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On a comparison of trace formulas for $GU(1,2)$ *and* $GU(3)$

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Introduction

The purpose of this paper is to prove an equality between alternating sums of traces of Hecke operators acting on certain spaces of automorphic forms on $GU(1,2)$ and $GU(3)$. In [9], [10], and [11], T. Ibukiyama investigated, in order to prove a part of Langlands' conjecture in [16] in a special case, certain "new forms" on two reductive groups to be compared. These new forms are defined by a parahoric subgroup at a finite place. His studies are mainly concerned with the comparison of such new forms on $Sp(2, \boldsymbol{R})$ and its compact twist. In [7], K. Hashimoto and Ibukiyama proved a comparison theorem, which is an equality between alternating sums of dimensions of certain spaces of automorphic forms on the above two groups. In this equality, each alternating sum expresses the dimension of the space of new forms with respect to the minimal parahoric subgroup. Our result below may be regarded as another partial realization of Ibukiyama's philosophy.

Let F be an imaginary quadratic number field, and set

$$H_1=\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}, \qquad H_2=\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix},$$

We consider two reductive groups G_1 and G_2 defined over $\boldsymbol{Q}$, whose sets of $\boldsymbol{Q}$-rational points are written as follows:

$$G_{i,\boldsymbol{Q}}=\{g \in GL(3, F) \mid {}^t\bar{g}H_ig=\nu(g)H_i,\ \nu(g) \in \boldsymbol{Q}^\times\} \quad (i=1, 2).$$

By a classical theorem of Landherr, these groups are isomorphic at any finite place l of $\boldsymbol{Q}$. Then we fix the isomorphism

$$\Theta_l\colon G_{1,l} \xrightarrow{\sim} G_{2,l}, \qquad \Theta_l(g)=\vartheta_l^{-1}g\vartheta_l,$$

where ϑ_l is an element of $GL(3, F\otimes_{Q}\boldsymbol{Q}_l)$ satisfying certain conditions.

The real reductive group $G_{1,\infty}$ acts on the symmetric domain

$$\mathscr{D}=\{(z, w)\in \boldsymbol{C}\times\boldsymbol{C} \mid 2\operatorname{Re}(z)-|w|^2>0\}.$$

In this paper, we treat only scalar valued holomorphic cusp forms for G_1. Let $r>4$ be an integer and $U_{1,f}$ an open compact subgroup of $G_{1,f}$ (the finite part of the adelized group). We denote by $\mathfrak{S}_r(U_{1,f})$ the space of holomorphic cusp forms on the adelized group $G_{1,A}$ of weight r with respect to $U_{1,f}$.

To the same r, we attach a $(1/2)(r-1)(r-2)$-dimensional irreducible representation ρ of $G_{2,\infty}$ so that there exist good relations between the corresponding orbital integrals on $G_{1,\infty}$ and $G_{2,\infty}$. (See 1.3 for the exact definition of ρ.) If $U_{2,f}$ is an open compact subgroup of $G_{2,f}$, we write $\mathfrak{S}_\rho(U_{2,f})$ for the space of automorphic forms on $G_{2,A}$ "of weight ρ", with respect to $U_{2,f}$. To be more precise, let M be the representation space of ρ. Then we put

$$\mathfrak{S}_\rho(U_{2,f})=\{f\colon G_{2,A}\longrightarrow M \mid f(\gamma gu)=f(g) \text{ for } \forall\gamma\in G_{2,Q},\ \forall u\in U_{2,f},$$
$$f(gg_\infty)=\rho(g_\infty)^{-1}f(g) \text{ for } \forall g_\infty\in G_{2,\infty}\}.$$

Now, let $\varphi_{i,f}$ be a $\boldsymbol{C}$-valued bi-$U_{i,f}$-invariant function on $G_{i,f}$ with compact support $(i=1, 2)$. We denote by $T_1(\varphi_{1,f})$ (resp. $T_2(\varphi_{2,f})$) the Hecke operator defined by $\varphi_{1,f}$ (resp. $\varphi_{2,f}$) acting on $\mathfrak{S}_r(U_{1,f})$ (resp. $\mathfrak{S}_\rho((U_{2,f}))$. By the standard argument of the Selberg trace formula, we can express the trace of each $T_i(\varphi_{i,f})$ as an integral. Here we note that the explicit formula for the dimension of $\mathfrak{S}_r(U_{1,f})$ was given by L. Cohn [3] in a special case. Some generalizations of this result were given by S. Kato [13], [14], and H. Kojima [15].

Our aim is to compare trace$(T_1(\varphi_{1,f}))$ and trace$(T_2(\varphi_{2,f}))$ for suitable $\varphi_{1,f}$ and $\varphi_{2,f}$. But we find that if $U_{1,f}$ or $U_{2,f}$ satisfies a certain (weak) assumption, there exist infinitely many $r>4$ such that the dimensions of $\mathfrak{S}_r(U_{1,f})$ and $\mathfrak{S}_\rho(U_{2,f})$ are not equal (Proposition 6.2). Therefore, following Ibukiyama's philosophy, we may consider the alternating sums of traces of Hecke operators attached to parahoric subgroups at a finite place. Namely, let p be a fixed finite place of $\boldsymbol{Q}$ *which decomposes in* F so that we have

$$G_{1,p}\cong G_{2,p}\cong \boldsymbol{Q}_p^\times\times GL(3, \boldsymbol{Q}_p).$$

From the system of standard parahoric subgroups of $GL(3, \boldsymbol{Q}_p)$, we define the system $U_{i,p}^\tau$ ($\tau=0, 1, 2, 01, 12, 20, 012$) of open compact subgroups of each $G_{i,p}$, satisfying $\Theta_p(U^\tau_{1,p})=U^\tau_{2,p}$ for any τ (see 5.1). Then we put

$$U^\tau_{i,f}=\prod_{l\neq p} U_{i,l}\times U^\tau_{i,p} \quad (i=1, 2)$$

where $U_{i,l}$ is an open compact subgroup of $G_{i,l}$ satisfying $\Theta_l(U_{1,l})=U_{2,l}$. (We assume that $U_{i,l}=G_i(\boldsymbol{Z}_l)$ for almost all l.) For i, τ as above, we take a bi-$U^\tau_{i,f}$-invariant function $\varphi^\tau_{i,f}$ on $G_{i,f}$ with compact support of the form

$$\varphi^\tau_{i,f}=\bigotimes_{l\neq p}\varphi_{i,l}\otimes\varphi^\tau_{i,p}$$

where $\varphi_{i,l}$ is a bi-$U_{i,l}$-invariant function on $G_{i,l}$ with compact support satisfying $\Theta^*_l(\varphi_{2,l})=\varphi_{1,l}$, and $\varphi^\tau_{i,p}$ is the characteristic function of $U^\tau_{i,p}$. Then our main result (Theorem 6.1) asserts that the equality

$$\sum_\tau \varepsilon(\tau)\mathrm{trace}(T_1(\varphi^\tau_{1,f}))=\sum_\tau \varepsilon(\tau)\mathrm{trace}(T_2(\varphi^\tau_{2,f}))$$

holds for all $r>4$, where

$$\varepsilon(\tau)=\begin{cases}+1 \cdots & \text{for} \quad \tau=0,\ 1,\ 2,\ 012\\ -1 \cdots & \text{for} \quad \tau=01,\ 12,\ 20.\end{cases}$$

Now, there is an essential difference between the above theorem and the comparison theorem in Hashimoto-Ibukiyama [7]. Namely, two $\boldsymbol{Q}$-groups compared in [7] are not isomorphic at the infinite place and one finite place, and the parahoric subgroups are introduced at that finite place, but our groups G_1 and G_2 are isomorphic at any finite place. We think that it is an interesting problem to find a representation-theoretical explanation of our theorem and the phenomenon described in Proposition 6.2.

Now, let us give a brief description of each section. In § 1, we define automorphic forms and Hecke operators for our two groups, and describe the integral representations of the traces of Hecke operators. In § 2, we study the conjugacy classes of each $G_{i,\boldsymbol{Q}}$. One of the main difficulties in comparisons of trace formulas comes from the difference between conjugacy and stable conjugacy, in many cases (see Langlands [17]). Namely, let G be a quasi-split group over an algebraic number field K and let G' be an inner form of G. For elements of G_K and G'_K, stable conjugacy means the conjugacy within $G_{\bar{K}}$ and $G'_{\bar{K}}$ respectively, where $\bar{K}$ denotes the algebraic closure of K. There exists a natural injection from the semi-simple stable conjugacy classes of G'_K into those of G_K. However, if conjugacy does not coincide with stable conjugacy this injection is not directly amenable to the comparison of local orbital integrals. In our case, it can be shown that there exists a "good" three to one correspondence from the conjugacy classes of $G_{1,\boldsymbol{Q}}$ contained in a stable conjugacy class of a certain type to the conjugacy classes of $G_{2,\boldsymbol{Q}}$ contained in the corresponding stable conjugacy class (Proposition 2.8). To prove this, we need parametrizations of conjugacy classes of each $G_{i,\boldsymbol{Q}}$ contained in a stable conjugacy class of the above type. After describing a rough classification of elements of $G_{i,\boldsymbol{Q}}$, we describe the parametrizations. Note that such parametrizations were studied by T. Asai in [2] for some classical groups over an algebraic number field. In § 3, we compare the contributions of central and "$\boldsymbol{Q}$-elliptic" elements to the trace formulas for both groups. Here, the above three to one correspondence plays an essential role. In § 4, we describe the contributions of "split hyperbolic" elements and non semi-simple elements of $G_{1,\boldsymbol{Q}}$ to the trace formula. In § 5, we prove the vanishing of the alternating sum of orbital integrals of $\varphi_{i,p}^{\tau}$ for a certain class of elements of $G_{i,p}$ (Proposition 5.1). This is another key point to prove the theorem. In § 6, we state our main result and some remarks.

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Notation. We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$, respectively, the ring of rational integers, the rational number field, the real number field, and the complex number field. In general, for an associative ring R with the identity element, we denote by $R^{\times}$ the group of all invertible elements of R. $M_n(R)$ stands for the ring of all square matrices of size n with entries in R, and we put $GL(n, R)=M_n(R)^{\times}$. R^n denotes the free R-module of all column vectors of size n with entries in R. If R is a separable algebra over a field K, we denote by $\mathrm{Tr}_{R/K}$ and $N_{R/K}$ respectively the reduced trace and norm from R to K. In § 4, we sometimes abbreviate $N_{C/R}(\ )$ to $N(\ )$. For each place v of $\boldsymbol{Q}$, we denote by $\boldsymbol{Q}_v$ the completion of $\boldsymbol{Q}$ at v; similar notation is used in § 2 for algebraic number fields. $\boldsymbol{Q}_A$ (resp. $\boldsymbol{Q}_A^{\times}$) stands for the adele ring of $\boldsymbol{Q}$ (resp. the idele group of $\boldsymbol{Q}$). For $x\in\boldsymbol{Q}_v^{\times}$ we denote by $|x|_v$ the module of x, and for $x=(x_v)\in \boldsymbol{Q}_A^\times$ we put $|x|_A=\prod_v |x_v|_v$. For $x\in \boldsymbol{C}$ we denote by $|x|$ the usual absolute value. For an algebraic group G defined over $\boldsymbol{Q}$, we denote by G_Q, G_v, G_A, and G_f, respectively, the group of $\boldsymbol{Q}$-rational points, the group of $\boldsymbol{Q}_v$-rational points, the adelized group, and the finite part of the adelized group. Whenever we treat a discrete topological group (like G_Q) we use the counting measure for it. If G is a group and $g\in G$, we write $G(g)$ (resp. $[g]_G$) for the centralizer of g in G (resp. the conjugacy class of G containing g), and if a subset S of G is stable under the conjugation, we denote by $S//G$ the set of all G-conjugacy classes in S. Sometimes we abbreviate $[g]_G$ to $[g]$ if there is no fear of confusion. $\boldsymbol{R}_+^\times$ stands for the group of all positive numbers. For $x\in \boldsymbol{R}^\times$ we denote by $\mathrm{sgn}(x)$ the signature of x. For $x\in \boldsymbol{C}$ we put $e[x]=\exp(2\pi i x)$ $(i=\sqrt{-1})$. By $\Gamma(s)$ we denote the gamma function.

§ 1. Automorphic forms and Hecke operators

1.1.

Let F be an imaginary quadratic number field, and H a non-degenerate Hermitian matrix of size three with entries in F. To a given pair (F, H), we attach the reductive group $GU(H)$ defined over $\boldsymbol{Q}$. We denote by $x\mapsto \bar{x}$ the non-trivial automorphism of $F/\boldsymbol{Q}$. For any $\boldsymbol{Q}$-algebra A, we extend this automorphism to the automorphism $z\mapsto \bar{z}$ of $F\otimes_Q A$ by $\overline{x\otimes y}=\bar{x}\otimes y$. Using this notation, the group of $\boldsymbol{Q}$-rational points of $GU(H)$ is written as follows:

$$GU(H)_Q=\{g\in GL(3, F)\,|\,{}^t\bar{g}Hg=\nu(g)H,\ \nu(g)\in \boldsymbol{Q}^\times\}.$$

We fix the imaginary quadratic number field F. Since the size is odd, the classical theorem on the equivalence of Hermitian forms (Landherr [18], see Lemma 2.5 below) then shows that there are only two isomorphism classes over $\boldsymbol{Q}$ of the groups of the above type; one is represented by $GU(H)$ for an indefinite H, and the other is represented by $GU(H)$ for a definite H. We then consider two groups, namely,

$$G_1=GU(H_1)\quad\text{for}\quad H_1=\begin{pmatrix} & & 1\\ & -1 & \\ 1 & & \end{pmatrix}$$

and

$$G_2 = GU(H_2) \quad \text{for} \quad H_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

We denote by S_i the semi-simple part of G_i:

$$S_i = \{g \in G_i \mid \det(g) = \nu(g) = 1\} \quad (i=1, 2).$$

Then, the $\boldsymbol{Q}$-rank of S_1 is one and S_2 is $\boldsymbol{R}$-annisotropic.

At any finite place l of $\boldsymbol{Q}$, G_1 and G_2 are isomorphic. For our later purpose, we fix the isomorphism. We write F_l for $F \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l$ and $\mathfrak{O}_l$ for $\mathfrak{O} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_l$ ($\mathfrak{O}$: the ring of integers of F). Since $\det(H_1) = \det(H_2)$, there exists $\vartheta_l \in GL(3, F_l)$ such that

$$^t\bar{\vartheta}_l H_1 \vartheta_l = H_2$$

(see Lemma 2.4 below). Taking such a matrix ϑ_l at each l, we get a sequence $\{\vartheta_l\}_{l \neq \infty}$. Here we may assume that ϑ_l belongs to $GL(3, \mathfrak{O}_l)$ for almost all l. We fix $\{\vartheta_l\}_{l \neq \infty}$ with this property, and define the sequence $\{\Theta_l\}_{l \neq \infty}$ of isomorphisms as follows:

$$(1.1) \qquad \Theta_l : G_{1,l} \xrightarrow{\sim} G_{2,l}; \qquad \Theta_l(g) = \vartheta_l^{-1} g \vartheta_l.$$

From now on, we identify $G_{1,l}$ with $G_{2,l}$ by means of Θ_l at every finite place l of $\boldsymbol{Q}$.

Let us consider open compact subgroups and Hecke algebras of $G_{i,f}$. We shall treat open compact subgroups of $G_{i,f}$ of the following form:

$$(1.2) \qquad U_{i,f} = \prod_{l \neq \infty} U_{i,l}; \quad U_{i,l}: \text{an open compact subgroup of } G_{i,l}.$$

We assume that $U_{i,l} = G_i(\boldsymbol{Z}_l)$ for almost all l in the above expression. For $U_{1,f}$ and $U_{2,f}$ of the above type, we say that they satisfy *assumption* (U_f) if $\Theta_l(U_{1,l}) = U_{2,l}$ at all finite place l. Once the open compact subgroup $U_{i,f}$ is fixed, we normalize the Haar measures dg_l of $G_{i,l}$ and dg_f of $G_{i,f}$ by

(1.3) the total volume of $U_{i,l}$ with respect to dg_l is one, $dg_f = \prod_{l \neq \infty} dg_l$.

We denote by $\mathscr{L}(G_{i,l}, U_{i,l})$ the Hecke algebra of $G_{i,l}$ with respect to $U_{i,l}$; namely, $\mathscr{L}(G_{i,l}, U_{i,l})$ is the set of bi-$U_{i,l}$-invariant functions on $G_{i,l}$ with compact support, and it forms a $\boldsymbol{C}$-algebra by the convolution with respect to dg_l. Let $\mathscr{L}(G_{i,f}, U_{i,f})$ be the restricted tensor product of $\{\mathscr{L}(G_{i,l}, U_{i,l})\}_{l\neq\infty}$. When $U_{1,f}$ and $U_{2,f}$ satisfy assumption (U_f), we say that $\varphi_{1,f}\in\mathscr{L}(G_{1,f}, U_{1,f})$ and $\varphi_{2,f}\in\mathscr{L}(G_{2,f}, U_{2,f})$ satisfy *assumption* (φ_f) if they have the form

$$\varphi_{i,f}=\bigotimes_{l\neq\infty}\varphi_{i,l}\quad(\varphi_{i,l}\in\mathscr{L}(G_{i,l}, U_{i,l}))\quad(i=1, 2)$$

and their factors satisfy $\Theta_l^*(\varphi_{2,l})=\varphi_{1,l}$ at all finite place l, where Θ_l^* means the pull-back determined by Θ_l.

Now, let us introduce some $\boldsymbol{Q}$-subgroups. Let $Z(G_i)$ be the center of G_i. The isomorphism defined over $\boldsymbol{Q}$

$$Z(G_1)\xrightarrow{\sim}Z(G_2);\qquad z\cdot 1_3\longmapsto z\cdot 1_3\quad(z\text{: a scalar})$$

is compatible with Θ_l at any finite place l, so we identify $Z(G_1)$ with $Z(G_2)$ by means of this isomorphism and write Z for both groups:

$$Z=Z(G_1)=Z(G_2).$$

Sometimes, we write z for $z\cdot 1_3\in Z$. By P we denote the parabolic subgroup of G_1 defined over $\boldsymbol{Q}$ whose set of $\boldsymbol{Q}$-rational point is

$$\left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\in G_{1,\boldsymbol{Q}}\right\}\quad(*\text{: an element of } F).$$

By M and N, we denote the following Levi part and the unipotent radical of P, respectively. Let k be a field of characteristic zero, and put $K=k\otimes_{\boldsymbol{Q}}F$. Then

$$\begin{aligned} M_k&=\left\{\begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}\in GL(3, K)\,\middle|\,\alpha\bar{\gamma}=N_{K/k}(\beta)\right\} \\ &=Z_k\times A_k\quad\text{(direct product)}, \end{aligned}$$

where

$$A_k=\{\lambda(\alpha)\,|\,\alpha\in K^\times\}$$

with

$$\lambda(\alpha)=\begin{pmatrix} \alpha & & \\ & 1 & \\ & & \bar{\alpha}^{-1} \end{pmatrix}. \tag{1.4}$$

In order to parametrize the elements of N_k, we fix an element ι of $F^\times$ which satisfies $\mathrm{Tr}_{F/Q}(\iota)=0$. Then

$$N_k=\{n(a, b) \mid a \in K, b \in k\}$$

with

$$n(a, b)=\begin{pmatrix} 1 & \bar{a} & \frac{1}{2}N_{K/k}(a)+\iota b \\ & 1 & a \\ & & 1 \end{pmatrix}. \tag{1.5}$$

1.2.

Let ∞ be the infinite place of $\boldsymbol{Q}$. The reductive Lie group $G_{1,\infty}$ is connected and acts on the symmetric domain $\mathscr{D}$ defined by

$$\mathscr{D}=\{(z, w) \in \boldsymbol{C}^2 \mid 2\,\mathrm{Re}(z)-|w|^2>0\}.$$

We define the action $g\langle\mathfrak{z}\rangle$ of $G_{1,\infty}$ on $\mathscr{D}$ and the scalar valued automorphy factor $j(g, \mathfrak{z})$ on $G_{1,\infty}\times\mathscr{D}$ by

$$g\begin{pmatrix} {}^t\mathfrak{z} \\ 1 \end{pmatrix}=j(g, \mathfrak{z})\begin{pmatrix} {}^t(g\langle\mathfrak{z}\rangle) \\ 1 \end{pmatrix} \quad (g \in G_{1,\infty},\ \mathfrak{z} \in \mathscr{D}).$$

We consider the point $\mathfrak{z}_0=(1/2, 0)$ as the origin of $\mathscr{D}$ and define the maximal compact subgroup U_∞ of $S_{1,\infty}$ by

$$U_\infty=\{g \in S_{1,\infty} \mid g\langle\mathfrak{z}_0\rangle=\mathfrak{z}_0\}. \tag{1.6}$$

Let $U_{i,f}$ be an open compact subgroup of $G_{1,f}$ of the form (1.2) and r a positive integer. We say that a $\boldsymbol{C}$-valued function f on $G_{1,A}$ is a holomorphic cusp form of weight r with respect to $U_{1,f}$ if f satisfies the following conditions:

(i) $f(\gamma g u)=f(g)$ for $\forall\gamma \in G_{1,Q}$, $\forall u \in U_{1,f}$.

