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# *On the stabilization of evolution equations by feedback with time-delay; An operator-theoretical approach*

Dedicated to Professor Seizô Itô on his 60th birthday

By Masahiro YAMAMOTO

## §1. Introduction.

We are concerned with an evolution equation

$$\frac{du(t)}{dt}+Au(t)=0 \qquad (t\geqq 0) \tag{1.1}$$

in a Hilbert space $X$ which is unstable in the sense that the operator norm $\|e^{-tA}\|_{X\to X}$ grows exponentially as $t\to\infty$. Then its stable modification has been considered by many authors (Sakawa and Matsushita [18], Nambu [14], Triggiani [24, 25], Suzuki and Yamamoto [19, 20, 21]). We can refer also to Nakagiri and Yamamoto [13] for functional differential equations. In terms of the operator theory, this stable system is stated as follows. That is, for given bounded linear operators $S: X\to \boldsymbol{R}^N$ and $T: \boldsymbol{R}^N\to X$, a new system called the feedback one, of the form

$$\frac{du(t)}{dt}+Au(t)=TSu(t) \qquad (t\geqq 0)\,. \tag{1.2}$$

This system is constructed to be stable so that

$$\|e^{-t(A-TS)}\|_{X\to X}\leqq Ce^{-\omega t} \qquad (t\geqq 0) \tag{1.3}$$

for some positive constants $C$ and $\omega$.

From the viewpoint of control theory, (1.2) is interpreted as follows; Some observation signal $Su$ observed at the time $t$ is fed back to the original system (1.1) promptly in the form $TSu$.

In this paper, from the practical point of view, we propose a modified feedback system $(1.2)_h$, where it takes some constant time $h\geqq 0$ to feed $Su$ back to the system (1.1):

$$(1.2)_h \qquad \begin{cases} \dfrac{du(t)}{dt}+Au(t)=TSu(t-h) & (t\geqq 0) \\ u(t)=\varphi(t) \qquad (-h\leqq t\leqq 0) . \end{cases}$$

Here the observation signal $Su$ gained at the time $t$ is actually fed back to (1.1) at the time $t+h$.

Our purpose is to give some consideration to the set of $h$ for which we can construct $T$ so that the resulting system $(1.2)_h$ is "stable".

This paper is composed of six sections. In § 2, we will, in some simple case, decide the set of $h$ stated above, and in § 3 we prove them. In particular, we shall see that there is a case where no stable system as $(1.2)_h$ exists. In § 4 we state some abstract results on the set of $h$ and in § 5 and § 6, we prove them.

## § 2. Stabilization of simple unstable mode.

We formulate our problem in a manner similar to the one in Suzuki and Yamamoto [19].

Let $X$ be a Hilbert space over $\boldsymbol{R}$ with an inner product $(\ ,\ )_X$ and let $-A$ be a generator of a $(C_0)$ semigroup $\{e^{-tA}\}_{t\geq 0}$ in $X$.

Let us call an evolution equation (2.1) a "free system":

$$(2.1) \qquad \frac{du(t)}{dt}+Au(t)=0 \qquad (t\geqq 0) .$$

We suppose that (2.1) is unstable and construct its stable modification in the following sense. Let $S: X\to\boldsymbol{R}^N$ and $T: \boldsymbol{R}^N\to X$ be bounded linear operators. We consider the equation

$$(2.2)_h \qquad \frac{du(t)}{dt}+Au(t)=TSu(t-h) \qquad (t\geqq 0)$$

with, as an initial condition,

$$(2.3)_h \qquad u(s)=\varphi(s) \qquad (-h\leqq s<0),\ u(0)=\varphi_0 ,$$

where $\varphi(s)\in L^2((-h,0);X)$ and $\varphi_0\in X$. Henceforth we denote by $L^2((-h,0);X)$ the Banach space of $X$-valued square integrable functions defined in $(-h,0)$.

Here $TS$ is a bounded linear operator on $X$ of finite rank. The system $(2.2)_h$ is called a "*feedback system* (*with time-delay*)" of (2.1). In view of their roles in practical devices, $S$ and $T$ may be called a "*sensor*" and a *"controller"*, respectively, and the pair $\langle T, S\rangle$ is called a *"feedback"* (Suzuki and Yamamoto [19]).

We define a Banach space $M_2^{(h)}$ by

$$M_2^{(h)}(X)=M_2^{(h)}\equiv L^2((-h,0)\,;\,X)\otimes X \tag{2.4}$$

with a norm

$$\|\tilde{\varphi}\|_{M_2^{(h)}}=\left(\int_{-h}^{0}\|\varphi(s)\|_X^2 ds\right)^{1/2}+\|\varphi_0\|_X \tag{2.5}$$

$$(\tilde{\varphi}=(\varphi(s),\varphi_0)\in M_2^{(h)})\,.$$

There exists a unique mild solution $u=u(t)$ to $(2.2)_h$ with the initial condition $(2.3)_h$: $(u(s), u(0))=(\varphi(s),\varphi_0)\in M_2^{(h)}$ (Nakagiri [10]). Here $u=u(t)\in C([0,\infty)\to X)$ is said to be a mild solution to $(2.2)_h$ with $(2.3)_h$, if $u=u(t)$ satisfies

$$u(t)=e^{-tA}\varphi_0+\int_0^t e^{-(t-s)A}Bu(s-h)ds \qquad (t\geqq 0)\,,$$

and $u(t)=\varphi(t)$ $(-h\leqq t<0)$, $u(0)=\varphi_0$. Then, as is shown in Borisovic and Turbabin [2] or Nakagiri [11], the mappings $\mathcal{S}_h(t)$ $(t\geqq 0)$ defined by

$$\begin{array}{rccc} \mathcal{S}_h(t): & M_2^{(h)} & \longrightarrow & M_2^{(h)} \\ & \Cup & & \Cup \\ & \tilde{\varphi}=(\varphi(s),\varphi_0) & \longmapsto & (u(t+s), u(t)) \end{array} \tag{2.6}$$

are a $(C_0)$ semigroup in $M_2^{(h)}$.

Below we simply say a "solution" for a mild one. We introduce

DEFINITION 2.1 (cf. [19]). Let $h\geqq 0$ be given. A feedback $\langle T, S\rangle$ is said to be *"(feedback) stabilizable (in X) with respect to* $\{e^{-tA}, h\}$" if the estimate

$$\|\mathcal{S}_h(t)\|_{M_2^{(h)}\to M_2^{(h)}}\leqq Ce^{-\varepsilon t} \qquad (t\geqq 0) \tag{2.7}$$

holds for some positive constants $C$ and $\varepsilon$.

REMARK 2.1. We have the estimate (2.7) if and only if there exist some positive constants $C'$ and $\varepsilon$ such that the estimate

$$\|u(t)\|_X\leqq C'e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h)}} \qquad (t\geqq 0) \tag{2.8}$$

holds for each solution $u$ to $(2.2)_h$ with $(2.3)_h$.

We are seeking to construct *a stabilizable feedback* $\langle T, S\rangle$ for a given $h\geqq 0$, in the case where (2.1) is *unstable*;

$$\inf\{\mathrm{Re}\,\lambda\,;\ \lambda\in\sigma(A)\}<0\,.$$

Here $\sigma(A)$ denotes the spectrum of $A$.

To this end, we introduce the following notions for the sensor $S$ and the controller $T$ according to Suzuki and Yamamoto [19]. See also Fattorini [4] and Sakawa [16, 17]. Let $Y\subset X$ be a closed linear subspace.

DEFINITION 2.2. A sensor $S: X\to \boldsymbol{R}^N$ is said to be "*Y-observable*" (in $X$) with respect to $e^{-tA}$ if the conditions $a\in Y$ and $Se^{-tA}a=0$ $(0\leqq t<\infty)$ imply $a=0$.

Henceforth $\bar{Z}$ denotes the closure of $Z\subset X$ in $X$.

DEFINITION 2.3. A controller $T: \boldsymbol{R}^N\to X$ is said to be "*Y-controllable*" (in $X$) with respect to $e^{-tA}$ if $\bar{Z}\supset Y$, where $Z=\cup_{t>0}Z_t$ with $Z_t=\left\{\int_0^t e^{-(t-s)A}Tf(s)ds\,;\,f\in L^2(0,\,T)^N\right\}$.

Here we note that $v(t)=\int_0^t e^{-(t-s)A}Tf(s)ds$ is a solution of $dv/dt+Av=Tf(t)$ $(t\geqq 0)$ with $v(0)=0$.

In this section, we restrict our consideration to the operator $A$ satisfying the following Assumptions 2.1 and 2.2.

ASSUMPTION 2.1. The operator $A$ is self-adjoint in $X$ with a compact resolvent.

Then, $\sigma(A)$ consists entirely of isolated eigenvalues $\lambda_i$ $(i=1, 2, \cdots)$ with finite multiplicities.

