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Certain pairs of finite groups with a common 2-subgroup[*)]

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In this paper we study certain pairs of finite groups (G, H) satisfying the following conditions:

(a) G and H have a common 2-subgroup S;
(b) both indices $q=|G:S|$ and $r=|H:S|$ are odd prime numbers;
(c) no nonidentity subgroup of S is normal both in G and in H;
(d) $C_G(O_2(G))\leqq O_2(G)$ and $C_H(O_2(H))\leqq O_2(H)$.

A familiar example of the pairs satisfying the conditions (a)-(d) is a pair (P_1, P_2) of nontrivial parabolic subgroups of certain (normal or twisted) Chevalley groups X of rank 2 defined over $GF(2)$ chosen so that P_1 and P_2 contain a common Borel subgroup B of X. In all such pairs (P_1, P_2), the common Borel subgroup is the common 2-subgroup, and the pair $(P_1\cap X', P_2\cap X')$ also satisfies the conditions (a)-(d) with respect to the common 2-subgroup $B\cap X'$, where X' is the commutator subgroup of X.

The main purpose of this paper is to show that every pair satisfying the conditions (a)-(d) with $\{q, r\}\neq\{3\}$ is isomorphic, either to a pair (P_1, P_2), or to a pair $(P_1\cap X', P_2\cap X')$ chosen as above in the twisted Chevalley group $X={}^2F_4(2)$. Here, by definition, two pairs (G_i, H_i) $(i=1,2)$ containing a common 2-subgroup S_i are isomorphic if there are isomorphisms $s: S_1\rightarrow S_2$, $g: G_1\rightarrow G_2$, and $h: H_1\rightarrow H_2$ such that $g(x)=s(x)=h(x)$ for all $x\in S_1$.

Since Fan [1] obtained the same result, we will describe our main result with emphasis on the differences between Fan's paper and ours. In a pioneering work [4], Goldschmidt considered the pairs (G, H) satisfying the conditions (a)-(c) with $\{q, r\}=\{3\}$, and determined all the isomorphism classes of such pairs by a graph-theoretical approach. Fan [1] pursues the graph-theoretical approach of Goldschmidt. Independently of the works of Goldschmidt and Fan, Gomi also considered the pairs satisfying the conditions (a)-(d) from a group-theoretical point of view and obtained a certain

*) This is the author's master's thesis at the University of Tokyo in 1985, under the supervision of Professor Kensaku Gomi.

description of the possible pairs [5]. Our aim is to pursue Gomi's group-theoretical approach. In order to state a relevant result in [5], we need the following definition.

(1.1) DEFINITION. Let (G, H) be a pair satisfying the conditions (a)–(d) with respect to a common 2-subgroup S.

(1) The pair (G, H) is of *${}^2F_4(2)'$-type* if $G/O_2(G)\cong\Sigma_3$, $H/O_2(H)\cong\boldsymbol{F}_{20}$ (the Frobenius group of order 20), and there is an H-composition series of $O_2(H)$

$$O_2(H)=R_0\geqq R_1\geqq R_2\geqq R_3=1$$

such that the subgroups Q_i $(i=0, 1, \cdots, 6)$ defined below form a G-composition series of $O_2(G)$:

$$Q_0=O_2(G)\ ;\ Q_1=\langle R_1^G\rangle\ ;\ Q_2=\cap R_3^G\ ;\ Q_3=\langle (R_1\cap Q_2)^G\rangle\ ;$$
$$Q_4=\cap R_1^G\ ;\ Q_5=\langle R_2^G\rangle\ ;\ Q_6=1\,.$$

(2) Let $S^*=(S\cap O^2(G))(S\cap O^2(H))$, $G^*=O^2(G)S^*$, and $H^*=O^2(H)S^*$. (Then the pair (G^*, H^*) necessarily satisfies the conditions (a)–(d) with respect to the common 2-subgroup S^*.) The pair (G, H) is of *${}^2F_4(2)$-type* if $|S:S^*|=2$ and the pair (G^*, H^*) is of ${}^2F_4(2)'$-type.

Our starting point is the following theorem in [5].

(1.2) THEOREM (Gomi [5]). *Let (G, H) be a pair satisfying the conditions* (a)–(d) *and assume $r\neq 3$. Then the pair (G, H) is of ${}^2F_4(2)'$-type or of ${}^2F_4(2)$-type. Furthermore, if the pair (G, H) is of ${}^2F_4(2)'$-type, then $|G|=2^{11}\cdot 3$ and $|H|=2^{11}\cdot 5$.*

Given (1.2), our first task is to prove that there is a unique isomorphism class of the pairs of ${}^2F_4(2)'$-type. We accomplish this by showing that every pair of ${}^2F_4(2)'$-type has a "standard presentation". Before stating this result precisely, let us consider presentations for ${}^2F_4(2)'$.

Tits gives in [10] a presentation for ${}^2F_4(2)$, and Parrott obtains in [8] a presentation for ${}^2F_4(2)'$ from the Tits presentation for ${}^2F_4(2)$ by the Reidemeister-Schreier method. Modifying Parrott's presentation slightly, we get the following presentation for ${}^2F_4(2)'$.

A presentation for ${}^2F_4(2)'$

Generators: σ_i $(i=1, \cdots, 8)$; τ_j $(j=2, 4, 6)$; ρ_k $(k=1, 8)$.

Relations:

$$\boldsymbol{R}_0 \begin{cases} \sigma_i^2=1 \ (i=1,\cdots,8)\,;\ [\sigma_i,\sigma_{i'}]=1 \ (|i-i'|\leqq 4)\,; \\ [\sigma_1,\sigma_6]=\sigma_3\,;\ [\sigma_1,\sigma_7]=\sigma_3\sigma_4\sigma_5\,;\ [\sigma_1,\sigma_8]=\sigma_2\sigma_3\sigma_4\tau_4\sigma_6\sigma_7\,; \\ [\sigma_2,\sigma_7]=[\sigma_3,\sigma_8]=\sigma_5\,;\ [\sigma_2,\sigma_8]=\sigma_4\sigma_6\,; \\ \tau_j^2=\sigma_{j-1}\sigma_j\sigma_{j+1} \ (j=2,4,6)\,;\ [\tau_2,\tau_6]=\sigma_2\tau_4\sigma_5\sigma_6\,; \\ [\tau_j,\tau_{j+2}]=\sigma_j\sigma_{j+1}\sigma_{j+2} \ (j=2,4)\,; \\ [\sigma_i,\tau_j]=\begin{cases} 1 & (|i-j|=0,1,3) \\ \sigma_{(i+j)/2} & (|i-j|=2)\,; \end{cases} \\ [\sigma_6,\tau_2]=\sigma_3\sigma_4\,;\ [\sigma_7,\tau_2]=\sigma_3\sigma_6\,;\ [\sigma_8,\tau_2]=\sigma_2\sigma_3\sigma_4\sigma_6\tau_6\,; \\ [\sigma_8,\tau_4]=\sigma_5\sigma_6\,;\ [\sigma_1,\tau_6]=\sigma_2\sigma_3\sigma_5\,;\ [\sigma_2,\tau_6]=\sigma_4\sigma_5\,; \end{cases}$$

$$\boldsymbol{R}_1^* \begin{cases} \rho_1^2=1\,;\ \rho_1\sigma_i\rho_1=\sigma_{10-i} \ (i=2,\cdots,8)\,; \\ (\sigma_1\rho_1)^5=1\,;\ \rho_1\tau_4\rho_1=\sigma_6\tau_6\,;\ \rho_1\tau_2\rho_1=(\tau_2^2\rho_1)^2\tau_2\,; \end{cases}$$

$$\boldsymbol{R}_8^* \begin{cases} \rho_8^2=1\,;\ \rho_8\sigma_i\rho_8=\sigma_{8-i} \ (i=1,\cdots,7)\,; \\ (\sigma_8\rho_8)^3=1\,;\ \rho_8\tau_2\rho_8=\sigma_6\tau_6\,;\ \rho_8\tau_4\rho_8=\sigma_4\tau_4\,; \end{cases}$$

$$\boldsymbol{R}^* \quad (\rho_1\rho_8)^8=1\,.$$

(We will write $\boldsymbol{R}_1=\boldsymbol{R}_0\cup\boldsymbol{R}_1^*$ and $\boldsymbol{R}_8=\boldsymbol{R}_0\cup\boldsymbol{R}_8^*$.)

Now we can state our first main result.

THEOREM 1. *Let (G,H) be a pair of ${}^2F_4(2)'$-type with respect to a common 2-subgroup S. Then there exist elements σ_i $(i=1,\cdots,8)$ and τ_j $(j=2,4,6)$ of S, an element ρ_1 of H, and an element ρ_8 of G, such that*

(1) $S=\langle\sigma_1,\cdots,\sigma_8,\tau_2,\tau_4,\tau_6\rangle$, $G=\langle S,\rho_8\rangle$, $H=\langle S,\rho_1\rangle$, *and*
(2) *the elements σ_i,τ_j,ρ_k satisfy the relations $\boldsymbol{R}_0$, $\boldsymbol{R}_1$, and $\boldsymbol{R}_8$.*

For simplicity, we will phrase this theorem as follows:

Every pair of ${}^2F_4(2)'$-type has standard generators $\{\sigma_i,\tau_j,\rho_k\}$ (subject to standard relations $\{\boldsymbol{R}_l\}$), or every pair of ${}^2F_4(2)'$-type has a standard presentation.

We can derive from Theorem 1 the uniqueness of the isomorphism classes of the pairs of ${}^2F_4(2)'$-type. To see this, we fix the following notation which we will use throughout the remainder of this section. Let $Y={}^2F_4(2)$ and define the elements

$$u_i \quad (i=1,\cdots,8) \quad \text{and} \quad r_k \quad (k=1,8)$$

of Y as in Section 4 of [10]. Define

$$\sigma_i=u_i^{\varepsilon_i} \qquad \varepsilon_i=\begin{cases} 1 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd,} \end{cases}$$

$$\tau_j = u_{j-1}u_{j+1},$$

$$\rho_k = r_k.$$

Then σ_i, τ_j and ρ_k are contained in Y', and satisfy the relations $\boldsymbol{R}_0$, $\boldsymbol{R}_1$, and $\boldsymbol{R}_8$. Also, define

$$B = \langle u_1, \cdots, u_8 \rangle, \quad P_1 = \langle B, r_1 \rangle, \quad P_8 = \langle B, r_8 \rangle,$$

$$U = \langle \sigma_1, \cdots, \sigma_8, \tau_2, \tau_4, \tau_6 \rangle, \quad M_1 = \langle U, \rho_1 \rangle, \quad M_8 = \langle U, \rho_8 \rangle.$$

Then we have $|B| = 2^{12}$, $|P_1| = 2^{12} \cdot 5$, and $|P_8| = 2^{12} \cdot 3$ because B is a Borel subgroup of Y, and P_1 and P_8 are the nontrivial parabolic subgroups of Y containing B such that $P_1/O_2(P_1) \cong Sz(2)$ and $P_8/O_2(P_8) \cong SL_2(2)$. Now, the relations $\boldsymbol{R}_l$ bound the orders of U, M_1, and M_8: $|U| \leqq 2^{11}$; $|M_1| \leqq 2^{11} \cdot 5$; $|M_8| \leqq 2^{11} \cdot 3$, while since $U = B \cap Y'$, $M_1 = P_1 \cap Y'$, $M_8 = P_8 \cap Y'$, and $|Y : Y'| = 2$, we have $|U| = 2^{11}$, $|M_1| = 2^{11} \cdot 5$, and $|M_8| = 2^{11} \cdot 3$. Hence $\boldsymbol{R}_0$, $\boldsymbol{R}_1$, and $\boldsymbol{R}_8$ are, respectively, the defining relations for U, M_1, and M_8 with respect to their generators σ_i, τ_j, and ρ_k. Therefore, if (G, H) is a pair of ${}^2F_4(2)'$-type, Theorem 1 yields that the pair (G, H) is isomorphic to the pair (M_8, M_1) because $|S| = 2^{11}$, $|G| = 2^{11} \cdot 3$, and $|H| = 2^{11} \cdot 5$.

The method used in our proof of Theorem 1 is analogous to the method which Parrott [8] uses to determine all simple groups X containing an involution whose centralizer H has the following properties.

