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On compact Riemannian manifolds admitting essential projective transformations

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

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§ 1. Introduction.

Let (M, g) be an m dimensional connected compact Riemannian manifold $(m\geqq 3)$. Let $C(M, g)$ (resp. $\mathcal{P}(M, g)$) be the group of conformal (resp. projective) transformations of (M, g), which is endowed with compact-open topology. We write $C(M, g)_0$ (resp. $\mathcal{P}(M, g)_0$) for the identity connected component of $C(M, g)$ (resp. $\mathcal{P}(M, g)$). We know that both $C(M, g)_0$ and $\mathcal{P}(M, g)_0$ are finite dimensional Lie transformation groups on M. It is well known that if $C(M, g)_0$ is not compact, then (M, g) is conformally equivalent to the standard sphere ([4]). On the other hand, the following has been a conjecture for a long time:

(1.1) CONJECTURE. *If $\mathcal{P}(M, g)_0$ is not compact, then (M, g) is projectively equivalent to either the standard sphere or the standard projective space.*

In this paper we shall prove a partial result on the above conjecture. To explain our result, let X be a C^∞ vector field on M. For $x\in M$, we define the *order* $o(X; x)$ of X at x as follows: Let $(x^1, \cdots, x^m)$ be a local coordinate system of M around x and $X=\sum a^i\partial/\partial x^i$ the local expression of X. We set

$$o(X; x)=\inf\{\textit{the order of zero of } a^i \text{ at } x\}.$$

Let $\mathfrak{p}(X, g)$ be the Lie algebra of infinitesimal projective transformations of (M, g). Then $\mathfrak{p}(M, g)$ is naturally identified with the Lie algebra of $\mathcal{P}(M, g)_0$. Moreover we know $o(X; x)\leqq 2$ for any $X\in\mathfrak{p}(M, g)$, $X\neq 0$, and any $x\in M$. We shall prove

(1.2) THEOREM. *If there exists a vector field X in $\mathfrak{p}(M, g)$ with $o(X, x)=2$ for some point x in M, then the compact connected Riemannian* *manifold (M, g) is projectively equivalent to either the standard sphere or the standard projective space.*

We remark that if $\mathcal{P}(M, g)_0$ is compact, then $o(X; x) \leqq 1$ for any $X \in \mathfrak{p}(M, g)$ and any $x \in M$. Thus we believe that our result is a substantial evidence supporting Conjecture 1.1 to be true. In this paper, we shall freely use the results proved in [3], [1], [2] and [5], although we recall all the definitions and the results which are needed in this paper.

§ 2. Projective spaces and projective groups.

Let $P^m(\boldsymbol{R})$ be a real projective space of dimension m with homogeneous coordinate system $[\xi^0 : \xi^1 : \cdots : \xi^m]$. The group of the projective transformations of $P^m(\boldsymbol{R})$, denoted by L, is $GL(m+1; \boldsymbol{R})$ modulo its center; the obvious action of $GL(m+1; \boldsymbol{R})$ on $\boldsymbol{R}^{m+1}$ induces the action of L on the projective space $P^m(\boldsymbol{R})$. We consider $\boldsymbol{R}^m$ as an open subset of $P^m(\boldsymbol{R})$ by the identification

$$(v^1, \cdots, v^m) \in \boldsymbol{R}^m \longmapsto [v^1 : \cdots : v^m : 1] \in P^m(\boldsymbol{R}). \tag{2.1}$$

Let o be the origin of $\boldsymbol{R}^m \subset P^m(\boldsymbol{R})$. Let L_0 be the isotropy subgroup of L at o so that $P^m(\boldsymbol{R}) = L/L_0$. Write $\mathfrak{l}$ for the Lie algebra of L and $\mathfrak{l}_0$ for the Lie subalgebra of $\mathfrak{l}$ corresponding to L_0. Then $\mathfrak{l}$ is naturally identified with $\mathfrak{sl}(m+1; \boldsymbol{R})$. Set

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline u & 0 \end{array} \right) \in \mathfrak{sl}(m+1; \boldsymbol{R});\ u \in \boldsymbol{R}^m \right\}$$

$$\mathfrak{g}_0 = \left\{ \left(\begin{array}{c|c} a & 0 \\ \hline 0 & B \end{array} \right) \in \mathfrak{sl}(m+1; \boldsymbol{R});\ a \in \boldsymbol{R} \right\}$$

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{array}{c|c} 0 & {}^t v \\ \hline 0 & 0 \end{array} \right) \in \mathfrak{sl}(m+1; \boldsymbol{R});\ v \in \boldsymbol{R}^m \right\}.$$

Then we have

$$\begin{aligned} &\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \\ &\mathfrak{l}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \\ &[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}. \end{aligned} \tag{2.2}$$

Here we set $\mathfrak{g}_i = 0$ if $i \neq -1, 0, 1$. We need a slightly different description of $\mathfrak{l}$. Let $(\boldsymbol{R}^m)^*$ be the dual space of $\boldsymbol{R}^m$; an element of $(\boldsymbol{R}^m)^*$ will be a row vector. Set

$$\mathfrak{l}' = \boldsymbol{R}^m \oplus \mathfrak{gl}(m; \boldsymbol{R}) \oplus (\boldsymbol{R}^m)^*.$$

