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Multiplicity free representations with respect to the restriction between $SL(2n, \boldsymbol{C})$ and $Sp(n, \boldsymbol{C})$

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§1. Introduction.

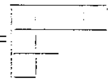
Multiplicity free representations of a group with respect to the restriction to a subgroup play important roles in the symmetry breaking theory. Let X be a physical system and G be its symmetry group. Then the set of states of X forms a representation space of G. To distinguish these states, we often break the symmetry from G to a subgroup H, and label (ρ, φ) for the states which are involved in the irreducible constituent ρ of G and φ of H. Here if ρ is not multiplicity free with respect to $G \supset H$, we can not settle the ambiguity of labeling problem from the knowledge of representations of H, cf. [1], [4], [5].

We shall study the embedding $SL(2n, \boldsymbol{C}) \supset Sp(n, \boldsymbol{C})$. Combinatorial discriptions of the decomposition rule of the restricted representations of irreducible representations of $SL(2n, \boldsymbol{C})$ to $Sp(n, \boldsymbol{C})$ into their irreducible constituents are given in [8], [9], and [10]. In this paper, we classify the multiplicity free irreducible finite dimensional representations of $SL(2n, \boldsymbol{C})$ with respect to this embedding (Theorem 3.1) using the results of [10]. The author is grateful to Professor Nagayoshi Iwahori for suggesting this problem to him, explaining its significance in application to physics. He is very obliged to the referee for pointing out an error in the original proof of Theorem 3.1 and suggesting its correction.

§2. Combinatorial notations. cf. [7], [10]

DEFINITION 2.1. A partition of n is a non increasing sequence of positive integers such that the total sum of them equals n. The diagram $Y(\lambda)$ of a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s)$ is the set of points (i, j) of $\boldsymbol{Z}^2$ such that $1 \leqq j \leqq \lambda_i$.

We usually figure it as the following example replacing the points by squares, and the first coordinate i is the row index increasing as one goes downwards.

Example. Let $\lambda=(4,2,1)$, then $Y(\lambda)=$ .

For a partition λ we define the transposed partition ${}^t\lambda$ of λ by $({}^t\lambda)_i=\#\{j \mid \lambda_j=i\}$.

We usually write λ for $Y(\lambda)$ as an abbreviation identifying a partition and its diagram.

A subdiagram μ of λ is a diagram corresponding to the partition μ such that $\mu_i \leqq \lambda_i$ for any i.

Let $\Gamma=\{1,2,\cdots,n,n',(n-1)',\cdots,2',1'\}$ be an ordered set with the order $1<2<\cdots<n<n'<(n-1)'<\cdots<2'<1'$. Elements of $\{1,2,\cdots,n\}$ (resp. $\{1',2',\cdots,n'\}$) are called positive symbols (resp. negative symbols.)

DEFINITION 2.2. *Tableau.* Let λ be a diagram. A tableau (Γ-tableau) T of shape λ is a map from λ into Γ. Graphically, T may be described by numbering each square of the diagram λ with the elements of Γ.

$T(i,j)$, the symbol written in (i,j)-place of T, is often written $t_{i,j}$ for abbreviation.

A tableau is called weakly normal if $t_{i,j} \leqq t_{i,j+1}$ for any i and j such that (i,j) and $(i,j+1)$ are involved in the shape of T.

DEFINITION 2.3. *Normalization.* Let T be a tableau of shape λ. Then there is a unique weakly normal tableau $\hat{T}$ of shape λ such that the set of symbols in the i-th row of $\hat{T}$ coinsides with that of T with multiplicities for any i. In the other words, $\hat{T}$ is the tableau got by arranging the symbols in each row of T into a non-decreasing sequence. $\hat{T}$ is called the normalization of T.

DEFINITION 2.4. *Standard tableau.* A tableau T is called standard if $t_{i,j}<t_{i,j+1}$ and $t_{i,j} \leqq t_{i+1,j}$ for any i and j.

DEFINITION 2.5. *Word.* Let T be a tableau. The word of T is the sequence $w(T)$ of the elements of Γ defined by reading the symbols in T from left to right in successive rows, starting with bottom to top.