(i) $J = O_2(H)$ has order 2^9 and is of class at least 3.
(ii) H/J is isomorphic to $\boldsymbol{F}_{20}$.
(iii) If P is a Sylow 5-subgroup of H, then $C_J(P) \leqq Z(J)$.

Using the conditions (i)-(iii) and the simplicity of X, Parrott constructs a certain subgroup G of X, determines the precise structure of G and H in terms of generators and relations, and shows that the generators of G and H together satisfy defining relations for ${}^2F_4(2)'$, thereby proving $X \cong {}^2F_4(2)'$. Now, let (G, H) be a pair of ${}^2F_4(2)'$-type. Then H satisfies the above conditions (i)-(iii). Of course, we have no information that G and H are contained in some simple group X and that H is the centralizer of an involution of X. But we are given a group G together with a group H from the beginning, and, using information on G and H, we get more than the conditions (i)-(iii). Therefore, it is not surprising that Parrott's method applies to the proof of Theorem 1 as well provided it is suitably modified.

In order to determine the isomorphism classes of the pairs of ${}^2F_4(2)$-type, we first prove the following theorem.

THEOREM 2. *Let X be a finite group having a pair of subgroups (G, H) of ${}^2F_4(2)'$-type with $C_X(R_2)=H$ where R_2 is as in* (1.1-1). *Then $X=\langle G, H\rangle$, and for any standard generators $\{\sigma_i, \tau_j, \rho_k\}$ of the pair (G, H), we have $(\rho_1\rho_8)^8=1$.*

As ${}^2F_4(2)'$ is simple and the relations $\boldsymbol{R}_0$, $\boldsymbol{R}_1$, $\boldsymbol{R}_8$, and $(\rho_1\rho_8)^8=1$ together form defining relations for ${}^2F_4(2)'$, Theorem 2 implies that X is isomorphic to ${}^2F_4(2)'$.

Our proof of Theorem 2 follows the same line as Parrott's proof of Lemma 8 in [8], but, in order to clarify the argument, we will use fusion lemmas of Glauberman and Goldschmidt instead of transfer lemmas of Thompson and Grün used in the Parrott's proof.

As is mentioned above, our proof of Theorems 1 and 2 is analogous to the proof of Parrott's theorem. In fact, given (1.2) and Theorems 1 and 2, Parrott's theorem will follow once we prove the following:

Let X be a simple group having an involution whose centralizer H satisfies the conditions (i)-(iii). *Then there exists a subgroup G of X such that the pair (G, H) satisfies the conditions* (a)-(d).

Now, we derive the following theorem from Theorem 2.

THEOREM 3. *Let (G, H) be a pair of ${}^2F_4(2)$-type with respect to a common 2-subgroup S, and embed G and H into the amalgamated product $X=G *_S H$. Define $S^*=(S\cap O^2(G))(S\cap O^2(H))$, $G^*=O^2(G)S^*$, $H^*=O^2(H)S^*$, and $X^*=\langle G^*, H^*\rangle$. Pick standard generators $\{\sigma_i, \tau_j, \rho_k\}$ of the pair (G^*, H^*) of ${}^2F_4(2)'$-type, and define $N=\langle((\rho_1\rho_8)^8)^{X^*}\rangle$. Then*

(1) *N is normal in X,*
(2) *$G\cap N=H\cap N=1$, and*
(3) *X/N is isomorphic to ${}^2F_4(2)$.*

This theorem establishes the uniqueness of the isomorphism classes of the pairs of ${}^2F_4(2)$-type. If (G, H) is a pair of ${}^2F_4(2)$-type, then, by Theorem 3, the pair (G, H) is isomorphic to the pair $(GN/N, HN/N)$ of subgroups of $X/N\cong {}^2F_4(2)$. As $|SN/N|=|S|=2^{12}$, SN/N is a Borel subgroup of X/N, and so $(GN/N, HN/N)$ is the pair of the nontrivial parabolic subgroups of X/N containing SN/N. Since $|GN/N|=|G|=2^{12}\cdot 3$ and $|HN/N|=|H|=2^{12}\cdot 5$, the pair (G, H) is isomorphic to the pair (P_8, P_1).

For the proof of (3) of Theorem 3, we need the fact that $\mathrm{Aut}({}^2F_4(2)')$ is isomorphic to ${}^2F_4(2)$. This fact already appeared in the literature, say [7], but we give a new proof based on (1.2).

Finally we state the following result characterizing the twisted Chevalley group ${}^2F_4(2)$.

THEOREM 4. *Let X be a finite group having a pair of subgroups (G, H) of ${}^2F_4(2)$-type. Assume that $H=C_X(z)$ for some involution z in H. Then X is isomorphic to ${}^2F_4(2)$.*

We can prove Theorem 4 by the same method as the method in the proof of Theorem 2, noticing the fact that all involutions of ${}^2F_4(2)$ are contained in ${}^2F_4(2)'$.

We have seen how we attain the main purpose of this paper. Probably, the greatest difference between Fan's paper and our paper consists in the treatment of the case $|S|=2^{12}$; Fan studies both the case $|S|=2^{11}$ and the case $|S|=2^{12}$ by means of generators and relations, while our argument for the case $|S|=2^{12}$ is almost free of computations of generators and relations. Also, our approach sheds some new light on the earlier result of Parrott, which have been necessary for the classification of the finite simple groups. Although Theorem 2 is somewhat weaker, as a characterization theorem, than Parrott's theorem, Theorem 2 is probably sufficient for the purpose of the classification of the finite simple groups. We remark, in this connection, that Theorem 2 is sufficient for Gomi's work on thin groups [6]. Taking it into consideration that the study of the pairs satisfying the conditions (a)-(d) has already been applied to the classification of the finite simple groups, we can hope that our paper will play a role in the revision program of the classification.

2

In this section we make a preliminary study of a pair of ${}^2F_4(2)'$-type. Throughout this section, (G, H) is a pair of ${}^2F_4(2)'$-type with respect to a common 2-subgroup S, and Q_i $(i=0, 1, \cdots, 6)$ and R_j $(j=0, 1, 2, 3)$ are as in (1.1-1).

The first lemma will be used to describe the actions of H/R_0 on R_3/R_1 and R_1/R_2.

(2.1) *Let F be a Frobenius group of order 20 acting faithfully on an elementary abelian 2-group E of order 16. Choose generators φ and f of F such that $\varphi^5=f^4=1$ and $\varphi^f=\varphi^2$. Then the following holds:*

(1) *Let $1\neq e_1\in C_E(f)$ and define $e_2=e_1^{\varphi}e_1^{\varphi^4}$, $e_3=e_1^{\varphi}e_1^{\varphi^2}$, and $e_4=e_1^{\varphi}$. Then*

$$e_1^{\varphi}=e_4,\ e_2^{\varphi}=e_1e_3e_4,\ e_3^{\varphi}=e_1e_2,\ e_4^{\varphi}=e_3e_4,$$
$$e_1^{f}=e_1,\ e_2^{f}=e_1e_2,\ e_3^{f}=e_2e_3,\ e_4^{f}=e_3e_4.$$

(2) *Let $E_i=\langle e_1, \cdots, e_i\rangle$ $(i=1, 2, 3, 4)$. Then*

$$E=E_4\geqq E_3\geqq E_2\geqq E_1\geqq E_0=1$$

is a unique $\langle f\rangle$-composition series of E.

(3) $E_3\cap E_1^{\varphi}=1$ *and* $E=E_3E_1^{\varphi}$.

(4) $E^{\#}=E-\{1\}$ *is divided into two F-orbits*

$$\cup(E_1^{\#})^{\langle\varphi\rangle} \quad \text{and} \quad \cup(E_2-E_1)^{\langle\varphi\rangle}.$$

PROOF. Note that $e_1e_1^{\varphi}e_1^{\varphi^2}e_1^{\varphi^3}e_1^{\varphi^4}=1$ because $C_E(\varphi)=1$. Then (1) follows from the definitions of e_i. Since φ acts irreducibly on E and $E_4^{\varphi}=E_4$, we have $E=E_4$. So (2) follows from the matrix representing f with respect to the basis $\{e_1, e_2, e_3, e_4\}$ of E. The remaining assertions follow from (1) and (2).

(2.2) *The following holds.*

(1) G/Q_0 *acts faithfully on* $Q_{i-1}/Q_i\cong \boldsymbol{E}_4$ $(i=1,2,4,6)$.

(2) $Q_2/Q_3\cong Q_4/Q_5\cong \boldsymbol{Z}_2$.

(3) H/R_0 *acts faithfully on* $R_{j-1}/R_j\cong \boldsymbol{E}_{16}$ $(j=1,2)$.

$$R_0/R_1\geqq (Q_0\cap R_0)/R_1\geqq Q_2R_1/R_1\geqq Q_3R_1/R_1\geqq 1,$$
$$R_1/R_2\geqq (Q_3\cap R_1)/R_2\geqq Q_4/R_2\geqq Q_5/R_2\geqq 1$$

are, respectively, the unique S/R_0-composition series of R_0/R_1 and of R_1/R_2.

(4) $R_2\cong \boldsymbol{Z}_2$ *and* $R_2=Z(H)$.

(5) $S=Q_0R_0$ *and S is of order* 2^{11}.

(6) $O_2(H)=R_0\geqq R_1\geqq R_2\geqq R_3=1$ *is the lower central series of* $O_2(H)$.

(7) $Q_3\cong R_1\cong \boldsymbol{E}_{32}$.

(8) H/R_0 *splits over* R_0.

(9) $C_H(Q_5)=Q_0$, $C_{Q_0}(Q_4)=Q_1$, *and* $C_{Q_0}(Q_3)=Q_3$.

(10) *Let* $r\in R_0-R_1$. *Then* $[R_1, r]=R_2$ *and* $|R_1:C_{R_1}(r)|=2$.

(11) *Let* $b\in Q_2-Q_3$. *Then* $C_{Q_3}(b)=Q_4$.

PROOF. First, we remark that $|Q_{i-1}/Q_i|=2$ or 4 and that $|R_{j-1}/R_j|=2$ or 16. Furthermore $|Q_{i-1}/Q_i|=4$ (resp. $|R_{j-1}/R_j|=16$) if and only if G/Q_0 (resp. H/R_0) acts faithfully on Q_{i-1}/Q_i (resp. R_{j-1}/R_j). Now, since $Q_5\neq R_2$, we have $|Q_5|=4$ and $|R_2|=2$. Hence $|R_1/R_2|=16$ because $R_2<Q_5<Q_4\leqq R_1$. Therefore $|Q_4/Q_5|=|(Q_3\cap R_1)/Q_4|=|R_1/(Q_3\cap R_1)|=|Q_2R_1/Q_2|=2$ by the definitions of Q_1 and Q_3. The definitions of Q_4 and Q_2 show that $Q_3\leqq R_1$ and $Q_1\not\leqq R_0$. Therefore $Q_4<Q_3\cap R_1<Q_3$, $Q_2<Q_2R_1<Q_1$, and $R_1<Q_3R_1<Q_2R_1\leqq R_0$. Thus $|Q_3/Q_4|=|Q_1/Q_2|=4$, $|Q_3R_1/R_1|=2$, and $|R_0/R_1|=16$. Consequently, we have $|R_0|=2^9$, $|S|=2^{11}$, $|Q_0|=2^{10}$, and $|Q_0/Q_3|=32$. Since $|Q_3R_1/Q_3|=2$, we have $|Q_1/Q_3|\leqq 8$ by the definition of Q_1, and so $|Q_2/Q_3|=|Q_2R_1/Q_3R_1|=2$ and $|Q_0/Q_1|=4$. If $R_0\leqq Q_0$, then $|Q_0/R_0|=2$ and so the definition of Q_2 shows that $|Q_0/Q_2|\leqq 8$, a contradiction. Therefore $R_0\not\leqq Q_0$, and hence $S=Q_0R_0$ and $|R_0/(Q_0\cap R_0)|=|(Q_0\cap R_0)/Q_2R_1|=2$ because $|S/Q_0|=2$. The uniqueness of S/R_0-

composition series of R_{j-1}/R_j follows from (2.1-2), and hence we have proved (1)-(5).

