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Jacobi forms and a Maass relation for Eisenstein series

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Introduction.

Let H_n be the Siegel upper half plane of degree n. A Jacobi form of degree n is a holomorphic function on $H_n\times \boldsymbol{C}^n$ with certain transformation properties with respect to $Sp(n, \boldsymbol{Z})\times \boldsymbol{Z}^{2n}$. A Jacobi form in the sense of Eichler-Zagier [E-Z], from our point of view, is that of degree one. In this paper we will show that some parts of [E-Z] can be generalized to the case of arbitrary degree n.

We shall explain briefly the contents of our paper. Let k, m be positive integers. A holomorphic function $\phi(\tau, z)$ on $H_n\times \boldsymbol{C}^n$ is a Jacobi form of weight k and index m if it satisfies the following transformation formulas:

(i) $\phi((a\tau+b)(c\tau+d)^{-1}, z(c\tau+d)^{-1})=\det(c\tau+d)^k\exp(2\pi\, imz(c\tau+d)^{-1}c^tz)\phi(\tau, z)$

for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $Sp(n, \boldsymbol{Z})$,

(ii) $\phi(\tau, z+\lambda\tau+\mu)=\exp(-2\pi\, im(\lambda\tau^t\lambda+2\lambda^tz))\phi(\tau, z)$ for any λ, μ in $\boldsymbol{Z}^n$,

(if $n=1$, we need a regularity condition at infinity).

As in the case of Siegel modular forms, we can prove the finite dimensionality of the space of Jacobi forms and the Koecher principle for Jacobi forms (§§ 1 and 2). We can define an operation of the Hecke algebra of $Sp(n, \boldsymbol{Z})$ on the space of Jacobi forms. If ϕ is a Jacobi form of weight k and index m, and if T is an element of the Hecke algebra with similitude ν, then $\phi|T$ is a Jacobi form of weight k and index $m\nu$ (§ 4). Let $E_k^{(n+1)}$ be the Siegel-Eisenstein series of weight k and of degree $n+1$. We denote the m-th Fourier-Jacobi coefficient of $E_k^{(n+1)}$ by $e_{k,m}$. For any natural number m there exists explicitly determined element $D(m)$ in the Hecke algebra, such that $e_{k,m}=e_{k,1}|D(m)$. When $n=1$ this is equivalent to the Maass relation for the Eisenstein series of degree two ([M]).

Notations. Throughout this paper n denotes a natural number and we call it the degree. Let $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. For any commutative ring A, $M_{p,q}(A)$ denotes the set of p by q matrices with coefficients in A, and $M_p(A)$ denotes $M_{p,p}(A)$.

We use ${}^tM$ and 1_n for the transpose of a matrix M and for the n by n identity matrix, respectively. We write J_n for a standard alternating matrix of degree $2n$, namely

$$J_n=\begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix},$$

where 0 is an n by n null matrix.

The Siegel upper half plane H_n of degree n is the set of symmetric n by n complex matrices with positive definite imaginary parts. The letter τ and z will always be reserved for variables in H_n and $\boldsymbol{C}^n$, respectively. Let G_n, Γ_n, S_n be the real symplectic group, the Siegel modular group and the group of similitudes, respectively; namely we define

$$G_n=Sp(n,\boldsymbol{R})=\{M\in M_{2n}(\boldsymbol{R})\,;\,{}^tMJ_nM=J_n\},$$

$$\Gamma_n=Sp(n,\boldsymbol{Z})=G_n\cap M_{2n}(\boldsymbol{Z}),$$

$$S_n=\{M\in M_{2n}(\boldsymbol{R})\,;\,{}^tMJ_nM=\nu J_n \quad \text{for some } \nu>0\}.$$

Let M be in S_n. If ${}^tMJ_nM=\nu J_n$, we write $\nu=\nu(M)$. For any $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in S_n and τ in H_n, we put $M\tau=(a\tau+b)(c\tau+d)^{-1}$. This defines an action of S_n on H_n.

The symbol $e(x)$ denotes $e^{2\pi ix}$ and $e^m(x)$ denotes $e(mx)$.

1. Jacobi group.

In this section we shall define the Jacobi group G_n^J and an action of G_n^J on the space of functions on $H_n\times\boldsymbol{C}^n$. In fact, for our later purpose, we extend the definition a little bit and treat a group S_n^J, which contains G_n^J as a normal subgroup.

