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Rational points on the modular curves $X_0^+(p^r)$

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Let $N \geqq 1$ be an integer and $X_0(N)$ be the modular curve defined over $\boldsymbol{Q}$ which corresponds to the modular group $\Gamma_0(N)$. The fundamental involution w_N of $X_0(N)$ is represented by the matrix $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Let $X_0^+(N)$ denote the quotient $X_0(N)/\{1, w_N\}$. All the $\boldsymbol{Q}$-rational points on $X_0(N)$ were determined [14] [5, 6, 7, 8] [15]. In this paper, we discuss the $\boldsymbol{Q}$-rational points on $X_0^+(p^r)$ for prime numbers p and integers $r \geqq 2$. When $r=2$, $X_0^+(p^2)$ is isomorphic to the modular curve $X_{\mathrm{split}}(p)$ cf. §1, [13] [16]. The author [16] discussed the $\boldsymbol{Q}$-rational points on $X_{\mathrm{split}}(p)$. The same idea as in [16] can be applied to the modular curves $X_0^+(p^r)$ for $r \geqq 2$ with genus $g_0^+(p^r) > 0$. For such a modular curve $X_0(p^r)$, the set of the $\boldsymbol{Q}$-rational points consists of only cusps [14] [7, 8] [15]. On the other hand, there are $\boldsymbol{Q}$-rational points of $X_0^+(p^r)$ which are represented by elliptic curves with complex multiplication. We call them the C. M. points. Let $n(p, r)$ denote the number of the $\boldsymbol{Q}$-rational points on $X_0^+(p^r)$ which are neither the cusps nor the C. M. points. We make use of quotients ($\neq\{0\}$) of the jacobian varieties $J_0^+(p^r)$ of $X_0^+(p^r)$ whose Mordell-Weil groups are of finite order. Let $J_0(p^r)$ be the jacobian variety of the modular curve $X_0(p^r)$. Then the quotient $J_0^-(p^r) = J_0(p^r)/(1+w_{p^r})J_0(p^r)$ becomes naturally a quotient of $J_0^+(p^{r+1})$ cf. §1 (1.2). Then our main result is as follows.

THEOREM (0.1). *Let p be a prime number and $r \geqq 2$ be an integer with $g_0^+(p^r) > 0$. Then $n(p, r) = 0$ for $p = 2$, 3, 7, 11, and $p \geqq 17$ with $\sharp J_0^-(p)(\boldsymbol{Q}) < \infty$.*

For the prime numbers p, $17 \leqq p < 300$, except for $p = 151$, 199, 227 and 277, it is known that $\sharp J_0^-(p)(\boldsymbol{Q}) < \infty$ [12] p. 40, [24] Table 5 pp. 135–141. For $p = 5$ and 13, we do not know whether the Mordell-Weil groups of $J_0^-(125)$ and $J_0^-(169)$ are finite or not (cf. [2]). The proof of main theorem above is essentially same as in [16]. One of the key steps is as follows. Let $\mathscr{X}_0(p^r)$ be the normalization of the projective j-line $\mathscr{X}_0(1) \simeq \boldsymbol{P}^1_{\boldsymbol{Z}}$ in

the function field of $X_0(p^r)$. Let y be a non cuspidal $\boldsymbol{Q}$-rational point on $X_0^+(p^r)$ with $g_0^+(p^r)>0$, and $x, x'=w_{p^r}(x)$ be the sections of the fibre of $X_0(p^r)$ at y. Then x, x' are not defined over $\boldsymbol{Q}$ [14] [7, 8] [15], so they are defined over a quadratic field k. Let $\mathfrak{p}$ be a prime of k lying over the rational prime p.

THEOREM (0.2). *Under the notation as above, $x\otimes\kappa(\mathfrak{p})$ and $x'\otimes\kappa(\mathfrak{p})$ are the sections of the smooth part $\mathscr{X}_0(p^r)^{\mathrm{smooth}}$ of $\mathscr{X}_0(p^r)$.*

The first two sections are preparations of the last section. In the first section, we give a review of the results on the modular curves $\mathscr{X}_0(p^r)$. In the second section, we prepare lemmas on elliptic curves. In the last section, we prove main theorem etc., and also give some related results.

Notation. For a prime number q, $\boldsymbol{Z}_q$, $\boldsymbol{Q}_q$ and $\boldsymbol{Q}_q^{ur}$ denote respectively the ring of q-adic integers, the q-adic completion of $\boldsymbol{Q}$ and the maximal unramified extension of $\boldsymbol{Q}_q$. Let K be a finite extension of $\boldsymbol{Q}$, $\boldsymbol{Q}_q$ or $\boldsymbol{Q}_q^{ur}$, and A be an abelian variety defined over K. Then $\mathscr{O}_K$ denotes the ring of integers of K, and $A_{/\mathscr{O}_K}$ denotes the Néron model of A over the base $\mathscr{O}_K$. Further $(A_{/\mathscr{O}_K}\otimes\bar{\boldsymbol{F}}_q)^0$ is the connected component of the unit section of the special fibre $A_{/\mathscr{O}_K}\otimes\bar{\boldsymbol{F}}_q$. For a quasi-finite flat group scheme $G/\mathscr{O}_K$, G^0 denotes the connected component of the unit section. For a subscheme Y of a modular curve $/\boldsymbol{Z}$, Y^h denotes the open subscheme of Y obtained by excluding the supersingular points on $Y\otimes\boldsymbol{F}_p$ for a fixed prime number p.

§1. Modular curves $X_0(p^r)$.

Let p be a prime number, $r\geqq 1$ be an integer and $X_0(p^r)$ be the modular curve $/\boldsymbol{Q}$ corresponding to the modular group $\Gamma_0(p^r)$. Then $X_0(p^r)$ is the coarse moduli space $/\boldsymbol{Q}$ of the generalized elliptic curves E with a cyclic subgroup A of order p^r [3]. The fundamental involution w_{p^r} of $X_0(p^r)$ is defined by the functor

$$(E, A)\longmapsto(E/A, E_{p^r}/A),$$

where $E_{p^r}=\ker(p^r: E\to E)$. Let $X_0^+(p^r)$ denote the quotient $X_0(p^r)/\langle w_{p^r}\rangle$. For the following pairs (p, r) $(r\geqq 2)$, $X_0^+(p^r)$ are not projective line:

p	r
2	$\geqq 6$

(1.0)
$$\begin{array}{rr} p & r \\ 3 & \geqq 4 \\ 5 & \geqq 3 \\ 7 & \geqq 3 \\ p \geqq 11 & \geqq 2. \end{array}$$

We know the following result.

THEOREM (1.1) ([14] [7, 8] [15]). *The rational points on* $X_0(p^r)$ *must be the cusps for any pair* (p, r) *in* (1.0).

There exists a covering of $X_0^+(p^{r+2})$ to $X_0^+(p^r)$, which is induced by the morphism of $X_0(p^{r+2})$ to $X_0(p^r)$ defined by

$$(E, A) \longmapsto (E/A, A_{p^{r+1}}/A_p) ,$$

where A_{p^i} is the unique cyclic subgroup of A of order p^i. Let $X_s(p^t) = X_{\mathrm{sp,Car}}(p^t)$ be the modular curve $/\boldsymbol{Q}$ which corresponds to the modular group

$$\Gamma_s(p^t) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \boldsymbol{SL}_2(\boldsymbol{Z}) \mid b \equiv c \equiv 0 \bmod p^t \right\}.$$

Then $X_s(p^t)$ is the coarse moduli space $/\boldsymbol{Q}$ of the generalized elliptic curves E with independent cyclic subgroups C_1 and C_2 of order p^t. The fundamental involution $w = w(p^t)$ of $X_s(p^t)$ is defined by

$$(E, C_1, C_2) \longmapsto (E, C_2, C_1).$$

Let $X_s^n(p^t) = X_{\mathrm{split}}(p^t)$ be the quotient $X_s(p^t)/\langle w \rangle$, which corresponds to the modular group $\left\langle \Gamma_s(p^t), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$. There exists a canonical isomorphism $(/\boldsymbol{Q})$ of $X_0(p^{2t})$ (resp. $X_0^+(p^{2t})$) to $X_s(p^t)$ (resp. $X_s^n(p^t)$) defined by

$$(E, A) \longmapsto (E/A_{p^t}, A/A_{p^t}, E_{p^t}/A_{p^t}).$$

For even integers $r \geqq 2$, we will make use of this isomorphism.

Let $J_0(p^r)$, $J_0^+(p^r)$ be the jacobian varieties of $X_0(p^r)$ and $X_0^+(p^r)$, respectively. Further, let $J_r = J_{p,r}$, $J_r^+ = J_{p,r}^+$ be respectively the "new part" of $J_0(p^r)$ and $J_0^+(p^r)$ (, i.e., under the canonical identification of the space of holomorphic cusp forms of weight 2 belonging to $\Gamma_0(p^r)$ $\left(\text{resp. } \left\langle \Gamma_0(p^r), \begin{pmatrix} 0 & -1 \\ p^r & 0 \end{pmatrix} \right\rangle\right)$ with the cotangent space of $J_0(p^r)$ (resp. $J_0^+(p^r)$), the cotangent space of J_r (resp. J_r^+) corresponds to the subspace spanned

by the new forms of level p^r ([1] [22] Chapter 7). Using the decomposition of the cotangent space of $J_0(p^r)$ by the action of w_{p^r}, we see that $J_0^+(p^r)$ is isogenous over $\boldsymbol{Q}$ to one of the following abelian varieties:

$$(1.2)\qquad \begin{cases} \prod_{\substack{1\leqq i<r \\ \text{odd}}} J_i^{(r+1-i)/2} \times \prod_{\substack{1<i<r \\ \text{even}}} J_i^{(r-i)/2} \times \prod_{\substack{1<i\leqq r \\ \text{even}}} J_i^+ & \text{if } r \text{ is even,} \\ \prod_{\substack{1<i<r \\ \text{even}}} J_i^{(r+1-i)/2} \times \prod_{\substack{1\leqq i<r \\ \text{odd}}} J_i^{(r-i)/2} \times \prod_{\substack{1\leqq i\leqq r \\ \text{odd}}} J_i^+ & \text{if } r \text{ is odd.} \end{cases}$$

Let $J_0^-(p^s)$ be the quotient $J_0(p^s)/(1+w_{p^s})J_0(p^s)$, where w_{p^s} is the automorphism of $J_0(p^s)$ induced by the involution w_{p^s} of $X_0(p^s)$. Let $\boldsymbol{T}$ be the subring of the ring of endomorphisms of $J_0(p^s)$ generated by Hecke operators T_m for $p\nmid m$ and w_{p^s}, and $\mathscr{I}$ be the ideal of $\boldsymbol{T}$ generated by $1+l-T_l$ and $1+w_{p^s}$ for prime numbers $l\neq p$. Put $\tilde{J}_0(p^s)=J_0(p^s)/(\cap_{n\geqq 1}\mathscr{I}^n)J_0(p^s)$ and $\tilde{J}_0(p^s)_{(m)}=J_0(p^s)/(\cap_{n\geqq 1}(m,\mathscr{I})^n)J_0(p^s)$ for integers $m\geqq 1$. For $p\leqq 7$ and $p=13$, $X_0(p)\simeq \boldsymbol{P}^1$ and for the other prime numbers p, $X_0(p)\neq \boldsymbol{P}^1$. Let 0 and ∞ be the $\boldsymbol{Q}$-rational cusps of $X_0(p^s)$ which are represented respectively by $(\boldsymbol{G}_m\times \boldsymbol{Z}/p^s\boldsymbol{Z}, \boldsymbol{Z}/p^s\boldsymbol{Z})$ and $(\boldsymbol{G}_m, \mu_{p^s})$, where $\mu_{p^s}=\mathrm{Spec}\, \boldsymbol{Z}[X]/(X^{p^s}-1)$. Then we know the following result.