Let $x\in G-S$ and take a Sylow 3-subgroup K of G contained in $\langle R_0, R_0^x\rangle$. If $R_0'\leqq R_2$, then K necessarily centralizes Q_2/Q_5, which contradicts (1). If $Z(R_0)\geqq R_1$, then K centralizes Q_4, also a contradiction. Therefore we have $R_0'\not\leqq R_2$ and $Z(R_0)\not\geqq R_1$, and hence $[R_j, R_0]=R_{j+1}$ $(j=0,1,2)$. So R_1 is abelian, and as $R_1=\langle Q_5^H\rangle$, R_1 is elementary abelian. Since Q_3 is the union of all G-conjugates of $Q_3\cap R_1$ by (1), Q_3 is also elementary abelian. This proves (6) and (7). Thus we have $C_{R_0}(R_1)=R_1$, and hence (10) holds by (4) and (6).

Let P be a Sylow 5-subgroup of H. Then $H=R_0N_H(P)$ and $R_0\cap N_H(P)=R_2$ by (3), (4), and the Frattini argument. Since $I(S)\not\leqq R_0$ by (7) and the definitions of Q_1 and Q_2, some involution in $S-R_0$ inverts P by Suzuki's lemma and Sylow's theorem. Therefore $N_H(P)/R_2$ splits over R_2. This proves (8).

Since $C_H(Q_5)\leqq C_H(Q_5/R_2)=S$ by (3), we have $C_H(Q_5)=C_S(Q_5)=Q_0$ by (1). By (7) and the definitions of Q_1 and Q_4, we have $Q_1\leqq C_{Q_0}(Q_4)\leqq Q_0$. On the other hand, we have $[Q_4, S]\not\leqq R_2$ and $[Q_4, R_0]\leqq R_2$ by (3) and (6). Hence $[Q_4, Q_0]\not\leqq R_2$ by (5), and so $C_{Q_0}(Q_4)=Q_1$. As $C_{Q_0}(Q_3)=C_{Q_1}(Q_3)\leqq C_{Q_1R_0}((Q_3\cap R_1)/R_2)=R_0$ by (3), $Q_3\leqq C_{Q_0}(Q_3)\leqq \cap R_0^G=Q_2$. Since $R_1=(Q_3\cap R_1)Q_5^h$ for some $h\in H$ by (2.1-3) and (3), and since $[Q_2\cap Q_0^h, Q_5^h]=1$, we have $[Q_2, Q_3\cap R_1]\neq 1$ by (10), and hence $C_{Q_0}(Q_3)=Q_3$. This proves (9), and (11) is a consequence of (9). This completes the proof.

In the next lemma, we study the fusion of involutions of S in G and in H.

(2.3) *The following holds.*

(1) $I(S)\leqq Q_0\cup R_0$.

(2) $I(Q_0)\leqq \cup R_1^G\leqq Q_1$. $(I(Q_2)\leqq Q_3.)$

(3) $I(R_0)\leqq \cup Q_3^H$.

(4) *If* $v\in Q_4-Q_5$, *then* $O_2(C_G(v))=C_{Q_0}(v)=Q_1$, $C_G(v)/Q_1\cong\Sigma_3$, *and* $C_H(v)=C_S(v)>Q_1$.

(5) *If* $u\in R_1-Q_1$, *then* $C_G(u)=C_S(u)\leqq R_0$.

(6) *Let* $y\in\cup(Q_2R_1-Q_3R_1)^H$. *Then* $y^2\in\cup(Q_4-Q_5)^H$ *and* $\{x\in R_0 \mid x^2\in y^2R_2\}=yR_1$.

PROOF. Since $S/R_0\cong \boldsymbol{Z}_4$, we have $I(S)\leqq Q_1R_0$ and $I(Q_0)\leqq Q_0\cap Q_1R_0$, so $I(Q_0)\leqq Q_1$ by (2.2-1). Pick an involution $t\in Q_1-R_0$. Then $C_{R_0/R_1}(t)=Q_2R_1/R_1$ by (2.2-3), and so $I(Q_1R_0/R_1)\leqq Q_1/R_1\cup R_0/R_1$. Thus $I(S)\leqq Q_1\cup R_0$. By (2.1-4) and (2.2-3), $(R_0/R_1)^{\#}$ is divided into two H-orbits

$$\cup((Q_3R_1/R_1)^{\#})^H \quad \text{and} \quad \cup(Q_2R_1/R_1-Q_3R_1/R_1)^H .$$

There is an involution in $Q_3R_1-R_1$ by (2.2-7), while there is a coset without involutions because $I(Q_0)\leqq Q_1$. Therefore we have $I(Q_2R_1)\leqq Q_3R_1$, and so $I(Q_2)\leqq Q_3$ and $I(R_0)\leqq \cup(Q_3R_1)^H$. Also, we have $I(Q_1)\leqq \cup(Q_3R_1)^G$ because $Q_1=\cup(Q_2R_1)^G$ by (2.2-1). Hence $I(Q_1)\leqq Q_3\cup(\cup R_1^G)$ and $I(R_0)\leqq R_1\cup(\cup Q_3^H)$ by (2.2-7) and (2.2-10). Since $R_1=\cup Q_4^H$ by (2.1-4) and (2.2-3), and since $Q_3=\cup(Q_3\cap R_1)^G$ by (2.2-1), we conclude that $I(Q_1)\leqq \cup R_1^G$ and that $I(R_0)\leqq \cup Q_3^H$. This proves (1)-(3).

Let $v\in Q_4-Q_5$. Then we have $C_{Q_0}(v)=Q_1$ by (2.2-9). As $|Q_0/Q_1|=|Q_1-Q_5|=4$ by (2.2-1) and (2.2-2), $G=Q_0C_G(v)$ and hence $C_G(v)/Q_1\cong G/Q_0\cong\Sigma_3$ and $Q_1=O_2(C_G(v))<C_S(v)$. Also, we have $C_H(v)=C_S(v)$ because $C_H(v)$ is a 2-group and $C_S(v)\geqq Q_1\not\leqq R_0$. Let $u\in R_1-Q_4$. Since $C_G(u)$ normalizes $\langle Q_4, u\rangle$ and $\langle Q_3, u\rangle$, we have $C_G(u)\leqq S$ by (2.2-1). Hence $C_G(u)=C_S(u)\leqq C_S(\langle u\rangle R_2/R_2)\leqq R_0$ by (2.2-3). This proves (4) and (5).

For the proof of (6), pick an element $\phi\in G$ of order 3 and an element $b\in C_{Q_2}(\phi)-Q_3$. Then $b^2\in Q_4-Q_5$ by (2) and (2.2-1). Let $y\in\cup(Q_2R_1-Q_3R_1)^H$. Since $y^h=bz$ for some $h\in H$ and $z\in R_1$, we have $(y^h)^2=b^2[b, z]\in Q_4-Q_5$, and so $y^2\in\cup(Q_4-Q_5)^H$. Also, we have $(yw)^2=y^2[y, w]\in y^2R_2$ for all $w\in R_1$, and hence the following mapping ξ is well-defined.

$$\begin{aligned} \xi : \cup(Q_2R_1/R_1-Q_3R_1/R_1)^H &\longrightarrow \cup(Q_4/R_2-Q_5/R_2)^H \\ yR_1 &\longrightarrow y^2R_2 . \end{aligned}$$

By (2.1-4) and (2.2-3), $(R_1/R_2)^{\#}$ is divided into two H-orbits

$$\cup((Q_5/R_2)^{\#})^H \quad \text{and} \quad \cup(Q_4/R_2-Q_5/R_2)^H ,$$

and as $|\cup(Q_2R_1/R_1-Q_3R_1/R_1)^H|=|\cup(Q_4/R_2-Q_5/R_2)^H|=10$, the above mapping ξ is bijective. Since $(Q_3R_1)^2\leqq R_2$, we have $\{x\in R_0 \mid x^2\in y^2R_2\}\leqq\cup(Q_2R_1-Q_3R_1)^H$, and hence $\{x\in R_0 \mid x^2\in y^2R_2\}=yR_1$ by the above remark. This completes the proof.

3

In this section, we prove Theorem 1 stated in Section 1. Let (G, H) be a pair of ${}^2F_4(2)'$-type with respect to a common 2-subgroup S, and let Q_i $(i=0, 1, \cdots, 6)$ and R_j $(j=0, 1, 2, 3)$ be as in (1.1-1). A major part of the proof is devoted to finding twelve elements g, h, v_i $(i=0, 1, 2, 3, 4)$, a, b, c, d, f satisfying the condition ($\sharp$) below and twenty-seven relations labeled (**S**-1)-(**S**-27).

$$(\ddagger)\quad \begin{aligned} &g\in G-S,\quad h\in H-S,\quad v_0\in R_2-\{1\},\quad v_1\in Q_5-R_2,\\ &v_2\in Q_4-Q_5,\quad v_3\in(Q_3\cap R_1)-Q_4,\quad v_4\in R_1-(Q_3\cap R_1),\\ &a\in(Q_3\cap R_1^g)-Q_4,\quad b\in Q_2-Q_3,\quad c\in(Q_0\cap Q_2^h)-R_1,\\ &d\in(Q_3^h\cap R_1^{gh})-Q_4^h,\quad f\in(Q_0\cap R_0^g)-Q_1R_0. \end{aligned}$$

In the course of the proof, however, we use two more elements $\varphi\in H-S$ and $\phi\in G-S$, and prove seven more relations ($\boldsymbol{T}$-1)-($\boldsymbol{T}$-7) involving them because the actions of G and H, respectively, on their chief factors are well described. In the final step of the proof, we construct a set of standard generators $\{\sigma_1,\cdots,\sigma_8,\tau_2,\tau_4,\tau_6,\rho_1,\rho_8\}$ by the aid of those twelve elements and twenty-seven relations.

Now, we begin the proof. First, there are elements $\varphi\in H-S$ and $f\in S-Q_1R_0$ satisfying

$$(\boldsymbol{S}\text{-}1)\quad f^4=1,\qquad (\boldsymbol{T}\text{-}1)\quad \varphi^5=1,\quad \varphi^f=\varphi^2$$

by (2.2-8). As $Q_3^\varphi\not\leqq Q_0$ by (2.1-3) and (2.2-3), we can take an element $d\in Q_3^\varphi-Q_0$. By (2.2-7), d is an involution, and so inverts some element $\phi\in G$ of order 3 by Suzuki's lemma. Thus,

$$(\boldsymbol{S}\text{-}2)\quad d^2=1,\qquad (\boldsymbol{T}\text{-}2)\quad \phi^3=1,\quad \phi^d=\phi^{-1}.$$

Replacing ϕ by ϕ^{-1}, if necessary, we have

$$f^2\in R_1^\phi$$

by (2.3-2), and hence

$$f\in S\cap C_G(f^2)\leqq S\cap R_0^\phi=Q_0\cap R_0^\phi$$

by (2.3-5).

Since $R_1=R_2\times[R_1,\varphi]$ by (2.2-3), (2.2-4), and (2.2-7), we can apply (2.1) to the actions of $\langle\varphi,f\rangle$ on R_1/R_2 and $[R_1,\varphi]$. Let $R_2=\langle v_0\rangle$ and define $v_1=v_0^\phi$, $v_2=v_1^\varphi v_1^{\varphi^4}$, $v_3=v_1^\varphi v_1^{\varphi^2}$, and $v_4=v_1^\varphi$. Then we have $v_1\in Q_5-R_2$,

$$(\boldsymbol{S}\text{-}3)\quad v_i^2=[v_j,v_k]=1\quad(i,j,k=0,1,2,3,4),$$

$$(\boldsymbol{T}\text{-}3)\quad v_0^\phi=v_1,\quad v_1^\phi=v_0v_1$$

by (2.2-1) and (2.2-7). Since $v_1\in R_2^\phi=[Q_4,f]\leqq[R_1,f]\leqq[R_1,\varphi]$ and $[Q_5,f]=1$ by (2.2-6) and (2.2-9), we have $v_2\in Q_4-Q_5$, $v_3\in(Q_3\cap R_1)-Q_4$, $v_4\in R_1-(Q_3\cap R_1)$,

$$(\boldsymbol{T}\text{-}4)\quad v_0^\varphi=v_0,\quad v_1^\varphi=v_4,\quad v_2^\varphi=v_1v_3v_4,\quad v_3^\varphi=v_1v_2,\quad v_4^\varphi=v_3v_4,$$

$$(\boldsymbol{S}\text{-}4)\quad [v_i,f]=\begin{cases} v_{i-1} & (i=2,3,4)\\ 1 & (i=0,1)\end{cases}$$

by (2.1-1), (2.1-2), (2.2-3), and (2.2-4). Consequently,

$$R_1 \cap Q_3^{\varphi} = \langle v_0, v_1v_2, v_1v_3, v_4 \rangle\,, \quad Q_4^{\varphi} = \langle v_0, v_1v_3, v_4 \rangle\,, \quad Q_5^{\varphi} = \langle v_0, v_4 \rangle\,,$$

and this will frequently be used throughout the remainder of this section. Since $d \in Q_3^{\varphi}$, we have

$$(\boldsymbol{S}\text{-}5) \qquad [d, v_i] = \begin{cases} v_0 & (i=1,2,3) \\ 1 & (i=0,4) \end{cases}$$

by (2.2-7) and (2.2-10).