As a set, we define S_n^J by $S_n^J=S_n\times\boldsymbol{R}^{2n}\times\boldsymbol{R}$, where S_n is the group of similitudes. We denote a general element of S_n^J by $[M,X,\kappa]$ in which $M\in S_n$, $X\in\boldsymbol{R}^{2n}$, $\kappa\in\boldsymbol{R}$. For $g_i=[M_i,X_i,\kappa_i]$ in S_n^J $(i=1,2)$, we define

$$g_1g_2=[M_1M_2,\,\nu(M_2)^{-1}X_1M_2+X_2,\,\nu(M_2)^{-1}\kappa_1+\kappa_2+\nu(M_2)^{-1}X_1M_2J_n{}^tX_2].$$

We recall that for $M\in S_n$, $\nu(M)$ is defined by ${}^tMJ_nM=\nu(M)J_n$. The above product makes S_n^J a group. We apply a similar procedure to $G_n=Sp(n,\boldsymbol{R})$ and $\Gamma_n=Sp(n,\boldsymbol{Z})$. By restricting the multiplication to $G_n^J=G_n\times\boldsymbol{R}^{2n}\times\boldsymbol{R}$

and $\Gamma_n^J=\Gamma_n\times \boldsymbol{Z}^{2n}\times \boldsymbol{Z}$, we obtain two subgroups G_n^J and Γ_n^J of S_n^J. We call G_n^J the Jacobi group of degree n.

The following remark makes the above definition understandable. Take $g=[M, X, \kappa]$ in S_n^J and write $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $X=(\lambda, \mu)$, in which a, b, c, d are $n\times n$ matrices and λ, μ are n-vectors. We put $\nu=\nu(M)$ and define g' by

$$g'=\begin{bmatrix} a & 0 & b & 0 \\ 0 & \nu & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1_n & 0 & 0 & {}^t\mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1_n & -{}^t\lambda \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then the correspondence $g\mapsto g'$ defines a homomorphism from S_n^J to S_{n+1}.

Let k and m be non-negative integers. Take $[M, X, \kappa]$ from S_n^J, and decompose M and X into $n\times n$ blocks $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and n-vectors (λ, μ), respectively. For any function $\phi(\tau, z)$ on $H_n\times \boldsymbol{C}^n$, we define

$$\begin{aligned}&(\phi|_{k,m}[M, X, \kappa])(\tau, z)\\ &=e^{m\nu}(\kappa+\lambda\tau{}^t\lambda+2\lambda{}^tz+\lambda{}^t\mu-(z+\lambda\tau+\mu)(c\tau+d)^{-1}c{}^t(z+\lambda\tau+\mu))\\ &\quad\times\det(c\tau+d)^{-k}\phi(M\tau, \nu(z+\lambda\tau+\mu)(c\tau+d)^{-1}),\end{aligned}$$

where we write $\nu=\nu(M)$. This is an action of S_n^J in the sense that for any $g_i=[M_i, X_i, \kappa_i]$, $i=1, 2$

$$(\phi|_{k,m}g_1)|_{k,m\nu}g_2=\phi|_{k,m}(g_1g_2),$$

where we write $\nu=\nu(M_1)$.

2. Jacobi form.

In this section we shall define the Jacobi form and prove some basic properties of Jacobi forms. Many of them are more or less well-known, but we include here for the sake of completeness. For more generalities we refer to [S].

Let k and m be positive integers.

2.1 DEFINITION. A holomorphic function ϕ on $H_n\times \boldsymbol{C}^n$ is called a Jacobi form of weight k and index m (and of degree n) if it satisfies the following two conditions:

i) $\phi|_{k,m}g=\phi$ for all g in Γ_n^J,

ii) ϕ has a Fourier expansion of the form

$$\phi(\tau, z)=\sum_{N,r} c(N, r)e(\mathrm{Tr}(N\tau)+r^{t}z),$$

where the summation is taken over all symmetric half integral matrices N of degree n and all integral n-vectors r. We require that $c(N, r)=0$ unless $4mN-{}^{t}rr$ is semi-positive.

The vector space of all such functions is denoted by $J_{k,m}(\Gamma_n)$.