ASSUMPTION 2.2. The first eigenvalue $\lambda_1$ is simple and

$$\lambda_1<0<\lambda_2<\lambda_3<\cdots\cdots\longrightarrow\infty\,. \tag{2.9}$$

It follows from Assumptions 2.1 and 2.2 that $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t\geqq 0}$ in $X$ (Tanabe [22], for example).

Here we consider the sensor $S$ and the controller $T$ given by

$$\begin{array}{cccc} S: & X & \longrightarrow & \boldsymbol{R} \\ & \Psi & & \Psi \\ & u & \longmapsto & (u,\,g)_X \end{array} \tag{2.10}$$

for some $g\in X$, and

$$
\begin{array}{ccc}
T: \boldsymbol{R} & \longrightarrow & X \\
\Cup & & \Cup \\
\eta & \longmapsto & \eta f
\end{array} \tag{2.11}
$$

for some $f \in X$, respectively. Henceforth we set $n_i = \dim \operatorname{Ker}(\lambda_i - A)$, fix an orthogonal basis of $\operatorname{Ker}(\lambda_i - A)$, and denote it by $\{\varphi_{ij}\}_{1 \leq j \leq n_i}$. Here we note $n_1 = 1$ by Assumption 2.2. Let $X_0$ be the one-dimensional linear subspace spanned by $\varphi_{11}$ and let $P$ be the orthogonal projection on $X_0$.

Now we state our results in this section.

THEOREM 2.1. *Let $S$ be $X_0$-observable with respect to $e^{-tA}$. Then there exists an $f \in X_0$ such that the feedback $\langle T, S \rangle$ is stabilizable with respect to $\{e^{-tA}, h\}$ if and only if*

$$
0 \leqq h < 1/|\lambda_1| . \tag{2.12}
$$

PROPOSITION 2.1. *Let $X^{(k)}$ be the linear subspace spanned by $\varphi_{11}$ and $\varphi_{k1}$ $(k > 1)$. Let us assume that $S$ is $X^{(k)}$-observable with respect to $e^{-tA}$. Then, if*

$$
0 \leqq h < \frac{1}{|\lambda_1|} - \frac{1}{|\lambda_k|} + \sqrt{\frac{1}{\lambda_1^2} + \frac{1}{\lambda_k^2}} , \tag{2.13}
$$

*there exists an $f \in X^{(k)}$ such that the feedback $\langle T, S \rangle$ is stabilizable with respect to $\{e^{-tA}, h\}$.*

REMARK 2.2. If the time-delay $h \geqq 0$ is greater than or equal to $1/|\lambda_1|$, we cannot stabilize the free system (2.1) by choosing any $T$ with $\mathcal{R}(T) \equiv T\boldsymbol{R} \subset X_0$. However, when the range space of $T$ is taken in larger class, it is possible to stabilize the free system for a time-delay $h \geqq 1/|\lambda_1|$. Namely, from Proposition 2.1, for given $h < 2/|\lambda_1|$, we can stabilize (2.1) provided that $S$ is $X^{(k)}$-observable with any $k > 1$. Furthermore we can prove the following:

*Let $X^{(k,m)}$ be the linear subspace spanned by $\varphi_{11}$, $\varphi_{k1}$ and $\varphi_{m1}$ $(k > m > 1)$. Let us assume that $S$ is $X^{(k,m)}$-observable with respect to $e^{-tA}$. Then, if*

$$
0 \leqq h < \frac{1}{|\lambda_1|} - \frac{1}{|\lambda_k|} - \frac{1}{|\lambda_m|} + \sqrt{\frac{1}{\lambda_1^2} + \frac{1}{\lambda_k^2} + \frac{1}{\lambda_m^2} + 2\sqrt{\frac{\lambda_1^2 + \lambda_k^2 + \lambda_m^2}{\lambda_1^2 \lambda_k^2 \lambda_m^2}}} , \tag{2.14}
$$

*there exists an $f \in X^{(k,m)}$ such that the feedback $\langle T, S \rangle$ is stabilizable with respect to $\{e^{-tA}, h\}$.*

We note that (2.13) implies (2.14).

REMARK 2.3. If $\lambda_1 = 0$, then Theorem 2.1 holds true by replacing (2.12) by $0 \leqq h < \infty$.

## § 3. Proof of Theorem 2.1 and Proposition 2.1.

For the proof of Theorem 2.1 and Proposition 2.1, we recall some known facts.

LEMMA 3.1. *The sensor* $S: X\to \boldsymbol{R}$ *given by* (2.10) *is* $X_0$*-observable with respect to* $e^{-tA}$ *if and only if*

$$(3.1)\qquad (\varphi_{11}, g)_X \neq 0 .$$

This lemma is a special case of Proposition 2 in [19].

The following lemma is well-known and can be proved by using Assumption 2.2 and the fact $\{\varphi_{ij}\}_{1\le i,1\le j\le n_i}$ is an orthonormal basis in $X$. (Assumption 2.1. See also Kato [8, p. 277], for instance.)

LEMMA 3.2. *Under Assumptions* 2.1 *and* 2.2 *in* § 2, *the estimate*

$$(3.2)\qquad \|(1-P)e^{-tA}\|_{X\to X}\leqq C_1 e^{-\lambda_2 t} \qquad (t\geqq 0)$$

*holds for some* $C_1>0$.

Now we proceed to

PROOF OF THEOREM 2.1. Since $T: \boldsymbol{R}\to X_0$ is expressed by

$$(3.3)\qquad T(\eta)=\eta(\alpha\varphi_{11}) \qquad (\eta\in \boldsymbol{R})$$

with some $\alpha\in \boldsymbol{R}$, the feedback system $(2.2)_h$ may be written as

$$(3.4)_h\qquad \frac{du(t)}{dt}+Au(t)=(u(t-h), g)_X\alpha\varphi_{11}$$

with

$$(3.5)_h\qquad u(s)=\varphi(s) \qquad (-h\leqq s<0), \quad u(0)=\varphi_0$$
$$(\tilde{\varphi}\equiv(\varphi(s), \varphi_0)\in M_2^{(h)}) .$$

Operating $P$ and $I-P$ on both hand sides, we see that the functions $u_0(t)=Pu(t)$ and $u_1(t)=(I-P)u(t)$ solve

$$(3.6)_h\qquad \frac{du_0(t)}{dt}+PAu_0(t)=(u_0(t-h), g)_X\alpha\varphi_{11}+(u_1(t-h), g)_X\alpha\varphi_{11} \quad (t\geqq 0)$$

with

$$(3.7)_h\qquad u_0(s)=P\varphi(s) \qquad (-h\leqq s<0), \quad u(0)=P\varphi_0 ,$$

and

$$\frac{du_1(t)}{dt}+(1-P)Au_1(t)=0 \qquad (t\geqq 0) \tag{3.8}$$

with

$$u_1(s)=(1-P)\varphi(s) \qquad (-h\leqq s<0), \quad u(0)=(1-P)\varphi_0\,, \tag{3.9}$$

respectively. Now we get by Lemma 3.2,

$$\begin{aligned}\|u_1(t)\|_X &\leqq C_1e^{-\lambda_2 t}\|(1-P)\varphi_0\|_X \\ &\leqq C_1e^{-\lambda_2 t}\|\tilde{\varphi}\|_{M_2^{(h)}}\,.\end{aligned} \tag{3.10}$$

Since we can write $u_0(t)=x(t)\varphi_{11}$ with some real-valued function $x(t)$, the equation $(3.6)_h$ with $(3.7)_h$ is equivalent to

$$\frac{dx(t)}{dt}+\lambda_1 x(t)=x(t-h)\alpha\beta+(u_1(t-h),\, g)_X\alpha \tag*{$(3.6)'_h$}$$

with

$$x(s)=(\varphi(s),\, \varphi_{11})_X \qquad (-h\leqq s<0), \quad x(0)=(\varphi_0,\, \varphi_{11})_X\,, \tag*{$(3.7)'_h$}$$

where we set

$$\beta=(\varphi_{11},\, g)_X\,. \tag{3.11}$$

Since $S$ is $X_0$-observable, we have by Lemma 3.1,

$$\beta\neq 0\,. \tag{3.12}$$

Let us consider an equation in $\mu$

$$\mu+\lambda_1-\alpha\beta e^{-\mu h}=0\,. \tag{3.13}$$

Then, in order to construct $T$ such that $\langle T, S\rangle$ is stabilizable in $(3.4)_h$, we have only to choose $\alpha$ so that there exists an $\varepsilon_1>0$ such that

$$\operatorname{Re}\mu<-\varepsilon_1 \tag{3.14}$$

for each root $\mu$ of (3.13). In fact, let us construct a semigroup $\mathcal{S}_h(t)$ for a one-dimensional differential-difference equation

$$\frac{dx(t)}{dt}+\lambda_1 x(t)=x(t-h)\alpha\beta\,, \tag*{$(3.15)_h$}$$

as in (2.6). Then, as can be shown along the line of Proposition 4.1 in

Travis and Webb [23] (cf. Nakagiri and Yamamoto [13]), it follows from (3.14) that

$$\|\mathcal{S}_h(t)\|_{M_2^{(h)}(\boldsymbol{R})\to M_2^{(h)}(\boldsymbol{R})}\leqq C_2e^{-\varepsilon_2 t} \tag{3.16}$$

for some positive constants $C_2$ and $\varepsilon_2$. Here the Banach space $M_2^{(h)}(\boldsymbol{R})\equiv L^2((-h,0)\,;\boldsymbol{R})\otimes\boldsymbol{R}$ is defined in a manner similar to (2.4) and (2.5). By Duhamel's principle (Nakagiri [11], cf. Hale [5]), the solution $x=x(t)$ to $(3.6)'_h$ with $(3.7)'_h$ is represented in the form

$$(x(t+s),\,x(t))=\mathcal{S}_h(t)\tilde{\phi}+\int_0^t\mathcal{S}_h(t-\tau)(0,\,(u_1(\tau-h),\,g)_X\alpha)d\tau \qquad (-h\leqq s<0,\,t\geqq 0)\,. \tag{3.17}$$