(ii) For any $g=g_\infty g_f$ $(g_\infty \in G_{1,\infty},\ g_f \in G_{1,f})$, $\nu(g_\infty)^{-r/2} j(g_\infty, \mathfrak{z}_0)^r f(g)$ depends only on g_f and $\mathfrak{z}=g_\infty\langle\mathfrak{z}_0\rangle$, and it is holomorphic as a function of $\mathfrak{z} \in \mathscr{D}$.

(iii) f is bounded on $G_{1,A}$.

We denote by $\mathfrak{S}_r(U_{1,f})$ the space of such functions.

Let us normalize the Haar measures on $Z_\infty\backslash G_{1,A}$ and $G_{1,A}$. As for $Z_\infty\backslash G_{1,\infty}$, we have easily

$$Z_\infty\backslash G_{1,\infty} \xrightarrow{\sim} \varDelta\backslash S_{1,\infty}; \quad \varDelta=Z_\infty \cap S_{1,\infty}.$$

Define the subgroup T of A_∞ by

$$T=\left\{\begin{pmatrix} s & & \\ & 1 & \\ & & s^{-1}\end{pmatrix}\middle| s\in \boldsymbol{R}^\times\right\}.$$

We normalize the Haar measure $d\dot{g}_\infty$ on $Z_\infty\backslash G_{1,\infty}$ as follows:

$$\int_{Z_\infty\backslash G_{1,\infty}} f(g_\infty)d\dot{g}_\infty=\int_{T\times N_\infty\times U_\infty} f(tnu)dtdndu \quad \text{for} \quad \forall f\in C_c^0(Z_\infty\backslash G_{1,\infty}),$$

where dt, dn, and du are Haar measures of T, N_∞, and U_∞, respectively, normalized by

(1.7)
$$dt=s^{-1}ds \quad \text{for} \quad t=\begin{pmatrix} s & & \\ & 1 & \\ & & s^{-1}\end{pmatrix}; \ ds \text{ is the Euclidean measure of } \boldsymbol{R},$$
$dn=|\iota|dadb$ for $n=n(a, b)$; da (resp. db) is the Euclidean measure of $\boldsymbol{C}$ (resp. $\boldsymbol{R}$),
the total volume of U_∞ with respect to du is one.

On Z_∞ we define the Haar measure dz_∞ by

(1.8) $$dz_\infty=s^{-1}dsd\theta \quad \text{for} \quad z_\infty=s\cdot e[\theta]\cdot 1_3 \quad (s>0,\ \theta\in\boldsymbol{R})$$

where ds and $d\theta$ are the Euclidean measure of $\boldsymbol{R}$. We define the Haar measure dg_∞ of $G_{1,\infty}$ by

$$d\dot{g}_\infty=\frac{dg_\infty}{dz_\infty}.$$

As for $G_{1,f}$, we have defined dg_f by (1.3). Then we define the Haar measures $d\dot{g}$ of $Z_\infty\backslash G_{1,A}$ and dg of $G_{1,A}$ by

$$d\dot{g}=d\dot{g}_\infty\times dg_f, \qquad dg=dg_\infty\times dg_f.$$

Now, we shall introduce the Bergman kernel of the Hilbert space of holomorphic functions on $Z_\infty\backslash G_{1,\infty}$. Let r be a positive integer. We denote by $\mathscr{H}_r^2$ the space of $\boldsymbol{C}$-valued function f on $G_{1,\infty}$ such that

(i) $\nu(g_\infty)^{-r/2}j(g_\infty, \mathfrak{z}_0)^r f(g_\infty)$ depends only on $\mathfrak{z}=g_\infty\langle\mathfrak{z}_0\rangle$, and it is holomorphic on $\mathscr{D}$ as a function of $\mathfrak{z}$,

(ii) $|f|$ is square integrable on $Z_\infty\backslash G_{1,\infty}$ with respect to a Harr measure.

It is known that if $r>2$, $\mathscr{H}_r^2 \neq \{0\}$. In this case, the Bergman kernel is calculated in Cohn [3]. With respect to our measure $d\dot{g}_\infty$, it is reformulated as follows: Let $\varphi_{1,\infty}$ be the $\boldsymbol{C}$-valued function on $G_{1,\infty}$ defined by

(1.9) $$\varphi_{1,\infty}(g)=C_r\nu(g)^{r/2}H_1(g\mathfrak{z}_0, \mathfrak{z}_0)^{-r}; \quad C_r=\pi^{-2}(r-1)(r-2).$$

Here, we write $H_1(x, y)$ for ${}^t\bar{y}H_1x$ ($x, y \in \boldsymbol{C}^3$), $\mathfrak{z}_0$ for ${}^t(1/2, 0, 1)$. Then we have

$$\varphi_{1,\infty}(g_2^{-1}g_1) \in \mathscr{H}_r^2; \quad \text{as a function of } g_1,$$
$$\int_{Z_\infty\backslash G_{1,\infty}} \varphi_{1,\infty}(g_2^{-1}g_1)f(g_2)dg_2=f(g_1) \quad \text{for} \quad \forall f \in \mathscr{H}_r^2.$$

PROPOSITION 1.1. *When $r>4$, the following statements hold for $\forall\varphi_{1,f} \in \mathscr{L}(G_{1,f}, U_{1,f})$.*

(i) *The series $\sum_{\gamma\in G_{1,\boldsymbol{Q}}}(\varphi_{1,\infty}\otimes\varphi_{1,f})(g_2^{-1}\gamma g_1)$ converges absolutely and uniformly for any (g_1, g_2) staying in a compact subset of $(Z_\infty\backslash G_{1,A})\times(Z_\infty\backslash G_{1,A})$, and as a function of g_1, it belongs to $\mathfrak{S}_r(U_{1,f})$.*

(ii) *If we put*

$$T_1(\varphi_{1,f})f(g_1)=\int_{Z_\infty G_{1,\boldsymbol{Q}}\backslash G_{1,A}} \sum_{\gamma\in G_{1,\boldsymbol{Q}}} (\varphi_{1,\infty}\otimes\varphi_{1,f})(g_2^{-1}\gamma g_1)f(g_2)d\dot{g}_2$$

for $f\in\mathfrak{S}_r(U_{1,f})$, the integral converges, and $T_1(\varphi_{1,f})$ becomes an endomorphism of the finite dimensional vector space $\mathfrak{S}_r(U_{1,f})$.

(iii) *The trace of the operator $T_1(\varphi_{1,f})$ on $\mathfrak{S}_r(U_{1,f})$ is given by*

(1.10) $$\operatorname{trace}(T_1(\varphi_{1,f}))=\int_{Z_\infty G_{1,\boldsymbol{Q}}\backslash G_{1,A}} \sum_{\gamma\in G_{1,\boldsymbol{Q}}} (\varphi_{1,\infty}\otimes\varphi_{1,f})(g^{-1}\gamma g)d\dot{g}.$$

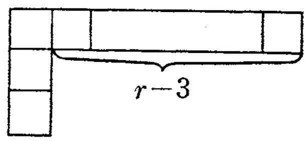
One can prove the above proposition by the argument of Godement [4]. (See also [3], [13], [15].)

1.3.

The reductive Lie group $G_{2,\infty}$ can be decomposed as follows:

$$G_{2,\infty}=\boldsymbol{R}_+^\times \times U(3) \quad \text{(direct product)},$$
$$U(3)=\{g\in GL(3, \boldsymbol{C}) \mid {}^t\bar{g}g=1_3\}.$$

Let r be as in 1.2. We denote by ρ_r the finite dimensional irreducible representation of $U(3)$ whose Young diagram is



We understand that the representation space M of ρ_r is realized as the space of column vectors. We denote by ρ the representation of $G_{2,\infty}$ defined by $\rho=1\otimes\rho_r$, where 1 means the unit representation of $\boldsymbol{R}_+^\times$.

Let $U_{2,f}$ be an open compact subgroup of $G_{2,f}$ of the form (1.2) and r, ρ as above. We say that an M-valued function f on $G_{2,A}$ is an automorphic form of weight ρ with respect to $U_{2,f}$ if f satisfies the following conditions:

(i) $f(\gamma gu)=f(g)$ for $\forall\gamma\in G_{2,Q}$, $\forall u\in U_{2,f}$,

(ii) $f(gg_\infty)=\rho(g_\infty)^{-1}f(g)$ for $\forall g_\infty\in G_{2,\infty}$.

We denote by $\mathfrak{S}_\rho(U_{2,f})$ the space of such functions.

We normalize the Haar measures on $Z_\infty\backslash G_{2,A}$ and $G_{2,A}$ as follows: On $Z_\infty\backslash G_{2,\infty}$, we define the Haar measure $d\dot{g}_\infty$ by

(1.11) the total volume of $Z_\infty\backslash G_{2,\infty}$ with respect to $d\dot{g}_\infty$ is one.

On Z_∞ we define the Haar measure dz_∞ by (1.8) and on $G_{2,\infty}$ we define the Haar measure dg_∞ by

$$d\dot{g}_\infty=\frac{dg_\infty}{dz_\infty}.$$

On $G_{2,f}$, we have defined dg_f by (1.3). We define the Haar measures $d\dot{g}$ of $Z_\infty\backslash G_{2,A}$ and dg of $G_{2,A}$ by

$$d\dot{g}=d\dot{g}_\infty\times dg_f,\qquad dg=dg_\infty\times dg_f.$$

Let $\varphi_{2,\infty}$ be the $\boldsymbol{C}$-valued function on $G_{2,\infty}$ defined by

$$\varphi_{2,\infty}(g)=\mathrm{trace}(\rho(g)^{-1}). \tag{1.12}$$

The explicit formula for $\varphi_{2,\infty}$ is well-known (cf. Weyl [24]).

PROPOSITION 1.2. *The following statements hold for* $\forall\varphi_{2,f}\in\mathscr{L}(G_{2,f}, U_{2,f})$.

(i) *For any compact subset* X *of* $(Z_\infty\backslash G_{2,A})\times(Z_\infty\backslash G_{2,A})$, *there are at most finitely many* $\gamma\in G_{2,Q}$ *such that* $(\rho^{-1}\otimes\varphi_{2,f})(g_2^{-1}\gamma g_1)\neq 0$ *for some* $(g_1, g_2)\in X$.

(ii) *If we put*

$$T_2(\varphi_{2,f})f(g_1)=\int_{Z_\infty G_{2,Q}\backslash G_{2,A}}\sum_{\gamma\in G_{2,Q}}(\rho^{-1}\otimes\varphi_{2,f})(g_2^{-1}\gamma g_1)f(g_2)d\dot{g}_2$$

for $f\in\mathfrak{S}_\rho(U_{2,f})$, *the integral converges, and* $T_2(\varphi_{2,f})$ *becomes an endomorphism of the finite dimensional vector space* $\mathfrak{S}_\rho(U_{2,f})$.

(iii) *The trace of the operator* $T_2(\varphi_{2,f})$ *on* $\mathfrak{S}_\rho(U_{2,f})$ *is given by*

$$\text{(1.13)}\qquad \operatorname{trace}(T_2(\varphi_{2,f}))=\int_{Z_\infty G_{2,Q}\backslash G_{2,A}}\sum_{\gamma\in G_{2,Q}}(\varphi_{2,\infty}\otimes\varphi_{2,f})(g^{-1}\gamma g)d\dot{g}.$$

Since $Z_\infty\backslash G_{2,\infty}$ is compact, the above proposition is obvious.

§ 2. Conjugacy classes

2.1.

In this subsection, we describe a rough classification of the conjugacy classes of $G_{1,Q}$ and $G_{2,Q}$.

DEFINITION. For an element γ of $G_{1,Q}$, we say that γ is ***Q**-elliptic* (resp. *elliptic*) if $[\gamma]_{G_{1,Q}}\cap P_Q=\emptyset$ (resp. $[\gamma]_{G_{1,\infty}}\cap P_\infty=\emptyset$). We say that γ is *non split hyperbolic* if γ is **Q**-elliptic but not elliptic. We say that γ is *split hyperbolic* if $[\gamma]_{G_{1,Q}}\cap(M_Q-Z_Q)\neq\emptyset$.

There are some differences between our terminology and that of Cohn [3]. Note that a singular torsion element is not necessarily elliptic in our terminology.

Let us introduce some **Q**-subgroups of G_1. We denote by $Z(N)$ the center of N:

$$Z(N)_Q=\{n(0,b)\,|\,b\in\boldsymbol{Q}\}.$$

By M^*, we denote the **Q**-subgroup of M whose set of **Q**-rational points is

$$M_Q^*=\left\{\begin{pmatrix}\alpha & & \\ & \beta & \\ & & \alpha\end{pmatrix}\in GL(3,F)\,\middle|\,N_{F/Q}(\alpha)=N_{F/Q}(\beta)\right\}.$$

By J, we denote the **Q**-subgroup of G_1 whose set of **Q**-rational points is

$$J_Q=\left\{\begin{pmatrix}* & 0 & * \\ 0 & * & 0 \\ * & 0 & *\end{pmatrix}\in G_Q\right\}\quad (*\text{: an element of } F).$$

If we put

$$\tilde{w}=\begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}, \tag{2.1}$$

this element generates the Weyl group W of G_1 with respect to the $\boldsymbol{Q}$-split torus contained in M:

$$W=\langle\tilde{w}\rangle.$$

PROPOSITION 2.1. (i) *The group $G_{1,Q}$ is the disjoint union of the following subsets. Each subset is invariant under $G_{1,Q}$-conjugacy.*

(a) Z_Q.

(b) *The set of singular elliptic elements. We write $E_1(sing.)$ for this set.*

(c) *The set of regular elliptic elements whose characteristic polynomials are reducible in F. We write $E_1(red.)$ for this set.*

(d) *The set of elliptic elements whose characteristic polynomials are irreducible in F. We write $E_1(irred.)$ for this set.*

(e) *The set of singular non split hyperbolic elements. We write $Nh(sing.)$ for this set.*

(f) *The set of regular non split hyperbolic elements. We write $Nh(reg.)$ for this set.*

(g) *The set of singular split hyperbolic elements. We write $Sh(sing.)$ for this set.*

(h) *The set of regular split hyperbolic elements. We write $Sh(reg.)$ for this set.*

(i) *The set of non semi-simple elements whose minimal polynomials are of degree two. We write $Ns(1)$ for this set.*

(j) *The set of non semi-simple elements whose minimal polynomials have the form $(x-\alpha)^3$ $(\alpha\in F^{\times})$. We write $Ns(2)$ for this set.*

(k) *The set of non semi-simple elements whose minimal polynomials have the form $(x-\alpha)^2(x-\beta)$ $(\alpha,\beta\in F^{\times},\ \alpha\neq\beta)$. We write $Ns(3)$ for this set.*

(ii) *The elements of $Sh(sing.)$, $Sh(reg.)$, $Ns(1)$, $Ns(2)$, and $Ns(3)$ are uniquely expressed in the following form:*

$$\gamma\in Sh(sing.)\Longleftrightarrow\gamma=\delta^{-1}\mu\delta;\quad \mu\in M_Q^*-Z_Q,\ \delta\in J_Q\backslash G_{1,Q},$$
$$\gamma\in Sh(reg.)\Longleftrightarrow\gamma=\delta^{-1}\mu\delta;\quad \mu\in M_Q-M_Q^*,\ \delta\in WM_Q\backslash G_{1,Q},$$
$$\gamma\in Ns(1)\Longleftrightarrow\gamma=\delta^{-1}\mu\nu\delta;\quad \mu\in Z_Q,\ \nu\in Z(N)_Q-\{1\},\ \delta\in P_Q\backslash G_{1,Q},$$
$$\gamma\in Ns(2)\Longleftrightarrow\gamma=\delta^{-1}\mu\nu\delta;\quad \mu\in Z_Q,\ \nu\in N_Q-Z(N)_Q,\ \delta\in P_Q\backslash G_{1,Q},$$

$\gamma \in Ns(3) \Longleftrightarrow \gamma = \delta^{-1}\mu\nu\delta;\quad \mu \in M_Q^* - Z_Q,\ \nu \in Z(N)_Q - \{1\},\ \delta \in M_Q Z(N)_Q \backslash G_{1,Q}.$

(iii) *The group $G_{2,Q}$ is the disjoint union of the following subsets. Each subset is invariant under $G_{2,Q}$-conjugacy.*

(a′) Z_Q.

(b′) *The set of singular elements which are not contained in Z_Q. We write E_2(sing.) for this set.*

(c′) *The set of regular elements whose characteristic polynomials are reducible in F. We write E_2(red.) for this set.*

(d′) *The set of elements whose characteristic polynomials are irreducible in F. We write E_2(irred.) for this set.*

The above proposition can be proved by the definition and elementary calculation of matrices, so we omit the proof. But, for reader's convenience, we summarize the rules of compositions in P.

LEMMA 2.2. *The rules of compositions in P are as follows:*

$$n(a_1, b_1)n(a_2, b_2) = n\Big(a_1 + a_2,\ b_1 + b_2 + \frac{\iota^{-1}}{2}(\bar{a}_1 a_2 - a_1 \bar{a}_2)\Big),$$

$$n(a, b)^{-1} = n(-a, -b),$$

$$\lambda(\alpha)\lambda(\beta) = \lambda(\alpha\beta),$$

$$\lambda(\alpha)^{-1} = \lambda(\alpha^{-1}),$$

$$n(a, b)^{-1} n(a_0, b_0) n(a, b) = n(a_0, b_0) n(0, \iota^{-1}(\bar{a}_0 a - a_0 \bar{a})),$$

$$\lambda(\alpha)^{-1} n(a_0, b_0) \lambda(\alpha) = n(\bar{\alpha}^{-1} a_0, N(\alpha)^{-1} b_0),$$

$$n(a, b)^{-1}\lambda(\alpha_0) n(a, b) = \lambda(\alpha_0) n\Big((1 - \bar{\alpha}_0^{-1})a,\ (1 - N(\alpha_0)^{-1})b + \frac{\iota^{-1}}{2}(\bar{\alpha}_0^{-1} - \alpha_0^{-1})N(a)\Big);$$

especially, if $\lambda(\alpha_0) \in M^*$ (i.e. $N(\alpha_0) = 1$),

$$n(a, b)^{-1}\lambda(\alpha_0) n(a, b) = \lambda(\alpha_0) n\Big((1 - \alpha_0)a,\ \frac{\iota^{-1}}{2}(\alpha_0 - \bar{\alpha}_0)N(a)\Big).$$

The above lemma will be used also in § 4.

2.2.

In this subsection, we construct a certain good three to one correspondence from the set of conjugacy classes of $G_{1,Q}$ contained in a stable conjugacy class in E_1(irred.) to the set of conjugacy classes of $G_{2,Q}$ contained in the corresponding stable conjugacy class in E_2(irred.)

For this purpose, we need to parametrize the conjugacy classes in each E_i(irred.). The following arguments to give a parametrization is found in Asai [2], while our situation is far more restricted than that of [2] and our final result is not described explicitly in [2].

We denote by ρ the non-trivial automorphism of $F/\boldsymbol{Q}$ in this subsection (but this will be denoted by "the bar" in the other places). Let n be a positive *odd* integer and fix $H \in GL(n, F)$ satisfying ${}^tH^\rho = H$. We define the non-degenerate ρ-hermitian space (V, H) over F by $V = F^n$ and $H(x, y) = {}^ty^\rho Hx$ for $x, y \in V$. Put $G = GU(H)$:

$$G_{\boldsymbol{Q}} = \{g \in GL(n, F) \mid {}^tg^\rho Hg = \nu(g)H,\ \nu(g) \in \boldsymbol{Q}^\times\}.$$

In this odd dimensional case, it is known that $\nu(G_{\boldsymbol{Q}}) = N_{F/\boldsymbol{Q}}(F^\times)$ (cf. Shimura [22], 2.4). It follows that $G_{\boldsymbol{Q}} = Z_{\boldsymbol{Q}}\boldsymbol{G}_{\boldsymbol{Q}}$, where Z is the center of G and $\boldsymbol{G}$ is the kernel of $\nu(\ )$. Therefore, the $G_{\boldsymbol{Q}}$-conjugacy class containing g coincides with the $\boldsymbol{G}_{\boldsymbol{Q}}$-conjugacy class containing g for any $g \in G_{\boldsymbol{Q}}$.

Let ν be an element of $N_{F/\boldsymbol{Q}}(F^\times)$, and let $f = f(X)$ be a monic element of the polynomial ring $F[X]$ of degree n. In the following, we assume

(2.2) f is irreducible in F and each root of f in $\boldsymbol{C}$ has absolute value ν.

