, \cdots, m-k-1)$. Let $\{g_j(x)\}_{j=0}^\infty$ be the Taylor coefficients of $u(t, x)$, that is, $u(t, x)\sim\sum_{j=0}^\infty g_j(x)t^j/j!$. Then, for any $s\in\boldsymbol{Z}_+$, $s\geqq m-k$, we can express $u(t, x)$ in the form

$$u(t, x)=\sum_{j=0}^{s-1}\frac{t^j}{j!}(Rg_j)(t, x)+t^s u_s(t, x)$$

for some $u_s(t, x)\in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$, and obtain the following relation

$$L_s u_s=f_s$$

for some $f_s(t, x)\in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ like in (2.4). In addition, we can see the following:

(i) $u(t, x)$ [resp. $f(t, x)$] is a regular distribution, if and only if $u_s(t, x)$ [resp. $f_s(t, x)$] is a regular distribution.

(ii) When $u(t, x)$, $u_s(t, x)$, $f(t, x)$ and $f_s(t, x)$ are regular distributions, we have

$$\partial WF(u) \subset \partial WF(u_s) \cup \partial WF(f) \cup \bigcup_{j=0}^{m-k-1} WF(g_j),$$

$$WF(u|_{t>0}) = WF(u_s|_{t>0}),$$

$$\partial WF(f_s) \subset \partial WF(f) \cup \bigcup_{j=0}^{m-k-1} WF(g_j),$$

$$WF(f|_{t>0}) = WF(f_s|_{t>0}).$$

Hence, to obtain Theorem 2 it is sufficient to prove the following result.

THEOREM 7. *Let L_s be the operator in* (2.2) *and assume that s is sufficiently large. Let $u(t, x)$, $f(t, x) \in C^\infty([0, T], \mathcal{D}'(\boldsymbol{R}^n))$ such that $L_s u = f$, and assume that $f(t, x)$ is a regular distribution. Then, $u(t, x)$ is also a regular distribution and the following inclusions hold:*

(1) $\partial WF(u) \subset \partial WF(f)$.

(2) $WF(u|_{t>0}) \subset \{(t, x, \tau, \xi) \mid t>0, (t, x, \tau, \xi) \in WF(f)\}$

$$\cup \bigcup_{i=1}^{m} \Big\{(t, x, t^\nu \lambda_i(t, x, \xi), \xi) \mid t>0, \exists s, \frac{s}{t} \in (0, 1), \exists (y, \eta) \in T^*\boldsymbol{R}^n \setminus 0,$$
$$x = x^{(i)}(t, s, y, \eta), \xi = \xi^{(i)}(t, s, y, \eta), (s, y, s^\nu \lambda_i(s, y, \eta), \eta) \in WF(f)\Big\}$$
$$\cup \bigcup_{i=1}^{m} \{(t, x, t^\nu \lambda_i(t, x, \xi), \xi) \mid t>0, \exists (y, \eta) \in T^*\boldsymbol{R}^n \setminus 0,$$
$$x = x^{(i)}(t, 0, y, \eta), \xi = \xi^{(i)}(t, 0, y, \eta), (y, \eta) \in \partial WF(f)\},$$

where $(x^{(i)}(t, s, y, \eta), \xi^{(i)}(t, s, y, \eta))$ is the solution of (1.4).

Since the boundedness of the dependence domain is also valid for $L_s u = f$, in the proof of Theorem 7 we may assume that $u(t, x)$, $f(t, x) \in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))$. Hence, by the reduction in (2.4)$\sim$(2.7), to obtain Theorem 7 it is sufficient to prove the following proposition.

PROPOSITION 5. *Let $\mathcal{P}_s$ be the operator in* (4.1) *and assume that s is sufficiently large. Let $u(t, x)$, $f(t, x) \in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m$ be such that $\mathcal{P}_s u - f \in C^\infty([0, T] \times \boldsymbol{R}^n)^m$, and assume that $f(t, x)$ is a regular distribution. Then, $u(t, x)$ is also a regular distribution and the following inclusions hold:*

(1) $\partial WF(u) \subset \partial WF(f)$.