THEOREM (1.3) (Mazur [12]). *For $p\geqq 17$ or $p=11$, the natural morphism $J_0(p)\to\tilde{J}_0(p)$ induces an isomorphism of the cuspidal subgroup $C=\langle \mathrm{cl}((0)-(\infty))\rangle$ of order $n=\mathrm{num}((p-1)/12)$ to the Mordell-Weil group of $J_0^-(p)$, and $\tilde{J}_0(p)$ is an optimal quotient of $J_0^-(p)$. Further the natural morphisms $J_0(p)(\boldsymbol{Q})_{\mathrm{tor}}\to J_0^-(p)(\boldsymbol{Q})_{\mathrm{tor}}\to\tilde{J}_0(p)(\boldsymbol{Q})$ are isomorphisms.*

(1.4) For the pairs $(p, s)=(2, 5)$, $(3, 3)$ and $(7, 2)$, $X_0(p^s)$ are elliptic curves with finite Mordell-Weil groups of order 4, 3 and 2 ([24] Table 1 pp. 81–113), respectively, and $J_0(p^s)=J_0^-(p^s)=\tilde{J}_0(p^s)$. For $(p, s)=(13, 2)$, $\tilde{J}_0(169)_{(7)}$ is an optimal quotient of $J_0^-(169)$ with finite Mordell-Weil group [2] §3. We do not know whether $J_0^-(125)$ has a quotient ($\neq\{0\}$) with finite Mordell-Weil group (see loc.cit.).

We will make use of the following morphisms for $r\geqq 2$. Let $\pi=\pi_{r,s}$ be the natural morphism of $X_0(p^r)$ to $X_0(p^s)$ defined by $(E, A)\mapsto(E, A_{p^s})$ for an integer s, $1\leqq s\leqq r-1$. Let $f=f_{r,s}$ be the morphism of $X_0(p^r)$ to $J_0(p^s)$ defined by $f(x)=\mathrm{cl}((w_{p^s}\pi(x))-(\pi w_{p^r}(x)))$, i.e.,

$$f\colon (E, A)\longmapsto \mathrm{cl}((E/A_{p^s}, E_{p^s}/A_{p^s})-(E/A, (E_{p^s}+A)/A)).$$

Then f induces a morphism $f^+=f^+_{r,s}$ of $X_0^+(p^r)$ to $J_0^-(p^s)$, which is defined by the following diagram:

$$\begin{array}{ccc} X_0(p^r) & \xrightarrow{\text{can.}} & X_0^+(p^r) \\ f\downarrow & \subset & \downarrow f^+ \\ J_0(p^s) & \xrightarrow{\text{can.}} & J_0^-(p^s). \end{array}$$

We will make use of f and f^+ in the following cases:

(1.5)

p	r	s
2	$\geqq 6$	5
3	$\geqq 4$	3
5	$\geqq 4$	3
7	$\geqq 3$	2
11	$\geqq 2$	1
13	$\geqq 3$	2
$p \geqq 17$	$\geqq 2$	1.

For a triple (p, r, s) as above, let $\tilde{f}=\tilde{f}_{r,s}$ be the morphism of $X_0(p^r)$ to $\tilde{J}_0(p^s)$ (for $p\neq 5, 13$), $\tilde{J}_0(125)_{(5)}$ and $\tilde{J}_0(169)_{(7)}$ induced by $f=f_{r,s}$. Let $\mathscr{X}_0(p^r)$, $\mathscr{X}_s(p^t)$ and $\mathscr{X}_s^n(p^t)$ be respectively the normalizations of the projective j-line $\mathscr{X}_0(1)\simeq \boldsymbol{P}_{\boldsymbol{Z}}^1$ in $X_0(p^r)$, $X_s(p^t)$ and $X_s^n(p^t)$. Further put $\mathscr{X}_0^+(p^r)=\mathscr{X}_0(p^r)/\langle w_{p^r}\rangle$. Then $\mathscr{X}_0^+(p^r)\otimes \boldsymbol{Z}[1/p]$ is smooth over $\boldsymbol{Z}[1/p]$, since $\mathscr{X}_0(p^r)\otimes \boldsymbol{Z}[1/p]$ is smooth [3] VI §6 and the fundamental involution $w_{p^r}\otimes \boldsymbol{F}_2$ (for $p\neq 2$) has at most finitely many fixed points. Denote also by $\pi=\pi_{r,s}$ the natural morphism of $\mathscr{X}_0(p^r)$ to $\mathscr{X}_0(p^s)$, and by f, $\tilde{f}$ (resp. f^+) the morphisms of the smooth part $\mathscr{X}_0(p^r)^{\text{smooth}}$ (resp. $\mathscr{X}_0^+(p^r)^{\text{smooth}}$) to the Néron models $J_0(p^s)_{/\boldsymbol{Z}}$ and $\tilde{J}_0(p^s)_{/\boldsymbol{Z}}$ (for $p\neq 5, 13$), $\tilde{J}_0(125)_{(5)/\boldsymbol{Z}}$ and $\tilde{J}_0(169)_{(7)/\boldsymbol{Z}}$ (resp. $J_0^-(p^s)_{/\boldsymbol{Z}}$).

(1.6) Let $0=\begin{pmatrix}0\\1\end{pmatrix}$, $\infty=\begin{pmatrix}1\\0\end{pmatrix}$ be the $\boldsymbol{Q}$-rational cuspidal sections of $\mathscr{X}_0(p^r)$ represented by the pairs $(\boldsymbol{G}_m\times \boldsymbol{Z}/p^r\boldsymbol{Z}, \boldsymbol{Z}/p^r\boldsymbol{Z})$ and $(\boldsymbol{G}_m, \mu_{p^r})$, respectively. Let (t, i) be a pair of integers t and i satisfying the following conditions: $1\leqq t\leqq r-1$, $(i, p)=1$ and $1\leqq i\leqq p^{m(t)}-1$ for $m(t)=\min\{t, r-t\}$. Let $\zeta=\zeta_{p^r}$ be a primitive p^r-th root of unity. For each pair (t, i) as above, let $\begin{pmatrix}i\\p^t\end{pmatrix}$ be the cuspidal section of $\mathscr{X}_0(p^r)$ represented by $(\boldsymbol{G}_m\times \boldsymbol{Z}/p^{r-t}\boldsymbol{Z}, \langle(\zeta^i, 1)\rangle)$ (over $\boldsymbol{Z}[1/p]$), where $\langle(\zeta^i, 1)\rangle$ is the subgroup of $\boldsymbol{G}_m\times \boldsymbol{Z}/p^{r-t}\boldsymbol{Z}$ generated by the section $(\zeta^i, 1)$. These cuspidal sections $\begin{pmatrix}i\\p^t\end{pmatrix}$ are $\boldsymbol{Q}(\zeta_{p^{m(t)}})$-rational and are conjugate over $\boldsymbol{Q}$ (for the fixed integer t). The fundamental

involution w_{p^r} exchanges 0 with ∞, and $\begin{pmatrix} i \\ p^t \end{pmatrix}$ with $\begin{pmatrix} j \\ p^{r-t} \end{pmatrix}$ for an integer j congruent to $-i^{-1} \bmod p^{m(t)}$. Let $\pi_{r+1}=\pi_{r+1,r}$ be the natural morphism of $\mathscr{X}_0(p^{r+1})$ to $\mathscr{X}_0(p^r)$: $(E, A)\mapsto(E, A_{p^r})$. Then π_{r+1} is isomorphic along ∞ and $\begin{pmatrix} i \\ p^r \end{pmatrix}\otimes \boldsymbol{Z}[1/p]$. The ramification indices of π_{r+1} at 0 and at $\begin{pmatrix} i \\ p^t \end{pmatrix}\otimes \boldsymbol{Z}[1/p]$ for $1\leqq t\leqq r-1$ are all p. Further π_{r+1} sends 0 to 0, $\begin{pmatrix} i \\ p^t \end{pmatrix}$ (for $1\leqq t\leqq r-1$) to $\begin{pmatrix} i \\ p^t \end{pmatrix}$, and ∞ and $\begin{pmatrix} i \\ p^r \end{pmatrix}$ to ∞.

(1.7) The irreducible components of $\mathscr{X}_0(p^r)\otimes \boldsymbol{F}_p$ are all defined over $\boldsymbol{F}_p$, and they intersect each other at the supersingular points on $\mathscr{X}_0(p^r)\otimes \boldsymbol{F}_p$ [3] VI §6. Let E_0 and E_r (resp. E_t for $1\leqq t\leqq r-1$) be the irreducible components of $\mathscr{X}_0(p^r)\otimes \boldsymbol{F}_p$ such that $0\otimes \boldsymbol{F}_p\in E_0$ and $\infty\otimes \boldsymbol{F}_p\in E_r$ $\left(\text{resp. } \begin{pmatrix} i \\ p^t \end{pmatrix}\otimes \boldsymbol{F}_p\in E_t\right)$. For a subscheme Y of a modular curve $X/\boldsymbol{Z}$, Y^h denotes the open subscheme $Y\backslash\{\text{supersingular points on } Y\otimes \boldsymbol{F}_p\}$ of Y. The following facts are induced by the construction of the fine moduli stack $\mathscr{M}^h_{\Gamma(p^r)}$ [3] V, and they will be explained below. The irreducible components E_0^h and E_r^h are smooth over $\boldsymbol{F}_p$. Further E_0^h (resp. E_r^h) is the coarse moduli space $/\boldsymbol{F}_p$ of the isomorphism classes of the generalized elliptic curves E with a cyclic subgroup scheme A which is isomorphic to $\boldsymbol{Z}/p^r\boldsymbol{Z}$ (resp. μ_{p^r}) for the étale topology. When $p=2$, E_1^h and E_{r-1}^h are also smooth over $\boldsymbol{F}_2$. For an integer i, $1\leqq i\leqq r/2$, the multiplicity of E_i^h is $p^{r-i}(p-1)$. The fundamental involution w_{p^r} exchanges E_i with E_{r-i}. When $r=2t$ is even, w_{p^r} fixes E_t. Let $\pi_r=\pi_{r,r-1}$ be the natural morphism of $\mathscr{X}_0(p^r)$ to $\mathscr{X}_0(p^{r-1})$: $(E, A)\mapsto(E, A_{p^{r-1}})$. Then $\pi_r(E_i)=E_i$ for the integers i, $0\leqq i\leqq r-1$, and $\pi_r(E_r)=E_{r-1}$. The restriction of π_r to E_r^h: $E_r^h\to E_{r-1}^h$ is an isomorphism and its restriction to E_0^h: $E_0^h\to E_0^h$ is radicial of degree p. When $p=2$, its restriction to E_{r-1}^h: $E_{r-1}^h\to E_{r-1}^h$ is also an isomorphism and its restriction to E_1^h: $E_1^h\to E_1^h$ (for $r\geqq 3$) is radicial of degree 2. Further we know the following facts. Let $\mathscr{M}_{\Gamma(p^r)}$ be the fine moduli stack which corresponds to the finite adèlic modular group

$$\Gamma_0(p^r)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \mathrm{GL}_2(\hat{\boldsymbol{Z}})\mid c\equiv 0 \bmod p^r\right\},$$

where $\hat{\boldsymbol{Z}}=\varprojlim_n \boldsymbol{Z}/n\boldsymbol{Z}$ [3]. Let E be a semistable elliptic curve over a scheme S with a cyclic subgroup A of rank$(/S)$ p^r. Let x be an object of $\mathscr{M}_{\Gamma_0(p^r)}(S)$ represented by the pair (E, A). Then if $x\otimes \boldsymbol{F}_p$ define a

section of E_i^h for $1 \leqq i \leqq r-1$, then A admits the following exact sequence

$$0 \longrightarrow \mu_{p^i} \longrightarrow A \longrightarrow \boldsymbol{Z}/p^{r-i}\boldsymbol{Z} \longrightarrow 0.$$

Now we here explain the statements as above. For a primitive p^r-th root $\zeta = \zeta_{p^r}$ of unity and $\zeta: \boldsymbol{Z}/p^r\boldsymbol{Z} \to \mu_{p^r}$ be the homomorphism $/\boldsymbol{Z}[\zeta]$ which sends 1 to ζ over $\boldsymbol{Q}(\zeta)$. For a cyclic subgroup A of $\boldsymbol{Z}/p^r\boldsymbol{Z}$, define the finite flat group scheme $R(A)$ $(/\boldsymbol{Z}[\zeta])$ by "push out" in the following diagram [3] V:

$$\begin{array}{ccccc} A & \longrightarrow & (\boldsymbol{Z}/p^r\boldsymbol{Z})^2 & \longrightarrow & (\boldsymbol{Z}/p^r\boldsymbol{Z})/A \\ \downarrow \zeta & \subset & \downarrow g & \subset & \| \\ A \otimes \mu_{p^r} & \longrightarrow & R(A) & \longrightarrow & (\boldsymbol{Z}/p^r\boldsymbol{Z})/A. \end{array}$$