We denote by $G_{\boldsymbol{Q}}(\nu, f)$ the set of all elements of $G_{\boldsymbol{Q}}$ whose multiplicators and characteristic polynomials are ν and f, respectively. We define $G_v(\nu, f)$ similarly at any place v of $\boldsymbol{Q}$. Now, we put $A = F[X]/(f)$. By (2.2), the complex conjugation σ is well-defined as an automorphism of A and it satisfies

$$F^\sigma = F, \qquad \sigma|_F = \rho, \qquad XX^\sigma = \nu$$

(we abbreviate "$X \bmod (f)$" to X). Put $B = \{a \in A \mid a^\sigma = a\}$. Then B is a totally real number field and A is a totally imaginary quadratic extension of B.

Now, let (W, I) be any non-degenerate σ-hermitian space over A. We define the non-degenerate ρ-hermitian space $\mathrm{Tr}_{A/F}(W, I) = (\mathrm{Tr}_{A/F}(W), \mathrm{Tr}_{A/F}(I))$ over F as follows: $\mathrm{Tr}_{A/F}(W)$ is nothing but the F-module W given by the scalar restriction, and $\mathrm{Tr}_{A/F}(I)(x, y) = \mathrm{Tr}_{A/F}(I(x, y))$ for $x, y \in W$. Now, fix $g \in GL(n, F)$ with characteristic polynomial f. We define the A-module structure on V via $Xx = gx$ $(x \in V)$, and write $\widetilde{V}^g$ for this A-module. For $x, y \in \widetilde{V}^g$, there exists a unique element $\widetilde{H}^g(x, y)$ of A

such that the equality

$$H(ax, y)=\mathrm{Tr}_{A/F}(a\tilde{H}^g(x, y))$$

holds for any $a\in A$, because H is non-degenerate. It is easy to see that the map $(x, y)\mapsto\tilde{H}^g(x, y)$ becomes a non-degenerate σ-hermitian form on $\tilde{V}^g$ if and only if g belongs to $G_Q(\nu, f)$. Moreover, the set of conjugacy classes $G_Q(\nu, f)//G_Q$ is described as follows (cf. [2], Theorem 1.8).

LEMMA 2.3. *The set $G_Q(\nu, f)$ is not empty if and only if there exists a non-degenerate σ-hermitian space $(\tilde{V}, \tilde{H})$ of rank one over A such that $\mathrm{Tr}_{A/F}(\tilde{V}, \tilde{H})$ is isometric to (V, H). If this is the case, then the map $g\mapsto(\tilde{V}^g, \tilde{H}^g)$ induces the canonical bijection*

$$G_Q(\nu, f)//G_Q\xrightarrow{\sim}\left\{\begin{array}{l}\text{the isometric classes of } \sigma\text{-hermitian}\\ \text{spaces } (\tilde{V}, \tilde{H}) \text{ of rank one over } A\\ \text{s.t. } \mathrm{Tr}_{A/F}(\tilde{V}, \tilde{H})\cong(V, H)\end{array}\right\}.$$

We now consider the following problems:

Problem 1. To classify the isometric classes of non-degenerate ρ-hermitian spaces over F and non-degenerate σ-hermitian spaces (of rank one) over A.

Problem 2. To determine the isometric class of $\mathrm{Tr}_{A/F}(\tilde{V}, \tilde{H})$ from the isometric class of $(\tilde{V}, \tilde{H})$.

We can solve Problem 1 completely, thanks to the results of Landherr [18]. Let L be a totally real number field, finite over $\boldsymbol{Q}$, and let M be a totally imaginary quadratic extension of L. For any place w of L, we put $M_w=M\otimes_L L_w$. We denote by σ the non-trivial automorphism of M/L and that of M_w/L_w. For a non-degenerate σ-hermitian space (W, I) of finite rank over M, we define its discriminant $d(W, I)$ as the coset in $L^\times/N_{M/L}(M^\times)$ containing $\det(I(w_i, w_j))$, where $\{w_i\}$ is a M-basis of W. If (W, I) is a non-degenerate σ-hermitian space of finite rank over M_w, we define $d(W, I)\in L_w^\times/N_{M_w/L_w}(M_w^\times)$ similarly, and if w is an infinite place of L, we define the signature of (W, I) as usual. By [18], we have the following two lemmas.

LEMMA 2.4. (i) *Let w be a finite place of L. Then the isometric class of a non-degenerate σ-hermitian space (W, I) of finite rank over M_w is determined by the rank and the discriminant $d(W, I)\in L_w^\times/N_{M_w/L_w}(M_w^\times)$.*

(ii) *Let w be an infinite place of L. Then the isometric class of a non-degenerate σ-hermitian space (W, I) of finite rank over $M_w = \boldsymbol{C}$ is determined by the rank and the signature (p, q).*

LEMMA 2.5. *The isometric class of a non-degenerate σ-hermitian space (W, I) of finite rank over M is determined by the rank, the discriminant $d(W, I) \in L^\times / N_{M/L}(M^\times)$, and the signatures $\{(p_w, q_w)\}_w$ where w runs through all infinite places of L. They can take arbitrary values subject to the following condition:*

$$(d(W, I), M/L)_w = (-1)^{q_w} \quad \text{at any infinite place } w \text{ of } L.$$

Here, $(\ , M/L)_w$ is the local norm residue symbol.

The above two lemmas, together with the Hasse principle for the norm map in a cyclic extension, imply the Hasse principle for the equivalence of Hermitian spaces.

We now consider Problem 2 mentioned above. Let $(\tilde{V}, \tilde{H})$ be a non-degenerate σ-hermitian space of rank one over A. By Lemma 2.5, it is enough to determine $d(\mathrm{Tr}_{A/F}(\tilde{V}, \tilde{H})) \in \boldsymbol{Q}^\times / N_{F/\boldsymbol{Q}}(F^\times)$ and the signature of $\mathrm{Tr}_{A/F}(\tilde{V}, \tilde{H})$ at the infinite place of $\boldsymbol{Q}$ from $d(\tilde{V}, \tilde{H}) \in B^\times / N_{A/B}(A^\times)$. Now, fix a non-degenerate σ-hermitian space $(\tilde{V}_0, \tilde{H}_0)$ of rank one over A with $d(\tilde{V}_0, \tilde{H}_0) \in N_{A/B}(A^\times)$. Then the discriminant of $\mathrm{Tr}_{A/F}(\tilde{V}_0, \tilde{H}_0)$ does not depend on the choice of $(\tilde{V}_0, \tilde{H}_0)$, and we denote it by $\Delta_{A/F}$:

$$\Delta_{A/F} = d(\mathrm{Tr}_{A/F}(\tilde{V}_0, \tilde{H}_0)) \in \boldsymbol{Q}^\times / N_{F/\boldsymbol{Q}}(F^\times); \qquad d(\tilde{V}_0, \tilde{H}_0) \in N_{A/B}(A^\times).$$

Then we have the following relation (cf. [2], Lemma 3.4).

$$d(\mathrm{Tr}_{A/F}(\tilde{V}, \tilde{H})) = \Delta_{A/F} N_{B/\boldsymbol{Q}}(d(\tilde{V}, \tilde{H})). \tag{2.3}$$

For our purpose, it is better to reformulate this relation. Let v be a place of $\boldsymbol{Q}$, and let w be any place of B lying above v. Then $A_w = A \otimes_B B_w$ is a σ-stable component of $A_v = A \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v$, and we may regard σ as an involution of A_w / B_w. Let $(\tilde{V}, \tilde{H})_v = (\tilde{V}_v, \tilde{H})$ be the completion of $(\tilde{V}, \tilde{H})$ at v, which is a σ-hermitian space of rank one over A_v. Then the subspace $(A_w \tilde{V}_v, \tilde{H}|_{A_w \tilde{V}_v})$ of $(\tilde{V}, \tilde{H})_v$ is nothing but completion $(\tilde{V}, \tilde{H})_w$ of $(\tilde{V}, \tilde{H})$ at w, which is a σ-hermitian space of rank one over A_w, and we have the following decomposition:

$$(\tilde{V}, \tilde{H})_v = \bigoplus_{w \mid v} (\tilde{V}, \tilde{H})_w \quad \text{(orthogonal sum)}. \tag{2.4}$$

Put $s_w = d((\tilde{V}, \tilde{H})_w)$ $(\in B_w^\times / N_{A_w/B_w}(A_w^\times))$. We define Δ_{A_w/F_v} for A_w/F_v by

$$\Delta_{A_w/F_v}=d(\mathrm{Tr}_{A_w/F_v}((\tilde{V}_0, \tilde{H}_0)_w)) \in \boldsymbol{Q}_v^\times/N_{F_v/\boldsymbol{Q}_v}(F_v^\times),$$

where $(\tilde{V}_0, \tilde{H}_0)$ is defined as above, and Tr_{A_w/F_v} is defined for any σ-hermitian space over A_w like $\mathrm{Tr}_{A/F}$ of a σ-hermitian space over A. We now put

$$\varepsilon_w=(\Delta_{A_w/F_v}N_{B_w/\boldsymbol{Q}_v}(s_w), F/\boldsymbol{Q})_v.$$

Here we may write $\varepsilon_w=(d(\mathrm{Tr}_{A_w/F_v}((\tilde{V}, \tilde{H})_w), F/\boldsymbol{Q})_v$. Using (2.4), the above relation (2.3) is rewritten as

(2.5) $\qquad (d(\mathrm{Tr}_{A/F}(\tilde{V}, \tilde{H})), F/\boldsymbol{Q})_v=\prod_{w|v}\varepsilon_w$ at any place v of $\boldsymbol{Q}$.

Now, as for the signature of $\mathrm{Tr}_{A/F}(\tilde{V}, \tilde{H})$ at the infinite place of $\boldsymbol{Q}$, we easily have

(2.6) the signature of $\mathrm{Tr}_{A/F}(\tilde{V}, \tilde{H})=(p, n-p)$, where p=the number of infinite places of B with $\varepsilon_w=+1$.

Now, using the translation theorem in local class field theory, we have $\varepsilon_w=(\Delta_{A_w/F_v}, F/\boldsymbol{Q})_v(s_w, A/B)_w$. Therefore $d(\tilde{V}, \tilde{H})$ is completely determined by the system $\{\varepsilon_w\}_w$, where w runs through all places of B. Here, it is easy to see that $(\Delta_{A_w/F_v}, F/\boldsymbol{Q})_v=+1$ for almost all w and $\prod_w(\Delta_{A_w/F_v}, F/\boldsymbol{Q})_v=+1$. If A_w is not a field, we easily have $(\Delta_{A_w/F_v}, F/\boldsymbol{Q})_v=(-1, F/\boldsymbol{Q})_v$. Combining these properties of $(\Delta_{A_w/F_v}, F/\boldsymbol{Q})_v$ with the well-known properties of $(\ , A/B)_w$, we see

(2.7) the range of $\{\varepsilon_w\}_w$ for all $d(\tilde{V}, \tilde{H}) \in B^\times/N_{A/B}(A^\times)$ is characterized by: $\varepsilon_w=+1$ for almost all w, $\varepsilon_w=(-1, F/\boldsymbol{Q})_v$ $(v=w|_{\boldsymbol{Q}})$ if A_w is not a field, $\prod_w\varepsilon_w=+1$.

From (2.5), (2.6), and (2.7), we have the following

PROPOSITION 2.6. *Suppose that $n=\dim_F V$ is odd and (ν, f) satisfies (2.2). Then $G_{\boldsymbol{Q}}(\nu, f)$ is not empty. For $g \in G_{\boldsymbol{Q}}(\nu, f)$, define the A-module $\tilde{V}^g$ as in Lemma 2.3. For any place w of B, let $t_w \in \boldsymbol{Q}_v^\times/N_{F_v/\boldsymbol{Q}_v}(F_v^\times)$ be the discriminant of the ρ-hermitian space $(A_w\tilde{V}_v^g, H|_{A_w\tilde{V}_v^g})$ over F_v, where $v=w|_{\boldsymbol{Q}}$. Then the correspondence $g\mapsto\{(t_w, F/\boldsymbol{Q})_v\}_w$ induces the canonical bijection from $G_{\boldsymbol{Q}}(\nu, f)//G_{\boldsymbol{Q}}$ to the set of all $\{\varepsilon_w\}_w$'s subject to the following conditions:*

$$\varepsilon_w=\pm 1 \quad \textit{for any } w \quad \textit{and} \quad \varepsilon_w=+1 \qquad \textit{for almost all } w,$$

$\varepsilon_w = (-1, F/\boldsymbol{Q})_v \quad (v = w|_{\boldsymbol{Q}})$ *if* A_w *is not a field,*

$\prod_{w|v} \varepsilon_w = (d(V, H), F/\boldsymbol{Q})_v$ *at any place* v *of* $\boldsymbol{Q}$,

the number of infinite places w *with* $\varepsilon_w = +1$ *equals* p *if* (V, H) *has signature* $(p, n-p)$.

Next we consider $G_v(\nu, f)//G_v$ for a place v of $\boldsymbol{Q}$. Using the local version of Lemma 2.3 and Lemma 2.4, we easily have the following

PROPOSITION 2.7. *Suppose that* $n = \dim_F V$ *is odd and* (ν, f) *satisfies* (2.2). *Let* v *be a place of* $\boldsymbol{Q}$. *For* $g \in G_v(\nu, f)$ *and any place* w *of* B *lying above* v, *define* $t_w \in \boldsymbol{Q}_v^\times / N_{F_v/\boldsymbol{Q}_v}(F_v^\times)$ *as in Proposition* 2.6. *Then the correspondence* $g \mapsto \{(t_w, F/\boldsymbol{Q})_v\}_{w|v}$ *induces the canonical bijection from* $G_v(\nu, f)//G_v$ *to the set of all* $\{\varepsilon_w\}_{w|v}$*'s subject to the following conditions*:

$\varepsilon_w = \pm 1$ *for any* w,

$\varepsilon_w = (-1, F/\boldsymbol{Q})_v$ *if* A_w *is not a field,*

$\prod_{w} \varepsilon_w = (d(V, H), F/\boldsymbol{Q})_v$,

the number of w*'s with* $\varepsilon_w = +1$ *equals* p *if* v *is the infinite place and* (V, H) *has signature* $(p, n-p)$.

The above two propositions imply the Hasse principle for conjugacy classes in $G_{\boldsymbol{Q}}(\nu, f)$, namely, the injectivity of the canonical map $G_{\boldsymbol{Q}}(\nu, f)//G_{\boldsymbol{Q}} \to \prod_v G_v(\nu, f)//G_v$. This principle is proved in [2] for some classical groups over an algebraic number field, without any assumption on the characteristic polynomial (cf. [2], Theorem 4.7).

We now apply the above parametrizations of conjugacy classes to G_1 and G_2 $(n=3)$. In this case, (ν, f) satisfies (2.2) if and only if $G_{i,\boldsymbol{Q}}(\nu, f)$ is contained in E_i(irred.) for $i = 1, 2$. For a conjugacy class S of $G_{i,\boldsymbol{Q}}$ and a place v of $\boldsymbol{Q}$, let S_v denote the conjugacy class of $G_{i,v}$ containing S.

PROPOSITION 2.8. *Let* ν *be an element of* $N_{F/\boldsymbol{Q}}(F^\times)$ *and let* $f = f(X)$ *be a monic element of* $F[X]$ *of degree three. Suppose that* (ν, f) *satisfies* (2.2). *Then, there exists a unique surjection*

$$\Psi \colon G_{1,\boldsymbol{Q}}(\nu, f)//G_{1,\boldsymbol{Q}} \twoheadrightarrow G_{2,\boldsymbol{Q}}(\nu, f)//G_{2,\boldsymbol{Q}}$$

with the following properties: *For any* $D \in G_{2,\boldsymbol{Q}}(\nu, f)//G_{2,\boldsymbol{Q}}$, $\Psi^{-1}(D)$ *consists of three conjugacy classes, and if we write them as* C, C', *and* C'', *these three conjugacy classes and* D *satisfy*

$$\Theta_l(C_l)=\Theta_l(C_l')=\Theta_l(C_l'')=D_l \quad at\ any\ l\neq\infty,$$
$$\{C_\infty, C_\infty', C_\infty''\}=G_{1,\infty}(\nu, f)//G_{1,\infty},$$
$$\{D_\infty\}=G_{2,\infty}(\nu, f)//G_{2,\infty}.$$

PROOF. Let l be a finite place of $\boldsymbol{Q}$ and S an element of $G_{1,l}(\nu, f)//G_{1,l}$. Let $\{\varepsilon_w\}_{w|l}$ and $\{\varepsilon_w'\}_{w|l}$ be the invariants of S and $\Theta_l(S)$, respectively, defined in Proposition 2.7. Then, by the definition of Θ_l, we have $\varepsilon_w=\varepsilon_w'$ for each w. Therefore the existence of Ψ comes from Proposition 2.6 and Proposition 2.7. The uniqueness of Ψ is clear by these propositions. Q.E.D.

REMARK. In the above proposition, the assumption that f is irreducible in F is essential. Suppose that f is reducible in F (and separable). Then the image of $G_{i,Q}(\nu, f)//G_{i,Q}\to\prod_v G_{i,v}(\nu, f)//G_{i,v}$ is restricted by some "product formulas". From this fact, one sees that there is no correspondence like Ψ in this case.

§ 3. Contributions of central elements and $\boldsymbol{Q}$-elliptic conjugacy classes

3.1.

In § 1, we described the integral representations of traces of Hecke operators $T_1(\varphi_{1,f})$ and $T_2(\varphi_{2,f})$, acting on $\mathfrak{S}_r(U_{1,f})$ and $\mathfrak{S}_\rho(U_{2,f})$, respectively (see (1.10) and (1.13))· Hereafter, we assume that $r>4$ and compare the traces of $T_1(\varphi_{1,f})$ and $T_2(\varphi_{2,f})$.

If an element $\varphi_{i,f}$ of $\mathscr{L}(G_{i,f}, U_{i,f})$ is fixed, we set

$$\varphi_i=\varphi_{i,\infty}\otimes\varphi_{i,f} \quad (i=1, 2).$$

Then, the integrand of (1.10) is written as follows:

$$\sum_{\gamma\in G_{1,Q}}\varphi_1(g^{-1}\gamma g)=\sum_{z\in Z_Q}\varphi_1(z)+\sum_{\gamma:Q\text{-elliptic}}\varphi_1(g^{-1}\gamma g)+\sum_{\gamma:\text{ split hyp. or non semi-simple}}\varphi_1(g^{-1}\gamma g). \tag{3.1}$$

The integral of the first term on $Z_\infty G_{1,Q}\backslash G_{1,A}$ equals

$$\mathrm{vol}(Z_\infty G_{1,Q}\backslash G_{1,A})\sum_{z\in Z_Q}\varphi_1(z), \tag{3.2}$$

where vol( ) is the volume with respect to the fixed measure $d\dot{g}$.

By an argument similar to that of Cohn [3], p. 12, Lemma 4, we see that the integral of the second term equals

(3.3) $$\sum_{[\gamma]} \int_{Z_\infty G_1(\gamma)_{\boldsymbol{Q}} \backslash G_{1,\boldsymbol{A}}} \varphi_1(g^{-1}\gamma g) d\dot{g},$$

where $[\gamma]$ runs through all $G_{1,\boldsymbol{Q}}$-conjugacy classes of $\boldsymbol{Q}$-elliptic elements.

The integrand of (1.13) is written as follows:

(3.4) $$\sum_{\gamma \in G_{2,\boldsymbol{Q}}} \varphi_2(g^{-1}\gamma g) = \sum_{z \in Z_{\boldsymbol{Q}}} \varphi_2(z) + \sum_{\gamma \notin Z_{\boldsymbol{Q}}} \varphi_2(g^{-1}\gamma g).$$

The integral of the first term on $Z_\infty G_{2,\boldsymbol{Q}} \backslash G_{2,\boldsymbol{A}}$ equals

(3.5) $$\mathrm{vol}(Z_\infty G_{2,\boldsymbol{Q}} \backslash G_{2,\boldsymbol{A}}) \sum_{z \in Z_{\boldsymbol{Q}}} \varphi_2(z),$$

where vol( ) is the volume with respect to the fixed measure $d\dot{g}$, and the integral of the second term equals

(3.6) $$\sum_{[\gamma]} \int_{Z_\infty G_2(\gamma)_{\boldsymbol{Q}} \backslash G_{2,\boldsymbol{A}}} \varphi_2(g^{-1}\gamma g) d\dot{g},$$

where $[\gamma]$ runs through all $G_{2,\boldsymbol{Q}}$-conjugacy classes not contained in $Z_{\boldsymbol{Q}}$.

3.2.

In this subsection, we compare (3.2) and (3.5). An element of $Z_{\boldsymbol{Q}}$ will be regarded as an element of $F^\times$.

LEMMA 3.1. *For an element z of $Z_{\boldsymbol{Q}}$, we have*

(3.7) $$\varphi_{1,\infty}(z) = \pi^{-2}(r-1)(r-2)(z/|z|)^{-r}$$

and

(3.8) $$\varphi_{2,\infty}(z) = \frac{1}{2}(r-1)(r-2)(z/|z|)^{-r}.$$

By (1.9), we have (3.7) directly, and (3.8) is verified by the well-known formula for $\varphi_{2,\infty}$.