Then $\mathfrak{l}'$ is a Lie algebra with the following bracket operation: if $u, v \in \boldsymbol{R}^m$, $\xi, \eta \in (\boldsymbol{R}^m)^*$ and $A, B \in \mathfrak{gl}(m; \boldsymbol{R})$, then

$$[u, v]=0\,, \quad [\xi, \eta]=0\,,$$
$$[A, u]=Au\,, \quad [\xi, A]=\xi A\,,$$
$$[A, B]=AB-BA\,, \quad [u, \xi]=u\xi+\xi u I_m\,,$$

where I_m denotes the identity matrix of degree m. Define a mapping $\iota: \mathfrak{l}' \to \mathfrak{l}$ by

$$\iota(u \oplus A \oplus \xi)=\left(\begin{array}{c|c} 0 & 0 \\ \hline u & 0 \end{array}\right)+\left(\begin{array}{c|c} -\dfrac{1}{m+1}\operatorname{Tr} A & 0 \\ \hline 0 & A-\left(\dfrac{1}{m+1}\operatorname{Tr} A\right) I_m \end{array}\right)+\left(\begin{array}{c|c} 0 & \xi \\ \hline 0 & 0 \end{array}\right).$$

Then ι is a Lie algebra isomorphism. By this isomorphism ι, we identify $\mathfrak{l}$ with $\mathfrak{l}'$, that is, $\mathfrak{g}_{-1}=\boldsymbol{R}^m$, $\mathfrak{g}_0=\mathfrak{gl}(m; \boldsymbol{R})$ and $\mathfrak{g}_1=(\boldsymbol{R}^m)^*$.

§ 3. Projective structures.

Let M be a C^∞ manifold of dimension m. Let $\pi_k: P^k(M) \to M$ be the bundle of the k-th frames of M. It is a principal bundle with the structure group $G^k(m)$. Remark that $G^1(m)=GL(m; \boldsymbol{R})$ and $\pi_1: P^1(M) \to M$ is the usual frame bundle of M. We also remark $G^1(m) \subset G^2(m)$ in the natural fashion. Let $\pi_2^1: P^2(M) \to P^1(M)$ be the natural projection. If $f: M \to M$ is a diffeomorphism of M, then f induces the bundle isomorphism $f_{(k)}: P^k(M) \to P^k(M)$ with $\pi_k \circ f_{(k)}=f \circ \pi_k$. Write Θ for the canonical form of $P^2(M)$ (for the definition, see p. 224 of [3]). Since Θ is $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$-valued, we set $\Theta=(\Theta^i)+(\Theta^i_j)$. We remark $f^*_{(2)}\Theta^i=\Theta^i$ and $f^*_{(2)}\Theta^i_j=\Theta^i_j$. Write $\theta=(\theta^i)$ for the canonical form of the frame bundle $P^1(M)$. Then we have

$$(\pi_2^1)^*\theta^i=\Theta^i\,. \tag{3.1}$$

With respect to the inclusion $\boldsymbol{R}^m \subset P^m(\boldsymbol{R})$ described in (2.1), we have the mapping

$$\sigma \in L_0 \longrightarrow j^2(\sigma) \in G^2(m)\,.$$

Since this mapping is injective, we consider L_0 as a subgroup of $G^2(m)$. Moreover we have

$$GL(m; \boldsymbol{R})=G^1(m) \subset L_0 \subset G^2(m)\,. \tag{3.2}$$

The Lie subalgebra of $\mathfrak{l}_0$ corresponding to $GL(m; \boldsymbol{R})$ with respect to inclusion (3.2) is exactly $\mathfrak{g}_0$ in (2.2).

A principal subbundle P of $P^2(M)$ with structure group L_0 ($\subset G^2(m)$) is called a *projective structure* on M. A diffeomorphism $f: M \rightarrow M$ is called an automorphism of P if $f_{(2)}(P)=P$. We write $\mathrm{Aut}(M; P)$ for the group of the automorphisms of P. A projective Cartan connection on P is, by definition, a 1-form ω on P with values in the Lie algebra $\mathfrak{l}$ of L satisfying the following conditions:

$$\omega(A^*)=A \qquad \textit{for every} \quad A \in \mathfrak{l}_0, \tag{3.3}$$

where A^* is the fundamental vector field on P corresponding to A;

$$(R_\sigma)^*\omega = Ad(\sigma^{-1})\omega, \tag{3.4}$$

where $Ad(\sigma)$ denotes the adjoint representation of L_0 on $\mathfrak{l}$;

$$\omega(X) \neq 0 \qquad \textit{for every nonzero tangent vector } X \textit{ of } P. \tag{3.5}$$

Let P^L be the principal bundle over M obtained by enlarging the structure group of P to L, that is

$$P^L = P \underset{L_0}{\times} L .$$

Then P is a subbundle of P^L and a projective Cartan connection ω in P can be uniquely extended to a principal L connection form on P^L, which is denoted by ω'. According to the decomposition $\mathfrak{l}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ in (2.2), we set

$$\omega=\omega_{-1}\oplus\omega_0\oplus\omega_1 \quad \text{and} \quad \omega'=\omega'_{-1}\oplus\omega'_0\oplus\omega'_1 .$$

The curvature form Ω of the projective Cartan connection ω is defined by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] .$$

According to the decomposition $\mathfrak{l}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1=\boldsymbol{R}^m\oplus\mathfrak{gl}(m; \boldsymbol{R})\oplus(\boldsymbol{R}^m)^*$, we set

$$\begin{aligned}\Omega &= \Omega_{-1}\oplus\Omega_0\oplus\Omega_1 \\ &= (\Omega^i)\oplus(\Omega^i_j)\oplus(\Omega_j) .\end{aligned}$$

Using the canonical form $\Theta=(\Theta^i)\oplus(\Theta^i_j)$, we have the decomposition as follows:

$$\begin{cases} \Omega^i = \dfrac{1}{2}\sum K^i_{jk}\Theta^j\wedge\Theta^k , \\ \Omega^i_j = \dfrac{1}{2}\sum K^i_{jkl}\Theta^k\wedge\Theta^l , \\ \Omega_i = \dfrac{1}{2}\sum K_{ijk}\Theta^j\wedge\Theta^k , \end{cases} \tag{3.6}$$

where K^i_{jk}, K^i_{jkl} and K_{ijk} are functions on P (see Proposition 2 of [3]).