Here for $\tilde{\varphi}=(\varphi(s),\,\varphi_0)\in M_2^{(h)}$, we set $\phi(s)=(\varphi(s),\,\varphi_{11})_X$ $(-h\leqq s<0)$, $\phi_0=(\varphi_0,\,\varphi_{11})_X$ and $\tilde{\phi}=(\phi(s),\,\phi_0)\in M_2^{(h)}(\boldsymbol{R})$. Applying the estimates (3.10) and (3.16) in (3.17), we get

$$\|u_0(t)\|_X\leqq C_3|x(t)|\leqq C_4e^{-\varepsilon_3 t}\|\tilde{\varphi}\|_{M_2^{(h)}(X)} \qquad (t\geqq 0) \tag{3.18}$$

for some positive $C_3$, $C_4$ and $\varepsilon_3$. By combining (3.18) with (3.10), we have

$$\|u(t)\|_X\leqq Ce^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h)}(X)} \qquad (t\geqq 0) \tag{3.19}$$

for some positive $\varepsilon$. This means that $\langle T,S\rangle$ is stabilizable.

Now we return to the choice of $\alpha$ in (3.13). The condition (3.14) holds for each root $\mu$ of (3.13) if and only if we have

$$|\lambda_1|h<1 \tag{3.20.1}$$

and

$$|\lambda_1|h<-\alpha\beta h<\sqrt{U^2(-\lambda_1 h)+\lambda_1^2h^2} \tag{3.20.2}$$

(Hayes [6] and, Bellman and Cooke [1, p. 444]). Here for $a\neq 0$, we define a function $U(a)$ by the root $v$ of $v=a\tan v$ in $(0,\pi)$ and we set $U(0)=\pi/2$.

Let $h<1/|\lambda_1|$. Since $\beta\neq 0$, we can choose $\alpha$ satisfying (3.20.2). Thus we see the "if" part of this theorem.

Conversely we assume $h\geqq 1/|\lambda_1|$. Then there exists some root $\mu_0$ of (3.13) with nonnegative real part, no matter what we choose as $\alpha$. Putting

$$u(t)=\begin{cases} e^{\mu_0 t}\varphi_{11} & (t\geqq 0)\\ \varphi_{11} & (-h\leqq t<0)\,,\end{cases}$$

we see that $u(t)$ satisfies $(3.4)_h$ and that $\lim_{t\to\infty}\|u(t)\|_X\neq 0$. Namely the feedback $\langle T, S\rangle$ is not stabilizable for any controller $T: \boldsymbol{R}\to X_0$. This proves the "only if" part.

Next we proceed to

PROOF OF PROPOSITION 2.1. We can express $T$ by

$$\begin{array}{ccc} T: \boldsymbol{R}^2 & \longrightarrow & X^{(k)} \\ \Psi & & \Psi \\ (\eta_1, \eta_2) & \longmapsto & \eta_1\alpha\varphi_{11}+\eta_2\beta\varphi_{k1} \end{array}$$

with some $\alpha, \beta\in \boldsymbol{R}$. Since $S$ is $X^{(k)}$-observable, we get $(\varphi_{11}, g)_X\neq 0$ and $(\varphi_{k1}, g)_X\neq 0$ by Proposition 2 in [19] (cf. Remark 4.3 below). Therefore we may assume that

$$(\varphi_{11}, g)_X=(\varphi_{k1}, g)_X=1$$

without loss of generality. Let $P_k$ denote the orthogonal projection on $X^{(k)}$. Decomposing $(2.2)_h$ into the equations on $P_kX$ and $(I-P_k)X$ similarly, we have only to show the existence of some $\alpha, \beta$ such that each solution to $(3.21)_h$ decays exponentially as $t\to\infty$ to prove the proposition:

$$(3.21)_h \qquad \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t)\end{pmatrix}=\begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_k\end{pmatrix}\begin{pmatrix} x(t) \\ y(t)\end{pmatrix}+\begin{pmatrix} \alpha & \alpha \\ \beta & \beta\end{pmatrix}\begin{pmatrix} x(t-h) \\ y(t-h)\end{pmatrix}.$$

This will be accomplished if we choose $\alpha$, $\beta$ so that each root $\mu$ of the characteristic equation (3.22) for $(3.21)_h$ has negative real part;

$$(3.22) \qquad \begin{aligned} D_h(\mu\,;\,\alpha, \beta)&=\det\left(\mu-\begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_k\end{pmatrix}-e^{-\mu h}\begin{pmatrix} \alpha & \alpha \\ \beta & \beta\end{pmatrix}\right) \\ &=0\,. \end{aligned}$$

Due to Datko [3], the following fact holds true: *For given* $\alpha, \beta\in\boldsymbol{R}$, *consider the equation* (3.22) *and assume that*

(3.23) *each root* $\mu$ *of* $D_0(\mu\,;\,\alpha, \beta)=0$ *has negative real part.*

*We set*

$$(3.24) \qquad \begin{aligned} h_0(\alpha, \beta)=\inf\{h\geqq 0\,;\ &\textit{There exists some root } \mu \textit{ of} \\ &D_h(\mu\,;\,\alpha, \beta)=0 \textit{ on the imaginary axis.}\}. \end{aligned}$$

*Then we have* $h_0(\alpha, \beta)>0$ *and, for an arbitrary* $h$ *in* $0\leqq h<h_0(\alpha, \beta)$, *each*

*root of* $D_h(\mu\,;\,\alpha,\beta)=0$ *has negative real part.*

Here we apply this fact. As is easily shown, we have (3.23) if and only if the conditions

$$-\lambda_1-\lambda_k+\alpha+\beta<0 \tag{3.25.1}$$

and

$$\lambda_1\lambda_k-\lambda_1\beta-\lambda_k\alpha>0 \tag{3.25.2}$$

hold. We put

$$\eta=\frac{1}{|\lambda_1|}-\frac{1}{|\lambda_k|}+\sqrt{\frac{1}{\lambda_1^2}+\frac{1}{\lambda_k^2}}\,. \tag{3.26}$$

Thus, all we have to do is to show the existence $\alpha,\beta\in\boldsymbol{R}$ satisfying (3.25) and $h_0(\alpha,\beta)\geqq\eta-\varepsilon$ for each $\varepsilon>0$. In fact, we prove

PROPOSITION 3.1. *Let* $\alpha,\beta\in\boldsymbol{R}$ *satisfy*

$$\alpha+\beta=-\sqrt{\lambda_1^2+\lambda_k^2} \tag{3.27.1}$$

*and*

$$(\lambda_1-\lambda_k)\alpha+\lambda_1\sqrt{\lambda_1^2+\lambda_k^2}>-\lambda_1\lambda_k\,. \tag{3.27.2}$$

*Then* (3.25) *holds and we have*

$$\sup\{h_0(\alpha,\beta)\,;\,\alpha,\beta \textit{ satisfy } (3.27)\}\geqq\eta\,. \tag{3.28}$$

PROOF OF PROPOSITION 3.1. First we easily see that (3.25) holds. Next we prove (3.28). We put

$$a=(\lambda_1-\lambda_k)\alpha+\lambda_1\sqrt{\lambda_1^2+\lambda_k^2}\,. \tag{3.29}$$

To calculate $h_0(\alpha,\beta)$ in (3.24), let a real number $r$ satisfy (3.30) for a given $h>0$;

$$D_h(\sqrt{-1}\,r\,;\,\alpha,\beta)=0\,. \tag{3.30}$$

Then, by eliminating $h$ in $\mathrm{Re}\,D_h(\sqrt{-1}\,r\,;\,\alpha,\beta)=0$ and $\mathrm{Im}\,D_h(\sqrt{-1}\,r\,;\,\alpha,\beta)=0$, we see that $r$ is a real root of

$$(\lambda_1^2+\lambda_k^2)r^6+a^2r^4+(\lambda_1^2\lambda_k^4+\lambda_1^4\lambda_k^2-\lambda_k^2a^2-\lambda_1^2a^2)r^2+(\lambda_1^2\lambda_k^2a^2-a^4)=0\,. \tag{3.31}$$