Let $\mathscr{C}_A$ be the finite moduli space which represents the following functor: for a scheme S $/\boldsymbol{Z}[\zeta]$, $\mathscr{C}_A(S)$ is the set of the isomorphism classes of the generalized elliptic curves E $/S$ with an isomorphism $\alpha: E_{p^r} \simeq R(A) \times S$. Then $\mathscr{C}_A$ is an open subspace of $\mathscr{M}^h_{\Gamma(p^r)} (= M^h_{\Gamma(p^r)})$, which is a scheme if $p^r \geqq 3$ [3] V, VII p. 300). Let G be the schematic closure of $g(\boldsymbol{Z}/p^r\boldsymbol{Z} \times \{0\})$ in $R(A)$. Consider the following canonical morphisms

$$M^h_{\Gamma(p^r)} \xrightarrow{\psi} M^h_{\Gamma_0(p^r)} \xrightarrow{\pi_r} M^h_{\Gamma_0(p^{r-1})}.$$

Then ψ sends (E, α) to $(E, \alpha^{-1}(G \times S))$. Set $A_0 = \{0\} \times \boldsymbol{Z}/p^r\boldsymbol{Z}$ and $A_i = \langle (1, p^i) \rangle$ for $1 \leqq i \leqq r$. Then for $(E, \alpha) \in \mathscr{M}^h_{A_i}(S)$, $(G \times S) \cap g(A_i) \simeq \mu_{p^i}$ and $\alpha^{-1}(G \times S)$ admits the following exact sequence

$$0 \longrightarrow \mu_{p^i} \longrightarrow \alpha^{-1}(G \times S) \longrightarrow \boldsymbol{Z}/p^{r-i}\boldsymbol{Z} \longrightarrow 0.$$

Now assume $r \geqq 2$. The subgroup of $\Gamma_0(p^{r-1})$ consisting of the automorphisms which fixes $\mathscr{C}_{A_r}$ is $\Gamma_0(p^r)$, so that π_r induces an isomorphism of E_r^h **onto** E_{r-1}^h. When $p=2$, by the same way as above, we see that π_r induces an isomorphism of E_{r-1}^h onto E_{r-1}^h.

REMARK (1.8). The non cuspidal $\boldsymbol{F}_p$-rational point on E_r^h (resp. E_0^h) is represented by an elliptic curve E $/\boldsymbol{F}_p$ with the subgroup $A = (E_{p^r})^0 = \ker(F^r: E \to E^{(p^r)} = E)$ (resp. $A = \ker(V^r: E^{(p^r)} = E \to E)$), where F is the Frobenius map and V is the Verschiebung.

Let $N \geqq 1$ be an integer and $J_0(N)$ be the jacobian variety of $X_0(N)$. Let B be an abelian subvariety of $J_0(N)$ defined over $\boldsymbol{Q}$, $A = J_0(N)/B$ be

the (optimal) quotient and $a: J_0(N) \to A$ be the natural morphism. Let N' (resp. N'') be the product of all the prime divisors q of N with $q^2 \nmid N$ (resp. $q^2 | N$). Let $g: X \to Y$ be a morphism of schemes and x be a section of X. Denote by $\mathrm{Cot}_x(g): \mathrm{Cot}_{g(x)} Y \to \mathrm{Cot}_x X$ the morphism of the cotangent spaces along x and $g(x)$. The morphism g is a formal immersion along x if $g^*(\widehat{\mathcal{O}_{Y,g(x)}}) = \widehat{\mathcal{O}_{X,x}}$, where $\widehat{\mathcal{O}_{X,x}}$ and $\widehat{\mathcal{O}_{Y,g(x)}}$ are the completions of the local rings by the maximal ideals $m_{X,x}$ and $m_{Y,g(x)}$. A criterion of the formal immersion is seen in E. G. A. IV 17.44. This tells us that g is a formal immersion along x if and only if the following conditions are satisfied: (i) $g^*(\mathcal{O}_{Y,g(x)}/m_{Y,g(x)}) = \mathcal{O}_{X,x}/m_{X,x}$ and (ii) $\mathrm{Cot}_x(g)$ is a surjective morphism.

PROPOSITION (1.9) (Mazur [14] §3). *Under the notation as above, let q be a prime number not dividing $2N''$. Then* $\mathrm{Cot}_0(a \otimes \boldsymbol{F}_q): \mathrm{Cot}_0(A_{/\boldsymbol{Z}} \otimes \boldsymbol{F}_q) \to \mathrm{Cot}_0(J_0(N)_{/\boldsymbol{Z}} \otimes \boldsymbol{F}_q)$ *is surjective, where* 0 *are the unit sections of group schemes* $A_{/\boldsymbol{Z}} \otimes \boldsymbol{F}_q$ *and* $J_0(N)_{/\boldsymbol{Z}} \otimes \boldsymbol{F}_q$.

Let $\tilde{\mathscr{X}}_0(N) \to \mathrm{Spec}\, \boldsymbol{Z}$ be the minimal model of $X_0(N)$, q be a prime number not dividing N'' and $R = W(\bar{\boldsymbol{F}}_q)$ be the ring of integers of $\boldsymbol{Q}_q^{ur}$. Denote by ι the duality of Grothendieck [14] §2:

$$\iota: \mathrm{Cot}_0 J_0(N)_{/\boldsymbol{Z}[1/N']} \xrightarrow{\sim} H^0(\tilde{\mathscr{X}}_0(N) \otimes \boldsymbol{Z}[1/N''], \Omega) ,$$

where Ω is the sheaf of regular differentials [3] p. 162. Let x be a section: $\mathrm{Spec}\, R \to \mathscr{X}_0(N)^{\mathrm{smooth}}$, and b be the morphism of $X_0(N)$ to $J_0(N)$ defined by $z \mapsto \mathrm{cl}((z) - (x))$. Denote also by b the morphism of $\mathscr{X}_0(N)^{\mathrm{smooth}} \otimes R$ to $J_0(N)_{/R}$, which is induced by the universal property of the Néron model.

PROPOSITION (1.10) (Mazur [14] §2 Lemma (2.1)). *Under the notation as above, the following diagram is commutative up to sign:*

$$\begin{array}{ccc} \mathrm{Cot}_x \mathscr{X}_0(N) \otimes R & \xleftarrow{\mathrm{Cot}_x(b)} & \mathrm{Cot}_0 J_0(N)_{/R} \\ {\scriptstyle a_1}\uparrow & \nwarrow \quad \swarrow{\scriptstyle \sim}\,\iota & \\ & H^0(\tilde{\mathscr{X}}_0(N) \otimes R, \Omega) & \\ \omega = \sum a_n q^n \frac{dq}{q} \mapsto & & \end{array}$$

where $q = q_x$ is the local parameter along x.

LEMMA (1.11) ([16] (3.6)). *Let p be a prime number congruent to $1 \bmod 8$, and c be the natural morphism of $J_0(p)_{/Z}$ to $\tilde{J}_0(p)_{/Z}$. Then there exists a form $\omega \in c^*(\mathrm{Cot}_0 \tilde{J}_0(p)_{/Z_2})$ $(\subset H^0(\tilde{\mathscr{X}}_0(p) \otimes \boldsymbol{Z}_2, \Omega^1))$ whose value at the cuspidal section $0=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a unit of $\boldsymbol{Z}_2$.*

PROPOSITION (1.12) ([16] (2.5)). *Let $p \geqq 17$ or $p=11$ be a prime number and $\tilde{f}=\tilde{f}_{2,1}: \tilde{\mathscr{X}}_0(p^2) \otimes \boldsymbol{Z}[1/p] \to \tilde{J}_0(p)_{/\boldsymbol{Z}[1/p]}$ be the morphism for a triple $(p, r, s)=(p, 2, 1)$ defined before. Then $\tilde{f} \otimes \boldsymbol{Z}_q$ is a formal immersion along any cuspidal section of $\tilde{\mathscr{X}}_0(p^2) \otimes \boldsymbol{Z}_q$ for $q \neq 2, p$.*

LEMMA (1.13). *Let $(p, s)=(2, 5)$, (3.3) or (7, 2) be a pair as in (1.4). The natural morphism of $\tilde{\mathscr{X}}_0(p^s)^{\mathrm{smooth}}$ to the Néron model $J_0(p^s)_{/Z}$ is an open immersion.*

PROOF. The modular curves $X_0(p^s)$ for (p, s) as above are elliptic curves. We can easily see this lemma, using the minimal models [24] Table 1 pp. 81–113. The details for $p=2$ and 3 will be explained in §3. □

PROPOSITION (1.14). *Under the notation as above, let (p, r, s) be a triple as in (1.5). Let q be a prime number and $\tilde{f}=\tilde{f}_{r,s}$ be the morphism of $\tilde{\mathscr{X}}_0(p^r)^{\mathrm{smooth}}$ to $\tilde{J}_0(p^s)_{/Z}$ (for $p \neq 5, 13$), $\tilde{J}_0(125)_{(5)/Z}$ or $\tilde{J}_0(169)_{(7)/Z}$ defined before. Then $\tilde{f} \otimes \boldsymbol{Z}_q$ is a formal immersion along the cuspidal sections for the following cuspidal sections:*

(i) *For $q \neq 2$, along $0=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\infty=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} i \\ p \end{pmatrix}$ and $\begin{pmatrix} i \\ p^{r-1} \end{pmatrix}$.*
(ii) *For $p \geqq 17$ or $p=11$, and $q=p$, along 0 and ∞.*
(iii) *For $p \geqq 17$, $p \equiv 1 \bmod 8$, and $q=2$, along 0 and ∞.*
(iv) *For $p=2, 3$ and 7, and $q=2$, along 0 and ∞.*
(v) *For $p=q=2$, along $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2^{r-1} \end{pmatrix}$.*

PROOF. For the simplicity, we here denote by $\tilde{J}_0(125)$ and $\tilde{J}_0(169)$ the quotients $\tilde{J}_0(125)_{(5)}$ and $\tilde{J}_0(169)_{(7)}$. For a triple (p, r, s) and a rational prime q as in the statement of the above proposition, the cuspidal sections 0, ∞, $\begin{pmatrix} i \\ p \end{pmatrix}$ and $\begin{pmatrix} i \\ p^{r-1} \end{pmatrix}$ are contained in the smooth part $\tilde{\mathscr{X}}_0(p^r)^{\mathrm{smooth}}$ $(/\boldsymbol{Z}_q)$ (1.7). The fundamental involution w_{p^r} exchanges 0 with ∞, and $\begin{pmatrix} i \\ p \end{pmatrix}$ with $\begin{pmatrix} j \\ p^{r-1} \end{pmatrix}$ for an integer j congruent to $-i^{-1} \bmod p$. Therefore it suffices to consider the sections ∞ and $C_i=\begin{pmatrix} i \\ p^{r-1} \end{pmatrix}$. Denote

by q_∞, q_i the local parameters along the cuspidal sections ∞ and C_i of $\mathscr{X}_0(p^r)\otimes R$ for $R=W(\bar{F}_q)$, respectively. Further denote by q'_0, q'_∞ and q'_i the local parameters along the cuspidal sections 0, ∞ and $\binom{i}{p}$ of $\mathscr{X}_0(p^s)\otimes R$, respectively. Let $u=u_{r,s}$ be the morphism of $\mathscr{X}_0(p^s)^{\text{smooth}}$ to $\tilde{J}_0(p^s)_{/Z}$ defined by

$$\begin{aligned} u\colon \mathscr{X}_0(p^s)^{\text{smooth}} &\longrightarrow J_0(p^s)_{/Z} \xrightarrow{\text{can.}} \tilde{J}_0(p^s)_{/Z}. \\ z &\longmapsto \mathrm{cl}((z)-(0)) \end{aligned}$$