We know that there exists a unique projective Cartan connection $\omega_P=\omega_{-1}\oplus\omega_0\oplus\omega_1$ with the following properties:

$$(3.7)\qquad \begin{cases} \omega_{-1}=(\Theta^i) \quad \text{and} \quad \omega_0=(\Theta^i_j)\,, \\ \sum_i \Omega^i_i=0\,, \\ \sum_i K^i_{jil}=0\,. \end{cases}$$

We call this unique projective Cartan connection ω_P the *normal projective Cartan connection* of the projective structure P. From the uniqueness, we have

$$(3.8)\qquad f^*_{(2)}\omega_P=\omega_P$$

for each $f\in \mathrm{Aut}(M;P)$. Each $f\in \mathrm{Aut}(M;P)$ extends uniquely to the bundle automorphism $f'_{(2)}$ of P^L. Then we have

$$(3.9)\qquad (f'_{(2)})^*\omega'_P=\omega'_P\,.$$

It is known that there exists a unique $\mathfrak{g}_0$-valued 2-form Φ^i_j on $P^1(M)$ such that $(\pi^2_1)^*\Phi^i_j=\Omega^i_j$. In view of (3.1) and (3.6), there exist the functions H^i_{jkl} on $P^1(M)$ such that $K^i_{jkl}=H^i_{jkl}\cdot\pi^1_2$. In particular (Ω^i_j) uniquely determines the tensor field W_P on M of type (1,3). We call W_P the *projective curvature tensor of Weyl*. We know if $W_P\equiv 0$, then the projective structure P is flat ([3]). We remark here

$$H^i_{jkl}\cdot f_{(1)}=H^i_{jkl}$$

for each $f\in \mathrm{Aut}(M;P)$. Or equivalently we have

$$(3.10)\qquad f^*W_P=W_P$$

for each $f\in \mathrm{Aut}(M;P)$.

From (3.3) and (3.5), we know that for each $v\in \boldsymbol{R}^m=\mathfrak{g}_{-1}$, there exists uniquely a vector field $B(v)$ on P satisfying

$$\Theta_{-1}(B(v))=v\,,\quad \Theta_0(B(v))=0 \quad\text{and}\quad \omega_1(B(v))=0\,.$$

We call $B(v)$ the *standard horizontal vector fields* corresponding to v.

§ 4. Development.

In this section we fix a C^∞ manifold M of dimension m and a projective structure P on M. We write $\omega_P=\Theta_{-1}+\Theta_0+\omega_1$ for the normal projective

Cartan connection of P. As explained in the previous section, the Cartan connection ω_P naturally defines the principal L-connection $\omega'_P=\Theta'_{-1}+\Theta'_0+\omega'_1$ on $P^L=P\underset{L_0}{\times}L$.

Fix $x\in M$ and $p\in P$ with $\pi_2(p)=x$. Let I be an open interval of $\boldsymbol{R}$ with $a\in I$. Let $c:I\to M$ be a regular C^∞ curve with $c(a)=x$. Let $c^*(t;a,p)$ be the horizontal lift of $c(t)$ through p in P^L with respect to the principal L-connection ω'_P, that is

$$\omega'_P\Big(\frac{d}{dt}c^*(t;a,p)\Big)=0 \quad \text{and} \quad c^*(a;a,p)=p\,. \tag{4.1}$$

Now choose any C^∞ curve $\rho:I\to L$ such that $\rho(a)=e$ and $c^*(t;a,p)\cdot\rho(t)\in P$ for any $t\in I$. Set $\bar{c}(t;a,p)=\rho(t)\cdot o\in P^m(\boldsymbol{R})$. Then $\bar{c}(t;a,p)$ is a C^∞ curve in $P^m(\boldsymbol{R})$ whose definition is independent of the particular choice of the curve $\rho(t)$. The curve $\bar{c}(t;a,p)$ is called the *development of the curve* $c(t)$ *at* $t=a$ *with initial point* p. Take $b\in I$ and $q\in P$ such that $\pi_2(q)=c(b)$. There exists uniquely $\tau\in L_0$ such that $q=c^*(b;a,p)\cdot\rho(b)\cdot\tau$. Then $c^*(t;b,q)=c^*(t;a,p)\cdot\rho(b)\cdot\tau$ and $c^*(t;b,q)\cdot\tau^{-1}\cdot\rho(b)^{-1}\cdot\rho(t)\tau\in P$. Hence we have

$$\bar{c}(t;b,q)=\tau^{-1}\cdot\rho(b)^{-1}\cdot\bar{c}(t;a,p)\,. \tag{4.2}$$

In particular we know that the curve $\bar{c}(t;b,q)$ is contained in a straight line in $P^m(\boldsymbol{R})$ if and only if $\bar{c}(t;a,p)$ is. Keeping this fact in mind, we call the regular curve $c(t)$ a *straight line* of P if the development $\bar{c}(t;a,p)$ is contained in a straight line through o in $P^m(\boldsymbol{R})$. Moreover if $\bar{c}(t;a,q)$ is a straight line with affine parameter t for some $q\in\pi_2^{-1}(x)$, then we call the curve $c(t)$ a *straight line with projective parameter.*

In the rest of this paper, we sometimes write $c^*(t)$ (resp. $\bar{c}(t)$) for $c^*(t;a,p)$ (resp. $\bar{c}(t;a,p)$) if there is no danger of confusion.