PROPOSITION 3.2. *Assume that the pair $(U_{1,f}, U_{2,f})$ satisfies assumption (U_f) and the pair $(\varphi_{1,f}, \varphi_{2,f})$ satisfies assumption (φ_f). Then we have*

$$\mathrm{vol}(Z_\infty G_{1,\boldsymbol{Q}} \backslash G_{1,\boldsymbol{A}}) \varphi_1(z) = \mathrm{vol}(Z_\infty G_{2,\boldsymbol{Q}} \backslash G_{2,\boldsymbol{A}}) \varphi_2(z)$$

for $\forall z \in Z_{\boldsymbol{Q}}$.

PROOF. By assumption (φ_f), we have $\varphi_{1,f}(z) = \varphi_{2,f}(z)$ for $\forall z \in Z_{\boldsymbol{Q}}$. From this fact and Lemma 3.1, we have only to prove the next equality between the volumes:

(3.9) $$\mathrm{vol}(Z_\infty G_{2,Q}\backslash G_{2,A})/\mathrm{vol}(Z_\infty G_{1,Q}\backslash G_{1,A})=2\pi^{-2}.$$

We prove the above equality by using the theory of Tamagawa numbers. The Tamagawa number of the semi-simple group S_i over $\boldsymbol{Q}$ is defined by

$$\tau(S_i)=\int_{S_{i,Q}\backslash S_{i,A}}\omega_{i,A},$$

where ω_i is a $\boldsymbol{Q}$-rational invariant differential form on S_i of degree 8 (note that $\dim S_i=8$) and $\omega_{i,A}$ is the Tamagawa measure associated to ω_i $(i=1, 2)$. It is known that $\tau(S_1)=\tau(S_2)=1$ (see Weil [23]). If we set, for $g\in G_i$,

$$\phi_i(g)=(\det(g), \nu(g)) \quad (i=1, 2),$$

we have $\phi_1(G_{1,k})=\phi_2(G_{2,k})$ for any field k of characteristic zero (see Shimura [22]). Moreover, by assumption (U_f), we have $\phi_1(U_{1,l})=\phi_2(U_{2,l})$ at any finite place l. Then we have

(3.10) $$\begin{aligned}&\mathrm{vol}(Z_\infty G_{2,Q}\backslash G_{2,A})/\mathrm{vol}(Z_\infty G_{1,Q}\backslash G_{1,A})\\ &=\Big(\int_{S_{2,\infty}}\omega_{2,\infty}\Big)^{-1}\prod_{l\neq\infty}\Big(\int_{S_{2,l}\cap U_{2,l}}\omega_{2,l}\Big)^{-1}\Big/\frac{d\dot{g}_\infty}{\omega_{1,\infty}}\prod_{l\neq\infty}\Big(\int_{S_{1,l}\cap U_{1,l}}\omega_{1,l}\Big)^{-1}.\end{aligned}$$

Now, we calculate the right hand side of the above equality for suitable ω_1 and ω_2. Let $\mathscr{S}_i$ be the Lie algebra of $\boldsymbol{Q}$-rational left invariant vector fields on S_i $(i=1, 2)$. By the usual method, $\mathscr{S}_i$ is identified with a matrix Lie algebra:

$$\mathscr{S}_i=\{X\in M_3(F)\,|\,{}^t\bar{X}H_i+H_iX=0\}.$$

Now, put

(3.11) $$C=\begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 1 & 0\\ 1 & 0 & -1\end{pmatrix}$$

and define the $\boldsymbol{Q}$-basis $E_1^i, \cdots, E_8^i$ of $\mathscr{S}_i$ as follows:

$$C^{-1}E_1^1C=\begin{pmatrix}\iota & & \\ & -\iota & \\ & & 0\end{pmatrix},\quad C^{-1}E_2^1C=\begin{pmatrix}\iota & & \\ & 0 & \\ & & -\iota\end{pmatrix},\quad C^{-1}E_3^1C=\begin{pmatrix}0 & & \\ & 0 & 1\\ & -1 & 0\end{pmatrix},$$

$$C^{-1}E_4^1C=\begin{pmatrix}0 & & \\ & 0 & \iota \\ & \iota & 0\end{pmatrix},\quad C^{-1}E_5^1C=\begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix},\quad C^{-1}E_6^1C=\begin{pmatrix} & & \iota \\ & 0 & \\ -\iota & & \end{pmatrix},$$

$$C^{-1}E_7^1C=\begin{pmatrix}0 & 1 & \\ -1 & 0 & \\ & & 0\end{pmatrix},\quad C^{-1}E_8^1C=\begin{pmatrix}0 & \iota & \\ -\iota & 0 & \\ & & 0\end{pmatrix};$$

$$E_j^2=C^{-1}E_j^1C\quad (j=1,2,3,4),\qquad E_5^2=\begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix},\qquad E_6^2=\begin{pmatrix} & & \iota \\ & 0 & \\ \iota & & \end{pmatrix},$$

$$E_7^2=\begin{pmatrix}0 & 1 & \\ -1 & 0 & \\ & & 0\end{pmatrix},\qquad E_8^2=\begin{pmatrix}0 & \iota & \\ \iota & 0 & \\ & & 0\end{pmatrix}.$$

Here, ι is the fixed element of $F^\times$ defined in 1.1. We denote by dE_j^i the dual of E_j^i and put

$$\omega_i=dE_1^i\wedge\cdots\wedge dE_8^i\quad (i=1,2).$$

Let l be a finite place. The restriction of Θ_l to $S_{1,l}$ is an isomorphism from $S_{1,l}$ to $S_{2,l}$, and we again write Θ_l for this restriction. Now, we claim that $\Theta_l^*(\omega_{2,l})=\omega_{1,l}$. It is clear that the validity of this equality does not depend on the choice of Θ_l (namely, the choice of ϑ_l). Then our claim can be verified by a strightforward calculation, if one chooses a suitable ϑ_l such as

$$\vartheta_l=C\begin{pmatrix}1 & \\ & V_l\end{pmatrix};\quad C \text{ is defined by (3.11), and } V_l \text{ satisfies } {}^t\bar{V}_lV_l=-1_2.$$

Therefore, the right hand side of (3.10) becomes

$$\left(\int_{S_{2,\infty}}\omega_{2,\infty}\right)^{-1}\Big/\frac{d\dot{g}_\infty}{\omega_{1,\infty}}.$$

In order to calculate the above ratio, we use the result of Macdonald [19]. We define the $\boldsymbol{R}$-basis $F_1^i,\cdots,F_8^i$ of $\mathscr{S}_{i,\infty}$ by

$$F_j^i=\begin{cases}E_j^i & \cdots\text{for}\quad j=3,5,7;\ i=1,2,\\ \dfrac{\sqrt{-1}}{\iota}E_j^i & \cdots\text{otherwise,}\end{cases}$$

and write dF_j^i for the dual of F_j^i. If we denote by $\omega_{i,\infty}^*$ the measure defined by $dF_1^i\wedge\cdots\wedge dF_8^i$, we have $\omega_{i,\infty}=|\iota|^{-5}\omega_{i,\infty}^*$ $(i=1, 2)$. Using the result of [19], we see that

$$\int_{S_{2,\infty}}\omega_{2,\infty}^*=2\pi^5.$$

As for $S_{1,\infty}$, the Iwasawa decomposition of $\mathscr{S}_{1,\infty}$ shows that

$$\frac{d\dot{g}_\infty}{\omega_{1,\infty}^*}=\left(\int_{U_\infty}|dF_1^1\wedge\cdots\wedge dF_4^1|\right)^{-1},$$

where U_∞ is the maximal compact subgroup of $S_{1,\infty}$ defined by (1.6) and the integrand is the measure on U_∞ defined by $dF_1^1\wedge\cdots\wedge dF_4^1$. Then we have, using the result of [19] again,

$$\frac{d\dot{g}_\infty}{\omega_{1,\infty}^*}=2^{-2}\pi^{-3}.$$

From these results, we see that the right hand side of (3.10) equals $2\pi^{-2}$. This implies (3.9). Q.E.D.

3.3.

In this subsection, we compare a part of (3.3) and a part of (3.6). In (3.3) and (3.6), we have

$$\text{(3.12)}\qquad \int_{Z_\infty G_i(\gamma)_Q\backslash G_{i,A}}\varphi_i(g^{-1}\gamma g)d\dot{g}$$
$$=\mathrm{vol}(Z_\infty G_i(\gamma)_Q\backslash G_i(\gamma)_A)\int_{G_i(\gamma)_A\backslash G_{i,A}}\varphi_i(g^{-1}\gamma g)\frac{dg}{dt}\quad (i=1, 2),$$

where dt is a Haar measure of the unimodular group $G_i(\gamma)_A$ and vol( ) is the volume with respect to dt. Recall that when $i=1$, $[\gamma]$ is contained in E_1(sing.), E_1(red.), E_1(irred.), Nh(sing.) or Nh(red.), and when $i=2$, $[\gamma]$ is contained in E_2(sing.), E_2(red.) or E_2(irred.).

When $[\gamma]$ is contained in $E_i(\text{red.})\cup E_i(\text{irred.})$, we normalize the Haar measure $dt_{\infty,i}$ of $G_i(\gamma)_\infty$ as follows: There are isomorphisms induced from inner automorphisms of $G_{i,\infty}$;

$$G_1(\gamma)\xrightarrow{\sim}CDC^{-1}\qquad\text{and}\qquad G_2(\gamma)\xrightarrow{\sim}D,$$

where C is the matrix defined by (3.11) and D is the group defined by

$$D=\left\{t\begin{pmatrix} e[\theta_1] & & \\ & e[\theta_2] & \\ & & e[\theta_3]\end{pmatrix}\,\middle|\, t\in \boldsymbol{R}^\times,\ \theta_1,\ \theta_2,\ \theta_3\in\boldsymbol{R}\right\}.$$

The pull-backs of $dt/t\wedge d\theta_1\wedge d\theta_2\wedge d\theta_3$ by these isomorphisms determine a unique measure on $G_i(\gamma)_\infty$, and we denote this measure by $dt_{\infty,i}$.

LEMMA 3.3. *Assume that* $[\gamma]$ *is contained in* $E_i(red.)\cup E_i(irred.)$.

(i) *When* $i=1$, *let* $\lambda_1, \lambda_2, \lambda_3\in \boldsymbol{C}$ *be the eigenvalues of* γ *such that* $\operatorname{sgn}(H_1(v_{\lambda_1}))=+1$, *where* v_{λ_1} *is a* λ_1*-eigenvector. Put* $\mu_k=|\lambda_k|/\lambda_k$ $(k=1, 2, 3)$. *Then we have*

$$\int_{G_1(\gamma)_\infty\backslash G_{1,\infty}}\varphi_{1,\infty}(g_\infty^{-1}\gamma g_\infty)\frac{dg_\infty}{dt_{\infty,1}}=\frac{\mu_1^{r-1}}{(\mu_1-\mu_2)(\mu_1-\mu_3)}\cdot\mu_1\mu_2\mu_3. \tag{3.13}$$

(ii) *When* $i=2$, *let* $\lambda_1, \lambda_2, \lambda_3\in\boldsymbol{C}$ *be the eigenvalues of* γ, *and put* $\mu_k=|\lambda_k|/\lambda_k$ $(k=1, 2, 3)$. *Then we have*

$$\int_{G_2(\gamma)_\infty\backslash G_{2,\infty}}\varphi_{2,\infty}(g_\infty^{-1}\gamma g_\infty)\frac{dg_\infty}{dt_{\infty,2}} \tag{3.14}$$
$$=\left\{\frac{\mu_1^{r-1}}{(\mu_1-\mu_2)(\mu_1-\mu_3)}+\frac{\mu_2^{r-1}}{(\mu_2-\mu_3)(\mu_2-\mu_1)}+\frac{\mu_3^{r-1}}{(\mu_3-\mu_1)(\mu_3-\mu_2)}\right\}\cdot\mu_1\mu_2\mu_3.$$

A formula essentially equivalent to (3.13) is proved in Cohn [3] (p. 72, Lemma 18). Our formula (3.13) can be proved by the same way. The second formula (3.14) is verified by the explicit formula for $\varphi_{2,\infty}$ directly.

PROPOSITION 3.4. *Assume that the pair* $(U_{1,f}, U_{2,f})$ *satisfies assumption* (U_f) *and the pair* $(\varphi_{1,f}, \varphi_{2,f})$ *satisfies assumption* (φ_f). *Let* Ψ *be the map defined in Proposition* 2.8. *For* $[\gamma_0]$ *contained in* $E_2(irred.)$, *put* $\Psi^{-1}([\gamma_0])=\{[\gamma_1], [\gamma_2], [\gamma_3]\}$. *Then we have*

$$\sum_{k=1}^{3}\int_{Z_\infty G_1(\gamma_k)_{\boldsymbol{Q}}\backslash G_{1,A}}\varphi_1(g^{-1}\gamma_k g)d\dot{g}=\int_{Z_\infty G_2(\gamma_0)_{\boldsymbol{Q}}\backslash G_{2,A}}\varphi_2(g^{-1}\gamma_0 g)d\dot{g}.$$

PROOF. For $k=1, 2, 3$, the map

$$c_0+c_1\gamma_k+c_2\gamma_k^2\longmapsto c_0+c_1\gamma_0+c_2\gamma_0^2 \quad (c_0,\ c_1,\ c_2\text{: scalar})$$

gives an isomorphism ψ_k defined over $\boldsymbol{Q}$ from $F[\gamma_k]$ to $F[\gamma_0]$. Let l be a finite place of $\boldsymbol{Q}$. By Proposition 2.8, there exists, for each k, an inner automorphism η of $G_{2,l}$ such that $\eta\circ\Theta_l(\gamma_k)=\gamma_0$. Extending $\eta\circ\Theta_l$ to an inner automorphism of $M_3(F_l)$, we have $\psi_k=\eta\circ\Theta_l$ on $F[\gamma_k]$; it follows in particular that ψ_k maps $G_1(\gamma_k)$ onto $G_2(\gamma_0)$. Therefore, if we normalize the Haar measure $dt_{l,1}$ of $G_1(\gamma_k)_l$ (resp. the Haar measure $dt_{l,2}$ of $G_2(\gamma_0)_l$) by

$$\int_{G_1(\gamma_k)_l\cap U_{1,l}} dt_{l,1}=1 \quad \left(\text{resp. } \int_{G_2(\gamma_0)_l\cap\eta(U_{2,l})} dt_{l,2}=1\right),$$

we have

$$\int_{G_1(\gamma_k)_l\backslash G_{1,l}} \varphi_{1,l}(g_l^{-1}\gamma_k g_l)\frac{dg_l}{dt_{l,1}}=\int_{G_2(\gamma_0)_l\backslash G_{2,l}} \varphi_{2,l}(g_l^{-1}\gamma_0 g_l)\frac{dg_l}{dt_{l,2}}.$$

Then Lemma 3.3, together with the above formula, implies

$$\sum_{k=1}^{3}\int_{G_1(\gamma_k)_A\backslash G_{1,A}} \varphi_1(g^{-1}\gamma_k g)\frac{dg}{dt_1}=\int_{G_2(\gamma_0)_A\backslash G_{2,A}} \varphi_2(g^{-1}\gamma_0 g)\frac{dg}{dt_2},$$

where we put $dt_i=dt_{\infty,i}\times\prod_{l\neq\infty} dt_{l,i}$ $(i=1, 2)$. As for the comparison of $\mathrm{vol}(Z_\infty G_1(\gamma_k)_Q\backslash G_1(\gamma_k)_A)$ and $\mathrm{vol}(Z_\infty G_2(\gamma_0)_Q\backslash G_2(\gamma_0)_A)$, note that $\psi_k^*(dt_{v,2})=dt_{v,1}$ holds not only for a finite place v but also $v=\infty$. Hence we see that these two volumes are equal. Therefore, by (3.12), we have the statement. Q.E.D.

COROLLARY. *Under the same assumptions, we have*

$$\sum_{[\gamma]\subset E_1(irred.)}\int_{Z_\infty G_1(\gamma)_Q\backslash G_{1,A}} \varphi_1(g^{-1}\gamma g)d\dot{g}=\sum_{[\gamma]\subset E_2(irred.)}\int_{Z_\infty G_2(\gamma)_Q\backslash G_{2,A}} \varphi_2(g^{-1}\gamma g)d\dot{g}.$$

3.4.

In this subsection, we describe the analytic aspects of the contributions of conjugacy classes contained in E_1(sing.), E_2(sing.) or Nh(reg.). (Since the orbital integral at the infinite place has a same form for an element of Nh(sing.) and that of Sh(sing.), we describe the contributions from Nh(sing.) in § 4.)

LEMMA 3.5. *Assume that $[\gamma]$ is contained in E_1(sing.). Let λ_1 (resp. λ_2) be the eigenvalue of γ with multiplicity one (resp. with multiplicity two) and dt_∞ a Haar measure of $G_1(\gamma)_\infty$. Then we have*

$$\int_{G_1(\gamma)_\infty\backslash G_{1,\infty}} \varphi_{1,\infty}(g_\infty^{-1}\gamma g_\infty)\frac{dg_\infty}{dt_\infty}=\text{const.}\times\frac{\mu_1^{r-1}}{(\mu_1-\mu_2)^2}\cdot\mu_1\mu_2^2, \tag{3.15}$$

where $\mu_k=|\lambda_k|/\lambda_k$ $(k=1, 2)$, and "const." *is a positive number which depends only on dt_∞.*

Again, the above formula is a reformulation of a formula in [3] (p. 72, Lemma 18).

LEMMA 3.6. *Assume that* $[\gamma]$ *is contained in* $E_2(sing.)$. *Let* λ_1 *(resp.* λ_2*) be the eigenvalue of* γ *with multiplicity one (resp. with multiplicity two) and* dt_∞ *a Haar measure of* $G_2(\gamma)_\infty$. *Then we have*

$$(3.16)\qquad \int_{G_2(\gamma)_\infty\backslash G_{2,\infty}} \varphi_{2,\infty}(g_\infty^{-1}\gamma g_\infty)\frac{dg_\infty}{dt_\infty}$$
$$=\text{const.}\times\frac{\mu_1\mu_2^2}{(\mu_1-\mu_2)^2}\{(r-2)\mu_2^{r-1}-(r-1)\mu_1\mu_2^{r-2}+\mu_1^{r-1}\},$$

where $\mu_k=|\lambda_k|/\lambda_k$ $(k=1,2)$, *and* "const." *is a positive number which depends only on* dt_∞.

The above formula is verified by the explicit formula for $\varphi_{2,\infty}$.

LEMMA 3.7. (i) *Let* $\mu=z\lambda(\alpha)$ $(z,\alpha\in\boldsymbol{C}^\times,\ |\alpha|\neq 1)$ *be an element of* $M_\infty-M_\infty^*$. *Then we have*

$$(3.17)\qquad \varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)$$
$$=C_r\left(\frac{|z|}{z}\right)^r\left\{\frac{1}{2}(\alpha+\bar{\alpha}^{-1})+\frac{1}{2}(\alpha+\bar{\alpha}^{-1}-2)|a|^2+(\alpha-\bar{\alpha}^{-1})\iota b\right\}^{-r}$$

for $n_\infty=n(a,b)$, *and*

$$(3.18)\qquad \int_{G_{1,\infty}(\mu)\backslash G_{1,\infty}}\varphi_{1,\infty}(g_\infty^{-1}\mu g_\infty)\frac{dg_\infty}{dt_\infty}=\int_{N_\infty}\varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)dn_\infty=0$$

where dt_∞ *is a Haar measure of* $G_{1,\infty}(\mu)=M_\infty$.

(ii) *Assume that* $[\gamma]$ *is contained in* $Nh(reg.)$. *Then we have*

$$\int_{Z_\infty G_1(\gamma)_Q\backslash G_{1,A}}\varphi_1(g^{-1}\gamma g)d\dot{g}=0.$$

PROOF. The first formula (3.17) is verified from (1.9) by a straightforward calculation. Then, by the Iwasawa decomposition of $G_{1,\infty}$, we have

$$\int_{G_{1,\infty}(\mu)\backslash G_{1,\infty}}\varphi_{1,\infty}(g_\infty^{-1}\mu g_\infty)\frac{dg_\infty}{dt_\infty}$$
$$=c\int_{N_\infty}\varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)dn_\infty$$
$$=c'\int_{\boldsymbol{C}\times\boldsymbol{R}}\left\{\frac{1}{2}(\alpha+\bar{\alpha}^{-1})+\frac{1}{2}(\alpha+\bar{\alpha}^{-1}-2)|a|^2+(\alpha-\bar{\alpha}^{-1})\iota b\right\}^{-r}dadb,$$

where c, c' are constants. In the last expression, the integrand has a form $c_1(b-c_2)^{-r}$ $(c_1 \in \boldsymbol{C},\ c_2 \in \boldsymbol{C}-\boldsymbol{R})$ as a function of b. Hence we have (i). Statement (ii) is obvious from (i). Q.E.D.