Let $w(T)=a_1, a_2, \cdots, a_m$ be the word of a tableau T. For a symbol i and a positive integer p, $c_T(i,p)$ denotes the number of occurrence of i in the subsequence $a_1, a_2, \cdots, a_p$ of $w(T)$. For a positive symbol i and a positive integer p, $d_T(i,p)$ denotes $c_T(i,p)-c_T(i',p)$.

DEFINITION 2.6. *$Sp(n)$-tableau.* Let λ be a diagram. An $Sp(n)$-tableau of shape λ is a standard tableau T of shape λ satisfying the following conditions 1) and 2) for any positive symbol $i=1, 2, \cdots, n-1$ and any positive integer p.

1) $c_T(i, p) \geqq c_T(i+1, p)$.
2) $d_T(i, p) \geqq d_T(i+1, p) \geqq 0$.

Note. We shall call these conditions "L-R property" because of the resembrance of the Littlewood-Richardson rule which we meet in the representation theory of general linear groups and symmetric groups.

Note. In [2], DeConcini defined "symplectic tableau" which is different from our $Sp(n)$-tableau.

REMARK. From the condition 1), the positive part of a $Sp(n)$-tableau must be a canonical tableau, cf. [10].

DEFINITION 2.7. *Weight of a $Sp(n)$-tableau.* Let T be a $Sp(n)$-tableau of shape λ where λ is a partition of m. Then from the condition 2), we can regard the sequence $D_T := (d_T(1, m), d_T(2, m), \cdots, d_T(n, m))$ as a diagram. D_T is called the weight of T.

Example. $\begin{matrix} 1 & 2 & 3 & 2' \\ 1 & 2 & & \end{matrix}$ is an $Sp(n)$-tableau of shape $(4, 2)$ for $n \geqq 3$. Its weight is $(2, 1, 1)$.

§ 3.

Let us recall the parametrization of the irreducible representations of special linear groups and symplectic groups by means of the Young diagrams, cf. [10], [11].

Via the highest weight theory, there is a canonical one-to-one correspondence between the set of equivalence classes of the irreducible representations of $SL(n, \boldsymbol{C})$ (resp. $Sp(n, \boldsymbol{C})$) and the set of diagrams such that the length of their first columns are not longer than $n-1$ (resp. n). We shall denote $\rho_\lambda^{(n)}$ (resp. $\rho_\lambda^{Sp(n)}$) for the irreducible representation of $SL(n, \boldsymbol{C})$ (resp. $Sp(n, \boldsymbol{C})$) corresponding to the diagram λ with respect to the above correspondence.

Let $\rho_\lambda^{(2n)}$ be an irreducible representation of $SL(2n, \boldsymbol{C})$. Let us consider its restricted representation $\rho_\lambda^{(2n)} \downarrow_{Sp(n, \boldsymbol{C})}^{SL(2n, \boldsymbol{C})}$ to $Sp(n, \boldsymbol{C})$ which is canonically imbedded in $SL(2n, \boldsymbol{C})$. Let us denote $d_{\lambda, \mu}$ for the multiplicity of the irreducible representation $\rho_\mu^{Sp(n)}$ in $\rho_\lambda^{(2n)} \downarrow_{Sp(n, \boldsymbol{C})}^{SL(2n, \boldsymbol{C})}$ as an irreducible constituent.

We say that $\rho_\lambda^{(2n)}$ is $Sp(n)$-multiplicity free if and only if $d_{\lambda,\mu} \leqq 1$ for any μ.

THEOREM 3.1. *The irreducible representation* $\rho_\lambda^{(2n)}$ *of* $SL(2n, \boldsymbol{C})$ *is* $Sp(n)$*-multiplicity free if and only if the transposed diagram* ${}^t\lambda$ *of* λ *appears in Table* 1 *below.*

Table 1.

a)	$(3^i, 2^j, 1^k) = (3, 3, \cdots, 3, 2, \cdots, 2, 1, \cdots, 1)$	$i, j, k > 0$
b)	$((p+1)^i, p^j)$	$i, j > 0$
c)	$(p^i, 1^j)$	$i, j \geqq 0$
d)	$((2n-1)^i, p^j)$	$i, j > 0$
e)	$((2n-1)^i, (2n-2)^j, (2n-3)^k)$	$i, j, k \geqq 0$
f)	$((2n-1)^i, (2n-2)^j, 1^k)$	$i, j, k \geqq 0$
g)	$((2n-1)^i, 2^j, 1^k)$	$i, j, k \geqq 0$

We shall prove the theorem using the following result in [10].