(4.3) LEMMA. *The development* $\bar{c}(t)$ *is a regular curve in* $P^m(\boldsymbol{R})$.

PROOF. Since $\pi_2(c^*(t)\cdot\rho(t))=\pi_2(c^*(t))=c(t)$ and $c(t)$ is a regular curve, we have

$$0=\Theta_{-1}\Big(\frac{d}{dt}(c^*(t)\cdot\rho(t))\Big)=\Theta'_{-1}\Big(\frac{d}{dt}(c^*(t)\cdot\rho(t))\Big)\,.$$

On the other hand, we have

$$\frac{d}{dt}(c^*(t)\rho(t))=\Big(\frac{d}{dt}c^*(t)\Big)\cdot\rho(t)+\Big(\rho(t)^{-1}\frac{d\rho(t)}{dt}\Big)^*c^*(t)\rho(t)\,.$$

Hence

$$0\neq \Theta'_{-1}\Big(\frac{d}{dt}c^*(t)\cdot\rho(t)\Big)+\Theta'_{-1}\Big(\Big(\rho(t)^{-1}\frac{d\rho(t)}{dt}\Big)^*_{c(t)\rho(t)}\Big).$$

$$=Ad(\rho(t)^{-1})\Theta'_{-1}\Big(\frac{d}{dt}c^*(t)\Big)+\Big(\text{the } \mathfrak{g}_{-1}\text{-component of } \rho(t)^{-1}\frac{d\rho(t)}{dt}\Big).$$

$$=\text{the } \mathfrak{g}_{-1}\text{-component of } \rho(t)^{-1}\frac{d\rho(t)}{dt}.$$

Clearly we have

$$\frac{d\bar{c}(t)}{dt}=0 \Longleftrightarrow \rho(t)^{-1}\frac{d\bar{c}(t)}{dt}=0$$

$$\Longleftrightarrow \rho(t)^{-1}\frac{d\rho(t)}{dt}\in \mathfrak{l}_0 \ (=\mathfrak{g}_0\oplus\mathfrak{g}_1)$$

$$\Longleftrightarrow \text{the } \mathfrak{g}_{-1}\text{-component of } \rho(t)^{-1}\frac{d\rho(t)}{dt}=0.$$

Hence we have $d\bar{c}(t)/dt\neq 0$. q. e. d.

(4.4) LEMMA. *Let f be in* Aut$(M;P)$ *with $f(x)=x$. Let σ_f be the unique element of L_0 such that $f_{(2)}(p)=p\cdot\sigma_f$. Then the curve $\sigma_f\cdot\bar{c}(t;a,p)$ is the development of the curve $f(c(t))$ at $t=a$ with $c(a)=p$.*

PROOF. Let $f'_{(2)}$ be the bundle isomorphism of P^L induced by $f_{(2)}$. From (3.9), we have $(f'_{(2)})^*\omega'_P=\omega'_P$. Hence $f'_{(2)}(c^*(t))\cdot\sigma_f^{-1}$ is the horizontal lift of $f(c(t))$ through p. Then we have

$$f'_{(2)}(c^*(t))\cdot\sigma_f^{-1}\cdot\sigma_f\cdot\rho(t)=f'_{(2)}(c^*(t))\cdot\rho(t)=f_{(2)}(c^*(t)\cdot\rho(t))\in P.$$

Thus the development of $f(c(t))$ at x with the initial condition p is $\sigma_f\cdot\rho(t)\cdot o=\sigma_f\cdot\bar{c}(t)$. q. e. d.

(4.5) LEMMA. *Let $\varphi: J\to I$ be a diffeomorphism with $\varphi(b)=a$. Then we have $\overline{(c\circ\varphi)}(t;b,p)=\bar{c}(\varphi(t);a,p)$.*

PROOF. We have $(c\circ\varphi)^*(t;b,p)=c^*(\varphi(t);a,p)$ and $(c\circ\varphi)^*\rho(\varphi(t))\in P$. Since $\bar{c}(\varphi(t);a,p)=\rho(\varphi(t))o$, our assertion is true. q. e. d.

(4.6) PROPOSITION. *Fix $x\in M$ and $p\in P$ with $\pi_2(p)=x$. There exist an open ball U of o in $\boldsymbol{R}^m\subset P^m(\boldsymbol{R})$, an open neighbourhood V of x in M and a diffeomorphism $E: U\to V$ satisfying the following conditions:*

(1) $E(o)=x$;

(2) *For the curve $c(t)=E(tv)$, we have $\bar{c}(t;o,p)=tv$.*

(3) *For any $v\in U$, $c(t)=E(tv)$ is a straight line of P with affine parameter such that $\dot{c}(o)=\pi_2(B(v)_p)$.*