The morphism $\tilde{f}=\tilde{f}_{r,s}$ is defined by the following diagram:

$$\begin{array}{ccccccc} \mathscr{X}_0(p^r)^{\text{smooth}} & & & \overset{\tilde{f}}{\searrow} & & & \\ \downarrow{\scriptstyle w_{p^s}\pi\times\pi w_{p^r}} & & & \subset & & & \\ \mathscr{X}_0(p^s)^{\text{smooth}}\times\mathscr{X}_0(p^s)^{\text{smooth}} & \longrightarrow & J_0(p^s)_{/Z}\times J_0(p^s)_{/Z} & \longrightarrow & J_0(p^s)_{/Z} & \xrightarrow{\text{can.}} & \tilde{J}_0(p^s)_{/Z}. \end{array}$$

$$(z_1, z_2)\longmapsto(\mathrm{cl}((z_1)-(0)), \mathrm{cl}((z_2)-(0))) \quad (x, y)\longmapsto x+y$$

Then $\mathrm{Cot}_C(\tilde{f})=\mathrm{Cot}_C(uw_{p^s}\pi)-\mathrm{Cot}_C(u\pi w_{p^r})$ for any cuspidal section C: $\mathrm{Spec}\, R\to\mathscr{X}_0(p^r)^{\text{smooth}}$. For $q\neq p$, by (1.12), it suffices to consider for $r\geqq 3$. Then for $q\neq p$ and $r\geqq 3$,

$$\begin{aligned} &(w_{p^s}\pi)^*(q'_0)=q_\infty\times(\text{a unit}) \\ &(\pi w_{p^r})^*(q'_0)=(q)^{p^{r-s}}\times(\text{a unit}) \end{aligned} \qquad \text{along } \infty,$$

$$\begin{aligned} &(w_{p^s}\pi)^*(q'_0)=q_i\times(\text{a unit}) \\ &(\pi w_{p^r})^*(q'_\infty)=(q_i)^{p^{r-1}}\times(\text{a unit}) \end{aligned} \qquad \text{along } C_i \text{ if } s=1,$$

$$\begin{aligned} &(w_{p^s}\pi)^*(q'_0)=q_i\times(\text{a unit}) \\ &(\pi w_{p^r})^*(q'_j)=(q_i)^{p^{r-s}}\times(\text{a unit}) \end{aligned} \qquad \text{along } C_i \text{ if } s\geqq 2,$$

where j is an integer congruent to $-i^{-1} \bmod p$. Therefore $\mathrm{Cot}_\infty(\tilde{f})\otimes R$ and $\mathrm{Cot}_{C_i}(\tilde{f})\otimes R$ are surjective for $q\neq p$ (cf. (1.9), (1.10). (1.11), (1.12)). For $q=p$, (1.7) shows that $\mathrm{Cot}_\infty(u\pi w_{p^r})\otimes\bar{F}_p$ and $\mathrm{Cot}_{C_i}(u\pi w_{p^r})\otimes\bar{F}_p$ are 0-maps, and $\mathrm{Cot}_\infty(uw_{p^s}\pi)\otimes\bar{F}_p$ and $\mathrm{Cot}_{C_i}(uw_{p^s}\pi)\otimes\bar{F}_p$ are not 0-maps (see loc.cit.). Thus $\mathrm{Cot}_\infty(f)\otimes R$ and $\mathrm{Cot}_{C_i}(f)\otimes R$ are surjective for $q=p$. □

§2. Elliptic curves.

In this section, we prepare some lemmas on elliptic curves and finite flat group schemes. Throughout this section, K denotes a finite

extension of $\boldsymbol{Q}_p^{ur}$ of degree e_K, and $R=\mathcal{O}_K$ is the ring of integers of K. For an elliptic curve E with a finite subgroup A defined over K, $A_{/R}$ denotes the schematic closure of A in the Néron model $E_{/R}$. Then $A_{/R}$ is a quasi finite flat subgroup scheme [20] §2. The finite flat group schemes of type $(p, \cdots, p)$ over $\boldsymbol{Z}_p$ are classified in [19] [20]. Let v be the valuation of K such that $v(K^\times)=\boldsymbol{Z}$. Let G be a finite flat group scheme of rank p over $R=\mathcal{O}_K$, then $G\simeq \mathrm{Spec}\, R[X]/(X^p-\delta X)$ for $\delta\in R$ with $0\leqq v(\delta)\leqq e_K$. If $v(\delta)=0$ (resp. $v(\delta)=e_K$), then $G\simeq(\boldsymbol{Z}/p\boldsymbol{Z})_{/R}$ (resp. $G\simeq\mu_{p/R}$) (see loc.cit.).

THEOREM (2.1) ([20] §3 (3.3.2), [19]). *Let* $\mathrm{Spec}\, R[X]/(X^p-\delta_i X)$ *be finite flat group schemes of rank* p *over* R. *Let* g *be a homomorphism of* G_1 *to* G_2 *such that* $g\otimes K: G_1\otimes K\to G_2\otimes K$ *is an isomorphism. Then* $v(\delta_1)\equiv v(\delta_2) \bmod (p-1)$. *Further*

(i) *If* $e_K<p-1$, *then* g *is an isomorphism.*

(ii) *If* $e_K=p-1$ *and* g *is not an isomorphism, then* $G_1\simeq(\boldsymbol{Z}/p\boldsymbol{Z})_{/R}$ *and* $G_2\simeq\mu_{p/R}$.

LEMMA (2.2). *Let* E *be a semistable elliptic curve with a cyclic subgroup* A *of order* p^r *defined over* K *for* $r\geqq 2$. *Let* x *be the* R*-section of* $\mathscr{X}_0(p^r)$ *such that* $x\otimes K$ *is represented by the pair* (E, A). *Then*

(i) *If* $x\otimes\bar{\boldsymbol{F}}_p$ *is a section of* E_i^h, *then* K *contains a primitive* $p^{m(i)}$*-th root* $\zeta_{p^{m(i)}}$ *of unity for* $m(i)=\min\{i, r-i\}$.

(ii) *If* $x\otimes\boldsymbol{F}_p$ *is a supersingular point, then* $e_K\geqq p+1$.

PROOF. If $x\otimes\bar{\boldsymbol{F}}_p$ is a section of E_i^h, then A admits the following exact sequence (1.7):

$$0\longrightarrow\mu_{p^i\,/K}\longrightarrow A\longrightarrow(\boldsymbol{Z}/p^{r-i}\boldsymbol{Z})_{/K}\longrightarrow 0.$$

Since A is cyclic, K contains $\zeta_{p^{m(i)}}$. Now assume that $x\otimes\bar{\boldsymbol{F}}_p$ is a supersingular point. Let y be the R-section of $\mathscr{X}_s(p)\simeq\mathscr{X}_0(p^2)$ such that $y\otimes K$ is represented by the semistable elliptic curve $F=E/A_p$ with the independent cyclic subgroups $C_1=A_{p^2}/A_p$ and $C_2=E_p/A_p$ of order p. The schematic closure $C_{i/R}$ of C_i in the Néron model $F_{/R}$ are finite flat group schemes, and $C_{i/R}=\mathrm{Spec}\, R[X]/(X^p-\delta_i X)$ for $\delta_i\in R$ with $1\leqq v(\delta_i)\leqq e_K-1$. Consider the following morphisms of finite flat group schemes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{1/R} & \longrightarrow & (E_{/R})_p & \longrightarrow & (E_{/R})_p/(C_{1/R}) & \longrightarrow & 0 \text{ (exact).} \\ & & & & \cup & \nearrow_{f} & \| & & \\ & & & & C_{2/R} & & C & & \end{array}$$

Then $f\otimes K: C_2 \to E_p/C_1 \simeq C\otimes K$ is an isomorphism. Then

$$C \simeq \operatorname{Spec} R[X]/(X^p - \delta X)$$

for $\delta \in R$ with $1 \leqq v(\delta) \leqq e_K - 1$ and $v(\delta) \equiv v(\delta_2) \bmod (p-1)$. If $v(\delta) = v(\delta_2)$, then f is an isomorphism and $C_{1/R} \cap C_{2/R} = \{0\}$. For any supersingular elliptic curve $H/\bar{\boldsymbol{F}}_p$, $H_p \simeq \operatorname{Spec} \bar{\boldsymbol{F}}_p[X]/(X^{p^2})$ as schemes, so that $(E_{/R})_p \neq C_{1/R} \oplus C_{2/R}$. Therefore $v(\delta) \neq v(\delta_2)$ and $e_K \geqq p+1$. □

Let m be the maximal ideal of R and E be an elliptic curve with a cyclic subgroup A defined over K. Then E has semistable reduction over a finite extension of K. If the modular invariant $j(E) \not\equiv 0$, $1728 \bmod m$, then there exists a semistable elliptic curve F with a cyclic subgroup B defined over K such that $(E, A) \simeq (F, B)$ over a quadratic extension of K. If $p \geqq 5$ and $j(E) \equiv 0 \bmod m$ (resp. $j(E) \equiv 1728 \bmod m$), then there exists a semistable elliptic curve F with a cyclic subgroup B defined over a finite extension K' of K of degree 1 or 3 (resp. of degree 1 or 2) such that $(E, A) \simeq (F, B)$ over the quadratic extension of K'.

COROLLARY (2.3). *Under the notation as Lemma* (2.2), *put* $e' = 1$ *if* $j(x) \not\equiv 0$, $1728 \bmod m$, $e' = 3$ *if* $j(x) \equiv 0 \bmod m$ *and* $e' = 2$ *if* $j(x) \equiv 1728 \bmod m$. *Then*

(i) *If* $x \otimes \bar{\boldsymbol{F}}_p$ *is a section of* E_i^k, *then* $e_K e' \geqq p^{m(i)-1}(p-1)$ *if* $1 \leqq i \leqq r-1$.

(ii) *If* $x \otimes \bar{\boldsymbol{F}}_p$ *is a supersingular point and* $p \geqq 5$, *then* $e_K e' \geqq p+1$.

§3. Rational points on $X_0^+(p^r)$.

In this section, we will prove Theorem (0.1) and give other results. Let (p, r) be a pair in (1.0). Let y be a non cuspidal $\boldsymbol{Q}$-rational point on $X_0^+(p^r)$, and x, $x' = w_{p^r}(x)$ be the sections of the fibre $X_0(p^r)_y$ at y. Then x and x' are not defined over $\boldsymbol{Q}$ (1.1). They are defined over a quadratic field k, and $x' = x^\sigma$ for $1 \neq \sigma \in \operatorname{Gal}(k/\boldsymbol{Q})$. There is an elliptic curve E with a cyclic subgroup A of order p^r such that E and A are defined over k and that the pairs (E, A) and $(E/A, E_{p^r}/A)$ represent respectively x and x' [3] VI (3.2). Further the pair (E^σ, A^σ) is isomorphic /$\boldsymbol{C}$ to $(E/A, E_{p^r}/A)$. Denote also by x, x' (resp. y) the $\mathcal{O}_k$ (resp. $\boldsymbol{Z}$)-sections of $\mathscr{X}_0(p^r)$ (resp. $\mathscr{X}_0^+(p^r)$) with generic fibres x and x' (resp. y). Let $\mathfrak{p}$ be a prime of k lying over the rational prime p and e_k be the

ramification index of p in k. When $r=2t$ is even, the modular curves $\mathscr{X}_0(p^{2t})$ and $\mathscr{X}_0^+(p^{2t})$ are isomorphic respectively to $\mathscr{X}_s(p^t)$ and $\mathscr{X}_s^n(p^t)$. Let y' be the corresponding (to y) $\boldsymbol{Q}$-rational point on $X_s^n(p^t)$. Then there exists an elliptic curve E' defined over $\boldsymbol{Q}$ with independent cyclic subgroups C_1 and C_2 of order p^t such that the set $\{C_1, C_2\}$ is $\boldsymbol{Q}$-rational and that the pair $(E', \{C_1, C_2\})$ represents y' (see loc.cit.).