THEOREM 3.2 (Restriction rule. cf. [10]). *In the notation above* $d_{\lambda,\mu}$ *equals the number of* $Sp(n)$*-tableaux of shape* ${}^t\lambda$ *and weight* μ.

Now we shall prepare some notations.

DEFINITION 3.3. *Reverse sequence.* Let p, q and r be positive integers such that $n \geqq q \geqq p$ and $q \geqq r$. Then the reverse sequence with top p, peak q and bottom r is the sequence of symbols $p, p+1, \cdots, q, q', (q-1)', \cdots, (r+1)', r'$. The length of this sequence is $2q-p-r+2$.

DEFINITION 3.4. The complete reverse sequence of length s $(s \leqq 2n-1)$ is the sequence

$$1, 2, \cdots, p, p', (p-1)', \cdots, 2', 1' \quad \text{if } s = 2p \text{ is even,}$$

$$1, 2, \cdots, p, p', (p-1)', \cdots, 3', 2' \quad \text{if } s = 2p-1 \text{ is odd.}$$

DEFINITION 3.5. *Highest sequence.* The highest sequence of length s $(s \leqq 2n-1)$ is the sequence

$$1, 2, 3, \cdots, s \quad \text{if } s \leqq n, \text{ and}$$

$$1, 2, 3, \cdots, n, n', (n-1)', \cdots, (2n-s+1)' \quad \text{if } s > n.$$

DEFINITION 3.6. *Extension of an $Sp(n)$-tableau.* Let λ be a diagram and μ be its subdiagram such that $\lambda_i-\mu_i=p_i=2q_i$ is even and $q_i \geqq q_{i+1}$ for any i. Suppose T is an $Sp(m)$-tableau of shape μ with weight τ. Then we define the extension T^{λ} of T as follows. Let a_i be the maximal positive symbol occuring in the i-th row of T. We have a tableau T_0 of shape λ defined as $T_0(i,j)=T(i,j)$ if $j \leqq \mu_i$ and the rest part of its i-th row is a reverse sequence of length p_i with top a_i+1 and bottom $(a_i+1)'$ for any i. (Note: peak is a_i+q_i). T^{λ} is the normalization of T_0.

We set $s=q_1$ in the above definition. Then we get the following.

PROPOSITION 3.7. *T^{λ} is an $Sp(m+s)$-tableau with weight τ.*

PROOF. The standardness of T^{λ} follows from the fact that there is no negative symbol larger than $(a_i)'$ in the i-th row of the $Sp(m)$-tableau T. Since the reverse sequences occuring in 3.6 have no influence on the L-R property and weight, T^{λ} satisfies the L-R property and has weight τ. It is clear that there is no positive symbol larger than $m+s$ in T^{λ}.

DEFINITION 3.8. Let λ be a diagram and let $\lambda_1 \leqq 2n$. The complement diagram ${}^c\lambda$ of λ is the diagram defined by $({}^c\lambda)_i=2n-\lambda_{k-i}$, where $k=({}^t\lambda)_1$ is the length of the first column of λ.

LEMMA 3.9. *For any irreducible representation $\rho_{\lambda}^{(2n)}$ of $SL(2n, \boldsymbol{C})$,*

$$\rho_{\lambda}^{(2n)} \downarrow {}^{SL(2n,C)}_{Sp(n,C)} = \rho_{{}^c\lambda}^{(2n)} \downarrow {}^{SL(2n,C)}_{Sp(n,C)}.$$

PROOF. Let H be a maximal torus of $SL(2n, \boldsymbol{C})$. When we regard a representation of $SL(2n, \boldsymbol{C})$ as a representation of H, $m(\lambda, \theta)$ is defined as the multiplicity of an irreducible character θ of H in $\rho_{\lambda}^{(2n)}$. Then it is easy to show that $m({}^c\lambda, \theta)=m(\lambda, -\theta)$, where $-\theta(h)=\theta(h^{-1})$ for any element h of H. Then the lemma follows from the general theory on the representations of semisimple groups, cf. [6].