PROOF. For each $v\in \boldsymbol{R}^m=\mathfrak{g}_{-1}$, we write $B(v)$ for the standard horizontal vector field corresponding to v. Remark that for each $s\in \boldsymbol{R}$, we have $B(sv)=sB(v)$. Set $F(v,t)=\pi_2(\exp tB(v)\cdot p)$. This is a C^∞-map which is defined for $\|v\|\leqq 1$, $|t|<\varepsilon$. We remark that $B(sv)=sB(v)$ implies $F(sv,t)=F(v,st)$. Set $E(v)=F(v,1)$. Then E is a C^∞-map which is defined for $\|v\|<\varepsilon/2$. Clearly $E(o)=x$. We shall calculate the differential of E at o. Take any $v\in \boldsymbol{R}^m$. Setting $\pi_2^1(p)=\alpha$, we have

$$
\begin{aligned}
\frac{d}{dt}E(tv)|_{t=0} &= \frac{d}{dt}F(tv,1)|_{t=0} \\
&= \frac{d}{dt}F(v,t)|_{t=0} \\
&= \pi_{2*}(B(v)_p) \\
&= \pi_{1*}(\pi_2^1)_*(B(v)_p) \\
&= \alpha(\theta((\pi_2^1)_*(B(v)_P))) \\
\text{(from (3.1))} \quad &= \alpha(\Theta_1(B(v)_P)) \\
&= \alpha(v)\,.
\end{aligned}
$$

Thus if we identify $T(M)_x$ with $\boldsymbol{R}^m$ by the frame $\alpha:\boldsymbol{R}^m\cong T(M)_x$, we have $E_{*o}=\mathrm{id}$. To prove (2), set $r(t)=(\exp tB(v)\cdot p)\cdot\exp(-tv)$ and $q(t)=\exp tB(v)\cdot p$. Then we have

$$\frac{dr(t)}{dt}=B(v)_{q(t)}\cdot\exp(-tv)-v^*_{r(t)}\,.$$

Hence

$$\omega'_P\left(\frac{dr(t)}{dt}\right)=Ad(\exp tv)v-v=0\,, \tag{4.7}$$

remarking $\mathfrak{g}_{-1}$ is abelian. Thus $r(t)$ is a horizontal lift of the curve $\pi_2(q(t))=F(v,t)=E(tv)$ with $r(0)=p$. Hence $c^*(t;0,p)=r(t)$ and $\rho(t)=\exp(tv)$. Thus we have $\bar{c}(t;0,p)=\exp tv\cdot o=tv$. The assertion (3) is clearly true.

q. e. d.

We call (U,E,V) the *projective normal coordinate* of P at x with respect to $p\in(\pi_2)^{-1}(x)$.

§ 5. Projective equivalence of affine connections.

Let M be a C^∞ manifold of dimension m. In this section we make use of fact 3.2 in § 3. Let ∇ be an affine connection on M with zero torsion. Then ∇ naturally defines the bundle inclusion $\iota_\nabla : P^1(M) \hookrightarrow P^2(M)$ (see Proposition 10 of [3]). We write $P(\nabla)$ for the L_0-subbundle of $P^2(M)$ obtained by enlarging the structure group of $\iota_\nabla(P^1(M))$ to L_0. Thus the affine connection ∇ naturally defines the projective structure $P(\nabla)$ on M. Two affine connections ∇_1 and ∇_2 are called *projectively equivalent* if $P(\nabla_1)=P(\nabla_2)$. A diffeomorphism $f: M\to M$ is called a projective transformation of ∇ if ∇ and $f_*\nabla$ are projectively equivalent. The following facts are proved in Proposition 12 of [3]:

(5.1) FACTS. (1) *The correspondence $\nabla \mapsto P(\nabla)$ gives rise to the bijection between the projectively equivalent classes of affine connections with zero torsion on M and the projective structures on M.*

(2) *Let ∇_1 and ∇_2 be affine connections with zero torsion on M. Then the following three conditions are mutually equivalent:*

(a) *∇_1 and ∇_2 are projectively equivalent;*

(b) *∇_1 and ∇_2 define the same geodesics if the parametrization is disregarded;*

(c) *There exists a 1-form α such that $\nabla_{1X}Y-\nabla_{2X}Y=\alpha(X)Y+\alpha(Y)X$ for arbitrary vector fields X and Y.*

(3) *In particular, we know that a diffeomorphism $f: M\to M$ is a projective transformation of an affine connection ∇ with zero torsion if and only if $f\in \mathrm{Aut}(M; P(\nabla))$.*

Let P be a projective structure on M. An affine connection ∇ is said to belong to P if $P=P(\nabla)$.

(5.2) LEMMA. *Let P be a projective structure on M and ∇ an affine connection belonging to P. Let c; $(-\varepsilon, \varepsilon)\to M$ be a C^∞ regular curve with $c(0)=x$. Then $c(t)$ is a geodesic with respect to ∇ (the parametrization being disregarded) if and only if the curve $c(t)$ is a straight line of P.*

PROOF. Fix $p\in P$ such that $\pi_2(p)=x$. According to Proposition 12 in [3], we know that the following two conditions are mutually equivalent:

(5.3) *$c(t)$ is a geodesic with respect to ∇;*

(5.4) *There exists a standard horizontal vector field $B(v)$ on P corresponding to $v\in\mathfrak{g}_{-1}$ such that $c(t)$ and $\pi_2(\exp sB(v)\cdot p)$ coincide with each other if their parametrizations are disregarded.*