§ 4. Contributions of the other conjugacy classes of $G_{1,Q}$

4.1.

In this section, we shall study the integral of the third term of (3.1). Recall that the third term of (3.1) is written as follows:

$$\sum_{\gamma} \varphi_1(g^{-1}\gamma g);\quad \varphi_1=\varphi_{1,\infty}\otimes\varphi_{1,f},\ \gamma \text{ runs through split hyperbolic or non semi-simple elements of } G_{1,Q}.$$

By Proposition 2.1, the above expression is written as

$$(4.1)\qquad \sum_{\gamma\in Sh(sing.)} \varphi_1(g^{-1}\gamma g)+\sum_{\gamma\in Sh(reg.)} \varphi_1(g^{-1}\gamma g)+\sum_{\gamma\in Ns(1)} \varphi_1(g^{-1}\gamma g) + \sum_{\gamma\in Ns(2)} \varphi_1(g^{-1}\gamma g)+\sum_{\gamma\in Ns(3)} \varphi_1(g^{-1}\gamma g).$$

4.2.

In this subsection, we consider the integral of the first term of (4.1). By Proposition 2.1.(ii), we have

$$(4.2)\qquad \sum_{\gamma\in Sh(sing.)} \varphi_1(g^{-1}\gamma g)=\sum_{\mu\in M_Q^*-Z_Q}\ \sum_{\delta\in J_Q\backslash G_{1,Q}} \varphi_1(g^{-1}\delta^{-1}\mu\delta g).$$

LEMMA 4.1. *Let* $\mu=z\lambda(\alpha_0)$ $(z,\ \alpha_0\in F^\times,\ N_{F/Q}(\alpha_0)=1,\ \alpha_0\neq 1)$ *be an element of* $M_Q^*-Z_Q$.

(i) *The series* $\sum_{\delta\in J_Q\backslash G_{1,Q}} \varphi_1(g^{-1}\delta^{-1}\mu\delta g)$ *is integrable on* $Z_\infty G_{1,Q}\backslash G_{1,A}$ *and its integral* $I(\mu)$ *is expressed by*

$$I(\mu)=\mathrm{vol}(Z_\infty J_Q\backslash J_A)\int_{J_A\backslash G_{1,A}} \varphi_1(g^{-1}\mu g)\frac{dg}{dt},$$

where dt *is a Haar measure on* J_A *and* vol( ) *is the volume with respect to* dt.

(ii) *If a Haar measure* dt_∞ *on* J_∞ *is fixed, we have*

$$(4.3)\qquad \int_{J_\infty\backslash G_{1,\infty}} \varphi_{1,\infty}(g_\infty^{-1}\mu g_\infty)\frac{dg_\infty}{dt_\infty} =\text{const.}\times\frac{\mu_1\mu_2^2}{(\mu_1-\mu_2)^2}\{(r-1)\mu_1\mu_2^{r-2}-(r-2)\mu_2^{r-1}\},$$

where $\mu_1=|z|/z$, $\mu_2=|z|/\alpha_0 z$ *and* "const." *is a positive number which depends only on* dt_∞.

(iii) *The integral of* (4.2) *on* $Z_\infty G_{1,Q}\backslash G_{1,A}$ *equals* $\sum_{\mu\in M_Q^*-Z_Q} I(\mu)$, *which turns out to be a finite sum for* μ.

PROOF. If we put

$$g(x)=\begin{pmatrix}\sqrt{1-x^2} & & \\ & 1 & \\ & & \dfrac{1}{\sqrt{1-x^2}}\end{pmatrix}\begin{pmatrix}1 & \dfrac{1}{\sqrt{1-x^2}} & \dfrac{x^2}{2(1-x^2)} \\ & 1 & \dfrac{x}{\sqrt{1-x^2}} \\ & & 1\end{pmatrix} \quad (0\leqq x<1),$$

the set $J_\infty\backslash G_{1,\infty}/U_\infty$ can be identified with $\{g(x)\,|\,0\leqq x<1\}$. Then a straightforward calculation shows that, for $\mu=z\lambda(\alpha_0)$,

$$\int_{J_\infty\backslash G_{1,\infty}} |\varphi_{1,\infty}(g_\infty^{-1}\mu g_\infty)|\frac{dg_\infty}{dt_\infty}=c\int_0^1 |\alpha_0-t|^{-r}(1-t)^{r-3}dt$$

and

$$\int_{J_\infty\backslash G_{1,\infty}} \varphi_{1,\infty}(g_\infty^{-1}\mu g_\infty)\frac{dg_\infty}{dt_\infty}=c\left(\frac{|z|}{z}\right)^r\int_0^1 (\alpha_0-t)^{-r}(1-t)^{r-3}dt,$$

where c is a positive number which depends only on dt_∞. By these formulas, we have (i) and (ii). Statement (iii) can be easily verified by using the Iwasawa decomposition of $G_{1,f}$ with respect to $U_{1,f}$ (see (4.9) below). Q.E.D.

In § 3, we did not describe the orbital integrals at the infinite place for elements of Nh(sing.). But it is clear that they have the same form as the orbital integrals at the infinite place for elements of Sh(sing.). Therefore, we have the following

LEMMA 4.2. *Assume that* $[\gamma]$ *is contained in* Nh*(sing.). Let* λ_1 *(resp.* λ_2*) be the eigenvalue of* γ *with multiplicity one (resp. with multiplicity two) and* dt_∞ *a Haar measure on* $G_1(\gamma)_\infty$. *Then we have*

$$(4.4)\qquad \int_{G_1(\gamma)_\infty\backslash G_{1,\infty}} \varphi_{1,\infty}(g_\infty^{-1}\gamma g_\infty)\frac{dg_\infty}{dt_\infty}$$
$$=\text{const.}\times\frac{\mu_1\mu_2^2}{(\mu_1-\mu_2)^2}\{(r-1)\mu_1\mu_2^{r-2}-(r-2)\mu_2^{r-1}\},$$

where $\mu_k = |\lambda_k|/\lambda_k$ $(k=1, 2)$, *and* "const." *is a positive number which depends only on* dt_∞.

4.3.

In this subsection, we consider the integral of the second term of (4.1). Our method is that of Jacquet-Langlands [12], pp. 528–538, and J. Arthur [1]. By Proposition 2.1.(ii), we have

$$\sum_{\gamma \in Sh(reg.)} \varphi_1(g^{-1}\gamma g) = \sum_{\mu \in M_Q - M_Q^*} \sum_{\delta \in W M_Q \backslash G_{1,Q}} \varphi_1(g^{-1}\delta^{-1}\mu\delta g),$$

therefore,

$$\sum_{\gamma \in Sh(reg.)} \varphi_1(g^{-1}\gamma g) = \frac{1}{2} \sum_{\mu \in M_Q - M_Q^*} \sum_{\delta \in M_Q \backslash G_{1,Q}} \varphi_1(g^{-1}\delta^{-1}\mu\delta g). \tag{4.5}$$

Let us introduce some notation. We define the module δ of P_A by

$$\delta(p) = \frac{d(p^{-1}np)}{dn} \quad (dn\text{: a Haar measure of } N_A). \tag{4.6}$$

For $p = z\lambda(\alpha)$ $(z \in Z_A,\ \alpha \in F_A^\times,\ n \in N_A)$, we have

$$\delta(p) = |N_{F_A/Q_A}(\alpha)|_A^{-2}. \tag{4.7}$$

If we put

$$U_{1,A} = U_\infty \times U_{1,f}, \tag{4.8}$$

the Iwasawa decomposition of $G_{1,A}$ with respect to $U_{1,A}$ can be written in the following form:

$$G_{1,A} = \coprod_{i=1}^{\kappa} P_A \xi_i U_{1,A} \quad (\xi_i \in G_{1,f}). \tag{4.9}$$

For $g = p\xi_i u$ $(p \in P_A,\ 1 \leq i \leq \kappa,\ u \in U_{1,A})$, we put

$$\delta(g) = \delta(p). \tag{4.10}$$

Furthermore, we define the function Λ on N_A by

$$\Lambda(n) = \delta(\tilde{w}n)^{-1/2} \quad (n \in N_A).$$

(For the definition of $\tilde{w}$, see (2.1).)

Lemma 4.3. *Let* $\mu = z\lambda(\alpha_0)$ $(z,\ \alpha_0 \in F^\times,\ N_{F/Q}(\alpha_0) \neq 1)$ *be an element of* $M_Q - M_Q^*$.

(i) *The series* $\sum_{\delta\in M_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}\mu\delta g)$ *is integrable on* $Z_\infty G_{1,Q}\backslash G_{1,A}$ *and its integral* $I'(\mu)$ *is expressed by*

$$I'(\mu)=-\text{const.}\times\int_{N_\infty}\varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)\log\Lambda(n_\infty)dn_\infty\int_{M_f\backslash G_{1,f}}\varphi_{1,f}(g_f^{-1}\mu g_f)\frac{dg_f}{dm_f},$$

where dn_∞ *(resp.* dg_f*) is the Haar measure defined by* (1.7) *(resp. by* (1.3)*),* dm_f *is a Haar measure of* M_f*, and* "const." *is a positive number which depends only on* dm_f.

(ii) *We have*

$$\int_{N_\infty}\varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)\log\Lambda(n_\infty)dn_\infty=2\left(\frac{|z|}{z}\right)^r\frac{\eta^r}{(\eta-1)(|\eta|^2-1)},\tag{4.11}$$

where

$$\eta=\begin{cases}\alpha_0^{-1}\cdots if & N(\alpha_0)>1,\\ \bar{\alpha}_0\ \cdots if & N(\alpha_0)<1.\end{cases}$$

(iii) *The integral of* (4.5) *on* $Z_\infty G_{1,Q}\backslash G_{1,A}$ *equals* $(1/2)\sum_{\mu\in M_Q-M_Q^*}I'(\mu)$, *which turns out to be a finite sum for* μ.

PROOF. For a positive number t, we define the subset $X(t)$ of $G_{1,A}$, using the Iwasawa decomposition (4.9), as follows:

$$X(t)=\{g\in G_{1,A}\mid\delta(g)^{-1/2}\geqq t\}.$$

Let χ_t be the characteristic function of $X(t)$. Following Jacquet-Langlands [12] and Arthur [1], we put

$$(\text{A})=\sum_{\delta\in M_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}\mu\delta g)(1-\chi_t(\delta g)-\chi_t(\tilde{w}\delta g))$$

and

$$(\text{B})=\sum_{\delta\in M_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}\mu\delta g)(\chi_t(\delta g)+\chi_t(\tilde{w}\delta g)).$$

Then we have

$$\sum_{\delta\in M_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}\mu\delta g)=(\text{A})+(\text{B}).$$

First, we investigate the integral of (A). If we put

$$\delta g=mn\xi_i u\quad(m\in M_A,\ n\in N_A,\ 1\leqq i\leqq\kappa,\ u\in U_{1,A}),\tag{4.12}$$

we have

$$(\mathrm{A})=\sum_{\delta\in M_Q\backslash G_{1,Q}}\varphi_1(\xi_i^{-1}n^{-1}\mu n\xi_i)(1-\chi_t(m)-\chi_t(\tilde{w}mn)).$$

By definition, $\chi_t(\tilde{w}mn)$ can be written as follows:

$$\chi_t(\tilde{w}mn)=\begin{cases}1\cdots\text{if} & \delta(m)^{-1/2}\leqq\dfrac{\Lambda(n)}{t}\\ 0\cdots\text{if} & \delta(m)^{-1/2}>\dfrac{\Lambda(n)}{t}.\end{cases}$$

Using the fact that $\Lambda(n)$ is bounded on N_A (see Arthur [1], Lemma 1.3), we see that, for sufficiently large t,

$$1-\chi_t(m)-\chi_t(\tilde{w}mn)=\begin{cases}1\cdots\text{if} & \dfrac{\Lambda(n)}{t}\leqq\delta(m)^{-1/2}\leqq t\\ 0\cdots\text{otherwise.}\end{cases}$$

Hereafter, we assume that t is large enough so that the above formula is valid. Then, the integral of

$$\sum_{\delta\in M_Q\backslash G_{1,Q}}|\varphi_1(g^{-1}\delta^{-1}\mu\delta g)(1-\chi_t(\delta g)-\chi_t(\tilde{w}\delta g))|$$

on $Z_\infty G_{1,Q}\backslash G_{1,A}$ equals to

$$c\sum_{i=1}^{\kappa}\int_{N_A}|\varphi_1(\xi_i^{-1}n^{-1}\mu n\xi_i)|(2\log t-\log\Lambda(n))dn,$$

where $dn=dn_\infty\times dn_f$, dn_f a Haar measure on N_f, and c is a positive number which depends only on dn_f and dm_f. Here, note that $\varphi_{1,f}(n_f^{-1}\mu n_f)$ has compact support in N_f (see Lemma 2.2). Therefore we can conclude that the above integral converges provided that

$$\int_{N_\infty}|\varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)|dn_\infty<\infty \tag{4.13}$$

and

$$\int_{N_\infty}|\varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)|\log\Lambda(n_\infty)dn_\infty<\infty \tag{4.14}$$

are verified. According to (3.17), (4.13) is clear. On the other hand, a straightforward calculation shows that

$$\Lambda(n_\infty)=\left|\frac{1}{2}+\frac{1}{2}|a|^2+\iota b\right|^{-2}\quad\text{for}\quad n_\infty=n(a,\,b). \tag{4.15}$$

Then, (4.14) is easily verified by (3.17) and the above formula. Hence we have seen that (A) is integrable on $Z_\infty G_{1,Q}\backslash G_{1,A}$ and its integral is expressed by

$$c\sum_{i=1}^{\kappa}\int_{N_A}\varphi_1(\xi_i^{-1}n^{-1}\mu n\xi_i)(2\log t-\log \Lambda(n))dn.$$

Now, Lemma 3.7.(i), together with the fact that $\log \Lambda(n)=\log \Lambda(n_\infty)+\log \Lambda(n_f)$, implies that the above expression equals

$$-c\sum_{i=1}^{\kappa}\int_{N_\infty}\varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)\log \Lambda(n_\infty)dn_\infty\int_{N_f}\varphi_{1,f}(\xi_i^{-1}n_f^{-1}\mu n_f\xi_i)dn_f,$$

which we rewrite as

$$-c'\int_{N_\infty}\varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)\log \Lambda(n_\infty)dn_\infty\int_{M_f\backslash G_{1,f}}\varphi_{1,f}(g_f^{-1}\mu g_f)\frac{dg_f}{dm_f},$$

where c' is a positive number which depends only on dm_f.

Next we shall prove that (B) is integrable on $Z_\infty G_{1,Q}\backslash G_{1,A}$ and that its integral approaches 0 as t approaches $+\infty$. Put

$$(\mathrm{C}\colon \mu)=\sum_{\delta\in M_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}\mu\delta g)\chi_t(\delta g).$$

By Lemma 2.2, we have

$$(\mathrm{C}\colon \mu)=\sum_{\delta\in M_QZ(N)_Q\backslash G_{1,Q}}\sum_{\nu\in Z(N)_Q}\varphi_1(g^{-1}\delta^{-1}\mu\nu\delta g)\chi_t(\delta g).$$

Let us consider the integral of

$$\sum_{\delta\in M_QZ(N)_Q\backslash G_{1,Q}}\left|\sum_{\nu\in Z(N)_Q}\varphi_1(g^{-1}\delta^{-1}\mu\nu\delta g)\right|\chi_t(\delta g).$$

The integral of the above expression on $Z_\infty G_{1,Q}\backslash G_{1,A}$ equals the integral of

$$\left|\sum_{\nu\in Z(N)_Q}\varphi_1(g^{-1}\mu\nu g)\right|\chi_t(g)$$

on $Z_\infty M_QZ(N)_Q\backslash G_{1,A}$. Using the Iwasawa decomposition (4.9), this integral can be written, up to a constant multiple, as the sum of

$$\int_{(Z(N)_Q\backslash N_A)\times(A_Q\backslash A_A)}\left|\sum_{\nu\in Z(N)_Q}\varphi_1(\xi_i^{-1}\lambda^{-1}n^{-1}\mu\nu n\lambda\xi_i)\right|\chi_t(\lambda)\delta(\lambda)dnd\lambda \tag{4.16}$$

for $1\leqq i\leqq\kappa$, where dn (resp. $d\lambda$) is a Haar measure of N_A (resp. A_A). In order to majorize the above integral, we take the fundamental domain for $A_Q\backslash A_A$ in the following form:

$$(4.17)\qquad A_Q\backslash A_A=\coprod_{j=1}^{h(i)}\{\lambda(\alpha)\,|\,\alpha\in C^\times/\Gamma_{ij}\}\cdot\lambda_{ij}\cdot(A_f\cap\xi_i U_{1,f}\xi_i^{-1})$$
$$(\lambda_{ij}\in A_f,\ \Gamma_{ij}\text{: a finite subgroup of } C^\times).$$

Then (4.16) equals, up to a constant multiple, the sum of

$$(4.18)\quad \int_{C^\times}\int_{Z(N)_Q\backslash N_A}\Big|\sum_{\nu\in Z(N)_Q}\varphi_1(\xi_i^{-1}\lambda_{ij}^{-1}\lambda(\alpha)^{-1}n^{-1}\mu\nu n\lambda(\alpha)\lambda_{ij}\xi_i)\Big|\chi_t(\lambda_{ij}\lambda(\alpha))dn|\alpha|^{-6}d\alpha$$

for $1\leqq j\leqq h(i)$, where $d\alpha$ is the Euclidean measure of C. Now, we take the fundamental domain for $Z(N)_Q\backslash N_A$ in the following form:

$$Z(N)_Q\backslash N_A=\{n(a,0)\,|\,a\in F_A\}\cdot\{n(0,b)\,|\,b\in R/L_{ij}\}\cdot(Z(N)_f\cap\lambda_{ij}\xi_i U_{1,f}\xi_i^{-1}\lambda_{ij}^{-1})$$
$$(L_{ij}\text{: a lattice in } R).$$

In order to majorize (4.18), we may assume that $\varphi_{1,f}$ is the characteristic function of a right $U_{1,f}$-coset in $G_{1,f}$ since $\varphi_{1,f}$ is the linear combination of a finite number of such functions in general. We assume this. Applying Lemma 2.2 to $n(a,0)^{-1}\mu n(0,\beta)n(a,0)$, we see that $\varphi_{1,f}(\xi_i^{-1}\lambda_{ij}^{-1}\times n(a,0)^{-1}\mu n(0,\beta)n(a,0)\lambda_{ij}\xi_i)$ has compact support with respect to $a\in F_f$ and $\beta\in Q_f$. From this fact and the above assumption, (4.18) is written as a constant multiple of

$$\int_{C^\times\times C\times(R/L_{ij})}\Big|\sum_{k\in Z}\varphi_{1,\infty}(\lambda(\alpha)^{-1}n(a,b)^{-1}\mu n(0,pk+q)n(a,b)\lambda(\alpha))\Big|$$
$$\times\chi_t(\lambda_{ij}\lambda(\alpha))|\alpha|^{-6}d\alpha da db,$$

where p is a positive number and q is a real number. Here, a straightforward calculation shows that the integrand of the above integral equals, up to a constant multiple, the absolute value of

$$\sum_{k\in Z}N(\alpha)^{r-3}f(\alpha,a,b)^{-r}\chi_t(\lambda_{ij}\lambda(\alpha))$$

where

$$f(\alpha,a,b)=\frac{1+N(\alpha_0)^{-1}}{2}N(\alpha)+\frac{1-2\alpha_0^{-1}+N(\alpha_0)^{-1}}{2}N(a)+\iota(1-N(\alpha_0)^{-1})b$$
$$+\iota(pk+q).\quad (N(*)=|*|^2.)$$

By the well-known formula

$$(4.19)\qquad \sum_{k\in Z}(z+k)^{-r}=\frac{1}{(r-1)!}(-2\pi i)^{r}\sum_{n=1}^{\infty}n^{r-1}e[nz]\quad (\mathrm{Im}(z)>0),$$

we see that the integrand of the above integral is majorized by a function of the form

$$c_1N(\alpha)^{r-3}\sum_{n=1}^{\infty}n^{r-1}e[in(c_2N(\alpha)+c_3N(a))]$$

where c_1, c_2, c_3 are positive constants. Then it is easy to verify that the above integral is finite and it approaches 0 as t approaches $+\infty$. Hence we have proved that (C: μ) is integrable on $Z_\infty G_{1,Q}\backslash G_{1,A}$ and its integral approaches 0 as t approaches $+\infty$: We have the same result for (C: $\tilde{w}\mu\tilde{w}^{-1}$) and for (B)=(C: μ)+(C: $\tilde{w}\mu\tilde{w}^{-1}$).

By the above arguments on (A) and (B), we have (i).