Now we shall prove the theorem. First we show the following.

CLAIM 3.10. *We can make two different $Sp(n)$-tableaux of shape ν with a same weight if ν does not exist in Table 1.*

Table 2 is the table of fundamental Sp-tableaux we shall use to prove this claim.

Table 2.

$A_{4,2}=\begin{array}{llll}1&2&3&3'\\1&1'\end{array}$	$\bar{A}_{4,2}=\begin{array}{llll}1&2&2'&1'\\1&2\end{array}$
$A_{5,2}=\begin{array}{lllll}1&2&3&4&4'\\1&1'\end{array}$	$\bar{A}_{5,2}=\begin{array}{lllll}1&2&3&2'&1'\\1&2\end{array}$
$A_{5,3}=\begin{array}{lllll}1&2&3&4&4'\\1&2&2'\end{array}$	$\bar{A}_{5,3}=\begin{array}{lllll}1&2&3&3'&2'\\1&2&3\end{array}$
$A_{6,3}=\begin{array}{llllll}1&2&3&4&5&5'\\1&2&2'\end{array}$	$\bar{A}_{6,3}=\begin{array}{llllll}1&2&3&4&3'&2'\\1&2&3\end{array}$
$B_{4,3,2}=\begin{array}{llll}1&2&3&1'\\1&2&2'\\1&2\end{array}$	$\bar{B}_{4,3,2}=\begin{array}{llll}1&2&3&3'\\1&2&3\\1&1'\end{array}$
$B_{5,4,3}=\begin{array}{lllll}1&2&3&4&2'\\1&2&3&3'\\1&2&3\end{array}$	$\bar{B}_{5,4,3}=\begin{array}{lllll}1&2&3&4&4'\\1&2&3&4\\1&2&2'\end{array}$
$D_{4,3,1}=\begin{array}{llll}1&2&3&1'\\1&2&2'\\1\end{array}$	$\bar{D}_{4,3,1}=\begin{array}{llll}1&2&3&2'\\1&2&1'\\1\end{array}$
$D_{5,4,1}=\begin{array}{lllll}1&2&3&4&1'\\1&2&3&3'\\1\end{array}$	$\bar{D}_{5,4,1}=\begin{array}{lllll}1&2&3&4&3'\\1&2&3&1'\\1\end{array}$
$E_{4,3,2,1}=\begin{array}{llll}1&2&3&1'\\1&2&2'\\1&2\\1\end{array}$	$\bar{E}_{4,3,2,1}=\begin{array}{llll}1&2&3&2'\\1&2&1'\\1&2\\1\end{array}$

A sub-index indicates the shape of a tableau. X_ν and $\bar{X}_\nu$ are Sp-tableau of a same shape ν and have a same weight (X is A, B, D or E).

PROOF OF THE CLAIM.

Case 1. Suppose there exists an i such that $\nu_i \geqq \nu_{i+1}+2$ and $\nu_{i+1}\neq 0$ or 1, and $\nu_i \neq 2n-1$. Then we can construct an $Sp(n)$-tableau of shape (ν_i, ν_{i+1}) as an extension of suitable $A_{p,q}$ in Table 2. In particular we shall use $A_{p,q}$ such that $p\equiv \nu_i \mod 2$ and $q\equiv \nu_{i+1} \mod 2$.

Now writing a highest sequence in any j-th row such that j is smaller than i, a complete reverse sequence in any k-th row such that k is larger than $i+1$, we get an $Sp(n)$-tableau of shape ν from the $Sp(n)$-tableau of shape (ν_i, ν_{i+1}) above. On the other hand, we have another $Sp(n)$-tableau of shape ν with the same weight from $\bar{A}_{p,q}$ similarly.