We recall some elementary facts about Theta functions. Let a and b be rational n-vectors. The Theta function $\theta_{a,b}(\tau, z)$ with characteristic (a, b) is defined by

$$\theta_{a,b}(\tau, z)=\sum_{q\in \boldsymbol{Z}^n} e((1/2)(q+a)\tau^{t}(q+a)+(q+a)^{t}(z+b)).$$

The right hand side converges absolutely and uniformly on any compact subset of $H_n\times \boldsymbol{C}^n$.

2.2 DEFINITION. Fix a point τ in H_n. Let m be a positive integer. We define $R_m(\tau)$ to be the vector space of all holomorphic functions f on $\boldsymbol{C}^n$ satisfying

$$f(z+\lambda\tau+\mu)=e^{m}(-(1/2)\lambda\tau^{t}\lambda-\lambda^{t}z)f(z)$$

for all integral n-vectors λ and μ.

2.3 THEOREM ([I]). *Let us fix a point τ in H_n and a positive integer m. Then the functions $\{\theta_{r,0}(m\tau, mz)\}$, $r=(r_1, \cdots, r_n)$, $mr_i\in \boldsymbol{Z}$, $0\leqq r_i<1$, $i=1, \cdots, n$ form a basis of $R_m(\tau)$. In particular the dimension of $R_m(\tau)$ is m^n.*

2.4 PROPOSITION. *Let ϕ be a Jacobi form of weight k and index m. There exist uniquely determined holomorphic functions f_r on H_n satisfying*

$$\phi(\tau, z)=\sum_{r} f_r(\tau)\theta_{r,0}(2m\tau, 2mz), \tag{1}$$

in which the summation is taken over all rational n-vectors $r=(r_1, \cdots, r_n)$ with $2mr_i\in \boldsymbol{Z}$, $0\leqq r_i<1$ for $i=1, \cdots, n$.

PROOF. Proposition 2.4 follows easily from Proposition 2.3 and Lemma 3.4 in [S].

For any integral n-vector r, we have

$$\theta_{a+r,b}(\tau, z)=\theta_{a,b}(\tau, z).$$

Therefore by an abuse of notation, we may write $\theta_{a,0}(\tau, z)$ for any representative a of $\boldsymbol{Q}^n$ modulo $\boldsymbol{Z}^n$. We need the following transformation formula for the Theta function.

2.5 PROPOSITION ([I]). *Let* $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a matrix in* Γ_n. *Then for any* r *in* $(1/2m)\boldsymbol{Z}^n$ *modulo* $\boldsymbol{Z}^n$, *we have*

$$\begin{aligned}(3)\qquad &\theta_{r,0}(2mM\tau, 2mz(c\tau+d)^{-1}) \\ &= e^m(z(c\tau+d)^{-1}c^tz)\det(c\tau+d)^{1/2}\sum_s u_{rs}(M)\theta_{s,0}(2m\tau, 2mz),\end{aligned}$$

in which the summation is taken over a complete system of representatives of $(1/2m)\boldsymbol{Z}^n$ *modulo* $\boldsymbol{Z}^n$, *and* $(u_{rs}(M))_{r,s}$ *is a constant unitary matrix of degree* $(2m)^n$ *depending on the choice of* $\det(c\tau+d)^{1/2}$.

Let $\phi(\tau, z)$ be a Jacobi form. Then by Proposition 2.4, there corresponds a family $\{f_r(\tau)\}_r$ of holomorphic functions on H_n. If we apply the condition of the Jacobi form for $[M, 0, 0]\in\Gamma_n^J$, $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we obtain

$$\begin{aligned}(4)\qquad &\sum_r f_r(M\tau)\theta_{r,0}(2mM\tau, 2mz(c\tau+d)^{-1}) \\ &= e^m(z(c\tau+d)^{-1}c^tz)\det(c\tau+d)^k\sum_r f_r(\tau)\theta_{r,0}(2m\tau, 2mz).\end{aligned}$$

Apply Proposition 2.5 to the left hand side and compare the coefficient of $\theta_{s,0}(2m\tau, 2mz)$, then for any s in $(1/2m)\boldsymbol{Z}^n$ modulo $\boldsymbol{Z}^n$ we get

$$(5)\qquad \sum_r f_r(M\tau)u_{rs}(M)=\det(c\tau+d)^{k-1/2}f_s.$$

We may say that $\{f_r(\tau)\}_r$ is a vector valued modular form of weight $k-1/2$ (with respect to the automorphic factor $\{u_{rs}\}$).