Let $\beta = v_4 v_4^{\psi} v_4^{\psi^2}$. Then $\beta \in Q_2$ by ($\boldsymbol{T}$-2) and (2.2-1), and so $\beta^{\psi} = \beta^{v_4} \in \beta\langle v_0 \rangle$ by ($\boldsymbol{T}$-2) and (2.2-10). Define

$$b = \begin{cases} \beta & \text{if} \quad \beta^{\psi} = \beta^{v_4} = \beta \\ \beta v_1 & \text{if} \quad \beta^{\psi} = \beta^{v_4} = \beta v_0\,. \end{cases}$$

Then we have $b \in Q_2$, and

$$(\boldsymbol{T}\text{-}5) \qquad b^{\psi} = b$$

by ($\boldsymbol{S}$-3) and ($\boldsymbol{T}$-3), and hence $b^4 = 1$ by (2.2-2) and (2.2-7). Also, we have $b^d = b^{-1}$ because $\beta^d = v_4 v_4^{\psi^2} v_4^{\psi} = (\beta^{v_4})^{-1}$ by ($\boldsymbol{S}$-3), ($\boldsymbol{S}$-5), and ($\boldsymbol{T}$-2). Thus

$$(\boldsymbol{S}\text{-}6) \qquad b^d = b^{-1} = b^3.$$

If $b \in Q_3$, then $b \in Q_4$ by ($\boldsymbol{T}$-2), ($\boldsymbol{T}$-5), and (2.2-1), and so $\beta \in Q_4$. Therefore $[v_4, v_4^{\psi}] = (v_4 v_4^{\psi})^2 = (\beta v_4^{\psi^2})^2 = 1$ by (2.2-7), but this is a contradiction because $C_S(v_4) \leqq R_0$ by (2.3-5). Thus we have

$$b \in Q_2 - Q_3\,.$$

Since $b^2 \neq 1$ by (2.3-2), we have

$$b^2 = [b, d] \in (Q_4 \cap Q_3^{\varphi}) - Q_5 = v_1 v_2 \langle v_0 \rangle$$

by ($\boldsymbol{S}$-6), ($\boldsymbol{T}$-5), and (2.2-1). Suppose $b^2 = v_0 v_1 v_2$. Take an element $\gamma \in (Q_0 \cap Q_2^{\varphi}) - R_1$ and let $\varphi' = \varphi^{\gamma}$ and $f' = f^{\gamma}$. Then we have $\varphi' \in H - S$, $d \in Q_3^{\varphi'} - Q_0$, and $f' \in (Q_0 \cap R_0^{\psi}) - Q_1 R_0$. Define v_i' $(i=0,1,2,3,4)$ and b' using φ', ψ, d, and f' just as we have defined v_i $(i=0,1,2,3,4)$ and b using φ, ψ, d, and f, and replace φ, f, v_i, and b by φ', f', v_i', and b', respectively. Then all the relations ($\boldsymbol{S}$-1)-($\boldsymbol{S}$-6) and ($\boldsymbol{T}$-1)-($\boldsymbol{T}$-5) continue to hold, because ($\boldsymbol{S}$-3)-($\boldsymbol{S}$-6) and ($\boldsymbol{T}$-3)-($\boldsymbol{T}$-5) are derived from ($\boldsymbol{S}$-1), ($\boldsymbol{S}$-2), ($\boldsymbol{T}$-1), and ($\boldsymbol{T}$-2) which clearly hold. Moreover, we have $v_1' = v_1$, $v_2' = v_0 v_2$, and $b' = b$ since

$$[\gamma, v_i]=\begin{cases} v_0 & (i=2) \\ 1 & (i=0,1,3,4) \end{cases}$$

by (2.2-9) and (2.2-10). So we get $(b')^2=b^2=v_0v_1v_2=v_1'v_2'$. Therefore, we will assume

$(\boldsymbol{S}\text{-}7)\qquad b^2=v_1v_2.$

Hence we have

$(\boldsymbol{T}\text{-}6)\qquad v_2^{\phi}=v_0v_2$

by ($\boldsymbol{T}$-3) and ($\boldsymbol{T}$-5).

Since $b^2\in(Q_3^{\varphi}\cap R_1)-Q_4^{\varphi}$ by ($\boldsymbol{S}$-7), either d or $d'=db^2$ is contained in $(Q_3^{\varphi}\cap R_1^{\phi\varphi})-Q_4^{\varphi}$. Furthermore, all the relations involving d (i. e. ($\boldsymbol{S}$-2), ($\boldsymbol{S}$-5), ($\boldsymbol{S}$-6), and ($\boldsymbol{T}$-2)) remain true if we replace d by d'. Therefore, we will assume

$$d\in(Q_3^{\varphi}\cap R_1^{\phi\varphi})-Q_4^{\varphi}.$$

Since $d^b=db^2$ by ($\boldsymbol{S}$-6), we have

$$b\notin C_{S^{\varphi}}(Q_3^{\varphi}/Q_4^{\varphi})=Q_0^{\varphi}=C_{S^{\varphi}}(Q_5^{\varphi})=C_{S^{\varphi}}(v_4)$$

by (2.2-1). Therefore

$$(\boldsymbol{S}\text{-}8)\qquad [b, v_i]=\begin{cases} v_0 & (i=3,4) \\ 1 & (i=0,1,2) \end{cases}$$

by (2.2-10) and (2.2-11). Using the definition of b and the relations ($\boldsymbol{S}$-3), ($\boldsymbol{S}$-4), ($\boldsymbol{S}$-7), ($\boldsymbol{S}$-8), ($\boldsymbol{T}$-3), and ($\boldsymbol{T}$-5), we calculate as follows.

$$\begin{aligned}[v_4, f^2v_4^{\phi}]&=[v_4, v_4^{\phi}]v_2=(v_4v_4^{\phi})^2v_2=(\beta v_4^{\phi^2})^2v_2\\ &=(bv_4^{\phi^2})^2v_2=b^2[b, v_4]^{\phi^2}v_2=v_0.\end{aligned}$$

Thus $f^2v_4^{\phi}\in(Q_3\cap R_1^{\phi})-Q_4$ and so

$$v_1=[b, f^2v_4^{\phi}]=[b, v_4]^{\phi}[b, f^2]^{v_4^{\phi}}=v_1[b, f^2]^{v_4^{\phi}}$$

by ($\boldsymbol{S}$-8), ($\boldsymbol{T}$-3), ($\boldsymbol{T}$-5), (2.2-10), and (2.2-11). Hence we have

$(\boldsymbol{T}\text{-}7)\qquad [b, f^2]=1.$

Now we apply (2.1) to the action of $\langle\varphi, f\rangle$ on R_0/R_1. Since $b\in(Q_2-Q_3)-Q_0^{\varphi}$ and $d\in Q_3^{\varphi}$, (2.2-3) shows that the cosets bR_1 and dR_1, respectively, correspond to $e_2\in(E_2-E_1)-E_3^{\varphi}$ and $e_4\in E_1^{\varphi}$ in (2.1-1). Therefore the coset $[bd, f]R_1$ correspond to $[e_2e_4, f]=e_1e_3\in E_3\cap E_2^{\varphi}$, and hence we have

$[bd, f] \in (Q_0 \cap Q_1^\varphi) - R_1$. Since $[(bv_1v_2)d, f] = [bdv_1v_2, f] = [bd, f]^{v_1v_2}v_1 \in [bd, f]v_1\langle v_0\rangle$ by (**S**-4), (**S**-5), and (2.2-10), and since $v_1\langle v_0\rangle \leqq R_1 - Q_3^\varphi \leqq (Q_0 \cap Q_1^\varphi) - Q_2^\varphi$, either $[bd, f]$ or $[(bv_1v_2)d, f]$ is contained in $(Q_0 \cap Q_2^\varphi) - R_1$. Furthermore, since $bv_1v_2 = b^{-1}$ by (**S**-6) and (**S**-7), all the relations (**S**-1)-(**S**-8) and (**T**-1)-(**T**-7) continue to hold if we replace b by bv_1v_2. Therefore, we will assume

$$[bd, f] \in (Q_0 \cap Q_2^\varphi) - R_1 .$$

Now, we define a by

(**S**-9) $\quad [b, f] = a .$

Since $b \in Q_2 - Q_3 \leqq R_0^\phi$ and $f \in (Q_0 \cap R_0^\phi) - Q_1R_0$, we have $a \in (Q_3 \cap R_1^\phi) - R_1 = (Q_3 \cap R_1^\phi) - Q_4$ by (2.2-3). Therefore,

(**S**-10) $\quad a^2 = 1 ,$ $\qquad$ (**S**-11) $\quad [a, b] = v_1 ,$

(**S**-12) $\quad [a, v_i] = \begin{cases} v_0 & (i=4) \\ 1 & (i=0, 1, 2, 3) \end{cases}$

by (2.2-7), (2.2-10), and (2.2-11). Furthermore, we have

(**S**-13) $\quad [a, f] = 1$

by (**S**-9), (**S**-10), and (**T**-7).

Since $a \notin Q_0^\varphi = C_{S^\varphi}(Q_3^\varphi/Q_4^\varphi)$ and $d \notin Q_0 = C_S(Q_3/Q_4)$ by (2.1-3) and (2.2-3), we have $[a, d] \in (Q_3 \cap Q_3^\varphi) - (Q_4 \cup Q_4^\varphi) = v_2v_3\langle v_0\rangle$. Suppose $[a, d] = v_0v_2v_3$, and define

$$d^* = dv_1v_3v_4 \in (Q_3^\varphi \cap R_1^{\phi\varphi}) - Q_4^\varphi .$$

Note that $O_2(N_G(\langle b\rangle)) = N_{Q_0}(\langle b\rangle)$ because $\langle d, \phi\rangle$ is a complement for Q_0 in G, and $\langle d, \phi\rangle \leqq N_G(\langle b\rangle)$ by (**S**-6) and (**T**-5). Since $d^* \notin Q_0$ by the definition of d^*, and since d^* inverts b by (**S**-6) and (**S**-8), we have $d^* \notin O_2(N_G(\langle b\rangle))$, and so d^* inverts some element $\phi^* \in N_G(\langle b\rangle)$ of order 3 by Suzuki's lemma. We have $\phi^* \in C_G(b)$ because $|N_G(\langle b\rangle) : C_G(b)| = 2$. Replacing ϕ^* by ϕ^{*-1}, if necessary, we have $R_j^{\phi^*} = R_j^\phi$, and so $a \in (Q_3 \cap R_1^{\phi^*}) - Q_4$, $d^* \in (Q_3^\varphi \cap R_1^{\phi^*\varphi}) - Q_4^\varphi$, and $f \in (Q_0 \cap R_0^{\phi^*}) - Q_1R_0$. Moreover, all the relations involving d and ϕ (i. e. (**S**-2), (**S**-5), (**S**-6), (**T**-2), (**T**-3), (**T**-5), and (**T**-6)) continue to hold if we replace d and ϕ by d^* and ϕ^*, respectively. Further, we have $[a, d^*] = v_2v_3$ by (**S**-12), and $[bd^*, f] = [bd, f]^{v_1v_3v_4}[v_1v_3v_4, f] = [bd, f]v_2v_3 \in (Q_0 \cap Q_2^\varphi) - R_1$ because we have assumed $[bd, f] \in (Q_0 \cap Q_2^\varphi) - R_1$. Therefore, we will assume

(**S**-14) $\quad [a, d] = v_2v_3 ,$

as well as $[bd, f] \in (Q_0 \cap Q_2^\varphi) - R_1$.