For this $r$, we can give $h$ by

$$h=J(r\,;\,a)\equiv-\frac{1}{r}\tan^{-1}\frac{-\sqrt{\lambda_1^2+\lambda_k^2}\,r^3+(\lambda_1\lambda_k\sqrt{\lambda_1^2+\lambda_k^2}-a(\lambda_1+\lambda_k))r}{(a-\sqrt{\lambda_1^2+\lambda_k^2}(\lambda_1+\lambda_k))r^2-a\lambda_1\lambda_k}, \tag{3.32}$$

from (3.30). Conversely, for the $h$ given by (3.32), the equation (3.30) has a real root. Therefore we get

$$h_0(\alpha,\beta)=\max\{\{\min J(r\,;\,a)\,;\ r \textit{ is a real root of } (3.31)\},0\}. \tag{3.33}$$

Now the equation obtained by substituting $a=-\lambda_1\lambda_k$ in (3.31) has three distinct roots 0, $\pm\dfrac{\lambda_1\lambda_k}{\sqrt{\lambda_1^2+\lambda_k^2}}\sqrt{-1}$ and 0 is a 4-fold root. Therefore, by a theorem on the continuity of roots of an algebraic equation (Knopp [9, p. 122], for instance), we see that if $a+\lambda_1\lambda_k>0$ is sufficiently small, then the equation (3.31) in $r$ has six distinct roots $r_j(a)$ $(1\leqq j\leqq 6)$ such that

$$r_5(a),\ r_6(a)\notin \boldsymbol{R}, \tag{3.34}$$

$$\lim_{a\to-\lambda_1\lambda_k} r_j(a)=0 \qquad (1\leqq j\leqq 4), \tag{3.35}$$

and

$$\lim_{a\to-\lambda_1\lambda_k} r_5(a)=\frac{\lambda_1\lambda_k}{\sqrt{\lambda_1^2+\lambda_k^2}}\sqrt{-1}, \quad \lim_{a\to-\lambda_1\lambda_k} r_6(a)=\frac{-\lambda_1\lambda_k}{\sqrt{\lambda_1^2+\lambda_k^2}}\sqrt{-1}. \tag{3.36}$$

Then we have

$$\begin{aligned}
&\sup\{h_0(\alpha,\beta)\,;\ \alpha,\beta \textit{ satisfy } (3.27)\}=\sup_{a>-\lambda_1\lambda_k} h_0(\alpha,\,-\sqrt{\lambda_1^2+\lambda_k^2}-\alpha)\\
&\geqq \limsup_{a\downarrow-\lambda_1\lambda_k} h_0(\alpha,\,-\sqrt{\lambda_1^2+\lambda_k^2}-\alpha)\\
&\geqq \limsup_{a\downarrow-\lambda_1\lambda_k}\ \min_{1\leq j\leq 4}\{\mathrm{Re}\,J(r_j(a)\,;\,a)\}, \text{ by (3.33) and (3.34)}\\
&=\lim_{\substack{a\downarrow-\lambda_1\lambda_k\\ r\to 0}} \mathrm{Re}\,J(r\,;\,a) \qquad , \text{ by (3.35)}\\
&=\eta\,.
\end{aligned}$$

This proves Proposition 3.1, which implies Proposition 2.1.

## §4. General results for stabilizability.

In this section, we state some abstract results for more general cases.
Namely, for $-A$, we make hypotheses similar to those in [19]. We recall that $-A$ is a generator of a $(C_0)$ semigroup in $X$.

ASSUMPTION 4.1. The operator $e^{-tA}$ is compact for $t>0$.

ASSUMPTION 4.2. The spectrum of $A$, $\sigma(A)$, is divided into two subsets $\Sigma_0$ and $\Sigma_1$. Furthermore $\Sigma_0$ consists entirely of isolated eigenvalues $\lambda_i$ $(1\leqq i\leqq l)$ with finite multiplicities $m_i$ and the relation

$$\max_{1\leq i\leq l} \mathrm{Re}\,\lambda_i \leqq 0 < \inf_{\lambda\in\Sigma_1} \mathrm{Re}\,\lambda \tag{4.1}$$

holds true.

We put

$$P=\frac{1}{2\pi\sqrt{-1}}\int_\Gamma (\lambda-A)^{-1}d\lambda \tag{4.2}$$

and

$$X_0=PX, \tag{4.3}$$

where $\Gamma$ is a Jordan curve surrounding $\Sigma_0$ and, at the same time, separates $\Sigma_0$ and $\Sigma_1$.

ASSUMPTION 4.3. The estimate

$$\|(1-P)e^{-tA}\|_{X\to X}\leqq C_1 e^{-\kappa_1 t} \qquad (t\geqq 0) \tag{4.4}$$

holds for some positive constants $C_1$ and $\kappa_1$.

Here we note that $-A$ considered in § 2 satisfies Assumptions 4.1–4.3. Let $\cdot^*$ denote the adjoint operator. We introduce

DEFINITION 4.1. Let $V$ and $W$ be subsets of $X$, and let $S: X\to \boldsymbol{R}^N$ and $T: \boldsymbol{R}^N\to X$ be the sensor and the controller, respectively. Then we define the sets $D(S, V)$ and $D(T, W)$ in $[0, \infty)$ by

(4.5) $D(S, V)=\{h\geqq 0$ ; *There exists some bounded linear operator* $T: \boldsymbol{R}^N\to V$ *such that* $\langle T, S\rangle$ *is stabilizable with respect to* $\{e^{-tA}, h\}.\}$

and

(4.6) $D(T, W)=\{h\geqq 0$ ; *There exists some bounded linear operator* $S: X\to \boldsymbol{R}^N$ *with* $\mathcal{R}(S^*)\subset W$ *such that* $\langle T, S\rangle$ *is stabilizable with respect to* $\{e^{-tA}, h\}.\}$,

respectively.

In other words, $h$ belonging to $D(S, V)$ or $D(T, W)$ is an *"admissible"* delay in the sense that we can stabilize the free system (2.1) by an appropriate $T$ or $S$. For example, Theorem 2.1 and Proposition 2.1 in § 2 can be stated in terms of this notation. That is,

THEOREM 2.1. *Let $S$ be $X_0$-observable with respect to $e^{-tA}$. Then we have*

$$D(S, X_0)=\left[0, \frac{1}{|\lambda_1|}\right).$$

PROPOSITION 2.1. *Let $S$ be $X^{(k)}$-observable with respect to $e^{-tA}$. Then we have*

$$D(S, X^{(k)})\supset\left[0, \frac{1}{|\lambda_1|}-\frac{1}{|\lambda_k|}+\sqrt{\frac{1}{\lambda_1^2}+\frac{1}{\lambda_k^2}}\right).$$

Here we recall that the subspaces $X_0$ and $X^{(k)}$ are defined in § 2.

Our main results in this section are stated as follows:

THEOREM 4.1. *The sets $D(S, V)$ and $D(T, W)$ are open in $[0, \infty)$.*

THEOREM 4.2. *Let $V\supset X_0$. The set $D(S, V)$ is not empty if and only if $S$ is $X_0$-observable with respect to $e^{-tA}$. Then $D(S, V)$ is an open set containing 0, the origin.*

We set

$$X_1=(1-P)X \tag{4.7}$$

and let $X_1^{\perp}$ denote the orthogonal complement of $X_1$, that is, $X_1^{\perp}=\{g\in X; (g, \varphi)_X=0 \text{ for each } \varphi\in X_1\}$.

THEOREM 4.3. *Let $W\supset X_1^{\perp}$. The set $D(T, W)$ is not empty if and only if $PT$ is $X_0$-controllable with respect to $e^{-tA}$. Then $D(T, W)$ is an open set containing 0.*

REMARK 4.1. When the time-delay $h$ is negligible, these theorems are nothing but Theorems 1 and 2 of [19].

REMARK 4.2. In Theorem 4.2, if the inclusion $V\supset X_0$ is not assumed, the set $D(S, V)$ may possibly be empty even if $S$ is $X_0$-observable. So is $D(T, W)$ in Theorem 4.3.

However we can show the "only if" parts of these theorems without these assumptions.

REMARK 4.3. As are shown in Propositions 2 and 3 in [19] etc., the $X_0$-observability of $S$ and the $X_0$-controllability of $PT$ are equivalent to the following rank conditions (4.8) and (4.9), respectively.