Conversely suppose given a family $\{f_r\,;\ r\in(1/2m)\boldsymbol{Z}^n \bmod \boldsymbol{Z}^n\}$ of holomorphic functions on H_n satisfying (5). If we define a function $\phi(\tau, z)$ on $H_n\times\boldsymbol{C}^n$ by (1), we obtain a Jacobi form of weight k and index m.

Thus we have proved

2.6 THEOREM ([S]). *The equation* (1) *gives an isomorphism between* $J_{k,m}(\Gamma_n)$ *and the space of vector valued modular forms satisfying the transformation formula* (5) *for any* M *in* Γ_n.

Using the Koecher principle for a vector valued automorphic form ([C], [F]), we have the Koecher principle for Jacobi forms.

2.7 THEOREM (Koecher principle). *Assume that* $n>1$. *Then any holomorphic function* $\phi(\tau, z)$ *on* $H_n\times\boldsymbol{C}^n$ *satisfying the condition*

$$\phi|_{k,m}g=\phi, \qquad \text{for all } g \text{ in } \Gamma_n^J$$

is a Jacobi form of weight k and index m.

2.8 COROLLARY. *The Jacobi forms form a bigraded ring.*

2.9 PROPOSITION. *Let ϕ be a Jacobi form of weight k and index m and λ, μ rational n-vectors. Then the function*

$$h(\tau)=e^m(\lambda\tau^t\lambda)\phi(\tau, \lambda\tau+\mu)$$

is a modular form of weight k with respect to a congruence subgroup of Γ_n (depending only on λ and μ).

Since this can be proved in the same way as was Theorem 1.3 in [E-Z], we omit the proof.

2.10 THEOREM. *The space $J_{k,m}(\Gamma_n)$ of Jacobi forms is finite dimensional.*

PROOF. We fix a point τ_0 in H_n. The functions of z on $\boldsymbol{C}^n$

$$\theta_{r,0}(2m\tau_0, 2mz): 2mr\in\boldsymbol{Z}^n,\ r=(r_1,\cdots,r_n),\ 0\leqq r_i<1,\ i=1,\cdots,n$$

are linearly independent and they are holomorphic, therefore they are linearly independent on $\boldsymbol{R}^n$. Since $\boldsymbol{Q}^n$ is dense in $\boldsymbol{R}^n$, we can find $(2m)^n$ vectors μ_i in $\boldsymbol{Q}^n$ satisfying

$$(6) \qquad \det(\theta_{r,0}(2m\tau_0, 2m\mu_i))_{r,i}\neq 0\,.$$

It follows from Proposition 2.9 that the functions $h_i(\tau)=\phi(\tau, \mu_i)$ belong to the space $M_k(\Gamma(\mu_i))$ of modular forms of weight k for the congruence subgroups $\Gamma(\mu_i)$ of Γ_n. We claim that the correspondence $\phi\mapsto\{h_i\}_i$ is injective. Since $M_k(\Gamma(\mu_i))$ is finite dimensional, our Theorem follows from the claim. Suppose that $h_i=0$ for all i. By Proposition 2.4, there exist holomorphic functions f_r on H_n such that

$$\phi(\tau, z)=\sum_r f_r(\tau)\theta_{r,0}(2m\tau, 2mz)\,.$$

If we take a sufficiently small neighborhood of τ_0, (6) holds there replacing τ_0 by τ. This implies that in the same neighborhood, $f_r(\tau)=0$ for all r. Since $f_r(\tau)$ is holomorphic, it is identically zero on H_n. Q.E.D.

3. Eisenstein series and other examples.

As in the theory of Siegel modular forms we will obtain our first examples of Jacobi forms by constructing Eisenstein series.