Now define $c=[bd,f]$. Then we have

($\boldsymbol{S}$-15) $\quad [d,f]=a^d c=av_2v_3c$

by ($\boldsymbol{S}$-9), ($\boldsymbol{S}$-10), and ($\boldsymbol{S}$-14). Moreover,

($\boldsymbol{S}$-16) $\quad [c,v_i]=\begin{cases} v_0 & (i=2) \\ 1 & (i=0,1,3,4), \end{cases}$

($\boldsymbol{S}$-17) $\quad [c,d]=v_4$

by (2.2-9) and (2.2-10).

The calculation using ($\boldsymbol{T}$-7) shows that

$$[d,f^2]=[bd,f^2]=[bd,f][bd,f]^f=cc^f=c^2[c,f].$$

Therefore, using ($\boldsymbol{S}$-4), ($\boldsymbol{S}$-13), and ($\boldsymbol{S}$-15),

$$\begin{aligned} c^2[c,f]=[d,f^2]=[d,f][d,f]^f&=av_2v_3c(av_2v_3c)^f \\ &=av_2v_3cav_1v_3c[c,f]. \end{aligned}$$

Thus, $1=c^{-1}av_2v_3cav_1v_3=[c,a]v_0v_1v_2$ by ($\boldsymbol{S}$-3), ($\boldsymbol{S}$-10), ($\boldsymbol{S}$-12), and ($\boldsymbol{S}$-16), and hence

($\boldsymbol{S}$-18) $\quad [a,c]=v_0v_1v_2.$

Next, using ($\boldsymbol{S}$-2), ($\boldsymbol{S}$-10), and ($\boldsymbol{S}$-14)-($\boldsymbol{S}$-18),

$$\begin{aligned} 1=[d^2,f]^d=[d,f][d,f]^d&=a^d cac^d=a^d ac[c,a]c^d \\ &=[d,a]c[c,a]c[c,d]=v_2v_3\cdot c\cdot v_0v_1v_2\cdot c\cdot v_4=c^2v_1v_3v_4. \end{aligned}$$

Therefore, we have

($\boldsymbol{S}$-19) $\quad c^2=v_1v_3v_4.$

Similarly, using ($\boldsymbol{S}$-2), ($\boldsymbol{S}$-6), and ($\boldsymbol{S}$-17),

$$1=[(bd)^2,f]^d=[bd,f]^b[bd,f]^d=c[c,b]cv_4=v_4c^{-1}[b,c]c^{-1}.$$

Therefore we have

($\boldsymbol{S}$-20) $\quad [b,c]=v_1v_3$

by ($\boldsymbol{S}$-16) and ($\boldsymbol{S}$-19).

Since $[c,f]\in Q_2-Q_3$ by (2.2-3), and since $[c,f]=c^2[d,f^2]\in C_{Q_2}(b)$ by ($\boldsymbol{S}$-6), ($\boldsymbol{S}$-8), ($\boldsymbol{S}$-19), and ($\boldsymbol{T}$-7), we have $[c,f]\in b\langle v_0,v_1,v_2\rangle$. Let $e=[c,f]=bx$ where $x\in\langle v_0,v_1,v_2\rangle$. Notice that $e^2=b^2=v_1v_2$. By ($\boldsymbol{S}$-4), ($\boldsymbol{S}$-16), and ($\boldsymbol{S}$-19),

$$v_2v_3=[f,c^2]=[f,c][f,c]^c=e^{-1}c^{-1}e^{-1}c=e^{-1}c^{-1}ev_1v_2c$$
$$=[e,c]v_0v_1v_2\,,$$

and so we have

$$v_0v_1v_3=[e,c]=[bx,c]=v_1v_3[x,c]\,.$$

Therefore $[x,c]=v_0$, and hence $x\in v_2\langle v_0, v_1\rangle$ by ($\boldsymbol{S}$-16). By ($\boldsymbol{S}$-2), ($\boldsymbol{S}$-5), and ($\boldsymbol{S}$-19),

$$[e,d]=[c^2[d,f^2],d]=[d,f^2,d]=[f^2,d]d[d,f^2]d$$
$$=[f^2,d]^2=(c^2e)^{-2}=e^{-2}=b^2\,.$$

On the other hand,

$$[e,d]=[bx,d]=[b,d]^x[x,d]=b^2[x,d]$$

by ($\boldsymbol{S}$-6). Thus $[x,d]=1$, and hence $x\in v_1v_2\langle v_0\rangle$ by ($\boldsymbol{S}$-5). Consequently, we have $[c,f]=c^2[d,f^2]\in bv_1v_2\langle v_0\rangle$.

Suppose $[c,f]=c^2[d,f^2]=bv_0v_1v_2$, and define $\varphi'=\varphi^{v_2v_3}$, $c'=c^{v_2v_3}=cv_0$, and $f'=f^{v_2v_3}=fv_1v_2$. Then $c'\in(Q_0\cap Q_2^{\varphi'})-R_1$, $d\in(Q_3^{\varphi'}\cap R_1^{\phi\varphi'})-Q_4^{\varphi'}$, $f'\in(Q_0\cap R_0^{\phi})-Q_1R_0$, $[c',f']=(c')^2[d,(f')^2]=bv_1v_2$, and all the relations ($\boldsymbol{S}$-1)-($\boldsymbol{S}$-20) and ($\boldsymbol{T}$-1)-($\boldsymbol{T}$-7) continue to hold if we replace φ, c, and f by φ', c', and f', respectively. Therefore, we may assume

($\boldsymbol{S}$-21) $\quad [c,f]=c^2[d,f^2]=bv_1v_2=b^{-1}$.

At this point, we obtained all the necessary relations for the generators v_i $(i=0,1,2,3,4)$, a,b,c,d, and f of S. Now, we turn to the determination of the action of $\langle\varphi,f\rangle$ on R_0.

Let $a^{\varphi}=dy$ where $y\in Q_4^{\varphi}=\langle v_0, v_1v_3, v_4\rangle$. Then

$$[a^{\varphi},f]=[dy,f]=[d,f]^y[y,f]=[d,f][d,f,y][y,f]$$
$$\equiv acv_2v_3[y,f] (\mathrm{mod}\langle v_0\rangle)$$

by ($\boldsymbol{S}$-14). Since

$$c^{f^2}=c[c,f^2]=c[c,f][c,f]^f=cb^{-1}(bv_1v_2)^f$$
$$=c[b,f]v_2=cav_2$$

by ($\boldsymbol{S}$-4), ($\boldsymbol{S}$-9), and ($\boldsymbol{S}$-21), we have

$$[a^{\varphi},f]^{f^2}\equiv a\cdot cav_2\cdot v_1v_2v_3\cdot[y,f]^{f^2}$$
$$\equiv cv_2v_3[y,f]^{f^2} \quad (\mathrm{mod}\langle v_0\rangle)$$

by (**S**-4) and (**S**-18). On the other hand, we have

$$[a^{\varphi}, f]=a^{\varphi}a^{\varphi^2} \quad \text{and} \quad [a^{\varphi}, f]^{f^2}=a^{\varphi^4}a^{\varphi^3}$$

by (**S**-10), (**S**-13), and (**T**-1). Therefore

$$aa^{\varphi}a^{\varphi^2}a^{\varphi^3}a^{\varphi^4}\equiv a\cdot acv_2v_3[y,f]\cdot[y,f]^{f^2}v_3v_2c^{-1}$$
$$\equiv[y,f,f^2] \quad (\mathrm{mod}\langle v_0\rangle).$$

Since $C_{R_0}(\varphi)\leqq\langle v_0\rangle$ by (2.2-3), we have $aa^{\varphi}a^{\varphi^2}a^{\varphi^3}a^{\varphi^4}\in\langle v_0\rangle$ and so $[y,f,f^2]\in\langle v_0\rangle$. Hence $y\in\langle v_0, v_1v_3\rangle$ by (**S**-4).

We have shown that $a^{\varphi}\in d\langle v_0, v_1v_3\rangle$. Suppose $a^{\varphi}\in dv_0\langle v_1v_3\rangle$, and define $\varphi'=\varphi^{v_1}$. Then we have $Q_i^{\varphi'}=Q_i^{\varphi}$, $R_j^{\psi\varphi'}=R_j^{\psi\varphi}$, and $a^{\varphi'}\in d\langle v_1v_3\rangle$. Furthermore, all the relations (**S**-1)-(**S**-21) and (**T**-1)-(**T**-7) continue to hold if we replace φ by φ'. Therefore, we may assume

$$a^{\varphi}\in d\langle v_1v_3\rangle.$$

Define $h=f^2\varphi$. Then we have $h\in H-S$, $Q_i^h=Q_i^{\varphi}$, $R_j^{\psi h}=R_j^{\psi\varphi}$, and furthermore,

(**S**-22) $\quad h^2=1, \quad v_0^h=v_0, \quad v_1^h=v_4, \quad v_2^h=v_1v_3v_4,$

(**S**-23) $\quad (f^2h)^5=1, \quad f^h=(f^2h)^2f$

by (**S**-1), (**S**-4), (**T**-1), and (**T**-4).

Since $[d,f^2]=c^2[c,f]=bv_2v_3v_4$ by (**S**-8), (**S**-19), and (**S**-21), we have

$$d^{\varphi}=d^{f^2h}=(dbv_2v_3v_4)^h=d^hb^hv_1v_2v_3$$

by (**S**-22). Notice that $a^{\varphi}=a^h$, that $(a^{\varphi})^{\varphi}=(a^{\varphi})^f$, and that $b^{\varphi}=b^h$ by (**S**-13), (**T**-1), and (**T**-7).

Case 1. If $a^{\varphi}=a^h=d$, then $d^h=a$ by (**S**-22), and $d^{\varphi}=d^f=d[d,f]=adc=acdv_4$ by (**S**-15) and (**S**-17). Therefore $b^{\varphi}=b^h=cdv_1v_2v_3v_4$.

Case 2. If $a^{\varphi}=a^h=dv_1v_3$, then $d^h=av_1v_2$ by (**S**-22), and $d^{\varphi}v_1v_2v_4=(dv_1v_3)^{\varphi}=(dv_1v_3)^f=acdv_1v_2v_3v_4$ by (**S**-4), (**S**-15), (**S**-17), and (**T**-4), so $d^{\varphi}=acdv_3$. Therefore $b^{\varphi}=b^h=cdv_0$.

We have $b^{\varphi}\in Q_2^{\varphi}$ because $b\in Q_2$, while $cdv_1v_2v_3v_4\notin Q_2^{\varphi}$ because $cdv_1v_3v_4\in Q_2^{\varphi}\not\ni v_2$. Thus, Case 2 occurs, and consequently we have

(**S**-24) $\quad a^h=dv_1v_3, \quad b^h=cdv_0.$

Finally, we determine the action of $\langle\psi, d\rangle$ on Q_0.

As $Q_0 \cap (Q_3R_1)^{\varphi} \leqq R_1$ by (2.1-3) and (2.2-3), both c and $f^{\phi^{-1}}$ are contained in $\cup (Q_2R_1 - Q_3R_1)^H$ by (2.1-4) and (2.2-3). Therefore

$$(f^2)^{\phi^{-1}} \in \cup (Q_4 - Q_5)^H$$

by (2.3-6). On the other hand, since $f^2 \in R_1^{\phi} - Q_4$, we have

$$(f^2)^{\phi^{-1}} \in C_{R_1}(b) - Q_4 = v_3v_4\langle v_0, v_1, v_2\rangle$$

by (**S**-8), (**T**-5), and (**T**-7). Therefore

$$(f^2)^{\phi^{-1}} \in v_1v_3v_4\langle v_0\rangle \cup v_2v_3v_4\langle v_0\rangle$$

by (**T**-4). Suppose $(f^2)^{\phi^{-1}} \in v_2v_3v_4\langle v_0\rangle$, and define $h' = h^{v_1v_3v_4}$, $v_i' = v_i^{v_1v_3v_4} = v_i$, $a' = a^{v_1v_3v_4} = av_0$, $b' = b^{v_1v_3v_4} = b$, $c' = c^{v_1v_3v_4} = c$, $d' = d^{v_1v_3v_4} = d$, and $f' = f^{v_1v_3v_4} = fv_2v_3$. Then we have $Q_i^{h'} = Q_i^h$, $R_j^{\phi h'} = R_j^{\phi h}$, and furthermore $(\phi, h', v_i', a', b', c', d', f')$ satisfies (**S**-1)-(**S**-24), (**T**-2), (**T**-3), (**T**-5)-(**T**-7), and $((f')^2)^{\phi^{-1}} \in v_1'v_3'v_4'\langle v_0'\rangle$. Therefore, we may assume $(f^2)^{\phi^{-1}} \in v_1v_3v_4\langle v_0\rangle = c^2\langle v_0\rangle$, and hence $f^{\phi^{-1}} \in cR_1$ by (2.3-6).