Let $\{\varphi_{ij}\,;\ 1\leqq j\leqq n_i\}$ be a basis of $\mathrm{Ker}(\lambda_i-A)$ and let $S$ be expressed in the form

$$\begin{array}{ccc} S: X & \longrightarrow & \boldsymbol{R}^N \\ \Uparrow & & \Uparrow \\ u & \longmapsto & ((u, g_1)_X, \cdots, (u, g_N)_X) \end{array}$$

with some $g_k\in X$ $(1\leqq k\leqq N)$. *Then $S$ is $X_0$-observable if and only if the condition*

$$\mathrm{rank}\, M_i=n_i \tag{4.8}$$

*holds true for $1\leqq i\leqq l$, where $M_i$ is an $N\times n_i$ matrix given by $M_i=((\varphi_{ij}, g_k)_X)_{1\leq k\leq N, 1\leq j\leq n_i}$.*

Throughout this paper, $\bar{\lambda}$ denotes the complex conjugate of a complex number $\lambda$.

Let $\{\varphi^*_{ij}\,;\ 1\leqq j\leqq n_i\}$ be a basis of $\mathrm{Ker}(\bar{\lambda}_i-A^*)$ and let $T$ be expressed in the form

$$\begin{array}{ccc} T: \boldsymbol{R}^N & \longrightarrow & X \\ \Uparrow & & \Uparrow \\ (\eta_1, \cdots, \eta_N) & \longmapsto & \sum_{k=1}^N \eta_k q_k \end{array}$$

with some $q_k\in X$ $(1\leqq k\leqq N)$. Here we note $\dim \mathrm{Ker}(\lambda_i-A)=\dim \mathrm{Ker}(\bar{\lambda}_i-A^*)=n_i$ (Kato [8], for example).

*Then $PT$ is $X_0$-controllable if and only if the condition*

$$\mathrm{rank}\, L_i=n_i \tag{4.9}$$

*holds true for $1\leqq i\leqq l$, where $L_i$ is an $N\times n_i$ matrix given by $L_i=((q_k, \varphi^*_{ij})_X)_{1\leq k\leq N, 1\leq j\leq n_i}$.*

## § 5. Proof of Theorem 4.1.

We treat the differential-difference equation

$$\frac{du(t)}{dt}+Au(t)=TSu(t-h) \qquad (t\geqq 0) \tag{5.1$_h$}$$

with

$$(5.2)_h \qquad u(s)=\varphi(s) \quad (-h\leqq s<0), \quad u(0)=\varphi_0,$$

from the point of view of the semigroup theory. Let $\mathcal{S}_h(t)$ $(t\geqq 0)$ be the semigroup described in § 2.

Henceforth, let $\mathcal{L}(Y, Z)$ denote the space of all bounded linear operators on a Banach space $Y$ to a Banach space $Z$.

Now we proceed to

PROOF OF THEOREM 4.1. We set

$$(5.3) \qquad D(S, T)\equiv\{h\geqq 0\,;\ \textit{The feedback } \langle T, S\rangle \textit{ is stabilizable with respect to } \{e^{-tA}, h\}.\}$$

for given $S\in\mathcal{L}(X, \boldsymbol{R}^N)$ and $T\in\mathcal{L}(\boldsymbol{R}^N, X)$. Since we have

$$(5.4) \qquad D(S, V)=\bigcup_{T\in\mathcal{L}(\boldsymbol{R}^N, X),\, \mathfrak{R}(T)\subset V} D(S, T)$$

and

$$(5.5) \qquad D(T, W)=\bigcup_{S\in\mathcal{L}(X, \boldsymbol{R}^N),\, \mathfrak{R}(S^*)\subset W} D(S, T)\,,$$

Theorem 4.1 follows from

PROPOSITION 5.1. *For given $S\in\mathcal{L}(X, \boldsymbol{R}^N)$ and $T\in\mathcal{L}(\boldsymbol{R}^N, X)$, the set $D(S, T)$ is open in $[0, \infty)$.*

PROOF OF PROPOSITION 5.1. Let $h_0\in D(S, T)$. Without loss of generality, we may assume $h_0>0$. Then we have

$$(5.6) \qquad \|\mathcal{S}_{h_0}(t)\|_{M_2^{(h_0)}\to M_2^{(h_0)}}\leqq Ce^{-\varepsilon t} \qquad (t\geqq 0)$$

for some positive constants $C$ and $\varepsilon$. We have only to show that for some positive constants $C'$ and $\varepsilon'$, the estimate

$$(5.7) \qquad \|\mathcal{S}_h(t)\|_{M_2^{(h)}\to M_2^{(h)}}\leqq C'e^{-\varepsilon' t} \qquad (t\geqq 0)$$

holds if $|h_0-h|$ is sufficiently small. To this end, we estimate solutions to $(5.1)_h$ by using Duhamel's principle and Gronwall's inequality. It is done as follows: Since $T$ is linear, we can express it in the form

$$(5.8) \qquad \begin{array}{ccc} T: \boldsymbol{R}^N & \longrightarrow & X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (\eta_1, \cdots, \eta_N) & \longmapsto & \sum_{k=1}^N \eta_k q_k \end{array}$$

by some $q_k\in X$ $(1\leqq k\leqq N)$. We rewrite $(5.1)_h$ by

$$\frac{du(t)}{dt}+Au(t)=TSu(t-h_0)+TS(u(t-h)-u(t-h_0)) \qquad (t\geqq 0)\,. \tag{5.9}$$

Then the solution $u=u(t)$ to $(5.1)_h$ with $(5.2)_h$ may be written in the form (5.10). (Proposition 5.4 in Nakagiri [11]. cf. Borisovic and Turbabin [2]. This formula is nothing but an integral representation of solutions.)

$$(u(t+\theta),\, u(t))=\mathcal{S}_{h_0}(t)\tilde{\varphi}-\int_0^t \mathcal{S}_{h_0}(t-s)((0,\, TSf(s)))ds \qquad (t\geqq 0)\,, \tag{5.10}$$

where we set

$$f(s)=u(s-h_0)-u(s-h) \tag{5.11}$$

and $\tilde{\varphi}=(\varphi(\theta),\, \varphi_0)\in M_2^{(h_0)}=L^2((-h_0, 0)\,;\, X)\otimes X$. Here we regard $\varphi(\theta)\in L^2((-h, 0)\,;\, X)$ as the element of $L^2((-h_0, 0)\,;\, X)$, by setting $\varphi(\theta)=0$ for $-h_0\leqq\theta\leqq -h$ if $h<h_0$ and restricting $\varphi(\theta)$ in $(-h_0, 0)$ if $h\geqq h_0$.

We estimate $u(t)$ in the case of $h<h_0$.

For the estimate of $\|u(t)\|_X$, we set

$$N(t)=\begin{cases} \max\limits_{0\leq\theta\leq t}\|u(\theta)\|_X+\left(\displaystyle\int_{t-2h_0}^0\|\varphi(\theta)\|_X^2 d\theta\right)^{1/2} & \text{for } h_0\leqq t\leqq 2h_0\,, \\ \max\limits_{t-2h_0\leq\theta\leq t}\|u(\theta)\|_X & \text{for } t>2h_0\,. \end{cases} \tag{5.12}$$

Henceforth $C_i$ $(i=1, 2, \cdots, 10)$ denote some positive constants bounded as $|h_0-h|\to 0$.

Then, since by (5.10) and (5.6), the estimate

$$\|u(t)\|_X\leqq C_1e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}}+\int_0^t C_1e^{-\varepsilon(t-s)}\|TS\|_{X\to X}\|f(s)\|_X ds \tag{5.13}$$

holds true for $t\geqq 0$, we get for $t>h_0$,

$$N(t)\leqq C_2e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}}+C_2\int_0^t e^{-\varepsilon(t-s)}\|f(s)\|_X ds\,. \tag{5.14}$$

In fact, by (5.13), having

$$\max_{\max(0,\,t-2h_0)\leq\theta\leq t}\|u(\theta)\|_X\leqq C_1e^{2h_0\varepsilon}e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}}+\int_0^t C_1e^{2h_0\varepsilon}e^{-\varepsilon(t-s)}\|f(s)\|_X ds$$

for $t>h_0$ and

$$\left(\int_{t-2h_0}^0\|\varphi(\theta)\|_X^2 d\theta\right)^{1/2}\leqq\|\tilde{\varphi}\|_{M_2^{(h_0)}}\leqq e^{2h_0\varepsilon}e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}}$$

for $h_0\leqq t\leqq 2h_0$, we see (5.14) by the definition (5.12) of $N(t)$. Here we have

LEMMA 5.1. *Let $|h_0-h|$ be sufficiently small. Then we have*

$$\int_0^{h_0} e^{-\varepsilon(t-s)}\|f(s)\|_X ds \leqq C_3 e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}}. \tag{5.15}$$

The proof of this lemma is given in Appendix I.

Now, assuming that $|h_0-h|$ is sufficiently small, we apply (5.15) in the second term at the right hand side in (5.14), so that we get

$$N(t) \leqq C_4 e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}} + C_4\int_{h_0}^{t} e^{-\varepsilon(t-s)}\|f(s)\|_X ds \tag{5.16}$$

for $t>h_0$.