Next we shall prove (ii). By (3.17) and (4.15) we have

$$\begin{aligned}&\varphi_{1,\infty}(n_\infty^{-1}\mu n_\infty)\log\Lambda(n_\infty)\\&\quad=C_r\left(\frac{|z|}{z}\right)^r\left\{\frac{1}{2}(\alpha_0+\bar{\alpha}_0^{-1})+\frac{1}{2}(\alpha_0+\bar{\alpha}_0^{-1}-2)|a|^2+(\alpha_0-\bar{\alpha}_0^{-1})\iota b\right\}^{-r}\\&\quad\quad\times\log\left|\frac{1}{2}+\frac{1}{2}|a|^2+\iota b\right|^{-2}\end{aligned}$$

for $n_\infty=n(a, b)$. Then we see that the integral of $\varphi_{1,\infty}(n(a, b)^{-1}\mu n(a, b))\times\log\Lambda(n(a, b))$ with respect to $b\in\boldsymbol{R}$ by the measure $|\iota|db$ (see (1.7)) can be written as

$$d_1\int_{-\infty}^{+\infty}(b-d_2)^{-r}\log(b^2+d_3)db$$

where

$$\begin{aligned}d_1&=-C_r\left(\frac{|z|}{z}\right)^r(\alpha_0-\bar{\alpha}_0^{-1})^{-r}i^{-r},\\d_2&=-\frac{1}{2i}\cdot\frac{1}{\alpha_0-\bar{\alpha}_0^{-1}}\{\alpha_0+\bar{\alpha}_0^{-1}+(\alpha_0+\bar{\alpha}_0^{-1}-2)|a|^2\},\\d_3&=\frac{(1+|a|^2)^2}{4}.\end{aligned}$$

By a straightforward calculation, the above integral becomes

$$-\frac{2\pi i d_1}{r-1}\left(-\frac{1+|a|^2}{2}i-d_2\right)^{-r+1}\cdots\text{if}\quad N(\alpha_0)>1,$$

$$\frac{2\pi i d_1}{r-1}\left(\frac{1+|a|^2}{2}i-d_2\right)^{-r+1}\cdots\text{if}\quad N(\alpha_0)<1.$$

Integrating the above expression with respect to $a\in\boldsymbol{C}$, we have (ii).

Statement (iii) can be easily verified by using the Iwasawa decomposition (4.9). Q.E.D.

4.4.

In this subsection, we consider the integrals of the third, the fourth, and the fifth terms of (4.1). By Proposition 2.1.(ii), we have

$$(4.20)\qquad \sum_{\gamma\in Ns(1)}\varphi_1(g^{-1}\gamma g)=\sum_{z\in Z_Q}\sum_{\nu\in Z(N)_Q-\{1\}}\sum_{\delta\in P_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}z\nu\delta g),$$

$$(4.21)\qquad \sum_{\gamma\in Ns(2)}\varphi_1(g^{-1}\gamma g)=\sum_{z\in Z_Q}\sum_{\nu\in N_Q-Z(N)_Q}\sum_{\delta\in P_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}z\nu\delta g),$$

and

$$(4.22)\qquad \sum_{\gamma\in Ns(3)}\varphi_1(g^{-1}\gamma g)=\sum_{\mu\in M_Q^*-Z_Q}\sum_{\nu\in Z(N)_Q-\{1\}}\sum_{\delta\in M_QZ(N)_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}\mu\nu\delta g).$$

Let us introduce notation. We denote by

$$Z(N)_Q-\{1\}/\sim$$

a set of representatives in $Z(N)_Q-\{1\}$ with respect to the M_Q-conjugacy. Namely, we can set

$$Z(N)_Q-\{1\}/\sim=\{n(0,\beta)\,|\,\beta\in\boldsymbol{Q}^\times/N_{F/Q}(F^\times)\}.$$

LEMMA 4.4. *Let z be an element of Z_Q.*

(i) *The series $\sum_{\nu\in Z(N)_Q-\{1\}}\sum_{\delta\in P_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}z\nu\delta g)$ is integrable on $Z_\infty G_{1,Q}\backslash G_{1,A}$ and its integral $I''(z)$ is expressed by*

$$I''(z)=\mathrm{vol}(Z_\infty M_Q^*N_Q\backslash M_A^*N_A)\sum_{\nu\in Z(N)_Q-\{1\}/\sim}\int_{M_A^*N_A\backslash G_{1,A}}\varphi_1(g^{-1}z\nu g)\frac{dg}{dt},$$

where dt is a Haar measure of $M_A^*N_A$ and vol( ) *is the volume with respect to dt.*

(ii) *If a Haar measure dt_∞ on $M_\infty^*N_\infty$ is fixed, we have*

$$(4.23)\qquad \int_{M_\infty^*N_\infty\backslash G_{1,\infty}}\varphi_{1,\infty}(g_\infty^{-1}z\nu g_\infty)\frac{dg_\infty}{dt_\infty}=-\mathrm{const.}\times\left(\frac{|z|}{z}\right)^r|\beta|^{-2}$$

for $\nu=n(0,\beta)$,

where "const." *is a positive number which depends only on* dt_∞.

(iii) *The integral of* (4.20) *on* $Z_\infty G_{1,Q}\backslash G_{1,A}$ *equals* $\sum_{z\in Z_Q} I''(z)$, *which turns out to be a finite sum for* z.

PROOF. First, we shall prove that the integral of

$$\sum_{\nu\in Z(N)_Q-\{1\}}\ \sum_{\delta\in P_Q\backslash G_{1,Q}} |\varphi_1(g^{-1}\delta^{-1}z\nu\delta g)|$$

on $Z_\infty G_{1,Q}\backslash G_{1,A}$ is finite. Note that $G_1(z\nu)=M^*N$ for any $z\in Z_Q$ and $\nu\in Z(N)_Q-\{1\}$. According to the Iwasawa decomposition (4.9), this integral is written, up to a constant multiple, as the sum of

$$\sum_{\nu\in Z(N)_Q-\{1\}} \int_{A_Q\backslash A_A} |\varphi_1(\xi_i^{-1}\lambda^{-1}z\nu\lambda\xi_i)|\delta(\lambda)d\lambda$$

for $1\leqq i\leqq\kappa$, where $d\lambda$ is a Haar measure of A_A. For each i, we define the fundamental domain for $A_Q\backslash A_A$ by (4.17). Then the above integral equals, up to a constant multiple, the sum of

$$\sum_{\nu\in Z(N)_Q-\{1\}} \int_{C^\times} |\varphi_1(\xi_i^{-1}\lambda_{ij}^{-1}\lambda(\alpha)^{-1}z\nu\lambda(\alpha)\lambda_{ij}\xi_i)|\,|\alpha|^{-6}d\alpha$$

for $1\leqq j\leqq h(i)$, where $d\alpha$ is the Euclidean measure of $\boldsymbol{C}$. Thus we have only to prove that the expression

$$\sum_{k\in \boldsymbol{Z}-\{0\}} \int_{C^\times} |\varphi_{1,\infty}(\lambda(\alpha)^{-1}zn(0,\ pk+q)\lambda(\alpha))|\,|\alpha|^{-6}d\alpha \tag{4.24}$$

has finite value for $p>0$, $q\in(\boldsymbol{R}-p\boldsymbol{Z})\cup\{0\}$. By Lemma 2.2, we have

$$\begin{aligned}\int_{C^\times} |\varphi_{1,\infty}(\lambda(\alpha)^{-1}zn(0,\ \beta)\lambda(\alpha))|\,|\alpha|^{-6}d\alpha &= \int_{C^\times} |\varphi_{1,\infty}(zn(0,\ |\alpha|^{-2}\beta))|\,|\alpha|^{-6}d\alpha \\ &= \pi|\beta|^{-2}\int_0^\infty |\varphi_{1,\infty}(zn(0,\ \mathrm{sgn}(\beta)b))|bdb,\end{aligned}$$

and by (1.9), we have

$$\varphi_{1,\infty}(zn(0,\ b))=C_r\left(\frac{|z|}{z}\right)^r(\iota b+1)^{-r} \quad \text{for} \quad b\in\boldsymbol{R}. \tag{4.25}$$

Then it is easy to verify that (4.24) has finite value. From these arguments, we see that the series

$$\sum_{\nu\in Z(N)_Q-\{1\}}\ \sum_{\delta\in P_Q\backslash G_{1,Q}} \varphi_1(g^{-1}\delta^{-1}z\nu\delta g)$$

is integrable on $Z_\infty G_{1,Q}\backslash G_{1,A}$ and its integral $I''(z)$ is expressed by

$$I''(z)=\sum_{\nu\in Z(N)_Q-\{1\}}\sum_{\delta\in P_Q\backslash G_{1,Q}}\int_{Z_\infty G_{1,Q}\backslash G_{1,A}}\varphi_1(g^{-1}\delta^{-1}z\nu\delta g)dg$$
$$=\sum_{\nu\in Z(N)_Q-\{1\}/\sim}\sum_{\delta\in M_Q^*N_Q\backslash G_{1,Q}}\int_{Z_\infty G_{1,Q}\backslash G_{1,A}}\varphi_1(g^{-1}\delta^{-1}z\nu\delta g)dg$$
$$=\sum_{\nu\in Z(N)_Q-\{1\}/\sim}\mathrm{vol}(Z_\infty M_Q^*N_Q\backslash M_A^*N_A)\int_{M_A^*N_A\backslash G_{1,A}}\varphi_1(g^{-1}z\nu g)\frac{dg}{dt}.$$

Thus we have (i).

As is seen from the above, the left hand side of (4.23) can be written as

$$c|\beta|^{-2}\int_0^\infty\varphi_{1,\infty}(zn(0,\ \mathrm{sgn}(\beta)b))bdb=cC_r\Big(\frac{|z|}{z}\Big)^r|\beta|^{-2}\int_0^\infty(\mathrm{sgn}(\beta)\iota b+1)^{-r}bdb$$

where c is a positive number which depends only on dt_∞ (see (4.25)). Then we can verify (ii) by a straightforward calculation.

Statement (iii) is obvious from the Iwasawa decomposition (4.9). Q.E.D.

As for the integral of (4.21), we have the following lemma, to which we shall give an outline of the proof (cf. Cohn [3], p. 32, Lemma 8).

LEMMA 4.5. *The series* (4.21) *is integrable on* $Z_\infty G_{1,Q}\backslash G_{1,A}$ *and its integral equals zero.*

PROOF (outline). Using the Iwasawa decomposition (4.9), we see that the series

$$\sum_{\nu\in N_Q-Z(N)_Q}\sum_{\delta\in P_Q\backslash G_{1,Q}}\varphi_1(g^{-1}\delta^{-1}z\nu\delta g) \tag{4.26}$$

identically equals zero for all but a finite number of $z\in Z_Q$. Hence it is enough to show that the above series is integrable on $Z_\infty G_{1,Q}\backslash G_{1,A}$ and its integral equals zero for any $z\in Z_Q$. Actually, we can show that the series

$$\sum_{\delta\in P_Q\backslash G_{1,Q}}\sum_{a_0\in F^\times}\Big|\sum_{b_0\in Q}\varphi_1(g^{-1}\delta^{-1}zn(a_0,\ b_0)\delta g)\Big|$$

is integrable on $Z_\infty G_{1,Q}\backslash G_{1,A}$, but we omit the proof since it is similar to that of Lemma 4.3 (majorization of (B)). Then, using (4.9), we see that (4.26) is integrable on $Z_\infty G_{1,Q}\backslash G_{1,A}$ and its integral equals, up to a constant multiple, the sum of

$$\int_{(N_Q\backslash N_A)\times(A_Q\backslash A_A)}\sum_{b_0\in Q}\varphi_1(\xi_i^{-1}\lambda^{-1}n^{-1}zn(a_0,\ b_0)n\lambda\xi_i)\delta(\lambda)dnd\lambda$$

for $a_0 \in F^{\times}$, $1 \leqq i \leqq \kappa$, where dn (resp. $d\lambda$) is a Haar measure of N_A (resp. A_A). By Lemma 2.2, we have

$$\lambda^{-1} n^{-1} z n(a_0, b_0) n \lambda = z n(\bar{\alpha}^{-1} a_0,\ N(\alpha)^{-1}\{b_0 + \iota^{-1}(\bar{a}_0 a - a_0 \bar{a})\})$$

for $\lambda = \lambda(\alpha)$, $n = n(a, b)$. Then using change of variables;

$$\alpha \longmapsto \bar{a}_0 \alpha, \qquad a \longmapsto a_0 a,$$

we see that the above integral does not depend on a_0. Therefore it in turn equals zero because of the integrability of (4.26). Q.E.D.

As for the integral of (4.22), we have the following

LEMMA 4.6. *Let $\mu = z\lambda(\alpha_0)$ (z, $\alpha_0 \in F^{\times}$, $N_{F/Q}(\alpha_0) = 1$, $\alpha_0 \neq 1$) be an element of $M_Q^* - Z_Q$.*

(i) *The series $\sum_{\nu \in Z(N)_Q - \{1\}} \sum_{\delta \in M_Q Z(N)_Q \backslash G_{1,Q}} \varphi_1(g^{-1}\delta^{-1}\mu\nu\delta g)$ is integrable on $Z_\infty G_{1,Q} \backslash G_{1,A}$ and its integral $I'''(\mu)$ is expressed by*

$$I'''(\mu) = \mathrm{vol}(Z_\infty M_Q^* Z(N)_Q \backslash M_A^* Z(N)_A) \times \lim_{s \downarrow 0} \sum_{\nu \in Z(N)_Q - \{1\}/\sim} \int_{M_A^* Z(N)_A \backslash G_{1,A}} \varphi_1(g^{-1}\mu\nu g)\delta(g)^s \frac{dg}{dt},$$

where dt is a Haar measure on $M_A^* Z(N)_A$ and vol( ) *is the volume with respect to dt.*

(ii) *If a Haar measure dt_∞ on $M_\infty^* Z(N)_\infty$ is fixed, we have*

$$(4.27) \qquad \int_{M_\infty^* Z(N)_\infty \backslash G_{1,\infty}} \varphi_{1,\infty}(g_\infty^{-1}\mu\nu g_\infty)\delta(g_\infty)^s \frac{dg_\infty}{dt_\infty} = -\mathrm{const.} \times \left(\frac{|z|}{z}\right)^r \frac{\alpha_0^{-r}}{1-\bar{\alpha}_0} \frac{\Gamma(r-2-2s)\Gamma(2s+1)}{\Gamma(r-2)} \times (\mathrm{sgn}(\beta)\iota)^{-2s-1}|\beta|^{-2s-1} \quad \text{for} \quad \nu = n(0, \beta),\ s > 0,$$

where "const." *is a positive number which depends only on dt_∞ and the analytic function x^{-2s-1} is so defined that it takes positive value for $x > 0$.*

(iii) *The integral of* (4.22) *on $Z_\infty G_{1,Q} \backslash G_{1,A}$ equals $\sum_{\mu \in M_Q^* - Z_Q} I'''(\mu)$, which turns out to be a finite sum for μ.*

PROOF. First, we shall prove that the integral of

$$(4.28) \qquad \sum_{\delta \in M_Q Z(N)_Q \backslash G_{1,Q}} \Big| \sum_{\nu \in Z(N)_Q - \{1\}} \varphi_1(g^{-1}\delta^{-1}\mu\nu\delta g) \Big|$$

on $Z_\infty G_{1,Q}\backslash G_{1,A}$ is finite. We use the Iwasawa decomposition (4.9) and the fundamental domain (4.17) for $A_Q\backslash A_A$. By an argument similar to that in the proof of Lemma 4.4, it is enough to show that

$$\int_{F_A\times C^\times}\left|\sum_{\nu\in Z(N)_Q-\{1\}}\varphi_1(\xi_i^{-1}\lambda_{ij}^{-1}\lambda(\alpha)^{-1}n(a,0)^{-1}\mu\nu n(a,0)\lambda(\alpha)\lambda_{ij}\xi_i)\right||\alpha|^{-6}dad\alpha$$

has finite value, where da is a Haar measure of F_A and $d\alpha$ is the Euclidean measure of $\boldsymbol{C}$. In order to majorize the above integral, we may and do assume that $\varphi_{1,f}$ is the characteristic function of a right $U_{1,f}$-coset in $G_{1,f}$. By Lemma 2.2, we have

$$n(a,0)^{-1}\mu\nu n(a,0)=\mu n\Big((1-\alpha_0)a,\frac{\iota^{-1}}{2}(\alpha_0-\bar{\alpha}_0)N(a)+\beta\Big)$$

for $\nu=n(0,\beta)$. Then we see that

$$\varphi_{1,f}(\xi_i^{-1}\lambda_{ij}^{-1}n(a,0)^{-1}\mu\nu n(a,0)\lambda_{ij}\xi_i)$$

has compact support with respect to $a\in F_f$ (note that $\alpha_0\neq 1$). Therefore we also see that this function equals zero unless ν, regarded as an element of $Z(N)_f$, belongs to a certain compact subset of $Z(N)_f$. From these facts and the above assumption, we see that the above integral has finite value provided that

$$\int_{C\times C^\times}\left|\sum_{k\in Z-\{0\}}\varphi_{1,\infty}(\lambda(\alpha)^{-1}n(a,0)^{-1}\mu n(0,pk+q)n(a,0)\lambda(\alpha))\right||\alpha|^{-6}dad\alpha$$
$$=\int_{C\times C^\times}\left|\sum_{k\in Z-\{0\}}\varphi_{1,\infty}(n(a,0)^{-1}\mu n(a,0)n(0,|\alpha|^{-2}(pk+q)))\right||\alpha|^{-4}dad\alpha$$

has finite value for any $p>0$ and $q\in(\boldsymbol{R}-p\boldsymbol{Z})\cup\{0\}$, where da is the Euclidean measure of $\boldsymbol{C}$. By (1.9), we have

$$\text{(4.29)}\qquad \varphi_{1,\infty}(n(a,0)^{-1}\mu n(a,0)n(0,|\alpha|^{-2}\beta))$$
$$=C_r\Big(\frac{|z|}{z}\Big)^r\alpha_0^{-r}[|\alpha|^2\{1+(1-\bar{\alpha}_0)|a|^2\}+\iota\beta]^{-r}|\alpha|^{2r}.$$

Then it is enough to prove that

$$I=\int_0^\infty\int_0^\infty\left|\sum_{k\in Z-\{0\}}[t\{1+(1-\bar{\alpha}_0)u\}+\iota(pk+q)]^{-r}\right|t^{r-2}dtdu$$

has finite value. If we put $\mathrm{sgn}(\mathrm{Im}(\iota))=\varepsilon$, $|\iota|^{-1}p^{-1}=v$ and $vt\,\mathrm{Re}(1+(1-\alpha_0)u)=x$, I equals a constant times

$$\int_{x=0}^{\infty}\int_{t=0}^{x/v}\left|\sum_{k\in \mathbf{Z}-\{0\}}[x+\{v(1-\mathrm{Re}(\alpha_0))^{-1}\,\mathrm{Im}(\alpha_0)(x-t)+k+\varepsilon p^{-1}q\}\sqrt{-1}]^{-r}\right|t^{r-3}dtdx.$$

Here, it is clear that the integrand is bounded when $x\to 0$. On the other hand, we have

$$\left|\sum_{k\in \mathbf{Z}-\{0\}}\{x+(y+k+\varepsilon p^{-1}q)\sqrt{-1}\}^{-r}\right|\leqq\left|\sum_{k\in \mathbf{Z}}\{x+(y+k+\varepsilon p^{-1}q)\sqrt{-1}\}^{-r}\right|+x^{-r}$$
$$(y=v(1-\mathrm{Re}(\alpha_0))^{-1}\,\mathrm{Im}(\alpha_0)(x-t)),$$

and (4.19) implies

$$\left|\sum_{k\in \mathbf{Z}}\{x+(y+k+\varepsilon p^{-1}q)\sqrt{-1}\}^{-r}\right|\leqq\frac{(2\pi)^r}{(r-1)!}\sum_{n=1}^{\infty}n^{r-1}e[n\sqrt{-1}x].$$

Then it is easy to see that the integrand of the above integral is integrable on $\{x\geqq R,\ 0\leqq t\leqq x/v\}$ for $R\gg 0$. Thus we have $I<+\infty$. From these arguments, we see that the integral of (4.28) on $Z_\infty G_{1,Q}\backslash G_{1,A}$ is finite, and we have

$$(4.30)\qquad \int_{Z_\infty G_{1,Q}\backslash G_{1,A}}\sum_{\delta\in M_Q Z(N)_Q\backslash G_{1,Q}}\ \sum_{\nu\in Z(N)_Q-\{1\}}\varphi_1(g^{-1}\delta^{-1}\mu\nu\delta g)dg$$
$$=\int_{Z_\infty M_Q Z(N)_Q\backslash G_{1,A}}\ \sum_{\nu\in Z(N)_Q-\{1\}}\varphi_1(g^{-1}\mu\nu g)dg.$$

Next, we shall prove that

$$(4.31)\qquad \sum_{\nu\in Z(N)_Q-\{1\}}|\varphi_1(g^{-1}\mu\nu g)|\delta(g)^s$$

is integrable on $Z_\infty M_Q Z(N)_Q\backslash G_{1,A}$ for any $s>0$, but as is seen from the preceding arguments we have only to show that

$$\sum_{k\in \mathbf{Z}-\{0\}}\int_{\mathbf{C}\times \mathbf{C}^\times}|\varphi_{1,\infty}(n(a,\,0)^{-1}\mu n(a,\,0)n(0,\,|\alpha|^{-2}(pk+q)))|\,|\alpha|^{-4-4s}da d\alpha$$

has finite value for any $p>0$ and $q\in(\mathbf{R}-p\mathbf{Z})\cup\{0\}$. The avobe expression equals

$$\sum_{k\in \mathbf{Z}-\{0\}}|pk+q|^{-1-2s}\int_{\mathbf{C}\times \mathbf{C}^\times}\varphi_{1,\infty}(n(a,\,0)^{-1}\mu n(a,\,0)n(0,\,|\alpha|^{-2}\,\mathrm{sgn}(pk+q)))|$$
$$\times|\alpha|^{-4-4s}da d\alpha.$$

From (4.29), we see that this last expression has finite value for any $s>0$. Hence we have shown the integrability of (4.31) and we have

$$\int_{Z_\infty M_Q Z(N)_Q\backslash G_{1,A}} \sum_{\nu\in Z(N)_Q-\{1\}} \varphi_1(g^{-1}\mu\nu g)\delta(g)^s dg$$
$$=\sum_{\nu\in Z(N)_Q-\{1\}} \int_{Z_\infty M_Q Z(N)_Q\backslash G_{1,A}} \varphi_1(g^{-1}\mu\nu g)\delta(g)^s dg$$
$$=\sum_{\nu\in Z(N)_Q-\{1\}/\sim} \sum_{\delta\in M_Q^* Z(N)_Q\backslash M_Q Z(N)_Q} \int_{Z_\infty M_Q Z(N)_Q\backslash G_{1,A}} \varphi_1(g^{-1}\mu\delta^{-1}\nu\delta g)\delta(g)^s dg$$
$$=\sum_{\nu\in Z(N)_Q-\{1\}/\sim} \int_{Z_\infty M_Q^* Z(N)_Q\backslash G_{1,A}} \varphi_1(g^{-1}\mu\nu g)\delta(g)^s dg$$

for any $s>0$ (note that $\mu\delta^{-1}=\delta^{-1}\mu$ for $\forall\delta\in M_Q Z(N)_Q$). Furthermore, the group $M^*Z(N)=G_1(\mu\nu)$ has the following property:

$$t\in M_A^* Z(N)_A \Longrightarrow \delta(t)=1.$$

Consequently we have

$$(4.32)\qquad \int_{Z_\infty M_Q Z(N)_Q\backslash G_{1,A}} \sum_{\nu\in Z(N)_Q-\{1\}} \varphi_1(g^{-1}\mu\nu g)\delta(g)^s dg$$
$$=\mathrm{vol}(Z_\infty M_Q^* Z(N)_Q\backslash M_A^* Z(N)_A)$$
$$\times \sum_{\nu\in Z(N)_Q-\{1\}/\sim} \int_{M_A^* Z(N)_A\backslash G_{1,A}} \varphi_1(g^{-1}\mu\nu g)\delta(g)^s \frac{dg}{dt}$$

for any $s>0$.