Now suppose $c(t)$ is a geodesic with respect to ∇. Set $r(s)=(\exp sB(v)\cdot p)\times\exp(-sv)$ and $q(s)=\exp sB(v)\cdot p$. Then we have

$$\frac{dr(s)}{ds}=B(v)_{q(s)}\cdot\exp(-sv)-v^*_{r(s)}.$$

Hence $\omega'_P(dr(s)/ds)=Ad(\exp(sv))v-v=0$ ($\mathfrak{g}_{-1}$ being abelian). Thus $r(s)$ is a horizontal lift of the curve $\gamma(s)=\pi_2(q(s))$ with $r(0)=p$. Hence $r(s)=\gamma^*(s\,;0,p)$ and $\gamma^*(s\,;0,p)\exp(sv)\in P$. Therefore we have $\bar{\gamma}(s\,;0,p)=\exp(tv)\cdot o$, which is a straight line through o in $P^m(\boldsymbol{R})$. Then from Lemma (4.5), we know $c(t)$ is a straight line of P. Conversely suppose $c(t)$ is a straight line of P. So $\bar{c}(t\,;a,p)$ is contained in a straight line through o, say, $\exp sv\cdot o$ ($v\in\mathfrak{g}_{-1}$) of $P^m(\boldsymbol{R})$. From Lemma (4.3), there exists a diffeomorphism $t=\varphi(s)$ such that

$$\bar{c}(\varphi(s)\,;a,p)=\exp(sv)\cdot o \quad\text{and}\quad \varphi(0)=a.$$

From Lemma (4.5), we have $c^*(\varphi(s)\,;a,p)\cdot\exp(sv)\in P$. Then

$$\begin{aligned}&\omega_P\Big(\frac{d}{ds}(c^*(\varphi(s)\,;a,p)\cdot\exp(sv))\Big)\\ &=\omega'_P\Big[\frac{d}{ds}(c^*(\varphi(s)\,;a,p)\cdot\exp(sv))+v^*_{c^*(\varphi(s);a.p)\exp(sv)}\Big]\\ &=v.\end{aligned}$$

Therefore the curve $c^*(\varphi(s)\,;a,p)\exp(sv)$ is the integral curve of the standard horizontal curve $B(v)$ through p. From (5.4), $c(\varphi(s))=\pi_2(\exp(sB(v))\cdot p)$ is a geodesic of ∇. q. e. d.

In the rest of this section we fix a projective structure P on M. We also fix $x\in M$.

(5.5) LEMMA. *Let f be in* $\mathrm{Aut}(M\,;P)$ *with $f(x)=x$. Let $c:(-\varepsilon,\varepsilon)\to M$ be a straight line of P with projective parameter such that $c(0)=x$ and $f_*(\dot{c}(0))\in\{\dot{c}(0)\}$. Suppose $\bar{c}(t\,;0,p)=tv\subset\boldsymbol{R}^m\subset P^m(\boldsymbol{R})$ $(v\in\boldsymbol{R}^m-\{o\})$. Then we have:*

(1) *$\sigma_f(v)\in\{v\}$, where $\sigma_f\in L_0$ is defined by $f_{(2)}(p)=p\cdot\sigma_f$.*

(2) *$f(c(t))=c(\varphi(t))$, where $\varphi(t)$ is defined by $\sigma_f(tv)=\varphi(t)v$. In particular φ is of the form $\varphi(t)=t(at+b)^{-1}$.*

PROOF. From Lemma 4.4, we have $\overline{(f\circ c)}(t\,;0,p)=\sigma_f(\bar{c}(t\,;0,p))$. Since $\bar{c}(t\,;0,p)$ is a straight line in $P^m(\boldsymbol{R})$ with affine parameter, the curve $\sigma_f(\bar{c}(t\,;0,p))$ is contained in a straight line in $P^m(\boldsymbol{R})$. Fix an affine connection ∇ with zero torsion belonging to P. From Lemma 5.2, both $c(t)$ and $f(c(t))$ are geodesics with respect to ∇ (the parametrizations being not necessarily affine parameters). Since $df(c(t))/dt|_{t=0}\in\{\dot{c}(0)\}$, there is a diffeomorphism $\varphi:(-\varepsilon,\varepsilon)\to(-\varepsilon,\varepsilon)$ with $\varphi(0)=0$ satisfying $f(c(t))=c(\varphi(t))$. From

Lemma 4.5, we obtain

$$\sigma_f(\bar{c}(t\,;0,p))=\bar{c}(\varphi(t)\,;0,p)\,.$$

Since $\bar{c}(s\,;0,p)=sv$, the above equality implies $\sigma_f(tv)=\varphi(t)v$. Therefore $\varphi(t)=t(at+b)^{-1}$ for some $a,b\in\boldsymbol{R}$. q. e. d.

(5.6) LEMMA. *Let $\{f_s\}\subset\mathrm{Aut}(M\,;P)$ be a 1-parameter subgroup such that $f_s(x)=x$ and $f_s=\mathrm{id}$ at x for each $s\in\boldsymbol{R}$. Let (U,E,V) be the projective normal coordinate of P at x with respect to $p\in(\pi_2)^{-1}(x)$. (1) f_s is written with respect to this coordinate system as follows:*

$$f_s(v)=\frac{v}{1+s\xi v}\qquad \textit{for some}\quad \xi\in\mathfrak{g}_1\ (=(\boldsymbol{R}^m)^*)\,.$$

(2) *If $\xi\neq 0$, we have $W_P=0$ on V.*

PROOF. It remains to prove assertion (2). Set $E=(x^1,\cdots,x^m)$. Take any $v\in U$ with $\xi v>0$. Then we have $f_s(v)\in U$ for any $s\geqq 0$, and $f_s(v)\to o$ as $s\to\infty$. Set $f_s(x)=(f_s^1(x),\cdots,f_x^m(x))$. From (1), we have

$$\frac{\partial f_s^j}{\partial x^i}(v)=\frac{\delta_i^j(1+s\xi v)-sv^j\xi^i}{(1+s\xi v)^2}$$

and

$$\frac{\partial f_{-s}^j}{\partial x^i}(f_s(v))=(\delta_i^j+sv^j\xi^i)(1+s\xi v)\,.$$