Next, using (5.10), we estimate $\|f(t)\|_X$. Henceforth, for $(\varphi(\theta), \varphi_0) \in M_2^{(h_0)}$, we set $\varphi_0 \equiv R(\varphi(\theta), \varphi_0)$. Then we have

$$\begin{aligned} f(t) &= R\mathcal{S}_{h_0}(t-h_0)(1-\mathcal{S}_{h_0}(h_0-h))\tilde{\varphi} \\ &\quad -\int_0^{t-h_0} R\mathcal{S}_{h_0}(t-h_0-s)(1-\mathcal{S}_{h_0}(h_0-h))((0, TSf(s)))ds \\ &\quad +\int_{t-h_0}^{t-h} R\mathcal{S}_{h_0}(t-h-s)((0, TSf(s)))ds \qquad \text{for } t>h_0. \end{aligned}$$

Thus, noting (5.6), (5.8) and the boundedness of $S$ and $T$, we get

$$\begin{aligned} \|f(t)\|_X &\leqq C_5 e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}} \\ &\quad + C_5\int_0^{t-h_0} e^{-\varepsilon(t-s)}\|Sf(s)\|_{R^N} \max_{1\leq k\leq N}\|(1-\mathcal{S}_{h_0}(h_0-h))((0, q_k))\|_{M_2^{(h_0)}} ds \\ &\quad + C_5\int_{t-h_0}^{t-h} e^{-\varepsilon(t-s)}\|TSf(s)\|_X ds \\ &\leqq C_6 e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}} \\ &\quad + C_6\int_0^{t-h_0} e^{-\varepsilon(t-s)}\delta(|h_0-h|)\|f(s)\|_X ds \\ &\quad + C_6\int_{t-h_0}^{t-h} e^{-\varepsilon(t-s)}\|f(s)\|_X ds \qquad \text{for } t>h_0. \end{aligned} \tag{5.17}$$

Here we put

$$\delta(|h_0-h|) = \max_{1\leq k\leq N}\|(1-\mathcal{S}_{h_0}(h_0-h))((0, q_k))\|_{M_2^{(h_0)}}. \tag{5.18}$$

Then, since $\mathcal{S}_{h_0}(t)$ is a $(C_0)$ semigroup, we have

$$\lim_{h\to h_0}\delta(|h_0-h|)=0\,. \tag{5.19}$$

Moreover we get for $t>h_0$,

$$C_6\int_{t-h_0}^{t-h} e^{-\varepsilon(t-s)}\|f(s)\|_X ds \leqq C_7 N(t)|h_0-h|^{1/2}, \tag{5.20}$$

if $|h_0-h|<1$. Although the proof of (5.20) is easy, it is given in Appendix II, for completeness. Therefore, by (5.17) and (5.20), we obtain

$$\begin{aligned}\|f(t)\|_X &\leqq C_7 e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}} + C_7 N(t)|h_0-h|^{1/2}\\ &\quad + C_7\delta(|h_0-h|)\int_0^t e^{-\varepsilon(t-s)}\|f(s)\|_X ds\end{aligned} \tag{5.21}$$

for $t>h_0$. By applying (5.15) in the third term at the right hand side in (5.21) provided that $|h_0-h|$ is sufficiently small, the estimate

$$\begin{aligned}\|f(t)\|_X &\leqq C_8 e^{-\varepsilon t}\|\tilde{\varphi}\|_{M_2^{(h_0)}} + C_8 N(t)|h_0-h|^{1/2}\\ &\quad + C_8\delta(|h_0-h|)\int_{h_0}^t e^{-\varepsilon(t-s)}\|f(s)\|_X ds\end{aligned} \tag{5.22}$$

follows for $t>h_0$.

By Gronwall's inequality in (5.22), we see

$$\begin{aligned}e^{\varepsilon t}\|f(t)\|_X &\leqq C_8 e^{C_8\delta(|h_0-h|)(t-h_0)}\|\tilde{\varphi}\|_{M_2^{(h_0)}} + C_8 N(t)e^{\varepsilon t}|h_0-h|^{1/2}\\ &\quad + \int_{h_0}^t e^{C_8\delta(|h_0-h|)(t-s)} C_8^2\delta(|h_0-h|)e^{\varepsilon s}N(s)|h_0-h|^{1/2}ds \quad \text{for } t>h_0.\end{aligned} \tag{5.23}$$

Then we substitute (5.23) into the second term at the right hand side of (5.16) and change the order of integration, so that we get

$$\begin{aligned}&\exp\{(\varepsilon - C_8\delta(|h_0-h|))t\}N(t)\\ &\leqq (C_4+(t-h_0)C_4C_8)\|\tilde{\varphi}\|_{M_2^{(h_0)}}\\ &\quad + C_4C_8|h_0-h|^{1/2}\int_{h_0}^t \exp\{(\varepsilon - C_8\delta(|h_0-h|))s\}N(s)ds\,.\end{aligned} \tag{5.24}$$

Here we also use

$$\int_{h_0}^t \exp(C_8\delta(|h_0-h|)(s-h_0))ds \leqq (t-h_0)\exp(C_8\delta(|h_0-h|)(t-h_0))\,.$$

Again we apply Gronwall's inequality in (5.24) to arrive at

$$\exp\{(\varepsilon - C_8\delta(|h_0-h|))t\}N(t)$$
$$\leqq (C_4+(t-h_0)C_4C_8)\|\tilde{\varphi}\|_{M_2^{(h_0)}}+C_4C_8|h_0-h|^{1/2}e^{C_4C_8|h_0-h|^{1/2}t}$$
$$\times\int_{h_0}^{t} e^{-C_4C_8|h_0-h|^{1/2}s}(C_4+(s-h_0)C_4C_8)ds\|\tilde{\varphi}\|_{M_2^{(h_0)}}$$
$$\leqq C_9(1+t)\|\tilde{\varphi}\|_{M_2^{(h_0)}}+e^{C_9|h_0-h|^{1/2}t}C_9(1+t)^2\|\tilde{\varphi}\|_{M_2^{(h_0)}}\,.$$

Therefore, if we take $h$ such that $|h_0-h|$ is sufficiently small, we see by (5.19)

$$N(t)\leqq C_{10}e^{-\varepsilon' t}\|\tilde{\varphi}\|_{M_2^{(h_0)}}\leqq C_{10}e^{-\varepsilon' t}\|\tilde{\varphi}\|_{M_2^{(h)}} \tag{5.25}$$

for some $\varepsilon'>0$. The estimate (5.25) implies (5.7).

In the case of $h>h_0$, we proceed similarly and so omit its proof.

## § 6. Proof of Theorems 4.2 and 4.3.

We show these theorems by an argument similar to the proof of Theorems 1 and 2 in [19].

PROOF OF THEOREM 4.2. Firstly we suppose that $S$ is $X_0$-observable. Then, by Theorem 1 in [19], there exists some bounded linear operator $T:\boldsymbol{R}^N\to X_0\subset V$ such that the feedback $\langle T, S\rangle$ is stabilizable with respect to $\{e^{-tA}, 0\}$. That is, we see that

$$0\in D(S, X_0)\,. \tag{6.1}$$

Since $D(S,X_0)\subset D(S, V)$, the set $D(S, V)$ is not empty.

Conversely we suppose that $D(S, V)$ is not empty, namely, that there exists $h\geqq 0$ such that

$$h\in D(S, V)\,. \tag{6.2}$$

Then there exists some bounded linear operator $T:\boldsymbol{R}^N\to V$ such that the feedback $\langle T, S\rangle$ is stabilizable with respect to $\{e^{-tA}, h\}$. We have to show that $S$ is $X_0$-observable. To this end, we assume that

$$a\in X_0 \quad\text{and}\quad Se^{-tA}a=0 \qquad (t\geqq 0) \tag{6.3}$$

and put

$$u(t)=\begin{cases} e^{-tA}a & (t\geqq 0)\\ 0 & (-h\leqq t<0)\,. \end{cases} \tag{6.4}$$

Then, since $a$ belongs to $X_0 \subset \mathscr{D}(A)$, we can view $u=u(t)$ as the solution to the feedback system $(2.2)_h$ ; that is,

$$\begin{cases} \dfrac{du(t)}{dt}+Au(t)=0=TSu(t-h) & (t \geqq 0) \\ u(t)=0 & -h \leqq t<0 \\ u(0)=a\,. \end{cases}$$

The stabilizability of $\langle T, S\rangle$ implies the estimate

$$\|u(t)\|_X \leqq C_1 e^{-\varepsilon t}\|a\|_X \qquad (t \geqq 0)\,. \tag{6.5}$$

On the other hand, in view of $a \in X_0$, we have

$$\|u(t)\|_X \geqq C_2\|a\|_X \qquad \text{as } t \rightarrow \infty\,. \tag{6.6}$$

Noting $\varepsilon>0$, we obtain $a=0$. This means the $X_0$-observability of $S$.