(2) $WF(u|_{t>0}) \subset \{(t, x, \tau, \xi) \mid t>0, (t, x, \tau, \xi) \in WF(f)\}$

$$\cup \bigcup_{i=1}^{m} \Big\{(t, x, t^{\nu}\lambda_i(t, x, \xi), \xi) \mid t>0, \exists s, \frac{s}{t} \in (0, 1), \exists (y, \eta) \in T^*\boldsymbol{R}^n \setminus 0,$$

$$x = x^{(i)}(t, s, y, \eta), \xi = \xi^{(i)}(t, s, y, \eta), (s, y, s^{\nu}\lambda_i(s, y, \eta), \eta) \in WF(f)\Big\}$$

$$\cup \bigcup_{i=1}^{m} \{(t, x, t^{\nu}\lambda_i(t, x, \xi), \xi) \mid t>0, \exists (y, \eta) \in T^*\boldsymbol{R}^n \setminus 0,$$

$$x = x^{(i)}(t, 0, y, \eta), \xi = \xi^{(i)}(t, 0, y, \eta), (y, \eta) \in \partial WF(f)\}.$$

PROOF. Let E be the right parametrix for $\mathcal{P}_s$ constructed in Theorem 4. Then, by the same argument as in the proof of Theorem 5.1 in [2] we can see the following: if $f(t, x) \in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m$ is a regular distribution, then $Ef(t, x)$ is also a regular distribution and the following inclusions hold:

(1) $\partial WF(Ef) \subset \partial WF(f)$.

(2) $WF(Ef|_{t>0}) \subset \{(t, x, \tau, \xi) \mid t>0, (t, x, \tau, \xi) \in WF(f)\}$

$$\cup \bigcup_{i=1}^{m} \Big\{(t, x, t^{\nu}\lambda_i(t, x, \xi), \xi) \mid t>0, \exists s, \frac{s}{t} \in (0, 1), \exists (y, \eta) \in T^*\boldsymbol{R}^n \setminus 0,$$

$$x = x^{(i)}(t, s, y, \eta), \xi = \xi^{(i)}(t, s, y, \eta), (s, y, s^{\nu}\lambda_i(s, y, \eta), \eta) \in WF(f)\Big\}$$

$$\cup \bigcup_{i=1}^{m} \{(t, x, t^{\nu}\lambda_i(t, x, \xi), \xi) \mid t>0, \exists (y, \eta) \in T^*\boldsymbol{R}^n \setminus 0,$$

$$x = x^{(i)}(t, 0, y, \eta), \xi = \xi^{(i)}(t, 0, y, \eta), (y, \eta) \in \partial WF(f)\}.$$

Hence, to obtain Proposition 5 we have only to show that $u(t, x), f(t, x) \in C^\infty([0, T], \mathcal{E}'(\boldsymbol{R}^n))^m$ and $\mathcal{P}_s u - f \in C^\infty([0, T] \times \boldsymbol{R}^n)^m$ imply

$$u - Ef \in C^\infty([0, T] \times \boldsymbol{R}^n)^m. \tag{6.1}$$

This is verified as follows. By $\mathcal{P}_s u - f \in C^\infty([0, T] \times \boldsymbol{R}^n)^m$ and $\mathcal{P}_s Ef - f \in C^\infty([0, T] \times \boldsymbol{R}^n)^m$ we have

$$\mathcal{P}_s(u - Ef) \in C^\infty([0, T] \times \boldsymbol{R}^n)^m. \tag{6.2}$$

Since $\mathrm{supp}(u)$, $\mathrm{supp}(f) \subset [0, T] \times K$ holds for some compact subset K of $\boldsymbol{R}^n$, by Lemma 8 we have

$$\mathrm{sing.\,supp}(u - Ef) \subset C(K, \lambda) \tag{6.3}$$

for some $\lambda>0$. Let L be a compact subset of $\boldsymbol{R}^n$ such that $C(K,\lambda)\subset[0,T]\times L$, and let $\varphi(x)\in C_0^\infty(\boldsymbol{R}^n)$ such that $\varphi(x)=1$ in a neighbourhood of L. Then, by (6.2) and (6.3) we have $\mathcal{P}_s\varphi(u-Ef)\in C^\infty([0,T]\times\boldsymbol{R}^n)^m$ and $\varphi(u-Ef)\in C^\infty([0,T],\mathcal{E}'(\boldsymbol{R}^n))^m$. Hence, by the part (2) of the Corollary in § 4 we obtain

$$\varphi(u-Ef)\in C^\infty([0,T]\times\boldsymbol{R}^n)^m. \tag{6.4}$$

(6.3) and (6.4) immediately yield (6.1). Thus, (6.1) is verified. Q. E. D.

Acknowledgements.

The fourth author thanks the National Research Council of Italy (CNR) for supporting his staying at the University of Bologna during the discussion on this work. The third author thanks the Department of Mathematics of the University of Tokyo for supporting his staying in Japan during the completion of this work.

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(Received July 8, 1986)

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