Case 2. Suppose there exists i and j such that $\nu_i = a+1 \neq 2n-1$,

$\nu_j=\nu_{i+1}=a\geqq 3$ and $\nu_{j+1}=1$. We can construct an $Sp(n)$-tableau T_0 of shape $(a+1, a, 1)$ as an extension of $D_{p+1,p,1}$ in Table 2 for $p\equiv a \mod 2$. Then we shall construct an $Sp(n)$-tableau T of shape ν as follows. We shall denote M^s for the sequence we shall write in the s-th row of T, which is defined by

M^s is the highest sequence for $s<i$,
M^i is the sequence written in the first row of T_0,
M^s is the sequence written in the second row of T_0 for $i<s\leqq j$,
M^s is the sequence consists of single element 1 for $s>j$.

On the other hand, we get another $Sp(n)$-tableau T_1 of shape $(a+1, a, 1)$ as an extension of $\bar{D}_{p+1,p,1}$. Then we have a tableau T' given by exchanging the j-th row and the i-th row of T with the first row and the second row of T_1. T and T' are different $Sp(n)$-tableaux of shape ν with a same weight.

Case 3. Next we shall consider the case $\nu_i-\nu_{i+1}\leqq 1$ if $\nu_{i+1}\neq 0$. Suppose there exists an i such that $\nu_i=a\geqq 2$ and $\nu_{i+1}=0$. From the hypothesis of claim, there exists k and m such that $\nu_k=a+1$, $\nu_{k+1}=a$, $\nu_m=a+2$ and $\nu_{m+1}=a+1$. We shall assume that $a+2<2n-1$.

Then we can construct two different $Sp(n)$-tableaux of shape ν with a same weight from $B_{p+2,p+1,p}$ and $\bar{B}_{p+2,p+1,p}$ in Table 2 for $p\equiv a \mod 2$. Since their construction is similar as in Case 2, we shall omit it.

Case 4. When $\nu_i-\nu_{i+1}\leqq 1$ for any i, then ν has rows of length 1, 2, 3, and 4 from the hypothesis. Then we can construct two different $Sp(n)$-tableaux of shape ν with a same weight easily from $E_{4,3,2,1}$ and $\bar{E}_{4,3,2,1}$.

We have seen that the claim holds when $\nu_1\neq 2n-1$. Hence to complete the proof, we shall assume that $\nu_1=2n-1$. We shall denote ν^* for the diagram made from ν by taking the all rows of length $2n-1$ off. It is clear that we have no problem when ν^* is not one of types a), b), and c). On the other hand, if ν^* is one of these three types, considering the complement diagram ${}^c\nu$ of ν, Lemma 3.9 assures us that $\rho_\nu^{(2n)}$ is not multiplicity free easily.

The rest to show is that the representations corresponding to the transposed diagrams in Table 1 are multiplicity free with respect to our restriction. Let T and S be $Sp(n)$-tableaux of a same shape ν and a same weight τ. We must show that if ν is found in Table 1, then $T=S$. Lemma 3.9 assures us that it suffices to check this for a), b), c), and g) in Table 1.

a) Let $\nu=(3^i, 2^j, 1^k)$. Then the number of occurrence of 1 and $1'$ in T are $i+j+k$ and $i+j+k-\tau_1$ respectively. Because of the L-R property, we can regard $T^{-1}(\Gamma-\{1,1'\})$ as a diagram, say μ. Extracting all squares of T on which 1 or $1'$ are written, we get a tableau T_1 of shape μ. For a example, if $T=\begin{array}{lll} 1 & 2 & 3 \\ 1 & 2 & 2' \\ 1 & 1' & \\ 1 & & \end{array}$, then $T_1=\begin{array}{ll} 2 & 3 \\ 2 & 2' \end{array}$. Shifting the symbols as $i\to i-1$ and $i'\to(i-1)'$, we see that the resulting tableau T_2 is also a $Sp(n)$-tableau. In the example above, $T_2=\begin{array}{ll} 1 & 2 \\ 1 & 1' \end{array}$. Similarly we get S_2 from S. It is clear from our construction that T_2 and S_2 have the same weight. Moreover, as remarked above, the number of 1 and $1'$ in T are both determined only by ν and τ. Hence, we also have $|T_2|=|S_2|$, here $|A|$ means the number of squares in the shape of a tableau A. Thus, in the present case, the desired conclusion follows from the next lemma.