LEMMA (3.1). *Let (p, r) be a pair in* (1.0), *and x, x', y and e_k be as above. If $e_k=2$, then $x\otimes\kappa(\mathfrak{p})=x'\otimes\kappa(\mathfrak{p})$ is a section of E_t if $r=2t$ is even, and it is a supersingular point if r is odd* (1.7). *If $x\otimes\kappa(\mathfrak{p})$ is a section of E_i^h for an integer $i\neq r/2$, then the rational prime p splits in k.*

PROOF. First note that $x'=w_{p^r}(x)=x^\sigma$ for $1\neq\sigma\in\mathrm{Gal}(k/\boldsymbol{Q})$ and that the irreducible components E_i are all $\boldsymbol{F}_p$-rational (1.7). If $e_k=2$, then $x\otimes\kappa(\mathfrak{p})=x^\sigma\otimes\kappa(\mathfrak{p})=x'\otimes\kappa(\mathfrak{p})$, so that $x\otimes\kappa(\mathfrak{p})=x'\otimes\kappa(\mathfrak{p})$ is a section of E_t if $r=2t$ is even, and it is a supersingular point if r is odd. Now assume that $x\otimes\kappa(\mathfrak{p})$ is a section of E_i^h for an integer $i\neq r/2$. Then $x\otimes\kappa(\mathfrak{p})\neq x'\otimes\kappa(\mathfrak{p})=x^\sigma\otimes\kappa(\mathfrak{p})$. If the rational prime p remains prime in k, then $x^\sigma\otimes\kappa(\mathfrak{p})=(x\otimes\kappa(\mathfrak{p}))^{(p)}$ becomes a section of $(E_i^h)^{(p)}=E_i^h$. Here $(x\otimes\kappa(\mathfrak{p}))^{(p)}$ and $(E_i^h)^{(p)}$ are the images of $x\otimes\kappa(\mathfrak{p})$ and E_i^h under the Frobenius map $F\colon \mathscr{X}_0(p^r)\otimes\boldsymbol{F}_p\to\mathscr{X}_0(p^r)\otimes\boldsymbol{F}_p$. But $x'\otimes\kappa(\mathfrak{p})=x^\sigma\otimes\kappa(\mathfrak{p})$ is a section of $E_{r-i}^h\neq E_i^h$. Therefore p splits in k. □

Now we prove the following theorem.

THEOREM (3.2). *Let (p, r) be a pair in* (1.0), *and x, $x'=x^\sigma$, y and e_k be as above. Then the rational prime p splits in k. Further $x\otimes\kappa(\mathfrak{p})$ and $x'\otimes\kappa(\mathfrak{p})$ are the sections of $E_0^h\cup E_r^h$ if $p\neq2$, and they are the sections $E_0^h\cup E_1^h\cup E_{r-1}^h\cup E_r^h$ if $p=2$.*

PROOF. If the modular invariant $j(x)\equiv0$ (resp. 1728) mod $\mathfrak{p}$, then $j(x')\equiv0$ (resp. 1728) mod $\mathfrak{p}$. Put $e'=1$ if $j(x)\not\equiv0$, 1728 mod $\mathfrak{p}$, $e'=3$ if $j(x)\equiv0$ mod $\mathfrak{p}$ and $p\geqq5$, and $e'=2$ if $j(x)\equiv1728$ mod $\mathfrak{p}$ and $p\geqq5$ cf. (2.3).

Case for $p\geqq11$. Corollary (2.3), applied to the inequality $e_ke'\leqq6<p-1$, shows that $x\otimes\kappa(\mathfrak{p})$ and $x'\otimes\kappa(\mathfrak{p})$ are the sections of $E_0^h\cup E_r^h$. Then by Lemma (3.1), the rational prime p splits in k.

Case for $p=7$. If $j(x)\not\equiv0$, 1728 mod $\mathfrak{p}$, then $e_ke'=e_k<p-1$. If $x\otimes\kappa(\mathfrak{p})$ is a supersingular point, then $j(x)\equiv1728$ mod $\mathfrak{p}$. Then $e_ke'\leqq4<p+1$. If $j(x)\equiv0$ mod $\mathfrak{p}$ and $e_ke'\geqq p-1$, then $e_k=2$ and r is even

(3.1). Then $e_k e' \leqq 6 < p(p-1)$. Therefore Corollary (2.3) and Lemma (3.1) give the result.

Let v be the normalized valuation of $\bar{\boldsymbol{Q}}_p$ such that $v(p)=1$, and regard $k_{\mathfrak{p}}$ as a subfield of $\bar{\boldsymbol{Q}}_p$.

Case for $p=5$ ($r \geqq 3$). The supersingular modular invariant in characteristic 5 is 0. The same argument as for $p=7$ gives the result, except for the case when $j(x) \equiv j(x') \equiv 0 \bmod \mathfrak{p}$ and $e_k=2$. For the remaining cases, it suffices to discuss for $r=3$ and 4. We make use of the equations of $X_0(5)$ and $X_0(25)$ [4] IV. The modular curve $X_0(5)$ is defined by the equation

$$j=(X^2+10X+5)^3/X \tag{3.2.1}$$

with the fundamental involution $w_5^*(X)=125/X$ [4] IV §3. The modular curve $X_0(25)$ is defined by the equation

$$j=g(Y)/Y(Y^4+5Y^3+3\cdot 5Y^2+5^2Y+5^2) \tag{3.2.2}$$

with $w_{25}^*(Y)=5/Y$, where $g(Y)=(Y^{10}+2\cdot 5Y^9+11\cdot 5Y^8+8\cdot 5^2Y^7+21\cdot 5^2Y^6+202\cdot 5Y^5+57\cdot 5^2Y^4+56\cdot 5^2Y^3+7\cdot 5^3Y^2+2\cdot 5^3Y+5)^3$ see loc. cit.

Case for $r=3$. Consider the following coverings

$$\begin{array}{ccc} X_0(125) & & (E,\ A) \\ \swarrow \quad \searrow & & \swarrow \quad \searrow \\ X_0(5) \qquad X_0(25) & & (E/A_5,\ A_{25}/A_5) \qquad (E/A_5,\ A/A_5). \end{array}$$

Let z_1, z_2 be the images of x under the above morphisms of $X_0(125)$ to $X_0(5)$ and $X_0(25)$, respectively. Then $j(z_1)=j(z_2)$ and the set $\{z_1, w_5(z_2)\}$ defines a $\boldsymbol{Q}$-rational point on $X_0^+(5)$. The points z_1, $w_5(z_1)$ are defined by $X=a$ and $X=125/a=a^\sigma$ for $a \in k^\times$ and $1 \neq \sigma \in \mathrm{Gal}(k/\boldsymbol{Q})$ (3.2.1). Then the valuation $v(a)=3/2$ and $v(j(z_1))=3/2$ (see loc.cit.), since the rational prime $p=5$ ramifies in k. The point z_2 is defined by $Y=b$ for $b \in k^\times$ (3.2.2). Using the equation (3.2.2) and the condition $v(j(z_2))>0$, we see that $v(b)=1/2$ and $v(j(z_2))=1/2$. This contradicts that $j(z_1)=j(z_2)$.

Case for $r=4$. We make use of the modular curve $X_s(25)$. Let z, z' ($=z^\sigma$ for $1 \neq \sigma \in \mathrm{Gal}(k/\boldsymbol{Q})$) be the images of x and $x'=x^\sigma$ under the isomorphism of $X_0(5^4)$ to $X_s(25)$: $(E, A) \mapsto (E/A_{25}, A/A_{25}, E_{25}/A_{25})$. Let w, w' be the images of z and z' under the natural morphism of $X_s(25)$ to $X_0(25)$: $(E, C_1, C_2) \mapsto (E, C_1)$. Then $w'=w^\sigma$. Since $v(j(w))>0$, using the

equation (3.2.2), we see that w is defined by $Y=a\in k^\times$ with $v(a)=1/2$. Put $a=\sqrt{5}\,b$ for $b\in\bar{\boldsymbol{Q}}_5$ with $v(b)=0$. Then

$$j(w)=\sqrt{5}\,\frac{(1+\sqrt{5}\,c)^3}{b(b^4+3b^2+1+\sqrt{5}\,d)}$$

for $c, d\in\bar{\boldsymbol{Q}}_5$ with $v(c)\geqq 0$ and $v(d)\geqq 0$. Since $v(j(w))>0$, $b^4+3b^2+1\not\equiv 0 \bmod (\sqrt{5})$. Let C be the curve defined by the equation

(3.2.3)
$$0=\frac{Y(Y^4+5Y^3+15Y^2+25Y+25)g(Z)-Z(Z^4+5Z^3+15Z^2+25Z+25)g(Y)}{Y-Z}$$

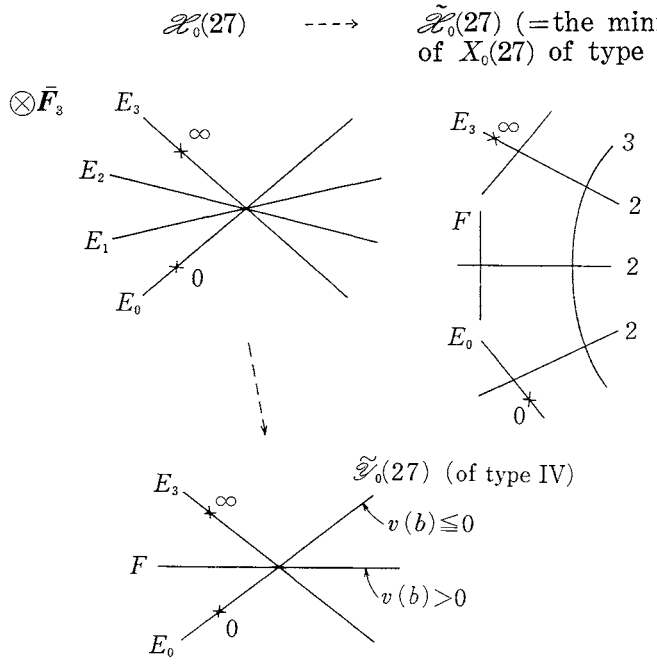
see loc.cit. The pair (w, w') of points defines a point on C defined by $(Y, Z)=(\sqrt{5}\,b, -\sqrt{5}\,b^\sigma)$. But the equation (3.2.3) gives the congruence $b^4+3b^2+1\equiv 0 \bmod (\sqrt{5})$. Thus we get a contradiction.

Proof for $p=3$ *and* $r\geqq 4$. The same argument as for $p=7$ gives the result, except for the case when $j(x)\equiv 0 \bmod \mathfrak{p}$. We first describe the $\boldsymbol{Q}$-rational points on $X_0^+(3^{2t})$ for $t\geqq 2$. The modular curve $X_0^+(81)$ is an elliptic curve and it is isogenous over $\boldsymbol{Q}$ to $X_0(27)$ [24] Table 5 pp. 135–141. Then $X_0^+(81)$ has three $\boldsymbol{Q}$-rational points [24] Table 1 pp. 81–113. Indeed, $X_0^+(3^{2t})$ $(t\geqq 2)$ has three $\boldsymbol{Q}$-rational points. One of them is a cusp and the others are C. M. points whose special fibres at the rational prime 3 are the sections of $(E_0\cup E_{2t})^h/\langle w_{3^{2t}}\rangle$. These C. M. points are represented by elliptic curves E and F $(/\boldsymbol{Q})$ which are isomorphic over $\boldsymbol{C}$ to $\boldsymbol{C}/\boldsymbol{Z}+\boldsymbol{Z}\sqrt{-2}$ and $\boldsymbol{C}/\boldsymbol{Z}+\boldsymbol{Z}(1+\sqrt{-11})/2$, respectively. Let c, $\bar{c}$ (resp. d, $\bar{d}$) be the endomorphisms of E (resp. F) corresponding to $1+\sqrt{-2}$ and $1-\sqrt{-2}$ (resp. $(1+\sqrt{-11})/2$ and $(1-\sqrt{-11})/2$) under the isomorphism $\mathrm{End}\,E\simeq\boldsymbol{Z}[\sqrt{-2}]$ (resp. $\mathrm{End}\,F\simeq\boldsymbol{Z}[(1+\sqrt{-11})/2]$). Put $A=\ker(c^t\colon E\to E)$, $\bar{A}=\ker(\bar{c}^t\colon E\to E)$, $B=\ker(d^t\colon F\to F)$ and $\bar{B}=\ker(\bar{d}^t\colon F\to F)$. Then the C. M. points are represented by the pairs $(E, \{A, \bar{A}\})$ and $(F, \{B, \bar{B}\})$.