Now, let s_0 be a positive constant and define the function $F(g)$ on $Z_\infty M_Q Z(N)_Q\backslash G_{1,A}$ by

$$F(g)=\begin{cases} \left|\sum_{\nu\in Z(N)_Q-\{1\}} \varphi_1(g^{-1}\mu\nu g)\right| & \cdots \text{if}\quad \delta(g)\leqq 1, \\ \left|\sum_{\nu\in Z(N)_Q-\{1\}} \varphi_1(g^{-1}\mu\nu g)\right|\delta(g)^{s_0} & \cdots \text{if}\quad \delta(g)>1. \end{cases}$$

By the preceding arguments, $F(g)$ is integrable on $Z_\infty M_Q Z(N)_Q\backslash G_{1,A}$. On the other hand, it is clear that

$$\left|\sum_{\nu\in Z(N)_Q-\{1\}} \varphi_1(g^{-1}\mu\nu g)\right|\delta(g)^s\leqq F(g)\quad \text{for}\quad \forall s\leqq s_0.$$

Combining these facts with (4.30) and (4.32), we have (i).

Next, let $\nu=n(0,\beta)$ be an element of $Z(N)_Q-\{1\}$. The left hand side of (4.27) is written as

$$cC_r\left(\frac{|z|}{z}\right)^r \alpha_0^{-r}|\beta|^{-1-2s}\int_{C\times C^\times}\{1+(1-\bar{\alpha}_0)|a|^2+\iota|\alpha|^{-2}\,\mathrm{sgn}(\beta)\}^{-r}|\alpha|^{-4-4s}da\,d\alpha,$$

where c is a positive number which depends only on dt_∞ (see (4.29)).

If we define the analytic function x^{-2s-1} by $0\leqq\arg(x)<2\pi$, a straightforward calculation shows that the above expression is written as the right hand side of (4.27). Hence we have (ii).

Statement (iii) is obvious from the Iwasawa decomposition (4.9).
Q.E.D.

§ 5. Alternating sums of orbital integrals at a finite place

5.1.

Let p be a finite place of $\boldsymbol{Q}$ which decomposes in F. In this section, we denote by G the group of $\boldsymbol{Q}_p$-rational points of G_i $(i=1,2)$. Then, there exists an isomorphism j defined over $\boldsymbol{Q}_p$:

$$j\colon G\xrightarrow{\sim}\boldsymbol{Q}_p^\times\times GL(3,\boldsymbol{Q}_p).$$

For a symbol τ; $\tau=0, 1, 2, 01, 12, 20, 012$, we define the open compact subgroup U^τ of G as follows:

$$\begin{aligned}
&U^0=j^{-1}(\Lambda\times GL(3,\boldsymbol{Z}_p)),\\
&U^1=j^{-1}(\Lambda\times\alpha GL(3,\boldsymbol{Z}_p)\alpha^{-1}),\\
(5.1)\qquad &U^2=j^{-1}(\Lambda\times\beta GL(3,\boldsymbol{Z}_p)\beta^{-1}),\\
&U^{01}=U^0\cap U^1,\qquad U^{12}=U^1\cap U^2,\qquad U^{20}=U^2\cap U^0,\\
&U^{012}=U^0\cap U^1\cap U^2,
\end{aligned}$$

where

$$\alpha=\begin{pmatrix}p&&\\&p&\\&&1\end{pmatrix},\qquad \beta=\begin{pmatrix}p&&\\&1&\\&&1\end{pmatrix}$$

and Λ is a fixed open compact subgroup of $\boldsymbol{Q}_p^\times$. By definition, we have

$$j(U^{01})=\Lambda\times\begin{pmatrix}\boldsymbol{Z}_p&\boldsymbol{Z}_p&p\boldsymbol{Z}_p\\\boldsymbol{Z}_p&\boldsymbol{Z}_p&p\boldsymbol{Z}_p\\\boldsymbol{Z}_p&\boldsymbol{Z}_p&\boldsymbol{Z}_p\end{pmatrix}^\times,$$

$$j(U^{012})=\Lambda\times\begin{pmatrix}\boldsymbol{Z}_p&p\boldsymbol{Z}_p&p\boldsymbol{Z}_p\\\boldsymbol{Z}_p&\boldsymbol{Z}_p&p\boldsymbol{Z}_p\\\boldsymbol{Z}_p&\boldsymbol{Z}_p&\boldsymbol{Z}_p\end{pmatrix}^\times,$$

and it is easy to see that U^0, U^1, U^2 are conjugate in G, and U^{01}, U^{12},

U^{20} are also conjugate in G.

Now, let φ^{τ} be the characteristic function of U^{τ}. For an element γ of G, we put

$$I^{\tau}(\gamma)=\int_{G(\gamma)\backslash G}\varphi^{\tau}(g^{-1}\gamma g)\frac{(dg)_{\tau}}{dt},$$

where the Haar measure $(dg)_{\tau}$ of G is normalized by

$$\int_{U^{\tau}}(dg)_{\tau}=1,$$

and dt is a Haar measure of the unimodular group $G(\gamma)$ which does not depend on τ. We define the signature $\varepsilon(\tau)$ by

$$\varepsilon(\tau)=\begin{cases}+1\cdots\text{for} & \tau=0,\ 1,\ 2,\ 012\\ -1\cdots\text{for} & \tau=01,\ 12,\ 20\end{cases} \tag{5.2}$$

and consider the alternating sum:

$$S(\gamma)=\sum_{\tau}\varepsilon(\tau)I^{\tau}(\gamma).$$

In this section, we shall prove the next

PROPOSITION 5.1. *Assume that an element γ of G does not belong to the center of G and the characteristic polynomial of the second factor of $j(\gamma)$ is reducible in $\boldsymbol{Q}_p$. Then we have $S(\gamma)=0$.*

Here, we add some remarks. Firstly, by the conjugations between U^{τ}'s we have

$$S(\gamma)=3I^{0}(\gamma)-3I^{01}(\gamma)+I^{012}(\gamma)\quad\text{for}\quad\forall\gamma\in G.$$

Secondly, we can reduce the above proposition to the similar statement for $GL(3,\boldsymbol{Q}_p)$. Namely, put

$$\boldsymbol{G}=GL(3,\boldsymbol{Q}_p)$$

and

$$\boldsymbol{U}^{0}=GL(3,\boldsymbol{Z}_p),\qquad \boldsymbol{U}^{01}=\begin{pmatrix}\boldsymbol{Z}_p & \boldsymbol{Z}_p & p\boldsymbol{Z}_p\\ \boldsymbol{Z}_p & \boldsymbol{Z}_p & p\boldsymbol{Z}_p\\ \boldsymbol{Z}_p & \boldsymbol{Z}_p & \boldsymbol{Z}_p\end{pmatrix}^{\times},\qquad \boldsymbol{U}^{012}=\begin{pmatrix}\boldsymbol{Z}_p & p\boldsymbol{Z}_p & p\boldsymbol{Z}_p\\ \boldsymbol{Z}_p & \boldsymbol{Z}_p & p\boldsymbol{Z}_p\\ \boldsymbol{Z}_p & \boldsymbol{Z}_p & \boldsymbol{Z}_p\end{pmatrix}^{\times}.$$

We denote by $\boldsymbol{\varphi}^{\tau}$ the characteristic function of $\boldsymbol{U}^{\tau}$ and set

$$\boldsymbol{I}^{\tau}(\gamma)=\int_{G(\gamma)\backslash G}\boldsymbol{\varphi}^{\tau}(g^{-1}\gamma g)\frac{(dg)_{\tau}}{dt} \quad (\tau=0,\ 01,\ 012)$$

for $\gamma\in\boldsymbol{G}$, where the Haar measure $(dg)_{\tau}$ of $\boldsymbol{G}$ is normalized by

$$\int_{U^{\tau}}(dg)_{\tau}=1$$

and the Haar measure dt of the unimodular group $\boldsymbol{G}(\gamma)$ is normalized by

$$\int_{G(\gamma)\cap U^{0}}dt=1.$$

Then, it is easy to see that the above proposition is reduced to the next

PROPOSITION 5.1′. *Assume that an element* γ *of* $\boldsymbol{G}$ *does not belong to the center of* $\boldsymbol{G}$ *and the characteristic polynomial of* γ *is reducible in* $\boldsymbol{Q}_p$. *Then we have*

$$3\boldsymbol{I}^{0}(\gamma)-3\boldsymbol{I}^{01}(\gamma)+\boldsymbol{I}^{012}(\gamma)=0.$$

5.2.

In this subsection, we shall prove Proposition 5.1′. First, let us consider the Iwasawa decomposition of $\boldsymbol{G}$. For $\sigma=01,\ 012$, define the parabolic subgroup $\boldsymbol{P}_{\sigma}$ of $\boldsymbol{G}$ by

$$\boldsymbol{P}_{01}=\left\{\begin{pmatrix} * & * & 0\\ * & * & 0\\ * & * & * \end{pmatrix}\in\boldsymbol{G}\right\},\qquad \boldsymbol{P}_{012}=\left\{\begin{pmatrix} * & 0 & 0\\ * & * & 0\\ * & * & * \end{pmatrix}\in\boldsymbol{G}\right\}$$

where $*$ is an element of $\boldsymbol{Q}_p$, and put

$$w_0=\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix},\quad w_1=\begin{pmatrix} & & 1\\ 1 & 0 & \\ 0 & 1 & \end{pmatrix},\quad w_2=\begin{pmatrix} & 1 & 0\\ & 0 & 1\\ 1 & & \end{pmatrix},$$

$$w_3=\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix},\quad w_4=\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix},\quad w_5=\begin{pmatrix} 1 & & \\ & 0 & 1\\ & 1 & 0 \end{pmatrix}.$$

Then it is easy to verify that

(5.3) $\quad \boldsymbol{G}=\coprod_{w\in W(\sigma,\tau)}\boldsymbol{P}_{\sigma}w\boldsymbol{U}^{\tau};\quad W(01,\ 0)=W(012,\ 0)=\{w_0\},$

$$W(01, 01)=\{w_0, w_5\}, \quad W(012, 01)=W(01, 012)=\{w_0, w_1, w_2\},$$
$$W(012, 012)=\{w_0, w_1, w_2, w_3, w_4, w_5\}.$$

Using the above decomposition, we see that if $\boldsymbol{G}(\gamma)$ is contained in $\boldsymbol{P}_\sigma$, the orbital integral $\boldsymbol{I}^\tau(\gamma)$ is expressed by

$$\boldsymbol{I}^\tau(\gamma)=\sum_{h\in U^0/U^\tau}\int_{G(\gamma)\backslash P_\sigma}\boldsymbol{\varphi}^\tau(h^{-1}p^{-1}\gamma ph)\frac{d_lp}{dt},$$

where d_lp is the left invariant Haar measure of $\boldsymbol{P}_\sigma$ normalized by

$$\int_{P_\sigma\cap U^0} d_lp=1. \tag{5.4}$$

In the right hand side of the above formula, we put $h=\tilde{p}w$ where $w\in W(\sigma, \tau)$ and $\tilde{p}\in(\boldsymbol{P}_\sigma\cap \boldsymbol{U}^0)/(\boldsymbol{P}_\sigma\cap w\boldsymbol{U}^\tau w^{-1})$. Note that the module of $\boldsymbol{P}_\sigma$ takes value one at any $\tilde{p}$. Hence we have

$$\boldsymbol{I}^\tau(\gamma)=\sum_{w\in W(\sigma,\tau)}\mathrm{vol}(\boldsymbol{P}_\sigma\cap w\boldsymbol{U}^\tau w^{-1})^{-1}\int_{G(\gamma)\backslash P_\sigma}\boldsymbol{\varphi}^\tau(w^{-1}p^{-1}\gamma pw)\frac{d_lp}{dt} \tag{5.5}$$

for any $\gamma\in\boldsymbol{G}$ such that $\boldsymbol{G}(\gamma)\subset\boldsymbol{P}_\sigma$, where vol( ) is the volume with respect to d_lp.

Now, let $\boldsymbol{N}_\sigma$ be the unipotent radical of $\boldsymbol{P}_\sigma$. We define the Levi part $\boldsymbol{M}_\sigma$ of $\boldsymbol{P}_\sigma$ by

$$\boldsymbol{M}_{01}=\left\{\begin{pmatrix} * & * & 0\\ * & * & 0\\ 0 & 0 & *\end{pmatrix}\in\boldsymbol{G}\right\}, \qquad \boldsymbol{M}_{012}=\left\{\begin{pmatrix} * & 0 & 0\\ 0 & * & 0\\ 0 & 0 & *\end{pmatrix}\in\boldsymbol{G}\right\}$$

where $*$ is an element of $\boldsymbol{Q}_p$. It is easy to see that

$$\boldsymbol{P}_\sigma\cap w\boldsymbol{U}^\tau w^{-1}=(\boldsymbol{M}_\sigma\cap w\boldsymbol{U}^\tau w^{-1})\cdot(\boldsymbol{N}_\sigma\cap w\boldsymbol{U}^\tau w^{-1}) \tag{5.6}$$

holds for any (σ, τ) and any $w\in W(\sigma, \tau)$. We normalize the Haar measures dn of $\boldsymbol{N}_\sigma$ and dm of $\boldsymbol{M}_\sigma$ by

$$\int_{N_\sigma\cap U^0} dn=\int_{M_\sigma\cap U^0} dm=1. \tag{5.7}$$

If an element γ of $\boldsymbol{G}$ satisfies $\boldsymbol{G}(\gamma)\subset\boldsymbol{M}_\sigma$, we put

$$\Delta_\sigma(\gamma)=|\det(Ad_{\mathscr{N}}(\gamma)-1_{\mathscr{N}})|_p,$$

where $\mathscr{N}$ is the Lie algebra of $\boldsymbol{N}$. Then, by Lemma 22 in Harish-

Chandra [5], we have

$$\int_{G(\gamma)\backslash P_\sigma} \boldsymbol{\varphi}^\tau(w^{-1}p^{-1}\gamma pw)\frac{d_l p}{dt} = \Delta_\sigma(\gamma)^{-1}\int_{(G(\gamma)\backslash M_\sigma)\times N_\sigma} \boldsymbol{\varphi}^\tau(w^{-1}m^{-1}\gamma mnw)\frac{dm}{dt}dn.$$

Combining this formula with (5.5) and (5.6), we have

$$(5.8)\qquad \boldsymbol{I}^\tau(\gamma) = \Delta_\sigma(\gamma)^{-1}\sum_{w\in W(\sigma,\tau)} \mathrm{vol}(\boldsymbol{M}_\sigma\cap w\boldsymbol{U}^\tau w^{-1})^{-1}\int_{G(\gamma)\backslash M_\sigma} \boldsymbol{\varphi}^\tau(w^{-1}m^{-1}\gamma mw)\frac{dm}{dt}$$

for any $\gamma\in\boldsymbol{G}$ such that $\boldsymbol{G}(\gamma)\subset\boldsymbol{M}_\sigma$, where vol( ) is the volume with respect to dm.

LEMMA 5.2. *If an element γ of $\boldsymbol{G}$ satisfies $\boldsymbol{G}(\gamma)\subset\boldsymbol{M}_{01}$, we have*

$$3\boldsymbol{I}^0(\gamma)-3\boldsymbol{I}^{01}(\gamma)+\boldsymbol{I}^{012}(\gamma)=0.$$

PROOF. Let w be an element of $W(01,\tau)$. Then a straightforward calculation shows:

(i) When $\tau=0$ and $w=w_0$ or $\tau=01$ and $w=w_0$, $\boldsymbol{M}_{01}\cap w\boldsymbol{U}^\tau w^{-1}$ equals

$$\left\{\begin{pmatrix} * & * & 0\\ * & * & 0\\ 0 & 0 & *\end{pmatrix}\right\}\cap GL(3,\boldsymbol{Z}_p),$$

(ii) in other cases, $\boldsymbol{M}_{01}\cap w\boldsymbol{U}^\tau w^{-1}$ equals

$$\left\{\begin{pmatrix} * & b & 0\\ * & * & 0\\ 0 & 0 & *\end{pmatrix}\middle|\, b\in p\boldsymbol{Z}_p\right\}\cap GL(3,\boldsymbol{Z}_p)$$

or

$$\left\{\begin{pmatrix} * & * & 0\\ c & * & 0\\ 0 & 0 & *\end{pmatrix}\middle|\, c\in p\boldsymbol{Z}_p\right\}\cap GL(3,\boldsymbol{Z}_p),$$

where $*$ is a p-adic integer. Note that two groups in (ii) are conjugate in $\boldsymbol{M}_{01}$. By these facts and (5.8), the required formula is verified directly. Q.E.D.

Next, suppose that an element γ of $\boldsymbol{G}$ satisfies $\boldsymbol{G}(\gamma)\subset\boldsymbol{P}_{012}$. In order to apply (5.5) to this case, we need the following lemma, which can be verified by a straightforward calculation.