LEMMA. *Let A and B be $Sp(n)$-tableaux of shape $(2^a, 1^b)$ and $(2^c, 1^d)$ respectively. Suppose $|A|=|B|$ and A and B have a same weight, then $A=B$.*

PROOF. Let x and y be the number of occurrence of $1'$ in A and B respectively. Because of the L-R property, $1'$ can occur only in the second column of A, and $2'$ can not occur in A. So the weight of A is $(a+b-x, a-x)$. By the same reason, the weight of B is $(c+d-y, c-y)$. So $x=y$ and the shape of A and B coincide. Hence $A=B$.

b) Let $\nu=((p+1)^i, p^j)$. First, we shall assume that p is even.

CLAIM 1. *In every row of T, the first p symbols of it forms a reverse sequence with top* 1.

Proof is easy from the L-R property.

Note let us consider the sub-tableau T_0 of T corresponding to the subdiagram $\mu=(p^{i+j})$ of ν. Then from Claim 1, T_0 is also an $Sp(n)$-tableau. We also see that the transposed diagram α^T of the weight of T_0 is an even partition. In other words, $(\alpha^T)_s=({}^tD_{T_0})_s$ is even for any s.

CLAIM 2. *For any row of T, its $p+1$-th symbol $T(i, p+1)$ is $((\alpha^T)_s)'$ for some $s\leqq r$, or $p+1$.*

Proof is easy.

From these claims, we see that ${}^t\tau={}^tD_T$ has just i odd rows. This construction is reversible, i. e. we can construct T from ν and τ as follows. We define a diagram γ by

$$\gamma_i=\begin{cases}({}^t\tau)_i & \text{if } ({}^t\tau)_i \text{ is even}\\ ({}^t\tau)_i-1 & \text{if } ({}^t\tau)_i \text{ is odd.}\end{cases}$$

It is easily seen that there exists at most one $Sp(n)$-tableau of shape (p^{i+j}) with weight γ such that any of its row is a reverse sequence. So T_0 must coincide with S_0 which is made from S similarly as T_0. Then T and S must coincide because they are standard. When p is odd, we can prove our statement similarly.

c) Let $\nu=(p^i, 1^j)$. Then the symbol $1'$ can occur only in the p-th column of T. The number of occurrence of $1'$ in T equals $k=i+j-\tau_1$. So an argument similar to the one used in a) shows that the number of $Sp(n)$-tableaux of shape ν with weight τ equals the number of $Sp(n-1)$-tableaux of shape $\tilde{\nu}=((p-1)^{i-k}, (p-2)^k)$ with weight $\tilde{\tau}=(\tau_2, \tau_3, \cdots, \tau_n)$. We knew that this number is at most 1 from the previous Case b).

g) Let $\nu=((2n-1)^i, 2^j, 1^k)$. We make $Sp(n)$-tableaux T_2 and S_2 in the same way as in a). Since any row of length $2n-2$ makes no influence on the weight of an $Sp(n-1)$-tableau, we shall consider the tableau T_3 (resp. S_3) made from T_2 (resp. S_2) by taking all rows of length $2n-2$ off. S_3 and T_3 are $Sp(n-1)$-tableaux. It suffice to show that $T_3=S_3$. Suppose that the shapes of T_3 and S_3 are $((2n-3)^a, 1^b)$ and $((2n-3)^c, 1^d)$ respectively. Since $|T_2|=|S_2|$ and $n\neq 1$, $(2n-2)(a-c)=(2n-3)(a-c)+(b-d)$. So $a-b=c-d$. On the other hand, it follows from the L-R property that the number of cells in the weight of T_3 (resp. S_3) equals $a+b$ (resp. $c+d$). In short, $|D_{T_3}|=a+b$ and $|D_{S_3}|=c+d$. Since $D_{T_3}=D_{S_3}$, the shape of T_3 coincides with that of S_3. Now we have $T_3=S_3$ from the result of c). Q. E. D.

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