Now consider the case for odd integers $r\geqq 5$. As was seen as above, it suffices to discuss the case for $r=5$ and $j(x)\equiv 0 \bmod \mathfrak{p}$. We make use of the minimal models of $X_0(27)$ over the basis $\boldsymbol{Z}_3$ and $\boldsymbol{Z}_3[\sqrt{-3}]$, and the equation of $X_0(9)$. The modular curve $X_0(9)$ is defined by the equation

(3.2.4)
$$j=g(X)/X(X^2+9X+27)$$

with $w_9^*(X)=27/X$, where $g(X)=(X^4+4\cdot 3X^3+2\cdot 27X^2+28\cdot 3X+3)^3$ [4] IV §2. Let z be the image of x under the morphism of $X_0(3^5)$ to $X_0(9)$ defined by $(E, A)\mapsto(E/A_3, A_{27}/A_3)$. Then z is defined by $X=a$ for $a\in k^\times$. Since $v(j(z))>0$, using the equation (3.2.4), we see that $v(a)=1/2$, 1, 3/2, 2 or 5/2. Let $\tilde{\mathscr{Y}}_0(27)\to\mathrm{Spec}\,\boldsymbol{Z}[(1+\sqrt{-3})/2]$ be the minimal model of $X_0(27)\otimes\boldsymbol{Q}(\sqrt{-3})$ ([24] Table 1 pp. 81-113).



LEMMA (3.2.5). *Let z be a $\boldsymbol{Q}_5^{ur}(\sqrt{-3})$-rational section of $\mathscr{X}_0(27)$ whose special fibre is the supersingular point. Denote also by z the $W(\bar{\boldsymbol{F}}_3)[\sqrt{-3}]$-section of the Néron model $\mathscr{E}=\tilde{\mathscr{Y}}_0(27)^{\mathrm{smooth}}$ defined by z. Then $z\otimes\bar{\boldsymbol{F}}_3$ is a section of the irreducible component F see above.*

PROOF. The modular curve $X_0(27)\otimes\boldsymbol{Q}(\sqrt{-3})$ is defined by the equation

$$\text{(3.2.6)}\qquad \begin{cases} Z^2=Y^4+4Y^3+6Y-3 \\ Y=(\sqrt[3]{3X^2+3X+1}-1)/(X+1) \end{cases}$$

with $w_{27}^*(Y)=Y$ and $w_{27}^*(Z)=-Z$ [4] IV §2. Here the function X in (3.2.6) is the same function in (3.2.4) see loc.cit. Blowing up along the super-

singular point $(Y, Z, \sqrt{-3})=(0, 0, 0)$, we get the minimal model $\tilde{\mathscr{Y}}_0(27)$. The section z of $X_0(27)$ is defined by $(X, Y, Z)=(a, b, c)$ for $a, b, c \in \boldsymbol{Q}_3^{ur}(\sqrt{-3})$. By the condition $v(j(z))>0$, we get $v(a)=1/2, 1, 3/2, 2$ or $5/2$ (3.2.4). Then by (3.2.6), $v(b)\geqq 1$. Further, if $z\otimes\bar{\boldsymbol{F}}_3$ is a section of $E_0\backslash\{\text{supersingular point}\}$, then $v(b)\leqq 0$. Therefore $z\otimes\bar{\boldsymbol{F}}_3$ is a section of $E_2\cup F$. But if $z\otimes\bar{\boldsymbol{F}}_3$ is a section of E_2 $(\subset\mathscr{E})$, then $w_3(z)\otimes\bar{\boldsymbol{F}}_3$ becomes a section of E_0. Thus $z\otimes\bar{\boldsymbol{F}}_3$ is a section of F. □

Let C be the schematic closure of the finite group $\langle \mathrm{cl}((0)-(\infty))\rangle$ in $\mathscr{E}$. Then by the construction of $\mathscr{E}$ (see above), we see that C is an étale subgroup of order 3 (cf. [24] Table 1 pp. 81–113). Let $f=f_{5,3}$ be the morphism of $\mathscr{X}_0(3^5)^{\mathrm{smooth}}$ to $J_0(27)_{/\boldsymbol{Z}}=J_0^-(27)_{/\boldsymbol{Z}}$ defined in §1. Then $f(x)\equiv m\,\mathrm{cl}((0)-(\infty))$, for an integer m, $0\leqq m\leqq 3$, since $J_0(27)(\boldsymbol{Q})=\langle \mathrm{cl}((0)-(\infty))\rangle$. Let $\mathscr{E}_0$ be the special fibre $\mathscr{E}\otimes\bar{\boldsymbol{F}}_3$ and $\mathscr{E}_0^0$ be the connected component of $\mathscr{E}_0$ of the unit section. Then $\mathscr{E}_0/\mathscr{E}_0^0\simeq \boldsymbol{Z}/3\boldsymbol{Z}$ and $\mathscr{E}_0/\mathscr{E}_0^0$ is generated by $\mathrm{cl}((0)-(\infty))\mathscr{E}_0^0$ (see the special fibre $\mathscr{E}_0$ as above). Then Lemma (3.2.5) shows that $f(x)\in\mathscr{E}_0^0$, so that $m=0$ and $f(x)=0$ (the unit section). Let E be an elliptic curve with a cyclic subgroup A of order 3^5 such that the pair (E, A) represents x. Then $(E/A_{27}, E_{27}/A_{27})\simeq (E/A, (E_{27}+A)/A)$ over $\boldsymbol{C}$. Therefore E is an elliptic curve with complex multiplication. Then for a $\boldsymbol{Q}$-rational point y on $X_0^+(3^5)$ and a section x of the fibre $X_0(3^5)_y$ at y, we see easily that $x\otimes\kappa(y)$ is a section of $E_0^h\cup E_5^h$.

Proof for $p=2$ *and* $r\geqq 6$. The same argument as for $p=7$ gives the result, except for the case when $j(x)\equiv 0 \bmod y$. We first describe the $\boldsymbol{Q}$-rational points on $X_0^+(2^t)$ for $t\geqq 3$. The modular curve $X_0^+(64)$ is an elliptic curve and it is isogenous over $\boldsymbol{Q}$ to $X_0(32)$ [24] Table 5 pp. 135–141. There are at most four $\boldsymbol{Q}$-rational points on $X_0^+(64)$ [24] Table 1 pp. 81–113. Indeed, $X_0^+(64)$ has four $\boldsymbol{Q}$-rational points. Two of them are cusps and the others are C. M. points. These special fibres at the rational prime 2 are sections of $(E_0\cup E_1\cup E_5\cup E_6)^h/\langle w_{64}\rangle$. We expain the corresponding C. M. points on $X_s^n(2^t)$ for $t\geqq 3$. Let E be an elliptic curve defined over $\boldsymbol{Q}$ which is isomorphic over $\boldsymbol{C}$ to $\boldsymbol{C}/\boldsymbol{Z}+\boldsymbol{Z}(1+\sqrt{-7})/2$. Let c, $\bar{c}$ be the endomorphisms of E corresponding to $(1+\sqrt{-7})/2$ and $(1-\sqrt{-7})/2$ under the isomorphism $\mathrm{End}\,E\simeq\boldsymbol{Z}[(1+\sqrt{-7})/2]$. Let e, $\bar{e}$ be generators of the cyclic subgroups $A=\ker(c^t: E\to E)$ and $\bar{A}=\ker(\bar{c}^t: E\to E)$, respectively. Put $B=\langle e+2^{t-1}\bar{e}\rangle$ and $\bar{B}=\langle \bar{e}+2^{t-1}e\rangle$. Then the C. M. points on $X_s^n(2^t)$ $(t\geqq 3)$ are represented by the pairs

$(E, \{A, \bar{A}\})$ and $(E, \{B, \bar{B}\})$.

Now consider the case for odd integers $r \geqq 7$. As noted as above, it suffices to consider the case when $r=7$ and $j(x) \equiv 0 \bmod \mathfrak{p}$. We make use of the equation of $X_0(8)$, $X_0(16)$ [4] IV and the minimal model of $X_0(32)$ [24] Table 1 pp. 81-113. The modular curve $X_0(16)$ is defined by the equation

$$j = g(X)/X(X+4)(X^2+4X+8)(X+2)^4 \tag{3.2.7}$$

with $w_{16}^*(X) = 8/X$, where $g(X) = (X^8+2^4X^7+7\cdot 2^4X^6+7\cdot 2^6X^5+69\cdot 2^4X^4+13\cdot 2^7X^3+11\cdot 2^7X^2+2^9X+2^4)^3$ [4] IV §1. The modular curve $X_0(32)$ is defined by the equation

$$C\colon\ Y^2 = X^3+6X^2+16X+16 \tag{3.2.8}$$

with $w_{32}^*(X) = 8(2X+4-Y)/X^2$ [4] IV §1. Here the function X in (3.2.8) is the same function in (3.2.7). The lemma below shows that if $v(j(x))>0$, then the rational prime 2 ramifies in k.

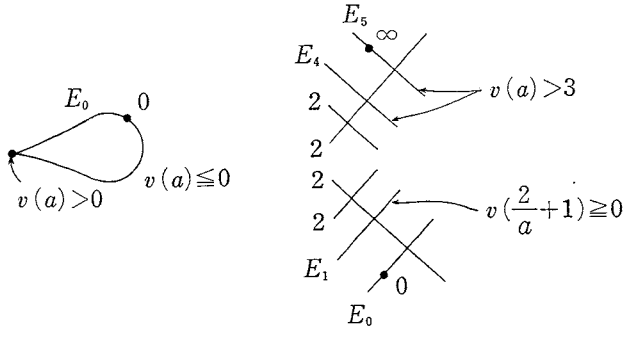
LEMMA (3.2.9). *Let z be a $\boldsymbol{Q}_2^{ur}$-rational point on $X_0(32)$. Then $v(j(z)) \leqq 0$.*

PROOF. The point z is defined by $(X, Y) = (a, b)$ for $0 \neq a, b \in \boldsymbol{Q}_2^{ur}$. Repeating the quadratic transformation, we get the minimal model $\tilde{\mathscr{X}}_0(32) \to \operatorname{Spec} \boldsymbol{Z}$ (3.2.7) (3.2.8) [24] Table 1 pp. 81-113:

$$C \quad \longleftarrow \quad \tilde{\mathscr{X}}_0(32) \qquad \text{(of type I*3)}$$

$\otimes \bar{\boldsymbol{F}}_2$

(3.2.10)



Using the equation (3.2.7), we see that if $v(j(z))>0$, then $v(a)=1$ or 2. Since x defines the $W(\bar{\boldsymbol{F}}_2)$-section of the Néron model $\tilde{\mathscr{X}}_0(32)^{\text{smooth}}$,

$v(a)\neq 2$ see (3.2.10). If $v(a)=1$, then by the equation (3.2.8), $v(b)\geqq 2$. The point $z'=w_{32}(z)$ is defined by $(X, Y)=(8(2a+4-b)/a^2, b')$ for $b'\in \boldsymbol{Q}_2^{ur}$ (3.2.8) and $v(j(z'))>0$. Then $v(8(2a+4-b)/a^2)\geqq 2$. Thus $v(j(z))\leqq 0$. □

The lemma below makes complete the proof of Theorem (3.2) for $p=2$.