Define $g = d\phi$. Then we have $g \in G - S$, $R_j^g = R_j^{\phi}$, and furthermore,

(**S**-25) $\quad g^2 = 1, \quad (gd)^3 = 1, \quad v_0^g = v_1, \quad v_2^g = v_0v_1v_2, \quad b^g = b^{-1}$

by (**S**-2), (**S**-5), (**T**-2), (**T**-3), (**T**-5), and (**T**-6). Since $f^g = f^{\phi^{-1}d} \in cR_1$, we have

$$a^g = [b, f]^g \in [b^{-1}, cR_1] \leqq \langle v_0, v_1v_3\rangle$$

by (**S**-7), (**S**-9), (**S**-20), and (**S**-25). Consequently, $[a^g, d] = 1$ by (**S**-5), and so $a = a^{g^2} = a^{g\phi} = a^{d\phi^{-1}}$. Therefore, $aa^{\phi}a^{\phi^2} = aa^da^g = v_2v_3a^g$, and since $aa^{\phi}a^{\phi^2} \in C_{Q_3}(\phi) = \langle v_1v_2\rangle$, we have $a^g \in v_2v_3\langle v_1v_2\rangle$. We conclude, therefore, that

(**S**-26) $\quad a^g = v_1v_3$.

Let $f^g = cz$ where $z \in R_1$. Then we have

$$\begin{aligned} v_0v_1v_2 = v_2^g &= [v_1v_3, f]^g = [a, cz] = [a, z][a, c]^z \\ &= [a, z]v_0v_1v_2, \end{aligned}$$

and so $z \in \langle v_0, v_1, v_2, v_3\rangle$ by (**S**-12). Also we have

$$\begin{aligned} v_1v_3 = a^g &= [b, f]^g = [b^{-1}, cz] = [b, z][b^{-1}, c]^z \\ &= [b, z]v_0v_1v_3, \end{aligned}$$

and so $z \in v_3\langle v_0, v_1, v_2\rangle$ by (**S**-8). Let $f^g = cv_3w$ where $w \in \langle v_0, v_1, v_2\rangle$. Then we have

$$[f, f^g]=[f, cv_3w]=[f, w][f, v_3]^w[f, c]^{v_3w}$$
$$=[f, w]bv_0v_2\,.$$

Also, since $[f^g, f]=[f, f^g]^{-1}=b^{-1}v_0v_2[w, f]$,

$$[f, f^g]=[f^g, f]^g=bv_0v_2[w, f]^g\,.$$

As $[f, w]=[w, f]\in\langle v_1\rangle$, it follows that $[w, f]=1$, and hence $w\in\langle v_0, v_1\rangle$, that is, $f^g\in cv_3\langle v_0, v_1\rangle$. Thus we have $(f^2)^g=c^2=v_1v_3v_4$.

Suppose $f^g\in cv_3v_1\langle v_0\rangle$, and define $f'=fv_0$. Then $(g, h, v_i, a, b, c, d, f')$ also satisfies all the relations ($\boldsymbol{S}$-1)-($\boldsymbol{S}$-26) and furthermore $(f')^g\in cv_3\langle v_0\rangle$. Therefore we may assume

$$f^g\in cv_3\langle v_0\rangle\,.$$

Since $f^d=f[f, d]=fc^{-1}a^d$, we have $(fa)^d=fc^{-1}$. Also, notice that $v_3^g=av_0$ and $v_4^g=f^2a$. If $f^g=cv_0v_3$, then $c^g=fav_0v_1$ and

$$f^{-1}=(f^{-1})^{gdgdgd}=(c^{-1}v_0v_3)^{dgdgd}=(c^{-1}v_3v_4)^{gdgd}=(fav_1)^{dgd}$$
$$=(fc^{-1}v_0v_1)^{gd}=(cv_0v_3f^{-1}a)^d=cv_1v_3cf^{-1}=v_1f^{-1}\,,$$

a contradiction. Therefore, we have

($\boldsymbol{S}$-27) $\quad f^g=cv_3\,.$

We have now got all the necessary relations between the elements g, h, v_i $(i=0, 1, 2, 3, 4)$, a, b, c, d, and f.

Recall $\boldsymbol{R}_0$, $\boldsymbol{R}_1$, and $\boldsymbol{R}_8$ in Section 1, and define

$$\sigma_1=f^2a\,,\quad \sigma_2=a\,,\quad \sigma_3=v_1\,,\quad \sigma_4=v_1v_2\,,$$
$$\sigma_5=v_0\,,\quad \sigma_6=v_1v_3\,,\quad \sigma_7=v_4\,,\quad \sigma_8=dv_1v_3\,,$$
$$\tau_2=fv_0v_1v_2\,,\quad \tau_4=abv_3\,,\quad \tau_6=cv_0v_1v_2\,,$$
$$\rho_1=hv_0\,,\quad \text{and}\quad \rho_8=gv_1v_2\,.$$

As $(g, h, v_i\ (i=0, 1, 2, 3, 4), a, b, c, d, f)$ satisfies ($\sharp$), we have $S=\langle v_0, v_1, v_2, v_3, v_4, a, b, c, d, f\rangle$, $G=\langle S, g\rangle$, and $H=\langle S, h\rangle$, and so $S=\langle\sigma_1, \cdots, \sigma_8, \tau_2, \tau_4, \tau_6\rangle$, $G=\langle S, \rho_8\rangle$, and $H=\langle S, \rho_1\rangle$ by the above definition. Furthermore, we can verify, using the relations ($\boldsymbol{S}$-1)-($\boldsymbol{S}$-27), that $\{\sigma_1, \cdots, \sigma_8, \tau_2, \tau_4, \tau_6, \rho_1, \rho_8\}$ satisfies all the relations $\boldsymbol{R}_0$, $\boldsymbol{R}_1$, and $\boldsymbol{R}_8$. This completes the proof of Theorem 1.

4

The purpose of this section is to prove Theorem 2 stated in Section 1. First, we record some general lemmas on fusion in a Sylow p-subgroup.

(4.1) (Glauberman [2, Th. 6.1]) *Let G be a finite group and S a Sylow p-subgroup of G. Assume that an abelian subgroup A is strongly closed in S with respect to G. Then $N_G(A)$ controls strong fusion in S.*

(4.2) (Goldschmidt) *Let G be a finite group, S a Sylow p-subgroup of G, Z a subgroup of $Z(S)$, and A the weak closure of Z in S with respect to G. Suppose A is abelian and strongly closed in S with respect to $C_G(Z)$. Then A is strongly closed in S with respect to G.*

PROOF. This is a consequence of (9.1) of [3].

(4.3) *Let G be a finite group, S a Sylow p-subgroup of G, and W a subgroup of $Z(S)$. Suppose W is weakly closed in S with respect to G and define $H=N_G(W)$. Then the following holds:*

(1) *H controls fusion in S.*
(2) *If a subgroup V of S satisfies the condition $N_H(V)\leqq S$, then $N_G(V)=N_S(V)O_{p'}(N_G(V))$.*

PROOF. Part (1) is well-known and an easy consequence of Sylow's theorem, so we omit the proof. For the proof of (2), we define $G_0=N_G(V)$ and $S_0=N_S(V)$. Then $S_0=N_H(V)=N_{G_0}(W)$. Since every p-subgroup of G containing W is contained in H, S_0 is a Sylow p-subgroup of G_0. Moreover $W\leqq Z(S_0)$ and W is weakly closed in S_0 with respect to G_0. Therefore S_0 controls G_0-fusion in S_0 by (1), and so G_0 has a normal p-complement by Tate's theorem [9]. This completes the proof.

Now we begin the proof of Theorem 2. Let (G, H) be a pair of ${}^2F_4(2)'$-type with respect to a common 2-subgroup S, and let Q_i $(i=0, 1, \cdots, 6)$ and R_j $(j=0, 1, 2, 3)$ be as in (1.1-1). Assume that G and H are contained in a finite group X such that $C_X(R_2)=H$.

We need more precise information on fusion in S than that in (2.3), and we record them here.

(4.4) *The following holds.*

(1) *$N_X(S)=S$ and so S is a Sylow 2-subgroup of X.*

(2) *There are exactly two conjugacy classes of involutions in X. If $z\in R_2-\{1\}$ and $v\in Q_4-Q_5$, then $z^X\cap Q_3=Q_5^\sharp$, $z^X\cap R_1=\cup(Q_5^\sharp)^H$, $z^X\cap S=\cup(\cup(z^G)^H)^G\leqq Q_1$, and $v^X\cap S\leqq\cup(\cup(v^H)^G)^H$.*

(3) $N_X(Q_5)=N_X(Q_4)=G$.

(4) *If* $Q_4^x \leqq Q_1$ *for some* $x \in X$, *then* $Q_4^x = Q_4^{hg}$ *for some* $h \in H$ *and* $g \in G$. *In particular, any two distinct X-conjugates of* Q_4 *in* Q_1 *intersect in* Q_5.

PROOF. Since $Z(S)=R_2$, we have $N_X(S)=N_X(S)\cap C_X(R_2)=N_H(S)=S$. So (1) holds. Let $z \in R_2-\{1\}$ and $v \in Q_4-Q_5$. By (2.1-4) and (2.2-3), we have $R_1^\sharp=(\cup(Q_5^\sharp)^H)\cup(\cup(Q_4-Q_5)^H)$. Also, we have $\cup(Q_5^\sharp)^H \leqq \cup(z^G)^H$ and $\cup(Q_4-Q_5)^H \leqq v^H$ by (2.2-1) and (2.3-4). Therefore $I(S)\leqq z^X \cup v^X$ by (2.3-1), (2.3-2), and (2.3-3). On the other hand, we have $3 \mid |C_X(v)|$ and $3 \nmid |C_X(z)|$ by (2.3-4) and the hypothesis of Theorem 2. Hence all involutions in X are divided into precisely two conjugacy classes, and furthermore $z^X \cap R_1=\cup(Q_5^\sharp)^H$. Since $Q_3\cap(\cup(Q_5^\sharp)^H)=Q_5^\sharp$ by (2.1-3) and (2.2-3), we have $z^X\cap(Q_3\cap R_1)=Q_5^\sharp$, and so $z^X\cap Q_3=Q_5^\sharp$ by (2.2-1). Therefore $z^X\cap R_0=z^X\cap(\cup Q_3^H)=\cup(z^X\cap Q_3)^H =\cup(Q_5^\sharp)^H\leqq Q_0$ by (2.3-3). Hence, we have $z^X\cap S=z^X\cap Q_0=z^X\cap(\cup R_1^G)= \cup(z^X\cap R_1)^G=\cup(\cup(Q_5^\sharp)^H)^G\leqq Q_1$. Also, since $v^X\cap Q_3\leqq\cup(v^H)^G$ by (2.2-1), we have $v^X\cap S=(v^X\cap Q_0)\cup(v^X\cap R_0)=(v^X\cap(\cup R_1^G))\cup(v^X\cap(\cup Q_3^H))=(\cup(v^X\cap R_1)^G)\cup (\cup(v^X\cap Q_3)^H)\leqq(\cup(v^H)^G)\cup(\cup(\cup(v^H)^G)^H)\leqq\cup(\cup(v^H)^G)^H$. This proves (2). As $R_2\leqq Q_5\leqq Q_4$, $C_X(Q_5)=C_H(Q_5)=Q_0$ and $C_X(Q_4)=C_{Q_0}(Q_4)=Q_1$ by (2.2-9). Since $|N_X(Q_5)/C_X(Q_5)|\leqq 6$, we have $N_X(Q_5)=G$. Since $N_X(Q_4)$ normalizes Q_5 by (2), $|Q_4-Q_5|=4$, and $Q_4=\langle Q_4-Q_5\rangle$, we have $|N_X(Q_4)/C_X(Q_4)|\leqq 24$ and hence $N_X(Q_4)=G$. This proves (3). Let $Q_5^x\leqq Q_4^x\leqq Q_1$ for some $x\in X$. If $Q_5^x\leqq Q_3$, then $Q_5^x=Q_5$ by (2). Therefore $x\in G$ by (3) and so $Q_4^x=Q_4$. Suppose $Q_5^x\nleqq Q_3$. Then $Q_5^x\cap(R_1^g-Q_3)\neq\emptyset$ for some $g\in G$ by (2.3-2). Take an element $w\in Q_5^{xg^{-1}}\cap(R_1-Q_3)$. Since $C_{Q_1}(w)\leqq Q_1\cap R_0=Q_2R_1$ by (2.3-4), and since $[Q_3, w]\neq 1$ by (2.2-9), we have $(Q_5^{xg^{-1}})^\sharp\leqq I(C_{Q_1}(w))\leqq R_1$ by (2.3-2), and so $(Q_5^{xg^{-1}})^\sharp\leqq z^X\cap R_1=\cup(Q_5^\sharp)^H$ by (2). If $z\notin Q_5^{xg^{-1}}$, then three involutions of $Q_5^{xg^{-1}}$ are contained separately in three distinct H-conjugates of Q_5, which is a contradiction. Thus we have $z\in Q_5^{xg^{-1}}$, and so $Q_5^{xg^{-1}}=Q_5^h$ for some $h\in H$. So $xg^{-1}h^{-1}\in G$ by (3), and hence $Q_4^x=Q_4^{hg}$. The remaining assertion also follows since $R_1\cap R_1^g=Q_4$ and $Q_4\cap Q_4^h=R_2$ for any $g\in G-S$ and $h\in H-S$. This proves (4).