Set

$$W_P\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k}=\sum_{l=1}^m W_{kij}^l\frac{\partial}{\partial x^l}\,.$$

Since f_s leaves W_P invariant, we have

$$\begin{aligned}&W_P\Big(\Big(\frac{\partial}{\partial x^i}\Big)_v,\Big(\frac{\partial}{\partial x^j}\Big)_v\Big)\Big(\frac{\partial}{\partial x^k}\Big)_v\\&=(f_s^*W_P)\Big(\Big(\frac{\partial}{\partial x^i}\Big)_v,\Big(\frac{\partial}{\partial x^j}\Big)_v\Big)\Big(\frac{\partial}{\partial x^k}\Big)_v\\&=f_{-s*}\Big(W_P\Big(f_{s*}\Big(\frac{\partial}{\partial x^i}\Big)_v,f_{s*}\Big(\frac{\partial}{\partial x^j}\Big)_v\Big)f_{s*}\Big(\frac{\partial}{\partial x^k}\Big)_v\Big)\,.\end{aligned}$$

Hence we have

$$W'^{l}_{kij}(v)=\frac{\partial f^{l}_{-s}}{\partial x^{a}}(f_s(v))W^{a}_{bcd}(f_s(v))\frac{\partial f^{b}_{s}}{\partial x^{k}}(v)\frac{\partial f^{c}_{s}}{\partial x^{i}}(v)\frac{\partial f^{d}_{s}}{\partial x^{j}}(v)$$

$$=W^{a}_{bcd}(f_s(v))(\delta^{l}_{a}+sv^{l}\xi^{a})(1+s\xi v)$$

$$\times\frac{\{\delta^{b}_{k}(1+s\xi v)-sv^{b}\xi^{k}\}\{\delta^{c}_{i}(1+s\xi v)-sv^{c}\xi^{i}\}\{\delta^{d}_{j}(1+s\xi v)-sv^{d}\xi^{j}\}}{(1+s\xi v)^{6}}.$$

Hence taking $s\rightarrow\infty$, we see $W'^{l}_{kij}(v)=0$. By the same argument, we have $W^{l}_{kij}(v)=0$ for any $v\in U$ with $\xi v<0$. Therefore $W^{l}_{kij}=0$ on U.

(5.7) LEMMA. *Let $\{f_s\}\subset\mathrm{Aut}(M;P)$ be a 1-parameter subgroup such that $f_s(x)=x$ and $(f_s)_*=\mathrm{id}$ at x for each $s\in\boldsymbol{R}$. Let ∇ be an affine connection with zero torsion on M which belongs to P and $c:\boldsymbol{R}\rightarrow M$ a geodesic with respect to ∇, $c(0)=x$. Then we have the following:*

(1) *There exists a 1-parameter group of diffeomorphisms φ_s of $\boldsymbol{R}$ such that $\varphi(0)=0$ and $f_s(c(t))=c(\varphi_s(t))$:*

(2) *If $\{\varphi_s\}$ is non trivial, then $W_P(c(t))=0$ for any $t\in\boldsymbol{R}$.*

PROOF. From Lemma 5.2, we know $f_s(c(t))$ is also a geodesic with respect to ∇, although the parametrization is not necessarily the affine parameter. Since $(f_s)_*=\mathrm{id}$ at x, we have $\{f_s\cdot c(t);t\in\boldsymbol{R}\}=\{c(t);t\in\boldsymbol{R}\}$. From this assertion (1) is true. To prove (2), let X be the vector field on M which generates $\{f_s\}$. Fix $p\in P$ with $\pi_2(p)=x$, and set $\alpha=\pi^1_2(p)$. Since $(f_s)_*=\mathrm{id}$ at x, we have $(f_s)_{(1)}(\alpha)=\alpha$ for any $s\in\boldsymbol{R}$. Hence there exists $\xi\in(\boldsymbol{R}^m)^*$ $(=\mathfrak{g}_1)$ such that $(f_s)_{(2)}(p)=p\cdot\exp(s\xi)$. Set $\sigma(s)=\exp(s\xi)$. From Lemma 4.4 and Lemma 4.5, we have

$$\sigma(s)\cdot\bar{c}(t;0,p)=\bar{c}(\varphi_s(t);0,p). \tag{5.8}$$

From Lemma 5.2, we know $\bar{c}(t;0,p)$ is contained in a straight line through o in $P^m(\boldsymbol{R})$. Hence $\{\varphi_s(t)\}$ is a 1-parameter group of projective transformations of $\boldsymbol{R}$. In particular, if we set $G=\{t\in\boldsymbol{R};\varphi_s(t)=t$ for any $s\in\boldsymbol{R}\}$, then G is a discrete subset of $\boldsymbol{R}$. Let $E:U\cong V$ be the projective normal coordinate of P with respect to p. Set $v=\dot{c}(0)\in\boldsymbol{R}^m$ $(=\mathfrak{g}_{-1})$ in terms of this coordinate system. We shall prove the following fact:

(5.9) $o(X;c(a))=2$ *for any $a\in G$ if $\{\varphi_s\}$ is non trivial.*

Since $\{\varphi_s\}$ is nontrivial, we have $\xi v\neq 0$. So assume $\xi v>0$. (The case $\xi v<0$ can be treated similarly.) So assume $G\neq\{0\}$. Take $a\in G$ such that $(0,a)\cap G=\emptyset$ or $(a,0)\cap G=\emptyset$. Assume $(0,a)\cap G=\emptyset$. Choose $b\in(0,a)$ so that $bv\in U$. Then $\varphi_s(b)\rightarrow a$ as $s\rightarrow-\infty$. From (5.6) we know $\bar{c}(a;0,p)=o$. Choose $\rho(t)\in L$ such that $\rho(0)=\mathrm{id}$ and $c^*(t;0,p)\rho(t)\in P$. Set $q=c^*(a;0,p)\rho(a)$.