Let k be an even positive integer. In the Siegel modular case we set

$$E_k(\tau)=\sum_{M\in\Gamma_{n,0}\backslash\Gamma_n}(1|_{k,m}[M,0,0])(\tau,0)\,, \tag{1}$$

where $\Gamma_{n,0}=\{M\in\Gamma_n\,;\ 1|_{k,m}[M,0,0]=1\}$ and 1 denotes the constant one function. It is well-known that for $k>n+1$ this converges absolutely and uniformly on any compact subset of H_n. Similarly we define

$$E_{k,m}(\tau,z)=\sum_{g\in\Gamma^J_{n,0}\backslash\Gamma^J_n}(1|_{k,m}g)(\tau,z)\,, \tag{2}$$

where

$$\begin{aligned}\Gamma^J_{n,0}&=\{g\in\Gamma^J_n\,;\ 1|_{k,m}g=1\}\\ &=\{[M,(\lambda,\mu),\kappa]\in\Gamma^J_n\,;\ M\in\Gamma_{n,0},\lambda=0\}\,.\end{aligned}$$

If the series converges, it defines a Jacobi form of weight k and index m. We call $E_{k,m}$ the Jacobi-Eisenstein series.

If M ranges over a complete set of representatives of $\Gamma_{n,0}\backslash\Gamma_n$ and λ moves over $\boldsymbol{Z}^n$, then $[M,(\lambda,0)M,0]$ ranges over a complete system of representatives of $\Gamma^J_{n,0}\backslash\Gamma^J_n$. It follows from the definition that

$$\begin{aligned}&\left(1|_{k,m}\left[\begin{bmatrix}a&b\\c&d\end{bmatrix},(\lambda a,\lambda b),0\right]\right)(\tau,z)\\ &=\det(c\tau+d)^{-k}e^m(-(z+\lambda b+\lambda a\tau)(c\tau+d)^{-1}c^t(z+\lambda b+\lambda a\tau)\\ &\quad+\lambda a\tau^t(\lambda a)+2\lambda a^tz+2\lambda a^t(\lambda b))\,.\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}&a-(a\tau+b)(c\tau+d)^{-1}c={}^t(c\tau+d)^{-1}\,,\\ &a\tau^ta-(a\tau+b)(c\tau+d)^{-1}c^t(a\tau+b)=(a\tau+b)(c\tau+d)^{-1}-a^tb\,.\end{aligned}$$

Hence the Eisenstein series is written explicitly as follows:

$$\begin{aligned}E_{k,m}(\tau,z)&=\sum_M\sum_\lambda\det(c\tau+d)^{-k}e^m(\lambda M\tau^t\lambda+2\lambda^t(c\tau+d)^{-1\,t}z-z(c\tau+d)^{-1}c^tz)\\ &=\sum_M\det(c\tau+d)^{-k}e^m(-z(c\tau+d)^{-1}c^tz)\theta_{0,0}(2mM\tau,2mz(c\tau+d)^{-1})\\ &=\sum_M(\theta_{0,0}|_{k,m}[M,0,0])(2m\tau,2mz)\,,\end{aligned}\tag{3}$$

in which $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ runs through a complete system of representatives of $\Gamma_{n,0}\backslash\Gamma_n$ and λ runs over $\boldsymbol{Z}^n$. On the other hand, it follows from Proposition 2.5 that for $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in Γ_n

$$\theta_{0,0}(2mM\tau, 2mz(c\tau+d)^{-1})$$
$$=\det(c\tau+d)^{1/2}e^m(z(c\tau+d)^{-1}c^tz)\sum_s u_{0s}(M)\theta_{s,0}(2m\tau, 2mz),$$

and the vector $(u_{0s}(M))_s$ is of norm one.

Thus we have proved the following

3.1 THEOREM. *Let k be an even integer with $k>n+2$. The Jacobi-Eisenstein series $E_{k,m}(\tau, z)$ converges absolutely and uniformly on any compact subset of $H_n\times\boldsymbol{C}^n$ and defines a Jacobi form of weight k and index m.*

The second example of Jacobi forms is the Fourier-Jacobi coefficients of Siegel modular forms. A holomorphic function F on H_n is called a Siegel modular form of degree $n>1$ and of weight k if it satisfies the transformation formula

$$F(M\tau)=\det(c\tau+d)^kF(\tau), \tag{4}$$

for all $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in Γ_n.