LEMMA (3.2.11). *Let K be a quadratic extension of $\boldsymbol{Q}_2^{ur}$. Let z be a K-rational point on $X_0(32)$ such that $z'=w_{32}(z)=z^\sigma$ for $1\neq\sigma\in \mathrm{Gal}(K/\boldsymbol{Q}_2^{ur})$. Then $v(j(z))\leqq 0$.*

PROOF. Suppose $v(j(z))>0$. Then by Lemma (3.2.9), z is not defined over $\boldsymbol{Q}_2^{ur}$. Let z_1 (resp. z_2) be the image of z under the morphism of $X_0(32)$ to $X_0(8)$ (resp. $X_0(16)$) defined by $(E, A)\mapsto(E/A_2, A_{16}/A_2)$ (resp. $(E, A)\mapsto(E/A_2, A/A_2)$). Then $j(z_1)=j(z_2)$ and $w_8(z_1)=z_1^\sigma$. The point z_2 is defined by $X=c$ for $c\in K^\times$ (3.2.7). Let π be a prime element of $\mathcal{O}_K$. Using the equation (3.2.7), we see that $v(c)=1/2$, 1, 3/2, 2 or 5/2. Further we get

If $v(c)=1/2$, then $v(j(z_2))=8+3/2$ for $n=2\cdot v((c/\pi)^8+16/\pi^8)$.
If $v(c)=1$, then $2\nmid c/2+1$ and $v(j(z_2))=4-2n$ for

$$n=2\cdot v\left(\frac{c}{2}+1\right).$$

If $v(c)=3/2$, then $4\nmid c^2/8+c/2+1$ and $v(j(z_2))=2-n/2$ for

$$n=2\cdot v\left(\frac{c^2}{8}+\frac{c}{2}+1\right).$$

If $v(c)=2$, then $4\nmid c/4+1$ and $v(j(z_2))=2-n/2$ for

$$n=2\cdot v\left(\frac{c}{4}+1\right).$$

If $v(c)=5/2$, then $v(j(z_2))=1$.

In all cases above, $v(j(z_2))\neq 3$. The modular curue $X_0(8)$ is defined by the equation

$$j=2^8(Z^4+8Z^3+20Z^2+16Z+1)^3/Z(Z+4)(Z+2)^2 \tag{3.2.12}$$

with $w_8^*(Z)=8/Z$ [4] IV §1. Then z_1 is defined by $Z=d$ for $d\in K^\times$. Since $w_8(z_1)=z_1^\sigma$, $d=8/d$ and $v(d)=3/2$. Then by the equation (3.2.12), $v(j(z_1))=3$. This contradicts that $j(z_1)=j(z_2)$. □

PROPOSITION (3.3). *Let (p, r) be a pair as in* (1.0), *x and $\mathfrak{p}$ be as at the beginning of this section. Let $f=f_{r,s}$ be the morphism of $\mathscr{X}_0(p^r)^{\text{smooth}}$ to the Néron model $J_0(p^s)_{/Z}$ defined in* §1. *If $x\otimes\kappa(\mathfrak{p})$ is a section of $E_0^h\cup E_r^h$, then $f(x)\otimes\kappa(\mathfrak{p})=0$ ($=$the unit section of $J_0(p^s)_{/Z}\otimes\boldsymbol{F}_p$).*

PROOF. The rational prime p splits in k (3.1). If $x\otimes\kappa(\mathfrak{p})$ is a cusp, then $x\otimes\kappa(\mathfrak{p})=0\otimes\kappa(\mathfrak{p})$ or $\infty\otimes\kappa(\mathfrak{p})$, and $f(x)\otimes\kappa(\mathfrak{p})=0$. If $x\otimes\kappa(\mathfrak{p})$ is not a cusp, then $x\otimes\kappa(\mathfrak{p})$ is represented by an elliptic curve $C/\kappa(\mathfrak{p})=\boldsymbol{F}_p$ with the cyclic subgroup $B=\ker(F^r: C\to C=C^{(p^r)})$ or $B=\ker(V^r: C=C^{(p^r)}\to C)$ (1.8). Let $\pi=\pi_{r,s}$ be the natural morphism of $\mathscr{X}_0(p^r)$ to $\mathscr{X}_0(p^s)$: $(E, A)\mapsto(E, A_{p^s})$ for a triple (p, r, s) as in (1.5). Then $w_{p^s}\pi(x)\otimes\kappa(\mathfrak{p})$, $\pi w_{p^r}(x)\otimes\kappa(\mathfrak{p})$ are represented by the pairs $(C/B_{p^s}, C_{p^s}/B_{p^s})$ and $(C/B, (C_{p^s}+B)/B)$, respectively. Using the Frobenius map or the Verschiebung, we see that the pair $(C/B_{p^s}, C_{p^s}/B_{p^s})$ is isomorphic to $(C/B, (C_{p^s}+B)/B)$ see [16] (3.3). Then $w_{p^s}\pi(x)\otimes\kappa(\mathfrak{p})=\pi w_{p^r}(x)\otimes\kappa(\mathfrak{p})$ and $f(x)\otimes\kappa(\mathfrak{p})=0$. □

PROPOSITION (3.4). *Let x and f be as in* (3.3) *above, and let f^+, $\tilde{f}$ be the morphisms defined in* §1 *for a triple (p, r, s) in* (1.5). *For $p\neq5$, if $f(x)\otimes\kappa(\mathfrak{p})=0$, then $\tilde{f}(x)=0$. If moreover the Mordell-Weil group of $J_0^-(p^s)$ is of finite order, then $f^+(y)=0$ for $y=\{x, w_{p^r}(x)\}$.*

PROOF. The $\boldsymbol{Q}$-rational section $\tilde{f}(x)$ of $\tilde{J}_0(p^s)$ (for $p\neq5$, 7) or $\tilde{J}_0(169)_{(7)}$ is of finite order (1.3) (1.4). For the simplicity, we here denote $\tilde{J}_0(169)$ instead of $\tilde{J}_0(169)_{(7)}$. For $p\neq2$, 5, $\tilde{f}(x)$ generates a finite étale subgroup of the Néron model $J_0(p^s)_{/Z[1/2]}$. The group of $\boldsymbol{Q}$-rational points $J_0(32)(\boldsymbol{Q})$ is generated by the class $\mathrm{cl}((0)-(\infty))$ of order 4 [24] Table 1 pp. 81–113. Further we know that $\mathrm{cl}((0)-(\infty))\otimes\boldsymbol{F}_2$ is also of order 4 see loc.cit. (3.2.10). Therefore $\tilde{f}(x)$ generates a finite étale subgroup of $J_0(p^s)_{/Z_p}$ (for $p\neq5$). Then the assumption $\tilde{f}(x)\otimes\boldsymbol{F}_p=0$ leads $\tilde{f}(x)=0$. If moreover the Mordell-Weil group of $J_0^-(p^s)$ is of finite order, then $f^+(y)$ generates a finite étale subgroup of $J_0^-(p^s)_{/Z_p}$ and $f^+(y)=0$. □

PROPOSITION (3.5). *Let x, y and f^+ be as in* (3.4) *above. If $p\neq37$ and $f^+(y)=0$, then y is a C. M. point.*

PROOF. Let (E, A) be a pair which represents the point x. Then $w_{p^s}\pi(x)$ and $\pi w_{p^r}(x)$ are represented by $(E/A_{p^s}, E_{p^s}/A_{p^s})$ and $(E/a, (E_{p^s}+A)/A)$, respectively. If $X_0^+(p^s)\simeq\boldsymbol{P}^1$, then $J_0^-(p^s)=J_0(p^s)$ and the assumption $f^+(y)=0$ leads that $w_{p^s}\pi(x)=\pi w_{p^r}(x)$. Then $E/A_{p^s}\simeq E/A$ over $\boldsymbol{C}$ and A/A_{p^s} is a cyclic subgroup of order $p^{r-s}\neq1$, so that E is an elliptic curve with complex multiplication. Now consider the case

when $X_0^+(p^s)\neq \boldsymbol{P}^1$. The condition $f^+(y)=0$ gives the linearly equivalence relation below

$$(\pi w_{p^s}(x))+(\pi(x))\sim(w_{p^s}\pi(x))+(w_{p^s}\pi w_{p^r}(x)).$$

If $\pi(x)=w_{p^s}\pi(x)$ or $w_{p^s}\pi w_{p^r}(x)$, then the same argument as above shows that x is a C. M. point. In the other case, $X_0(p^s)$ must have the hyperelliptic involution γ such that $\gamma\pi(x)=\pi w_{p^r}(x)$. We know that if $X_0^+(p^s)\neq \boldsymbol{P}^1$, then $X_0(p^s)$ is not hyperelliptic, except for $(p, s)\neq(37, 1)$ [17] [18]. □

THEOREM (3.6). *For the following pairs (p, r), $n(p, r)=0$:*

p	r
2	$\geqq 6$
3	$\geqq 4$
7	$\geqq 3$
11	$\geqq 2$
$p\geqq 17$	$\geqq 2$ *if* $p\neq 37$ *and* $\sharp J_0^-(p)(\boldsymbol{Q})<\infty$.

PROOF. Let y be a non cuspidal $\boldsymbol{Q}$-rational point on $X_0^+(p^r)$ and $x, x'=w_{p^r}(x)$ be the sections of the fibre $X_0(p^r)_y$ at y. For $p\geqq 3$, Theorem (3.2) and Proposition (3.3) show that $f(x)\otimes\kappa(\not{s})$ is the unit section of $J_0(p^s)_{/\boldsymbol{Z}}\otimes\boldsymbol{F}_p$. Then for any pair (p, r) $(p\geqq 3)$ as above, Proposition (3.4) and Proposition (3.5) show that y is a C. M. point. As was seen in the proof of Theorem (3.2), $X_0^+(2^{2t})(\boldsymbol{Q})$ $(t\geqq 3)$ consists of two cusps and two C. M. points. There remains the case for $X_0^+(2^r)$ for odd integers $r\geqq 7$.

LEMMA (3.6.1). *Let z_1, z_2 be $\boldsymbol{Q}$-rational points on $X_0^+(2^7)$. If $z_1\otimes\boldsymbol{F}_2=z_2\otimes\boldsymbol{F}_2$, then $z_1=z_2$. In particular, for a non cuspidal $\boldsymbol{Q}$-rational point z on $X_0^+(2^7)$, $z\otimes\boldsymbol{F}_2$ is not a cusp.*

PROOF. Let x_i and $x_i'=w_{2^7}(x_i)$ be the sections of the fibre $X_0(2^7)_{z_i}$ at z_i for $i=1, 2$. Then x_i and x_i' are defined over $\boldsymbol{Q}_2$ and they are the sections of $E_0^h\cup E_1^h\cup E_6^h\cup E_7^h$ (3.1) (3.2). If $z_1\otimes\boldsymbol{F}_2=z_2\otimes\boldsymbol{F}_2$, then changing x_2 by x_2', if necessary, we may assume that $x_1\otimes\boldsymbol{F}_2=x_2\otimes\boldsymbol{F}_2$. Let $f=f_{7,5}$ be the morphism of $\mathscr{X}_0(2^7)^{\text{smooth}}$ to $J_0(32)_{/\boldsymbol{Z}}$ defined in §1. Then $f(x_1)\otimes\boldsymbol{F}_2=f(x_2)\otimes\boldsymbol{F}_2$, and $f(x_1)$ and $f(x_2)$ are contained in the finite étale subgroup of order 4 generated by the class $\mathrm{cl}((0)-(\infty))$ [24] Table 1 pp. 81–113 (3.2.10). Therefore $f(x_1)=f(x_2)$. Then applying Proposition (1.14), we get $x_1=x_2$, hence $z_1=z_2$. □

Lemma (3.6.1) and Theorem (3.2) show that there are at most four $\boldsymbol{Q}$-rational points on $X_0^+(2^r)$ for any odd integer $r \geqq 7$. Two of them are the cusps $\{0, \infty\}$ and $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}11\\2^{r-1}\end{pmatrix}\right\}$. There is a $\boldsymbol{Q}$-rational point on $X_0^+(2^r)$ which is represented by elliptic curves with complex multiplication. Let E be an elliptic curve defined over $\boldsymbol{Q}$ which is isomorphic over $\boldsymbol{C}$ to $\boldsymbol{C}/\boldsymbol{Z}+\boldsymbol{Z}(1+\sqrt{-7})/2$. Let c, $\bar{c}$ be the endomorphisms of E which correspond to $(1+\sqrt{-7})/2$ and $(1-\sqrt{-7})/2$ under the isomorphism $\mathrm{End}\, E \simeq \boldsymbol{Z}[(1+\sqrt{-7})/2]$, respectively. Put $A=\ker(c^r: E \to E)$ and $\bar{A}=\ker(\bar{c}^r: E \to E)$. Then $(E/A, E_{2^r}/A) \simeq (E, \bar{A})$ over $\boldsymbol{C}$ and the set $\{(E, A), (E, \bar{A})\}$ represents the C. M. point on $X_0^+(2^r)$. The special fibre at the rational prime 2 of this C. M. point is a section of $(E_0 \cup E_r)^h/\langle w_{2^r}\rangle$. In the rest of the proof, we will show that $X_0^+(2^r)(\boldsymbol{Q})$ consists of these three points for any odd integer $r \geqq 7$. Suppose that there exists another $\boldsymbol{Q}$-rational point y on $X_0^+(2^r)$ and let $x, x'=w_{2^r}(x)$ be the sections of the fibre $X_0^+(2^r)_y$ at y. Then x and x' are defined over a quadratic field k in which the rational prime $p=2$ splits, and $x'=x^\sigma$ for $1 \neq \sigma \in \mathrm{Gal}(k/\boldsymbol{Q})$ (1.1) (3.1) (3.2). Let $\mathfrak{p}$ be a prime of k lying over the rational prime 2. Then by (3.2) and (3.6.1), $x \otimes \kappa(\mathfrak{p})$ and $x' \otimes \kappa(\mathfrak{p})$ are the sections of $E_1^h \cup E_6^h$. Let $z, z'=w_{32}(z)$ be the images of x and x' under the morphism of $X_0(2^r)$ to $X_0(32)$ defined by $(E, A) \mapsto (E/A_2, A_{64}/A_2)$. Then $z'=z^\sigma$ for $1 \neq \sigma \in \mathrm{Gal}(k/\boldsymbol{Q})$. These points z and z' are defined by $(X, Y)=(a, b)$ and $(X, Y)=(8(2a+4-b)/a^2, b')$ for $0 \neq a, b, b' \in k$ with $a^\sigma=8(2a+4-b)/a^2$ (3.2.7) (3.2.8). For a prime $\mathfrak{p}$ of k, $v_\mathfrak{p}$ denotes the normalized valuation of $k_\mathfrak{p}$.