Next we proceed to

PROOF OF THE "IF" PART OF THEOREM 4.3. We suppose that $PT$ is $X_0$-controllable. Then by Theorem 2 in [19], there exists some bounded linear operator $S: X \rightarrow \boldsymbol{R}^N$ with $\mathscr{R}(S^*) \subset X_1^{\perp}$ such that the feedback $\langle T, S\rangle$ is stabilizable with respect to $\{e^{-tA}, 0\}$. This implies that $0 \in D(T, X_1^{\perp})$, hence we see that $D(T, W) \supset D(T, X_1^{\perp})$ is not empty.

In order to show the "only if" part, we consider the formal adjoint system $(6.7)_h$ of the feedback system $(2.2)_h$ :

$$\frac{du(t)}{dt}+A^* u(t)=S^* T^* u(t-h) \qquad (t \geqq 0)\,. \tag{6.7$_h$}$$

We prepare some lemmas. Noting that the Hilbert space $X$ is reflexive, we see from Phillips [15]

LEMMA 6.1. *The adjoint operator $-A^*$ is also a generator of a $(C_0)$ semigroup in $X$ and the relation*

$$(e^{-tA})^*=e^{-tA^*} \tag{6.8}$$

*holds.*

We recall that $\mathcal{S}_h(t)$ is a $(C_0)$ semigroup given by (2.6). By Lemma 6.1, we can construct a $(C_0)$ semigroup $\tilde{\mathcal{S}}_h(t)$ for $(6.7)_h$ in $M_2^{(h)}$ by a similar way for $(2.2)_h$ ; that is,

$$\begin{array}{lcl} \tilde{\mathcal{S}}_h(t) : M_2^{(h)} & \longrightarrow & M_2^{(h)} \\ \Downarrow & & \Downarrow \\ \tilde{\varphi}=(\varphi(s), \varphi_0) & \longmapsto & (u(t+s), u(t)) , \end{array} \tag{6.9}$$

where $u(t)$ is the solution to $(6.7)_h$ with $u(s)=\varphi(s)$ $(-h\leqq s<0)$ and $u(0)=\varphi_0$. Let $G_h$ and $\tilde{G}_h$ be the generators of $\mathcal{S}_h(t)$ and $\tilde{\mathcal{S}}_h(t)$, respectively. Then we get

LEMMA 6.2. (i) *The relation $\overline{\sigma(G_h)}=\sigma(\tilde{G}_h)$ holds. That is, a complex number $\lambda$ belongs to $\sigma(G_h)$ if and only if $\bar{\lambda}$, the complex conjugate, belongs to $\sigma(\tilde{G}_h)$.* (ii) *We have the estimate*

$$\|\mathcal{S}_h(t)\|_{M_2^{(h)}\to M_2^{(h)}}\leqq Ce^{-\varepsilon t} \qquad (t\geqq 0) \tag{6.10}$$

*for some positive constants $C$ and $\varepsilon$ if and only if*

$$\|\tilde{\mathcal{S}}_h(t)\|_{M_2^{(h)}\to M_2^{(h)}}\leqq C'e^{-\varepsilon' t} \qquad (t\geqq 0) \tag{6.11}$$

*for some positive constants $C'$ and $\varepsilon'$.*

PROOF OF LEMMA 6.2. The proof of (i) is given in Nakagiri [12], Nakagiri and Yamamoto [13], in more general cases. However, for convenience, we prove (i) in Appendix III. As for (ii), recalling that $e^{-tA}$ is compact for $t>0$, we see by Schauder's theorem ([26], for example) that $e^{-tA^*}=(e^{-tA})^*$ is also a compact operator in $X$. Then, as in Proposition 4.1 of Travis and Webb [23] (cf. Nakagiri and Yamamoto [13]), we can show that

$$\sup\{\mathrm{Re}\,\lambda\,;\,\lambda\in\sigma(G_h)\}<0 \tag{6.10$'$}$$

and

$$\sup\{\mathrm{Re}\,\lambda\,;\,\lambda\in\sigma(\tilde{G}_h)\}<0 \tag{6.11$'$}$$

imply the estimates (6.10) and (6.11), respectively. Combining this fact with Corollary 1 in Hille and Phillips [7, p. 457], we see that the estimates (6.10) and (6.11) are equivalent to the relations (6.10)$'$ and (6.11)$'$, respectively. Therefore the part (ii) follows from the part (i).

Now we recall

$$P=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}(\lambda-A)^{-1}d\lambda$$

and

$$X_0=PX\,.$$

We set

$$X_0^* = P^* X,$$

where $P^*$ is the adjoint operator of $P$. Then the following duality between the observability and the controllability holds (Suzuki and Yamamoto [19]):

LEMMA 6.3. *The operator* $PT: \boldsymbol{R}^N \to X_0$ *is* $X_0$*-controllable in* $X$ *with respect to* $e^{-tA}$ *if and only if* $(PT)^*: X \to \boldsymbol{R}^N$ *is* $X_0^*$*-observable in* $X_0^*$ *with respect to* $e^{-t(PA)^*}$.

Thus, in order to prove the "only if" part, we have only to show that

$$T^* e^{-t(PA)^*} a = 0 \quad (0 \leqq t < \infty) \text{ and } a \in X_0^* \text{ imply } a = 0\,. \tag{6.12}$$

In (6.12) we note that $e^{-t(PA)^*} a = e^{-tA^*} a$ for $a \in X_0^*$. We put

$$u(t) = \begin{cases} e^{-tA^*} a & (t \geqq 0) \\ 0 & (-h \leqq t < 0)\,. \end{cases} \tag{6.13}$$

Then, since $a$ belongs to $X_0^* \subset \mathcal{D}(A^*)$, we can view $u = u(t)$ as the solution to the feedback system $(6.7)_h$:

$$\begin{cases} \dfrac{du(t)}{dt} + A^* u(t) = 0 = S^* T^* u(t-h) & (t \geqq 0) \\ u(t) = 0 \qquad -h \leqq t < 0 \\ u(0) = a\,. \end{cases}$$

The stabilizability of $\langle T, S \rangle$ and Lemma 6.2 (ii) imply the estimate

$$\|u(t)\|_X \leqq C_3 e^{-\varepsilon' t} \|a\|_X \qquad (t \geqq 0)\,. \tag{6.14}$$

On the other hand, by $a \in X_0^*$, we have

$$\|u(t)\|_X \geqq C_4 \|a\|_X \qquad (\text{as } t \to \infty)\,. \tag{6.15}$$

By (6.14) and (6.15), we see $a = 0$.

## Appendix I.

PROOF OF LEMMA 5.1. By setting $t \downarrow h_0$ in (5.14), we get

$$N(h_0) \leqq C_2 e^{-\varepsilon h_0} \|\tilde{\varphi}\|_{M_2^{(h_0)}} + C_2 \int_0^{h_0} e^{-\varepsilon(h_0 - s)} \|f(s)\|_X ds\,. \tag{I.1}$$

Since $u(s)=\varphi(s)$ $(-h_0\leqq s<0)$, we have

$$C_2\int_0^{h_0}e^{-\varepsilon(h_0-s)}\|f(s)\|_X ds\leqq C_2\int_0^{h_0}\|f(s)\|_X ds$$

$$\leqq C_2\int_0^{h_0}\|u(s-h_0)\|_X ds+C_2\int_0^{h_0}\|u(s-h)\|_X ds \qquad \text{(by (5.11))}$$

$$=C_2\int_{-h_0}^0\|u(s)\|_X ds+C_2\left(\int_0^h\|u(s-h)\|_X ds+\int_h^{h_0}\|u(s-h)\|_X ds\right)$$

$$=C_2\int_{-h_0}^0\|\varphi(s)\|_X ds+C_2\int_{-h}^0\|u(s)\|_X ds+C_2\int_0^{h_0-h}\|u(s)\|_X ds$$

$$\leqq 2C_2h_0^{1/2}\left(\int_{-h_0}^0\|\varphi(s)\|_X^2 ds\right)^{1/2}+C_2(h_0-h)\max_{0\leq s\leq h_0-h}\|u(s)\|_X$$

(by $h_0>h$ and Schwarz's inequality)

$$\leqq 2C_2h_0^{1/2}\|\tilde\varphi\|_{M_2^{(h_0)}}+C_2(h_0-h)N(h_0)$$

(by the definition of $N(t)$ : (5.12)).

That is, we get

$$C_2\int_0^{h_0}e^{-\varepsilon(h_0-s)}\|f(s)\|_X ds\leqq 2C_2h_0^{1/2}\|\tilde\varphi\|_{M_2^{(h_0)}}+C_2(h_0-h)N(h_0)\,. \tag{I.2}$$

Substituting (I.2) into (I.1), we obtain

$$N(h_0)\leqq(C_2+2C_2h_0^{1/2})\|\tilde\varphi\|_{M_2^{(h_0)}}+C_2(h_0-h)N(h_0)\,. \tag{I.3}$$

Let $|h_0-h|$ be sufficiently small such that $1-C_2(h_0-h)>0$. Then (I.3) implies (I.4):

$$N(h_0)\leqq\frac{C_2+2C_2h_0^{1/2}}{1-C_2(h_0-h)}\|\tilde\varphi\|_{M_2^{(h_0)}}\equiv C_2'\|\tilde\varphi\|_{M_2^{(h_0)}}\,. \tag{I.4}$$

Combining (I.4) with (I.2), we obtain

$$\int_0^{h_0}e^{\varepsilon s}\|f(s)\|_X ds\leqq e^{\varepsilon h_0}(2h_0^{1/2}+C_2'(h_0-h))\|\tilde\varphi\|_{M_2^{(h_0)}}\,, \tag{I.5}$$

by which we see (5.15). This proves Lemma 5.1.