3.2 THEOREM. *Let $\tau'=\begin{bmatrix} \tau & {}^tz \\ z & t \end{bmatrix}$ denote a general point in H_{n+1}. Let $F(\tau')$ be a Siegel modular form of weight k and degree $n+1$. We write the Fourier-Jacobi expansion* ([P]) *of F in the form*

$$F(\tau')=\sum_{m\geq 0}\phi_m(\tau, z)e(mt).$$

Then $\phi_m(\tau, z)$ is a Jacobi form of weight k and index m.

For the proof we refer to Theorem 6.1 of [E-Z].

Another important example of a Jacobi form is the Theta series defined by an even unimodular lattice.

3.3 THEOREM. *Let $T={}^tT$ be an even unimodular positive definite matrix of order $2k$. For any integral $2k$-vector y such that $yT^ty=2m$, we define $\theta_{T,y}$ by*

(5) $$\theta_{T,y}(\tau, z)=\sum_{X\in M_{2k,n}(Z)} e((1/2)\operatorname{Tr}({}^tXTX\tau)+yTX^tz).$$

Then $\theta_{T,y}$ is a Jacobi form of weight k and index m.

Since this can be proved in the same way as in [F], we omit the proof.

4. Hecke operator.

The Hecke algebra of the Siegel modular group acts on the space of Jacobi forms. Let M be an integral element in S_n. Decompose the double coset $\Gamma_n M\Gamma_n$ into left cosets:

$$\Gamma_n M\Gamma_n=\bigcup_i \Gamma_n M_i.$$

For any Jacobi form ϕ of weight k and index m, we define

(1) $$\phi|_{k,m}(\Gamma_n M\Gamma_n)=\nu(M)^{nk-n(n+1)/2}\sum_i \phi|_{k,m}[M_i, 0, 0].$$

It is clear that the right hand side does not depend on the choice of representatives M_i.

4.1 PROPOSITION. *Let M be an integral element of S_n and let ϕ be a Jacobi form of weight k and index m. Then the function $\phi|_{k,m}(\Gamma_n M\Gamma_n)$ on $H_n\times \boldsymbol{C}^n$ defined by* (1) *is a Jacobi form of weight k and index $m\nu(M)$.*

PROOF. We write $\psi=\phi|_{k,m}\Gamma_n M\Gamma_n$. We have to prove the transformation formulas. Let N be in Γ_n. If $\{M_i\}_i$ is a complete system of representatives of $\Gamma_n\backslash\Gamma_n M\Gamma_n$, so is the set $\{M_iN\}_i$. Since

$$(\phi|_{k,m}[M_i, 0, 0])|_{k,m\nu(M)}[N, 0, 0]=\phi|_{k,m}[M_iN, 0, 0],$$

we have $\psi|_{k,m\nu(M)}[N, 0, 0]=\psi$. On the other hand, since $\nu(M)XM_i^{-1}$ is integral for any X in $\boldsymbol{Z}^{2n}$, we have

$$\begin{aligned}(\phi|_{k,m}[M_i, 0, 0])|_{k,m\nu(M)}[1_{2n}, X, 0]&=(\phi|_{k,m}[1_{2n}, \nu(M)XM_i^{-1}, 0])|_{k,m}[M_i, 0, 0]\\ &=\phi|_{k,m}[M_i, 0, 0].\end{aligned}$$

When $n=1$, the condition at infinity follows from the explicit formula given below. Q. E. D.

In the rest of this section we shall fix a prime number p. Let $\delta_i=\delta_i(p)$ be a diagonal matrix of order n such that the first $n-i$ components are one and the rest are p. We set $H=\mathrm{GL}(n, \boldsymbol{Z})$ and

$$H_i(p)=H_i=(\delta_i^{-1}H\delta_i)\cap H.$$

It is well-known that the double coset

$$T(p)=\Gamma_n\begin{bmatrix}1_n & 0\\ 0 & p1_n\end{bmatrix}\Gamma_n$$

has the following decomposition into left cosets [F]:

$$T(p)=\bigcup_{0\leq i\leq n}\bigcup_{v}\bigcup_{b}\Gamma_n\begin{bmatrix}\delta_i v & b^t v^{-1}\\ 0 & p^t(\delta_i v)^{-1}\end{bmatrix},$$

in which v runs over a complete system of representatives of $H_i\backslash H$ and b has the form $\begin{bmatrix}b' & 0\\ 0 & 0\end{bmatrix}$ with $b'={}^tb'$ in $M_{n-i}(\boldsymbol{Z})$ and b' moves modulo p.