CLAIM. For $\mathfrak{p} \nmid 2$, $v_\mathfrak{p}(a)=0$ and $(v_\mathfrak{p}(a), v_{\mathfrak{p}'}(a))=(3, 0)$ or $(0, 3)$, where $\mathfrak{p}\mathfrak{p}'=2\mathcal{O}_k$.

PROOF. Using the equation (3.2.7), we see that $v_\mathfrak{p}(a)$, $v_\mathfrak{p}(a^\sigma)=v_{\mathfrak{p}'}(a)$ are equal to 0, 1, 2 or 3. The sections $z \otimes \kappa(\mathfrak{p})$ and $z' \otimes \kappa(\mathfrak{p})$ are contained in $E_0^h \cup E_5^h$. Then by (3.2.10), we see that $(v_\mathfrak{p}(a), v_{\mathfrak{p}'}(a))=(3, 0)$ or $(0, 3)$. The class $\mathrm{cl}((0)-(\infty))$ generates a finite étale subgroup scheme in the Néron model $J_0(32)_{/\boldsymbol{Z}[1/2]}$ and $J_0(32)(\boldsymbol{Q})=\langle \mathrm{cl}((0)-(\infty))\rangle$ [24] Table 1 pp. 81–113, (3.2.10). Applying (1.14), we see that $z \otimes \kappa(\mathfrak{p})$ is not a cusp for any prime $\mathfrak{p} \nmid 2$. Then by (3.2.7), $v_\mathfrak{p}(a)=0$ for all $\mathfrak{p} \nmid 2$. □

Changing x by x', if necessary, we may assume that $x \otimes \kappa(\mathfrak{p})$ is a section of E_1^h. Then by the claim above, $a\mathcal{O}_k=\mathfrak{p}'^3$, $a^\sigma\mathcal{O}_k=\mathfrak{p}^3$ and $aa^\sigma=$

± 8. By the relations tnat $a^\sigma = 8(2a+4-b)/a^2$ and (3.2.8), there are two possibilities: (i) $a=(-5\pm\sqrt{-7})/2$ and (ii) $a=(3\pm\sqrt{41})/2$. The first case corresponds to the C. M. point. By our assumption, the rational point y corresponds to the second case. The section x is represented by a pair (E, A) for an elliptic curve E with a cyclic subgroup A of order 2^7 defined over $k=\boldsymbol{Q}(\sqrt{41})$. Put $(F, B)=(E/A_2, A/A_2)$, and let λ, $\hat{\lambda}$ be the characters of the idèle group $k_A^\times$ induced by the Galois action of $\mathrm{Gal}(\bar{k}/k)$ on $A(\bar{k})$ and $E_{2^7}/A(\bar{k})$, respectively. Then $\lambda\hat{\lambda}$ is the cyclotomic character $\theta=\theta_{2^7}$ induced by the Galois action on $\mu_{2^7}(\bar{k})$. For each prime $\mathfrak{p}$ of k, let $\lambda_\mathfrak{p}$, $\hat{\lambda}_\mathfrak{p}$ be the restrictions of λ and $\hat{\lambda}$ to $\mathcal{O}_\mathfrak{p}^\times$ $(\subset k_A^\times)$.

CLAIM. Under the notation as above, $\lambda_\mathfrak{p}=1$ on $(\mathcal{O}_\mathfrak{p}^\times)^4$ and $\lambda_\mathfrak{p}^2=1$ for all primes $\mathfrak{p}\nmid 2$.

PROOF. For each prime $\mathfrak{p}$, let $I_\mathfrak{p}$ be the inertia subgroup of $\mathfrak{p}$. Let ρ be the representation of the Galois action on $F_4(\bar{k})$. The subgroup of 2-torsion points F_2 is decomposed into the direct sum $A_4/A_2\oplus E_2/A_2$. Then $\rho(\tau)^2=1$ for $\mathfrak{p}\nmid 2$ and $\tau\in I_\mathfrak{p}$. The elliptic curve E is isogenous over k to F, so that $\lambda_\mathfrak{p}^2=1$ for $\mathfrak{p}\nmid 2$. Since the modular invariant $j(x)\not\equiv 0 \bmod \mathfrak{p}$ and $k\hookrightarrow\boldsymbol{Q}_2$, E and F have good reduction over a quadratic extension K of $\boldsymbol{Q}_2^{ur}$. Let x_1 be the image of x under the morphism of $X_0(2^7)$ to $X_0(2^6)$: $(E, A)\mapsto(E/A_2, A/A_2)$. By our choice of x, $x_1\otimes\kappa(\mathfrak{p})$ is a section of E_0^h. Then the schematic closure $B_{/\mathcal{O}_K}$ of B in the Néron model $F_{/\mathcal{O}_K}$ is a finite étale group scheme of rank 2^6 (1.7). Then for $\tau\in(I_\mathfrak{p})^2$, $\lambda_\mathfrak{p}(\tau)=1$ or $1+2^6 \pmod{2^7}$. Therefore for $\tau\in(I_\mathfrak{p})^4$, $\lambda_\mathfrak{p}(\tau)=1$. □

Now we make complete the proof for $(p, r)=(2, 7)$. Let $u=32-5\sqrt{41}$ and $u'=32+5\sqrt{41}$ be the units of $\mathcal{O}_k=\boldsymbol{Z}[(1+\sqrt{41})/2]$. Then $\mathfrak{p}^5\| u-1$ or $\mathfrak{p}^5\| u'-1$. Changing u by u', if necessary, we may assume that $\mathfrak{p}^5\| u-1$. For $1\neq\sigma\in\mathrm{Gal}(k/\boldsymbol{Q})$, $x^\sigma=w_{2^7}(x)$ and $(E^\sigma, A^\sigma)\simeq(E/A, E_{2^7}/A)$ over a quadratic extension of k (, since $j(x)\neq 0$, 1728). Let λ^σ be the character of $k_A^\times$ induced by the Galois action on $A^\sigma(\bar{k})$. Then $\lambda^\sigma=\lambda\mu$ for a character μ of order 1 or 2. The conductor of $\mu_\mathfrak{p}=\mu|\mathcal{O}_\mathfrak{p}^\times$ divides $\mathfrak{p}^3$ and $\mu_\mathfrak{p}(u)=1$. Then under the identification of $\mathcal{O}_\mathfrak{p}^\times$ with $\boldsymbol{Z}_2^\times$, we get $\lambda_{\mathfrak{p}'}(u)=\lambda_{\mathfrak{p}'}^\sigma(u)=(u \bmod \mathfrak{p}^7)$. Therefore $1=\lambda(u)=\lambda_\mathfrak{p}(u)\lambda_{\mathfrak{p}'}(u)=(u \bmod \mathfrak{p}^7)$. This contradicts that $\mathfrak{p}^5\| u-1$. Thus we completed the proof. □

Further we get the following results.

THEOREM (3.7). *For $p\geqq 17$ and $r\geqq 2$,*

$$n(p, r) \leqq \dim J_0(p) - \dim \tilde{J}_0(p).$$

PROOF. Its proof is the same as that of Theorem (4.1) [16] §4. □

Let y be a non cuspidal $\boldsymbol{Q}$-rational point on $X_0^+(p^r)$ represented by an elliptic curve $E/\boldsymbol{Q}$ with a cyclic subgroup A of order p^r. For a rational prime q, if E has potentially good reduction at q, then we say that y has potentially good reduction at q. If y is a C. M. point, then y has potentially good reduction at any rational prime q.

THEOREM (3.8). *Let y be a non cuspidal $\boldsymbol{Q}$-rational point without C. M. on $X_0^+(p^r)$. Then y has potentially good reduction at q for any triple (p, r, q) below:*

All primes q for the pairs (p, r) in Theorem (0.1), *and*

p	r	q	
13	$\geqq 3$	$q \neq 2, 13$	
$p \geqq 17$	$\geqq 2$	all q	*if* $p \equiv 1 \bmod 8$
		$q \neq 2$	*otherwise.*

PROOF. By the existence of the canonical coverings of $X_0^+(p^{r+2})$ to $X_0^+(p^r)$ §1, it suffices to show the theorem (3.8) for the pairs $(p, r) = (13, 3)$, $(13, 4)$, and $(p, 2)$, $(p, 3)$ for $p \geqq 17$. Let y be a $\boldsymbol{Q}$-rational point on $X_0^+(p^r)$ for a pair (p, r) as above, and x be a section of the fibre $X_0^+(p^r)_y$ at y. Then x is defined over a number field of degree $\leqq 2$. Let $\tilde{f} = \tilde{f}_{r,k}$ be the morphism of $X_0(p^r)$ to $\tilde{J}_0(p)$ $(p \geqq 17)$ or to $\tilde{J}_0(169)_{(7)}$ $(p = 13)$ defined in §1. If x is a cusp, then $x = 0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (1.6). Then by (3.2), (3.3) and (3.4), we know that $\tilde{f}(x)$ is the unit section. Suppose that $x \otimes \kappa(\mathfrak{p}) = C \otimes \kappa(\mathfrak{p})$ for a cusp C and a prime $\mathfrak{p}$ of k. Changing x by $w_{p^r}(x)$, if necessary, we may assume that $C = \begin{pmatrix} i \\ p^m \end{pmatrix}$ for integers i and m, $0 \leqq m \leqq r/2$ (1.6). Then C is defined over $\boldsymbol{Q}(\zeta_{p^m})$, where ζ_{p^m} is a primitive p^m-th root of unity see loc.cit.. The class $\mathrm{cl}((C) - (\infty))$ is of finite order [10] §3 (3.2), and it generates a finite étale subgroup scheme of the Néron model $J_0(p^r)_{/Z[1/2p]}$ (2.1). First consider the case when $\mathfrak{p} \nmid 2p$. The condition that $\tilde{f}(x)$ is the unit section and the assumption on $\mathfrak{p}$ show that $\tilde{f}(C)$ is also the unit section. For $p = 13$, $r = 4$ and $C' = \begin{pmatrix} i \\ 169 \end{pmatrix}$, $\tilde{f}(C') = \mathrm{cl}((0) - (\infty))$, which is of order 7. Thus $m = 0$ or 1. Then Proposition (1.14) applied to x, C and f leads that $x = C$. If $\mathfrak{p} \mid p$ $(p \geqq 17)$, then by (3.2), $m = 0$. The rational prime 2 is of

order $\leqq 2$ in $\boldsymbol{Q}(\zeta_p)$, so that if $\ell \mid 2$, then $m=0$. Then Proposition (1.14) leads $x=C$. □

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