## Appendix II.

PROOF OF (5.20). We recall that $h_0>h$. Firstly let $t>2h_0$. Noting that $0<t-2h_0\leqq s-h_0<s-h\leqq t$ for $s\in[t-h_0, t-h]$, we see that

$$\|f(s)\|_X \leqq 2 \max_{s-h_0 \leq \theta \leq s-h} \|u(\theta)\|_X \leqq 2N(t) \qquad (s \in [t-h_0, t-h]),$$

which implies (5.20) for $t > 2h_0$.

Secondly let $h_0 < t \leqq 2h_0$. Then we have

$$\int_{t-h_0}^{t-h} e^{-\varepsilon(t-s)} \|u(s-h)\|_X ds \leqq |h_0 - h|^{1/2} \left( \int_{t-h_0}^{t-h} \|u(s-h)\|_X^2 ds \right)^{1/2},$$

by Schwarz's inequality. Therefore, noting $h_0 > h$, we get

$$\int_{t-h_0}^{t-h} e^{-\varepsilon(t-s)} \|u(s-h)\|_X ds \leqq |h_0 - h|^{1/2} \left( \int_{t-h_0-h}^{t-2h} \|u(s)\|_X^2 ds \right)^{1/2}$$

$$\leqq |h_0 - h|^{1/2} \left\{ \left( \int_0^t \|u(s)\|_X^2 ds \right)^{1/2} + \left( \int_{t-2h_0}^0 \|u(s)\|_X^2 ds \right)^{1/2} \right\}$$

$$\leqq |h_0 - h|^{1/2} \left\{ (2h_0)^{1/2} \max_{0 \leq s \leq t} \|u(s)\|_X + \left( \int_{t-2h_0}^0 \|u(s)\|_X^2 ds \right)^{1/2} \right\}$$

$$\leqq C_7' |h_0 - h|^{1/2} N(t).$$

As for $\int_{t-h_0}^{t-h} e^{-\varepsilon(t-s)} \|u(s-h_0)\|_X ds$, we estimate similarly. Thus we see (5.20) for $t > h_0$.

## Appendix III.

PROOF OF (i) OF LEMMA 6.2. We recall that $\mathcal{S}_h(t)$ (respectively, $\tilde{\mathcal{S}}_h(t)$) is the $(C_0)$ semigroup defined by (2.6) (respectively, (6.9)) for $(2.2)_h$ (respectively, $(6.7)_h$). As are easily shown (Nakagiri [11], for example), the generators $G_h$ and $\tilde{G}_h$ are given by (III. 1) and (III. 2), respectively:

$$G_h((\varphi(s), \varphi_0)) = \left( \frac{d\varphi(s)}{ds}, -A\varphi_0 + TS\varphi(-h) \right) \tag{III.1}$$

$$\mathcal{D}(G_h) = \{(\varphi(s), \varphi_0) \in M_2^{(h)} ; \varphi(0) = \varphi_0,$$

$$\varphi_0 \in \mathcal{D}(A) \text{ and } \varphi(s) \in H^1((-h, 0) ; X)\}$$

and

$$\tilde{G}_h((\psi(s), \psi_0)) = \left( \frac{d\psi(s)}{ds}, -A^*\psi_0 + S^*T^*\psi(-h) \right) \tag{III.2}$$

$$\mathcal{D}(\tilde{G}_h) = \{(\psi(s), \psi_0) \in M_2^{(h)} ; \psi(0) = \psi_0,$$

$$\psi_0 \in \mathcal{D}(A^*) \text{ and } \psi(s) \in H^1((-h, 0) ; X)\}.$$

Here $H^1((-h, 0); X)$ is the Sobolev space $W^{1,2}((-h, 0); X)$ on the interval $(-h, 0)$. Let $\bar{\lambda}$ denote the complex conjugate of a complex number $\lambda$.

Now we prepare

LEMMA III.1. *Let an operator $F_\lambda$ in $X$ be defined by*

$$F_\lambda\varphi = -A\varphi + e^{-\lambda h}TS\varphi - \lambda\varphi \tag{III.3}$$

$$\mathcal{D}(F_\lambda) = \mathcal{D}(A).$$

*If $\lambda \in \rho(G_h)$, the resolvent set of $G_h$, then $F_\lambda$ is a bijection. Furthermore $F_\lambda^*$, the adjoint of $F_\lambda$, is also a bijection.*

PROOF OF LEMMA III.1. Let $\lambda \in \rho(G_h)$. Firstly, for each $f \in X$, we have to show that there exists a $g \in \mathcal{D}(F_\lambda)$ such that $F_\lambda g = f$. Since $\lambda \in \rho(G_h)$, there exists a unique $\tilde{\varphi} = (\varphi(s), \varphi_0) \in \mathcal{D}(G_h)$ such that

$$(G_h - \lambda)\tilde{\varphi} \equiv \left(\frac{d\varphi(s)}{ds} - \lambda\varphi(s),\ -A\varphi_0 + TS\varphi(-h) - \lambda\varphi_0\right) = (0, f). \tag{III.4}$$

Then $g \equiv \varphi_0$ satisfies $F_\lambda g = f$.

In fact, from the first component of (III.4), we get, by $\varphi(0) = \varphi_0$,

$$\varphi(s) = e^{\lambda s}\varphi_0. \tag{III.5}$$

Substituting (III.5) into the second component of (III.4), we obtain

$$F_\lambda\varphi_0 = f. \tag{III.6}$$

Secondly, from the fact that $G_h - \lambda$ is an injection for $\lambda \in \rho(G_h)$, we see that $F_\lambda$ is an injection. Thus we prove that $F_\lambda$ is a bijection. Next we see by the closed range theorem ([26], for example) that $F_\lambda^*$ is also a bijection.

We return to the proof of (i) of Lemma 6.2. Supposing that $\lambda \in \rho(G_h)$, we have to show that $\bar{\lambda} \in \rho(\tilde{G}_h)$. To this end, by the open mapping theorem ([26], for example), we have only to prove that the map $\tilde{G}_h - \bar{\lambda}$ is a bijection in $M_2^{(h)}$. First we prove that $\tilde{G}_h - \bar{\lambda}$ is an injection. Let us assume that

$$(\tilde{G}_h - \bar{\lambda})((\psi(s), \psi_0)) = 0.$$

Then we see $\psi(s) = e^{\bar{\lambda}s}\psi_0$, so that we have $F_\lambda^*\psi_0 = 0$ in the same way as (III.6). Now, since $\lambda \in \rho(G_h)$, we see by Lemma III.1 that $F_\lambda^*$ is a bijection. Hence we get $\psi_0 = 0$.

Next we show that $\tilde{G}_h - \bar{\lambda}$ is a surjection. Since $F_\lambda^*$ is a bijection by $\lambda \in \rho(G_h)$ and Lemma III.1, for each $(\psi(s), \psi_0) \in M_2^{(h)}$, we can define

$$\varphi(s)=e^{\bar{\lambda}s}\left(\int_0^s e^{-\bar{\lambda}\theta}\psi(\theta)d\theta+(F_{\bar{\lambda}}^*)^{-1}\left\{-e^{-\bar{\lambda}h}S^*T^*\int_0^{-h}e^{-\bar{\lambda}\theta}\psi(\theta)d\theta+\psi_0\right\}\right)$$

and

$$\varphi_0=(F_{\bar{\lambda}}^*)^{-1}\left\{-e^{-\bar{\lambda}h}S^*T^*\int_0^{-h}e^{-\bar{\lambda}\theta}\psi(\theta)d\theta+\psi_0\right\}.$$

Then we see that $(\varphi(s), \varphi_0)\in\mathcal{D}(\tilde{G}_h)$ and

$$(\tilde{G}_h-\bar{\lambda})((\varphi(s), \varphi_0))=(\psi(s), \psi_0).$$

This implies that $\tilde{G}_h-\bar{\lambda}$ is a surjection. Thus we prove that $\overline{\rho(G_h)}\subset\rho(\tilde{G}_h)$. We can show $\overline{\rho(G_h)}\supset\rho(\tilde{G}_h)$ similarly and so omit its proof. These prove $\overline{\sigma(G_h)}=\sigma(\tilde{G}_h)$.

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Department of Mathematics
College of Arts and Sciences
University of Tokyo
Komaba, Meguro, Tokyo
153 Japan