Let ϕ be a Jacobi form of weight k and index m. Take the Fourier expansion of ϕ (see Definition 2.1):

$$\phi(\tau, z)=\sum_{N,r}c(N, r)e(\mathrm{Tr}(Nr)+r^t z),$$

in which N ranges over the set of n by n symmetric half-integral matrices and r runs over $\boldsymbol{Z}^n$. Fixing i and v we write

$$D(\delta_i v)=\sum_b \phi|_{k,m}\left[\begin{bmatrix}\delta_i v & b^t v^{-1}\\ 0 & p^t(\delta_i v)^{-1}\end{bmatrix}, 0, 0\right],$$

in which the summation with respect to b is taken over the set mentioned above. Using the obvious formula

$$\sum_{b' \bmod p}e((1/p)\mathrm{Tr}(Nb))=\begin{cases}p^{(n-i)(n-i+1)/2} & \text{if } \delta_i N\delta_i\equiv 0 \bmod p,\\ 0 & \text{otherwise,}\end{cases}$$

we have

$$\begin{aligned}D(\delta_i)&=\sum_{N,r}\sum_b \det(p\delta_i)^{-k}c(N, r)e(\mathrm{Tr}(N(\delta_i\tau+b)\delta_i p^{-1})+p^t(z\delta_i p^{-1}))\\ &=p^{-k(n-i)+(n-i)(n-i+1)/2}\sum_{\delta_i N\delta_i\equiv 0 \bmod p}c(N, r)e(p^{-1}\mathrm{Tr}(\delta_i N\delta_i\tau)+r\delta_i^t z)\\ &=p^{-k(n-i)+(n-i)(n-i+1)/2}\sum_{N,r}c(p\delta_i^{-1}N\delta_i^{-1}, r\delta_i^{-1})e(\mathrm{Tr}(N\tau)+r^t z),\end{aligned}$$

where we use a convention that if N is not half-integral or r is not integral then $c(N, r)=0$. More generally we have,

$$D(\delta_i v)=D(\delta_i)|_{k,mp}\left[\begin{bmatrix} v & 0 \\ 0 & {}^t v^{-1} \end{bmatrix}, 0, 0\right]$$
$$=p^{-k(n-i)+(n-i)(n-i+1)/2}\sum_{N,r} c(p\delta_i^{-1}\,{}^t v^{-1}Nv^{-1}\delta_i^{-1}, rv^{-1}\delta_i^{-1})e(\mathrm{Tr}(N\tau)+r^t z)\,.$$

Summing up over all i and v, we obtain the following

4.2 PROPOSITION. *Let ϕ be a Jacobi form of weight k and index m, having the Fourier expansion*

$$\phi(\tau, z)=\sum_{N,r} c(N, r)e(\mathrm{Tr}(N\tau)+r^t z)\,.$$

Then we have

$$(\phi|_{k,m}T(p))(\tau, z)$$
$$=\sum_{N,r}\sum_{0\leq i\leq n} p^{i(k-n)+i(i-1)/2}\sum_{v\in H_i\backslash H} c(p\delta_i^{-1}\,{}^t v^{-1}Nv^{-1}\delta_i^{-1}, rv^{-1}\delta_i^{-1})e(\mathrm{Tr}(N\tau)+r^t z)\,.$$

Since the double coset $T_{0,n}(p^2)=\Gamma_n(p1_{2n})\Gamma_n$ consists of just one left coset $\Gamma_n(p1_{2n})$, we get the following

4.3 PROPOSITION. *With the same assumptions and notations as above, we have*

$$(\phi|_{k,m}T_{0,n}(p^2))(\tau, z)=p^{nk-n(n+1)}\phi(\tau, pz)$$
$$=p^{nk-n(n+1)}\sum_{N,r} c(N, p^{-1}r)e(\mathrm{Tr}(N\tau)+r^t z)\,.$$

5. Maass relation.

In this section we shall closely examine the action of $T(p)$ on the Eisenstein series. Using a result of Böcherer [B], we get relations among the Fourier coefficients of Siegel-Eisenstein series. When $n=1$, these relations reduce to the Maass relations for the Siegel-Eisenstein series of degree two.