LEMMA 5.3. *The group* $\boldsymbol{P}_{012} \cap w_k \boldsymbol{U}^{\tau} w_k^{-1}$ *is written as* $V \cap GL(3, \boldsymbol{Z}_p)$, *where* V *and the volume* vol *of* $\boldsymbol{P}_{012} \cap w_k \boldsymbol{U}^{\tau} w_k^{-1}$ *with respect to the measure* $d_l p$ *(normalized by* (5.4)) *are given as follows.* (*We denote by* $*$ *a* p-*adic integer.*)

$$V=\left\{\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}\right\}; \quad \mathrm{vol}=1 \cdots \quad \textit{for} \quad (\tau, k)=(0, 0),\ (01, 0),\ (012, 0),$$

$$V=\left\{\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & p* & * \end{pmatrix}\right\}; \quad \mathrm{vol}=p^{-1} \cdots \quad \textit{for} \quad (\tau, k)=(01, 2),$$

$$V=\left\{\begin{pmatrix} * & 0 & 0 \\ p* & * & 0 \\ * & * & * \end{pmatrix}\right\}; \quad \mathrm{vol}=p^{-1} \cdots \quad \textit{for} \quad (\tau, k)=(012, 3),$$

$$V=\left\{\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & p* & * \end{pmatrix}\right\}; \quad \mathrm{vol}=p^{-1} \cdots \quad \textit{for} \quad (\tau, k)=(012, 5),$$

$$V=\left\{\begin{pmatrix} * & 0 & 0 \\ p* & * & 0 \\ p* & * & * \end{pmatrix}\right\}; \quad \mathrm{vol}=p^{-2} \cdots \quad \textit{for} \quad (\tau, k)=(01, 1),\ (012, 1),$$

$$V=\left\{\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ p* & p* & * \end{pmatrix}\right\}; \quad \mathrm{vol}=p^{-2} \cdots \quad \textit{for} \quad (\tau, k)=(012, 2),$$

$$V=\left\{\begin{pmatrix} * & 0 & 0 \\ p* & * & 0 \\ p* & p* & * \end{pmatrix}\right\}; \quad \mathrm{vol}=p^{-3} \cdots \quad \textit{for} \quad (\tau, k)=(012, 4).$$

Now, we write $\boldsymbol{P}$, $\boldsymbol{N}$, $\boldsymbol{M}$ for $\boldsymbol{P}_{012}$, $\boldsymbol{N}_{012}$, $\boldsymbol{M}_{012}$, respectively, and define the module $\delta(p)$ by

$$\delta(p)=\frac{d(p^{-1}np)}{dn}.$$

For an element m of $\boldsymbol{M}$, we have

$$\delta(m)=\left|\frac{\lambda}{\nu}\right|_p^2 \quad \text{for} \quad m=\begin{pmatrix}\lambda & & \\ & \mu & \\ & & \nu\end{pmatrix}.$$

We denote by $\phi(\ )$ the characteristic function of $\boldsymbol{Z}_p^\times$.

LEMMA 5.4. *If γ is an element of $\boldsymbol{G}$ of the form*

$$\text{(a)} \begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 1 & 0 & \alpha\end{pmatrix} \quad \textit{or} \quad \text{(b)} \begin{pmatrix}\alpha & 0 & 0\\ 1 & \alpha & 0\\ 0 & 1 & \alpha\end{pmatrix},$$

we have

$$3\boldsymbol{I}^0(\gamma)-3\boldsymbol{I}^{01}(\gamma)+\boldsymbol{I}^{012}(\gamma)=0.$$

PROOF. First, set

$$\gamma=\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 1 & 0 & \alpha\end{pmatrix}.$$

Then we have

$$\boldsymbol{G}(\gamma)=\left\{\begin{pmatrix}\lambda & 0 & 0\\ x & \mu & 0\\ z & y & \lambda\end{pmatrix}\in G\right\}=(\boldsymbol{M}\cap\boldsymbol{G}(\gamma))\cdot\boldsymbol{N}$$

and

$$\boldsymbol{M}\cap\boldsymbol{G}(\gamma)\backslash\boldsymbol{M}\cong\left\{\begin{pmatrix}\lambda & & \\ & 1 & \\ & & 1\end{pmatrix}\in\boldsymbol{M}\right\}.$$

Hence we have

$$\int_{G(\gamma)\backslash P}\boldsymbol{\varphi}^\tau(w^{-1}p^{-1}\gamma pw)\frac{d_lp}{dt}$$

$$=\int_{Q_p^\times}\boldsymbol{\varphi}^\tau\left(w^{-1}\begin{pmatrix}\lambda & & \\ & 1 & \\ & & 1\end{pmatrix}^{-1}\gamma\begin{pmatrix}\lambda & & \\ & 1 & \\ & & 1\end{pmatrix}w\right)\delta\left(\begin{pmatrix}\lambda & & \\ & 1 & \\ & & 1\end{pmatrix}\right)d^\times\lambda$$

$$=\int_{Q_p^\times}\boldsymbol{\varphi}^\tau\left(w^{-1}\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ \lambda & 0 & \alpha\end{pmatrix}w\right)|\lambda|_p^2d^\times\lambda,$$

where the Haar measure $d^{\times}\lambda$ of $\boldsymbol{Q}_p^{\times}$ is normalized by

$$\int_{\boldsymbol{Z}_p^{\times}} d^{\times}\lambda=1.$$

Combining the above formula with (5.5) and Lemma 5.3, we have

$$\boldsymbol{I}^0(\gamma)=(1-p^{-2})\phi(\alpha), \qquad \boldsymbol{I}^{01}(\gamma)=(2+p)\boldsymbol{I}^0(\gamma), \qquad \boldsymbol{I}^{012}(\gamma)=(3+3p)\boldsymbol{I}^0(\gamma).$$

Then we have the required formula.

Next, set

$$\gamma=\begin{pmatrix}\alpha & 0 & 0\\ 1 & \alpha & 0\\ 0 & 1 & \alpha\end{pmatrix}.$$

Then we have $\boldsymbol{G}(\gamma)=\boldsymbol{Z}\boldsymbol{N}'$, where $\boldsymbol{Z}$ is the center of $\boldsymbol{G}$ and

$$\boldsymbol{N}'=\left\{\begin{pmatrix}1 & 0 & 0\\ x & 1 & 0\\ z & x & 1\end{pmatrix}\in \boldsymbol{N}\right\}.$$

We also have $\boldsymbol{Z}\backslash\boldsymbol{M}\cong\boldsymbol{M}_1$ and $\boldsymbol{N}'\backslash\boldsymbol{N}\cong\boldsymbol{N}_1$, where

$$\boldsymbol{M}_1=\left\{\begin{pmatrix}\lambda & & \\ & \mu & \\ & & 1\end{pmatrix}\in\boldsymbol{M}\right\}, \qquad \boldsymbol{N}_1=\left\{\begin{pmatrix}1 & & \\ u & 1 & \\ 0 & 0 & 1\end{pmatrix}\in\boldsymbol{N}\right\}.$$

Therefore we have

$$\int_{G(\gamma)\backslash P}\boldsymbol{\varphi}^{\tau}(w^{-1}p^{-1}\gamma pw)\frac{d_l p}{dt}=\int_{N_1M_1}\boldsymbol{\varphi}^{\tau}(w^{-1}m_1^{-1}n_1^{-1}\gamma n_1m_1w)\delta(m_1)dn_1dm_1$$

$$=\int_{Q_p^{\times}\times Q_p^{\times}\times Q_p^{\times}}\boldsymbol{\varphi}^{\tau}\left(w^{-1}\begin{pmatrix}\alpha & 0 & 0\\ \lambda\mu^{-1} & \alpha & 0\\ \lambda u & \mu & \alpha\end{pmatrix}w\right)|\lambda|_p^2 d^{\times}\lambda d^{\times}\mu du,$$

where $d^{\times}\lambda$, $d^{\times}\mu$, du are normalized by

$$\int_{\boldsymbol{Z}_p^{\times}}d^{\times}\lambda=\int_{\boldsymbol{Z}_p^{\times}}d^{\times}\mu=\int_{\boldsymbol{Z}_p}du=1.$$

Combining this formula with (5.5) and Lemma 5.3, we have

$$\boldsymbol{I}^0(\gamma)=(1-p^{-1})\phi(\alpha), \qquad \boldsymbol{I}^{01}(\gamma)=3\boldsymbol{I}^0(\gamma), \qquad \boldsymbol{I}^{012}(\gamma)=6\boldsymbol{I}^0(\gamma).$$

Then we have the required formula. Q.E.D.

By Lemmas 5.2, 5.4 and the well-known classification of conjugacy classes of $\boldsymbol{G}$, we see that Proposition 5.1′ has been proved.

§ 6. The main result and remarks

6.1.

Now we can state our main result. Fix a finite place p of $\boldsymbol{Q}$ which decomposes in F. For $\tau=0, 1, 2, 01, 12, 20, 012$, and $i=1, 2$, let $U_{i,f}^{\tau}$ be an open compact subgroup of $G_{i,f}$. We assume that the systems $\{U_{1,f}^{\tau}\}$ and $\{U_{2,f}^{\tau}\}$ satisfy the following conditions:

(i) For each τ, $U_{1,f}^{\tau}$ and $U_{2,f}^{\tau}$ are of the form (1.2) and they satisfy assumption (U_f).

(ii) At any finite place $l\neq p$, the l-component of $U_{i,f}^{\tau}$ does not depend on τ $(i=1, 2)$.

(iii) At the place p, the system {p-component of $U_{i,f}^{\tau}$} has the form (5.1) $(i=1, 2)$.

Then we consider the systems $\{\varphi_{1,f}^{\tau}\}$ and $\{\varphi_{2,f}^{\tau}\}$ of spherical functions which satisfy the following conditions:

(i) For each τ, $\varphi_{i,f}^{\tau}$ is an element of $\mathscr{L}(G_{i,f}, U_{i,f}^{\tau})$ $(i=1, 2)$ and they satisfy assumption (φ_f).

(ii) At any finite place $l\neq p$, the l-component of $\varphi_{i,f}^{\tau}$ does not depend on τ $(i=1, 2)$.

(iii) At the place p, p-component of $\varphi_{i,f}^{\tau}$ is the characteristic function of $U_{i,p}^{\tau}$ $(i=1, 2)$.

By Proposition 1.1.(ii) and Proposition 1.2.(ii), we can define the Hecke operators $T_1(\varphi_{1,f}^{\tau})$ and $T_2(\varphi_{2,f}^{\tau})$, acting on the spaces $\mathfrak{S}_r(U_{1,f}^{\tau})$ and $\mathfrak{S}_\rho(U_{2,f}^{\tau})$, respectively (under the assumption $r>4$). We have described the relation between r and ρ in 1.3.

THEOREM 6.1. *Assumptions being as above, we have*

$$\sum_{\tau} \varepsilon(\tau)\mathrm{trace}(T_1(\varphi_{1,f}^{\tau}))=\sum_{\tau} \varepsilon(\tau)\mathrm{trace}(T_2(\varphi_{2,f}^{\tau}))$$

for $\forall r>4$, *where* $\varepsilon(\tau)$ *is the signature defined by* (5.2).

PROOF. In Proposition 2.1, we described a rough classification of elements of $G_{i,Q}$ $(i=1, 2)$, each class being stable under the conjugation. Then by Proposition 3.2, the contributions of Z_Q to both sides of the above formula are equal, and by the corollary of Proposition 3.4, the contribution of E_1(irred.) to the left hand side equals to the contribution of E_2(irred.) to the right hand side. Next, the contribution of Nh(reg.) to the left hand side vanishes by Lemma 3.7. Now, it is clear that any other element of $G_{i,Q}$ satisfies the assumption of Proposition 5.1 when it is regarded as an element of $G_{i,p}$. Thus we see that the contributions of E_1(sing.), E_1(red.), Nh(sing.), Sh(sing.), Sh(reg.), $Ns(1)$ and $Ns(2)$ to the left hand side and the contributions of E_2(sing.) and E_2(red.) to the right hand side are all equal to zero (see (3.12), Lemmas 4.1, 4.2, 4.3, 4.4, 4.5). As for the contribution of $Ns(3)$ to the left hand side, recall that the expression for $I'''(\mu)$ given in Lemma 4.6 was a constant times

$$\lim_{s\downarrow 0}\sum_{\beta\in Q^\times/N_{F/Q}(F^\times)}\int_{M_A^* Z(N)_A\backslash G_{1,A}}\varphi_1(g^{-1}\mu n(0,\beta)g)\delta(g)^s\frac{dg}{dt}. \tag{6.1}$$

But any $\beta\in \boldsymbol{Q}^\times/N_{F/Q}(F^\times)$ can be witten as $N_{F_p/Q_p}(\alpha)$ for some $\alpha\in F_p^\times$. Then we write

$$\begin{aligned}&\int_{M_A^* Z(N)_A\backslash G_{1,A}}\varphi_1(g^{-1}\mu n(0,\beta)g)\delta(g)^s\frac{dg}{dt}\\ &=\prod_{v\neq p}\int_{M_v^* Z(N)_v\backslash G_{1,v}}\varphi_{1,v}(g_v^{-1}\mu n(0,\beta)g_v)\delta(g_v)^s\frac{dg_v}{dt_v}\times\delta(\lambda(\alpha))^s\\ &\quad\times\int_{M_p^* Z(N)_p\backslash G_{1,p}}\varphi_{1,p}(\lambda(\alpha)^{-1}g_p^{-1}\lambda(\alpha)\mu n(0,1)\lambda(\alpha)^{-1}g_p\lambda(\alpha))\delta(\lambda(\alpha)^{-1}g_p\lambda(\alpha))^s\frac{dg_p}{dt_p}\end{aligned}$$

for $\varphi_1=\bigotimes_{v\neq p}\varphi_{1,v}\otimes\varphi_{1,p}$. Denote the last expression by

$$I(\beta,s)\int_{M_p^* Z(N)_p\backslash G_{1,p}}\varphi_{1,p}(g_p^{-1}\mu n(0,1)g_p)\delta(g_p)^s\frac{dg_p}{dt_p}.$$

Then, (6.1) equals

$$\lim_{s\downarrow 0}\Big(\sum_\beta I(\beta,s)\Big)\int_{M_p^* Z(N)_p\backslash G_{1,p}}\varphi_{1,p}(g_p^{-1}\mu n(0,1)g_p)\frac{dg_p}{dt_p}.$$

So we can apply Proposition 5.1 to the contribution of $Ns(3)$ to the left hand side and it in turn equals to zero. Hence we have the statement. Q.E.D.

6.2.

In this subsection, we add some remarks to the above theorem.

Let $U_{i,f}$ be an open compact subgroup of $G_{i,f}$ of the form (1.2) $(i=1, 2)$ and assume that they satisfy assumption (U_f). Since our groups G_1 and G_2 are isomorphic at *any* finite place of $\boldsymbol{Q}$, one might expect a formula like

$$\mathrm{trace}(T_1(\varphi_{1,f}))=\mathrm{trace}(T_2(\varphi_{2,f})) \quad \text{for} \quad \forall r>4$$

when $\varphi_{1,f}\in\mathscr{L}(G_{1,f}, U_{1,f})$ and $\varphi_{2,f}\in\mathscr{L}(G_{2,f}, U_{2,f})$ satisfy assumption (φ_f). But we can prove the following

PROPOSITION 6.2. *Let $U_{i,f}$ be an open compact subgroup of $G_{i,f}$ $(i=1, 2)$ and assume that $Nh(sing.)\cap(G_{1,\infty}\times U_{1,f})\neq\emptyset$ or $Sh(sing.)\cap(G_{1,\infty}\times U_{1,f})\neq\emptyset$ or $E_2(sing.)\cap(G_{2,\infty}\times U_{2,f})\neq\emptyset$. Then there exist infinitely many $r>4$ such that*

$$\dim_{\boldsymbol{C}}\mathfrak{S}_r(U_{1,f})\neq\dim_{\boldsymbol{C}}\mathfrak{S}_\rho(U_{2,f}).$$

PROOF. Suppose that $\dim_{\boldsymbol{C}}\mathfrak{S}_r(U_{1,f})=\dim_{\boldsymbol{C}}\mathfrak{S}_\rho(U_{2,f})$ for almost all $r>4$. Then we see that the contributions of $Z_{\boldsymbol{Q}}$ to the both sides of this equality coincide, because they are the main terms when $r\to\infty$. Next we compare the main terms of the contributions of non-central elements. By the results of § 3 and § 4, we see that the contributions of non-central elements of $G_{1,\boldsymbol{Q}}$ to $\dim_{\boldsymbol{C}}\mathfrak{S}_r(U_{1,f})$ is divided into the sum of quantities of the form

$$\sum_{\mu_1\neq\mu_2} a(\mu_1, \mu_2)\frac{\mu_1}{\mu_1-\mu_2}\mu_2^r r \quad (a(\mu_1, \mu_2)\geqq 0)$$

where μ_i runs through all roots of the unity in F $(i=1, 2)$ and a quantity which is bounded when $r\to\infty$. Similarly, the contributions of non-central elements of $G_{2,\boldsymbol{Q}}$ to $\dim_{\boldsymbol{C}}\mathfrak{S}_\rho(U_{2,f})$ is divided into the sum of quantities of the form

$$\sum_{\mu_1\neq\mu_2} b(\mu_1, \mu_2)\frac{-\mu_1}{\mu_1-\mu_2}\mu_2^r r \quad (b(\mu_1, \mu_2)\geqq 0),$$

μ_1 and μ_2 being as above, and a bounded quantity. (See Lemmas 3.6, 4.1, 4.2.) Hence we have an equality of the form

$$\sum_{\mu_1\neq\mu_2} c(\mu_1, \mu_2)\frac{\mu_1}{\mu_1-\mu_2}\mu_2^r=0 \quad (c(\mu_1, \mu_2)\geqq 0)$$

which is valid for any sufficiently large r. Note that there exists (μ_1, μ_2) such that $c(\mu_1, \mu_2)>0$ by the assumption. Then we have a contradiction by using the Vandermonde's determinant. Q.E.D.

In this paper, we treated a comparison of trace formulas for $G_1=GU(H_1)$ and $G_2=GU(H_2)$. By the same method, we can compare trace formulas for $U(H_1)$ and $U(H_2)$, and get a result completely analogous to the main theorem of this paper.

References

[1] Arthur, J., The Selberg trace formula for groups of F-rank one, Ann. of Math. **100** (1974), 326-385.

[2] Asai, T., The conjugacy classes in the unitary, symplectic, and orthogonal groups over an algebraic number field, J. Math. Kyoto Univ. **16** (1976), 325-350.

[3] Cohn, L., The dimension of spaces of automorphic forms on a certain two-dimensional complex domain, Mem. Amer. Math. Soc. No. 158, 1975.

[4] Godement, R., Généralités sur les formes modulaires I, II, Sém. H. Cartan, 1957/58.

[5] Harish-Chandra (Noted by G. van Dijk), Harmonic analysis on reductive $\mathfrak{p}$-adic groups, Lecture Notes in Math., vol. 162, Springer-Verlag, Berlin, 1970.

[6] Hashimoto, K. and T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **27** (1980), 549-601.

[7] Hashimoto, K. and T. Ibukiyama, On relations of dimensions of automorphic forms of $Sp(2, \boldsymbol{R})$ and its compact twist $Sp(2)$ (II), Advanced Studies in Pure Math. 7, Kinokuniya, Tokyo, 1985, 31-102.

[8] Hashimoto, K. and H. Koseki, Class numbers of definite unimodular hermitian forms over the rings of imaginary quadratic fields, to appear.

[9] Ibukiyama, T., On symplectic Euler factors of genus two, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **30** (1983), 587-614.

[10] Ibukiyama, T., On relations of dimensions of automorphic forms of $Sp(2, \boldsymbol{R})$ and its compact twist $Sp(2)$ (I), Advanced Studies in Pure Math. **7**, Kinokuniya, Tokyo, 1985, 7-30.

[11] Ibukiyama, T., On automorphic forms of $Sp(2, \boldsymbol{R})$ and its compact form $Sp(2)$, Sém. Delange Pisot-Poitou, 1982/83, Birkhäuser, Boston-Basel-Stuttgart, 125-134.

[12] Jacquet, H. and R. P. Langlands, Automorphic forms on $GL(2)$, Lecture Notes in Math. vol. 114, Springer-Verlag, Berlin, 1970.

[13] Kato, S., A dimension formula for a certain space of automorphic forms of $SU(p, 1)$, Math. Ann. **266** (1984), 457-477.

[14] Kato, S., A dimension formula for a certain space of automorphic forms of $SU(p, 1)$, II: the case of $\Gamma(N)$ with $N\geqq 3$, Tôhoku Math. J. (2) **37** (1985), 571-584.

[15] Kojima, H., Dimension of the spaces of vector valued automorphic forms on the unitary group $SU(2, 1)$, Tôhoku Math. J. (2) **36** (1984), 563-579.

[16] Langlands, R. P., Problems in the theory of automorphic forms, Lecture Notes in Math. vol. 170, Springer-Verlag, Berlin, 1970, 18-86.

[17] Langlands, R. P., Stable conjugacy; definitions and lemmas, Canad. J. Math. **31** (1979), 700-725.

[18] Landherr, W., Aquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper, Abh. Math. Sem. Univ. Hamburg **11** (1935), 245–248.
[19] Macdonald, I. G., The volume of a compact Lie group, Invent. Math. **56** (1980), 93–95.
[20] Shimizu, H., On traces of Hecke operators, J. Fac. Sci. Univ. Tokyo, Sec. 1 **10** (1963), 1–19.
[21] Shimizu, H., On zeta functions of quaternion algebras, Ann. of Math. **81** (1965), 166–193.
[22] Shimura, G., Arithmetic of unitary groups, Ann. of Math. **79** (1964), 364–409.
[23] Weil, A., Adeles and algebraic groups, Lecture Notes, Institute for Advanced Study, Princeton, 1959–60.
[24] Weyl, H., The Classical Groups, Princeton Univ. Press, Princeton, 1939.

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