Let $z=\sigma_5$ and $v=\sigma_4$. The standard presentation shows that $z\in I(S)\cap C_S(\sigma_1\rho_1)$ where $\sigma_1\rho_1$ is of order 5, and that $v\in I(S)\cap C_S(\sigma_8\rho_8)$ where $\sigma_8\rho_8$ is of order 3. Hence $z\in R_2-\{1\}$ by (2.2-3), $v\in Q_4-Q_5$ by (2.2-1) and (2.3-2), and z and v are representatives of the conjugacy classes of involutions of X by (4.4-2).

By the hypothesis of Theorem 2, we already have $C_X(z)=H$. Therefore we focus our attention on the determination of $C_X(v)$. In fact, we prove the following proposition.

(4.5) $\quad C_X(v)\leqq G$.

PROOF. Let $C=C_X(v)$, $C_0=C_G(v)$, $S_0=C_S(v)$, and $Z_0=Z(S_0)$. The proof is divided into several steps.

(a) *$Z_0=\langle z, v\rangle$ and S_0 is a Sylow 2-subgroup of C.*

PROOF. Since $C_X(Q_1)=Q_4$ and $C_X(Q_4)=Q_1<S_0$ by (2.2-9) and (2.3-4), we have $Z_0=\langle z, v\rangle$. Furthermore $N_C(S_0)\leqq N_C(Z_0)=C_C(Z_0)=C_H(v)=S_0$ because $z\not\sim v\sim vz$ in X by (4.4-2), and so S_0 is a Sylow 2-subgroup of C.

(b) *Q_4 is weakly closed in S_0 with respect to C.*

PROOF. Suppose $Q_4^x\leqq S_0$ for some $x\in C$. Then $Q_5^x\leqq Q_1$ by (4.4-2), so we have $Q_4^x=\langle Q_5, v\rangle^x=\langle Q_5^x, v\rangle\leqq Q_1$. Therefore $Q_4^x=Q_4$ by (4.4-4).

(c) *Q_4 is the weak closure of Z_0 in S_0 with respect to C.*

PROOF. Suppose $Z_0^x\leqq S_0$ for some $x\in C$. Then we have $Z_0^x\leqq Q_1$ by the same argument as in (b), and hence $Q_4\leqq C_X(Z_0^x)=S_0^x$ by (2.2-9). Therefore $Z_0^x\leqq Q_4^x=Q_4$ by (b).

(d) *Q_4 is strongly closed in S_0 with respect to C.*

PROOF. This follows from (4.2) since $C_C(Z_0)=S_0\trianglerighteq Q_4$.

(e) $S_0\cap C'O^2(C)=S_0\cap C_0'O^2(C_0)$.

PROOF. This follows from (4.1) and the focal subgroup theorem since $N_C(Q_4)=C\cap N_X(Q_4)=C\cap G=C_0$ by (4.4-3).

Here, we define $S_1=[Q_1, O^2(C_0)]Q_3$, $D_0=C_0'O^2(C_0)S_1$, and $D=C'O^2(C)S_1$.

(f) *S_1 is a Sylow 2-subgroup of D, $S_1\trianglelefteq D_0$, $Z(S_1)=Q_4$, and a Sylow 3-subgroup of D_0 acts irreducibly on S_1/Q_3, Q_3/Q_4, and $Q_4/\langle v\rangle$.*

PROOF. Let L be a Sylow 3-subgroup of C_0. Since $C_0/Q_1\cong\Sigma_3$ by (2.3-4), we have $O^2(C_0)\leqq Q_1L$. Therefore $O^2(C_0)\leqq[Q_1, L]Q_3L$ by the action of L on $Q_1/Q_3\cong \boldsymbol{E}_8$, and hence $S_0\cap C'O^2(C)=S_0\cap C_0'O^2(C_0)\leqq[Q_1, L]Q_3=[Q_1, O^2(C_0)]Q_3=S_1$. Thus S_1 is a Sylow 2-subgroup of D, $S_1\trianglelefteq D_0$, and L acts irreducibly on S_1/Q_3, Q_3/Q_4, and $Q_4/\langle v\rangle$. Hence we have $Z(S_1)=Q_4$ by (2.2-9).

(g) *If T is a subgroup of S_1 of order 4 containing v, then $N_D(T)\leqq S_1$.*

PROOF. Note that $N_{D_0}(T)\leqq S_1$ by (f). Since $Z(S_1)$ is weakly closed in S_1 with respect to D by (b) and (f), and since $N_D(Z(S_1))=D\cap N_X(Q_4)=D\cap G=D_0$ by (4.4-3), we have $N_D(T)=N_{S_1}(T)O(N_D(T))$ by (4.3-2). Also, since

$Q_5 \leqq N_D(T)$, $C_D(z) = D \cap C_C(z) = D \cap S_0 = S_1$ is a 2-group, and three involutions of Q_5 are conjugate in D, we have $O(N_D(T)) = 1$.

(h) $S_1 \trianglelefteq D$.

PROOF. Suppose false, and let $\bar{D} = D/\langle v \rangle$. Then $\bar{D}$ has a strongly embedded subgroup $N_{\bar{D}}(\bar{S}_1)$ by (g), and in particular all involutions of $\bar{S}_1$ are conjugate in $\bar{D}$. Therefore, for an involution $w \in (Q_3 \cap R_1) - Q_4$, we have $z \sim w$ or vw in D, which contradicts (d).

(i) $C \leqq G$.

PROOF. Since $D \leqq N_X(S_1) \leqq N_X(Z(S_1)) = N_X(Q_4) = G$ by (f), (h), and (4.4-3), we have $C = DS_0 \leqq G$.

This completes the proof of (4.5).

Now, we can complete the proof of Theorem 2.

Let $X_0 = \langle G, H \rangle$. We already have $N_X(S) = S \leqq X_0$, $C_X(z) = H \leqq X_0$, and $C_X(v) \leqq G \leqq X_0$. Moreover, all involutions of S are conjugate to z or v in X_0 by (4.4-2). Therefore X_0 is a strongly embedded subgroup of X, and hence $X = X_0$ because X has precisely two conjugacy classes of involutions.

Next we prove that $(\rho_1\rho_8)^8 = 1$.

If $\langle \rho_1, \rho_8 \rangle$ is a 2-group, then $(\rho_1\rho_8)^8 = 1$ because $(\Omega_1(S))^8 = (Q_1R_0)^8 = 1$. Thus we will assume that $\langle \rho_1, \rho_8 \rangle$ is not a 2-group, and argue for a contradiction.

The standard presentation shows that

$$z = \sigma_5 \sim \sigma_3 \sim \sigma_7 \sim \sigma_1 \sim \rho_1,$$

$$v = \sigma_4 \sim \sigma_6 \sim \sigma_2 \sim \sigma_8 \sim \rho_8.$$

Therefore $\rho_1 \not\sim \rho_8$ in X and the order of $\rho_1\rho_8$ is even. Assume $|\rho_1\rho_8| = 2n$ where $n \geqq 1$, and let $\omega = (\rho_1\rho_8)^n$. Then we have $\langle \rho_1, \rho_8 \rangle \leqq C_X(\omega)$ and $\omega \sim z$ or v.

Suppose first $\omega \sim v$. Then $C_X(\omega) \sim C = C_G(v)$, and neither ρ_1 nor ρ_8 is contained in $O_2(C_X(\omega))$. But this contradicts (4.4-2) because $O_2(C) = Q_1$ by (2.3-4).

Suppose next $\omega \sim z$. Then $C_X(\omega) \sim H = C_X(z)$, and so $\rho_1\rho_8$ centralizes some element of $C_X(\omega)$ of order 5. Since the Sylow 5-centralizer of H is cyclic of order 10, we have $(\rho_1\rho_8)^{10} = 1$, and hence $\omega = (\rho_1\rho_8)^5 = (\rho_8\rho_1)^5$. Then the standard presentation shows that

$$\sigma_2^{\omega}=\sigma_2^{(\rho_8\rho_1)^5}=\sigma_4^{-1}\rho_1\rho_8\rho_1\rho_8\rho_1\rho_8\rho_1\sigma_4\,,$$

$$\sigma_6^{\omega}=\sigma_6^{(\rho_1\rho_8)^5}=\sigma_4^{-1}\rho_8\rho_1\rho_8\rho_1\rho_8\rho_1\rho_8\rho_1\rho_8\sigma_4\,,$$

$$(\sigma_2\sigma_6)^2=1\,.$$

Therefore we have $(\rho_1\rho_8)^{16}=1$ and hence $(\rho_1\rho_8)^2=1$. But this is a contradiction because $\sigma_8^{(\rho_1\rho_8)^2}=\sigma_4\neq\sigma_8$.

This completes the proof of Theorem 2.

5

In this section we prove Theorems 3 and 4 stated in Section 1. Throughout this section, let (G, H) be a pair of ${}^2F_4(2)$-type with respect to a common 2-subgroup S, and define $S^*=(S\cap O^2(G))(S\cap O^2(H))$, $G^*=O^2(G)S^*$, and $H^*=O^2(H)S^*$.

For the proof of Theorem 3, embed G and H into the amalgamated product $X=G*_S H$, and define $X^*=\langle G^*, H^*\rangle$. Pick standard generators $\{\sigma_i, \tau_j, \rho_k\}$ of the pair (G^*, H^*) and define $N=\langle((\rho_1\rho_8)^8)^{X^*}\rangle$.

First of all, we notice that

$$X\neq X^* \quad\text{and}\quad X^*\cong G^*\underset{S^*}{*}H^*$$

by the uniqueness of the standard forms of elements in an amalgamated product. Thus we have

$$X^*/N\cong{}^2F_4(2)' \quad\text{and}\quad G^*\cap N=H^*\cap N=1$$

by the presentation for ${}^2F_4(2)'$ given in Section 1.

Now suppose $N\not\trianglelefteq X$. Take an element $x\in S-S^*$, and define $M=N\cap N^x$. We have $X^*/N^x\cong{}^2F_4(2)'$ and $G^*\cap N^x=H^*\cap N^x=1$ by the above remark. Let $\bar{X}^*=X^*/N^x$. Then the pair $(\bar{G}^*, \bar{H}^*)$ is of ${}^2F_4(2)'$-type and $\{\bar{\sigma}_i, \bar{\tau}_j, \bar{\rho}_k\}$ are its standard generators. Furthermore, we have $\bar{H}^*=C_{\bar{X}^*}(Z(\bar{S}^*))$ by the structure of the centralizer of an involution in ${}^2F_4(2)'$. Therefore we can apply Theorem 2 to $\bar{X}^*$ and the pair $(\bar{G}^*, \bar{H}^*)$, and get $(\bar{\rho}_1\bar{\rho}_8)^8=1$. This shows that $(\rho_1\rho_8)^8\in N\cap N^x=M$, a contradiction. Thus (1) holds.