Since $f_s(c(a))=c(a)$, there exists uniquely $\tau(s)\in L_0$ such that $(f_s)_{(2)}(q)=q\cdot\tau(s)$. We claim $\tau(s)=\rho(a)^{-1}\sigma(s)\rho(a)$. In fact,

$$\begin{aligned}(f_s)_{(2)}(c^*(t;0,p)\rho(t)) &= (f_s)'_{(2)}(c^*(t;0,p))\cdot\rho(t)\\ &= c^*(t;0,p)\sigma(s)\cdot\rho(t)\\ &= c^*(t;0,p)\cdot\rho(t)\cdot\rho(t)^{-1}\cdot\sigma(s)\cdot\rho(t).\end{aligned}$$

On the other hand

$$(f_s)_{(2)}(c^*(a;0,p)\rho(a))=q\cdot\tau(s).$$

Thus $\tau(s)=\rho(a)^{-1}\sigma(s)\rho(a)$. Since $\rho(a)\cdot o=c(\bar{a};0,u)=o$, we have $\rho(a)\in L_0$. Since $\sigma(s)\in\exp\mathfrak{g}_1$ and $\exp\mathfrak{g}_1$ is normal in L_0, we see $\tau(s)\in\exp\mathfrak{g}_1$. Then $(f_s)_{(1)}(\pi_2^1(q))=\pi_2^1((f_s)_{(2)}(q))=\pi_2^1(q\cdot\tau(s))=\pi_2^1(q)$. So we know $(f_s)_*=\mathrm{id}$ at $c(a)$, i.e., $o(X;c(a))=2$. Similarly for $o(X;c(a))=2$ in case $(a,0)\cap G=\emptyset$. Now replacing x with $c(a)$ and ξ with $Ad(\rho(a)^{-1})\xi$ where $(0,a)\cap G=\emptyset$ or $(a,0)\cap G=\emptyset$, we find $o(X;b)=2$ if $(a,b)\cap G=\emptyset$ $(a>0)$, or $(b,a)\cap G=\emptyset$ $(a<0)$. Repeating the same argument, we conclude $o(X;a)=2$ for any $a\in G$. Now set $G=\{A_n\}_{n=0,\pm1,\pm2,\cdots}$. Then $\{\varphi_s\}$ is transitive on (A_n,A_{n+1}). From (3.10) and Lemma 5.6, we know $W_P(c(t))=0$ for any $t\in(A_n,A_{n+1})$. Thus we have $W_P(c(t))=0$ for any $t\in\boldsymbol{R}$. q. e. d.

§ 6. Proof of Theorem 1.2.

We keep the notation in § 1. Let $\{f_s\}$ be the 1-parameter group of transformations generated by the vector field X.

(6.1) LEMMA. *The set $\{x\in M;o(X;x)=2\}$ consists of at most two points. In particular, the fundamental group $\pi_1(M)$ of M is isomorphic to either $\{1\}$ or $\boldsymbol{Z}_2$.*

PROOF. Suppose we had three distinct points x_1, x_2 and x_3 such that $o(X;x_i)=2$ $(i=1,2,3)$. Take any $y\in M$ such that $X(y)\neq 0$. Consider the orbit $I=\{f_s(y);s\in\boldsymbol{R}\}$, a 1-dimensional submanifold of M. Let $c_i:\boldsymbol{R}\to M$ be geodesics which pass through x_i and y $(i=1,2,3)$. Since, $(f_s)_*=\mathrm{id}$ at x_i $(i=1,2,3)$, we know f_s leaves each $c_i(\boldsymbol{R})$ invariant. Hence $I\subset c_i(\boldsymbol{R})$. In particular we have $c_1(\boldsymbol{R})=c_2(\boldsymbol{R})=c_3(\boldsymbol{R})$ for every point y. This is impossible.

Now let $\pi:M^*\to M$ be the universal covering of M. Let g^* and X^* be the lift of g and X to M^* respectively. Then we have $X^*\in p(M^*,g^*)$ and $o(X^*;y)=2$ if $o(X;\pi(y))=2$. q. e. d.

PROOF OF THEOREM 1.2. Let ∇ be the Levi-Civita connection of g and $P=P(\nabla)$ the projective structure to which ∇ belongs. Set $F=$

$\{y \in M ; X(y)=0\}$. Take any $y \in M-F$. Let $c(t)$ be a geodesic joining x and y. From Lemma 5.7, we have $W_P(c(t))=0$. Hence we have W_P vanishes on $M-F$. On the other hand, we know $M-F$ contains no interior point. Hence W_P vanishes on M. Let $\pi^* : M^* \to M$ be the universal covering of M. M^* is compact by Lemma 6.1. Let g^*, X^* and P^* be the natural lift of g, X and P respectively. Then we have $o(X^* ; y)=2$ for $y \in \pi^{-1}(x)$. Hence $W_P=0$ on M. From Theorem 16 of [3], we have our conclusion.

q. e. d.

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