Next suppose $N_0=G\cap N\neq 1$. Since $G^*\cap N=1$, we have $G=G^*\times N_0$ and $|N_0|=2$, and so $S=S^*\times N_0$. Therefore $H=H^*\times N_0$ because $H^*\cap N=1$. Hence G and H have a common normal subgroup N_0 in S, which contradicts the condition (c) in Section 1. Thus (2) holds.

Let $\bar{X}=X/N$. Since $|\bar{X}:\bar{X}^*|=2$, $\bar{X}^*\cong{}^2F_4(2)'$, and $|\bar{S}|=2^{12}$, we have that $\bar{S}$ is a Sylow 2-subgroup of $\bar{X}$. Hence $\bar{X}$ is embedded in $\mathrm{Aut}\,\bar{X}^*$ because $\bar{G}$ and $\bar{H}$ do not have a common normal 2-subgroup. Since $\mathrm{Aut}({}^2F_4(2)')\cong{}^2F_4(2)$ (see (5.1) below), we have $\bar{X}\cong{}^2F_4(2)$.

Now we prove the following proposition.

(5.1) $\mathrm{Aut}({}^2F_4(2)')\cong{}^2F_4(2)$.

PROOF. Let $L={}^2F_4(2)'$ and $A=\mathrm{Aut}\,L\geqq L$, and let U be a Sylow 2-subgroup of L. Note that there are exactly two maximal 2-local subgroups M_1 and M_2 of L containing U. Choose them so that the pair (M_1, M_2) is of ${}^2F_4(2)'$-type with respect to the common 2-subgroup U, and let Q_i $(i=0,1,\cdots,6)$ and R_j $(j=0,1,2,3)$ be as in (1.1-1).

By the Frattini argument, we have $A=LT$ where $T=N_A(U)$. Since $T\leqq N_A(M_k)$ and R_j are characteristic subgroups of M_2, we have $T\leqq N_A(R_j)$ $(j=0,1,2,3)$, and so $T\leqq N_A(Q_i)$ $(i=0,1,\cdots,6)$ by the definitions of Q_i. Therefore T stabilizes the normal series

$$U\geqq Q_1R_0\geqq R_0\geqq Q_0\cap R_0\geqq Q_2R_1\geqq Q_3R_1\geqq R_1\geqq Q_3\cap R_1\geqq Q_4\geqq Q_5\geqq R_2\geqq 1$$

of U, and hence $O^2(T)$ centralizes U. As $O_2(M_k)\leqq U$ and $C_{M_k}(O_2(M_k))\leqq O_2(M_k)$, $O^2(T)$ centralizes $L=\langle M_1, M_2\rangle$, and so $O^2(T)=1$. Thus we have that T is a Sylow 2-subgroup of A. Clearly the pair (M_1T, M_2T) satisfies the conditions (a)-(d) with respect to the common 2-subgroup T and $|M_2T:T|=5$. Hence we get $|T|\leqq 2^{12}$ by (1.2). Therefore we have $A\cong{}^2F_4(2)$.

This completes the proof of (5.1), and hence of Theorem 3.

Now we turn to the proof of Theorem 4. Assume that G and H are contained in a finite group X such that $C_X(z)=H$ for some involution z in H. We use the notation introduced in Section 1. Then Theorem 3 implies that the pair (G, H) is isomorphic to the pair (P_8, P_1). Identify the pairs (G, H) and (P_8, P_1) by this isomorphism, and take the elements $u_i\in S$ $(i=1,\cdots,8)$, $r_1\in H$, and $r_8\in G$. We will show that $X=\langle G, H\rangle$ and that $(r_1r_8)^8=1$, and appeal to the Tits presentation for ${}^2F_4(2)$. Note that the pair (G^*, H^*) corresponds to the pair (M_8, M_1) by the definitions of G^* and H^*. Consequently, since $I({}^2F_4(2))\leqq{}^2F_4(2)'$, we have

(##) $\quad I(S)\leqq S^*,\quad I(G)\leqq G^*,\quad I(H)\leqq H^*$.

Let Q_i^* $(i=0,1,\cdots,6)$ and R_j^* $(j=0,1,2,3)$ be, respectively, the normal subgroups of G^* and H^* as in (1.1-1), and let $X_0=\langle G, H\rangle$. Then, using (##), we have the following result analogous to (4.4).

(5.2) *The following holds.*

(1) $\langle z\rangle=\Omega_1(Z(S))=R_2^*$.

(2) $N_X(S)=S$ *and so* S *is a Sylow 2-subgroup of* X.

(3) *There are exactly two conjugacy classes of involutions in* X *and*

in X_0. *Furthermore,* $z^X \cap Q_3^* = (Q_5^*)^z$, $z^X \cap R_1^* = \cup ((Q_5^*)^z)^H$, *and* $z^X \cap S \leqq Q_1^*$.

(4) $N_X(Q_5^*) = N_X(Q_4^*) = G$.

(5) *If* $(Q_4^*)^x \leqq Q_1^*$ *for some* $x \in X$, *then* $(Q_4^*)^x = (Q_4^*)^{hg}$ *for some* $h \in H$ *and* $g \in G$. *In particular, any two distinct X-conjugates of* Q_4^* *in* Q_1^* *intersect in* Q_5^*.

PROOF. Since $S^* \cap Z(S) = R_2^*$, we have $z \in R_2^* = \Omega_1(Z(S))$ by (♯♯), and so $N_X(S) = N_X(S) \cap C_X(z) = N_H(S) = S$. Thus (1) and (2) hold. By (♯♯) and (4.4-2), G^* and H^* control fusion of involutions in S, and hence there are precisely two conjugacy classes of involutions in X and in X_0 as $3 \nmid |C_X(z)|$. Therefore we have $z^X \cap Q_3^* = (Q_5^*)^\sharp$, $z^X \cap R_1^* = \cup ((Q_5^*)^\sharp)^H$, and $z^X \cap S \leqq Q_1^*$. This proves (3). Notice that $Q_5^* = Z(Q_0^*) \trianglelefteq G$ and that $Q_4^* = Z(\Omega_1(Q_0^*)) \trianglelefteq G$. Since $|N_X(Q_5^*)/C_X(Q_5^*)| \leqq 6$, we have $C_G(Q_5^*) = O_2(G)$, and hence $C_X(Q_5^*) = C_H(Q_5^*) = O_2(G)$ as $C_{H^*}(Q_5^*) = Q_0^*$. Therefore $N_X(Q_5^*) = G$. Since $|N_X(Q_4^*)/C_X(Q_4^*)| \leqq 24$ and $C_{Q_0^*}(Q_4^*) = Q_1^* \leqq C_X(Q_4^*) \leqq O_2(G)$, we have $N_X(Q_4^*) = G$. This proves (4). Finally we have (5), arguing as in (4.4-4). This completes the proof.

Now let $v = u_4$. We have $z = u_5^2 \in R_2^*$ and $v \in Q_4^* - Q_5^*$. We will show that $C_X(v) \leqq G$ by the argument in the proof of (4.5).

Let $C = C_X(v)$, $C_0 = C_G(v)$, $S_0 = C_S(v)$, $Z_0 = Z(S_0)$, $C_0^* = C_{G^*}(v)$, $S_0^* = C_{S^*}(v)$, and $Z_0^* = Z(S_0^*) = \langle z, v \rangle$. Then $Z_0^* = \Omega_1(Z_0)$, and so $N_X(S_0) = N_X(S_0) \cap C_X(Z_0^*) = S_0$ because $C_X(Z_0^*) = C_H(v) = C_S(v) = S_0$. Therefore we have

(a′) S_0 *is a Sylow 2-subgroup of* C.

Furthermore, arguing as in (4.5), we have

(b′) Q_4^* *is weakly closed in* S_0 *with respect to* C, *and*
(c′) Q_4^* *is the weak closure of* Z_0^* *in* S_0 *with respect to* C.

Hence, by (4.1), (4.2), and (5.2-4), C_0 controls C-fusion in S_0. Thus we have

(e′) $C/O^2(C) \cong C_0/O^2(C_0)$.

Note that $O^2(C_0) = O^2(C_0^*)$, and define $S_1^* = [Q_1^*, O^2(C_0)]Q_3^*$, $D_0 = O^2(C_0)S_1^*$, and $D = O^2(C)S_1^*$. Then $D_0 \leqq C_0^*$ and hence, similarly as in (4.5), we have

(f′) S_1^* *is a Sylow 2-subgroup of* D, $S_1^* \trianglelefteq D_0$, $Z(S_1^*) = Q_4^*$, *and a Sylow 3-subgroup of* D_0 *acts irreducibly on* S_1^*/Q_3^*, Q_3^*/Q_4^*, *and* $Q_4^*/\langle v \rangle$.

Since $Z(S_1^*)$ is weakly closed in S_1^* with respect to D by (b′) and (f′), we can apply (4.3) and get the following.

(g′) *If T is a subgroup of S_1^* of order 4 containing v, then $N_D(T) \leqq S_1^*$.*

Therefore, as $D/\langle v\rangle$ does not have a strongly embedded subgroup, we have

(h′) $S_1^* \trianglelefteq D$,

and hence $C = DS_0 \leqq N_X(Z(S_1^*))S_0 = N_X(Q_4^*) = G$, as desired.

As all involutions in S are conjugate to z or v in X_0 by (5.2-3), X_0 is a strongly embedded subgroup of X, and hence $X = X_0$ by (5.2-3).

It now remains to show that $(r_1r_8)^8 = 1$. If $\langle r_1, r_8\rangle$ is a 2-group, then $(r_1r_8)^8 = 1$ because $(\Omega_1(S))^8 = (\Omega_1(S^*))^8 = 1$ by (##). Suppose $\langle r_1, r_8\rangle$ is not a 2-group. Note that $r_1 \sim z \nsim v \sim r_8$ in X by the Tits presentation for ${}^2F_4(2)$, and let $\omega = (r_1r_8)^n$ where $2n = |r_1r_8|$. Then $\langle r_1, r_8\rangle \leqq \Omega_1(C_X(\omega))$ and $\omega \sim z$ or v. As $\Omega_1(C_X(z)) \leqq H^*$ and $\Omega_1(C_X(v)) \leqq C_{G^*}(v)$ by (##), we reach a contradiction in either case by the same argument as in the proof of Theorem 2. Thus we have $(r_1r_8)^8 = 1$.

By the Tits presentation for ${}^2F_4(2)$, X is a homomorphic image of ${}^2F_4(2)$. As $|X| > |S| = 2^{12} > 2$, we conclude that X is isomorphic to ${}^2F_4(2)$.

This completes the proof of Theorem 4.

References

[1] Fan, P. S., Amalgams of prime index, to appear J. Algebra **98** (1986), 375-421.
[2] Glauberman, G., A sufficient condition for p-stability, Proc. London Math. Soc. (3) **25** (1972), 253-287.
[3] Goldschmidt, D. M., 2-Fusion in finite groups, Ann. of Math. (2) **99** (1974), 70-117.
[4] Goldschmidt, D. M., Automorphisms of trivalent graphs, Ann. of Math. (2) **111** (1980), 377-406.
[5] Gomi, K., Pairs of groups having a common 2-subgroup of prime indices, J. Algebra **97** (1985), 407-437.
[6] Gomi, K., On the thin finite simple groups, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **32** (1985), 143-164.
[7] Griess, R. L. and R. Lyons, The automorphism group of the Tits simple group ${}^2F_4(2)'$, Proc. Amer. Math. Soc. **52** (1975), 75-78.
[8] Parrott, D., A characterization of the Tits' simple group, Canad. J. Math. **24** (1972), 672-685.
[9] Tate, J., Nilpotent quotient groups, Topology **3** suppl. **1** (1964), 109-111.
[10] Tits, J., Algebraic and abstract simple groups, Ann. of Math. (2) **80** (1964), 313-329.

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