In the rest of this section we shall fix a prime number p. As in Section 4, let δ_i be the diagonal matrix with the first $n-i$ elements are one and the rest are p. We define an element δ_i' in S_n by $\delta_i'=\begin{bmatrix} p\delta_i^{-1} & 0 \\ 0 & \delta_i \end{bmatrix}$.

5.1 LEMMA. *We have the following decomposition of $T(p)$:*

$$T(p)=\bigcup_{0\leq i\leq n}\Gamma_{n,0}\delta_i'\Gamma_n\,.$$

PROOF. Obviously we get distinct double cosets for different i. Take a representative $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of a double coset in $\Gamma_{n,0}\backslash T(p)/\Gamma_n$. There exist matrices u in $\mathrm{GL}(n, \boldsymbol{Z})$ and v in $\mathrm{GL}(2n, \boldsymbol{Z})$ such that

$$(c, d)=u(0, \varepsilon)v\,,$$

where ε is a diagonal matrix with positive elements $\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_j|\varepsilon_{j+1}$, $j=1, \cdots, n-1$. By replacing the representative if necessary, we may assume that $u=1_n$. Decompose v into $n\times n$ blocks $v=\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$. Then from the relation $c^t d=d^t c$, we get $v_3{}^t v_4=v_4{}^t v_3$. Therefore replacing v_1, v_2 if necessary, we may assume that v is a symplectic matrix. It follows from the relation $a^t d-b^t c=p1_n$, that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} p\varepsilon^{-1} & x \\ 0 & \varepsilon \end{bmatrix}v\,,$$

where we put $x=-av_2+bv_1$. Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $T(p)$, $\begin{bmatrix} p\varepsilon^{-1} & x \\ 0 & \varepsilon \end{bmatrix}$ is also in $T(p)$. In particular there exists an index i such that $\varepsilon=\delta_i$. It follows from the definition of $T(p)$ that

$$x=\begin{bmatrix} s_{11} & ps_{12} \\ {}^t s_{12} & ps_{22} \end{bmatrix},$$

in which $s_{11}={}^t s_{11}$, s_{12} and $ps_{22}=p^t s_{22}$ are integral matrices of size $(n-i)\times(n-i)$, $(n-i)\times i$ and $i\times i$, respectively. Our Lemma follows from the following identity;

$$\begin{bmatrix} p\delta_i^{-1} & x \\ 0 & \delta_i \end{bmatrix}=\begin{bmatrix} 1_n & s_1 \\ 0 & 1_n \end{bmatrix}\delta_i'\begin{bmatrix} 1_n & s_2 \\ 0 & 1_n \end{bmatrix},$$

where we set

$$s_1=\begin{bmatrix} s_{11} & s_{12} \\ {}^t s_{12} & 0 \end{bmatrix} \quad \text{and} \quad s_2=\begin{bmatrix} 0 & 0 \\ 0 & ps_{22} \end{bmatrix}.$$

Q. E. D.

5.2 LEMMA. *For any* $0\leqq i\leqq n$, *we put*

$$H_i(p)=H_i=(\delta_i\,\mathrm{GL}(n, \boldsymbol{Z})\delta_i^{-1})\cap\mathrm{GL}(n, \boldsymbol{Z})\,,$$

$$g_p(n, i)=[H_n : H_i]=[H_n : H_{n-i}]\,.$$

Then we have

$$g_p(n, i)=\prod_{1\leqq\alpha\leqq i}(p^{n-i+\alpha}-1)(p^\alpha-1)^{-1},$$

and

$$\sum_{0\leq\alpha\leq n} p^{\alpha(\alpha+1)/2} g_p(n,\alpha)X^\alpha = \prod_{1\leq\alpha\leq n}(1+p^\alpha X).$$

Since this is well-known and easy to prove, we omit the proof.
To simplify the notation, we put for $M\in S_n$ and $\lambda\in\boldsymbol{Q}^n$

$$j(k, m\,; M, \lambda)=1|_{k,m}[1_{2n}, (\lambda, 0), 0][M, 0, 0].$$

Also for any $0\leqq i\leqq n$, we set

$$\Gamma_n(\delta_i)=\Gamma_n\cap(\delta_i'^{-1}\Gamma_{n,0}\delta_i'),$$

and

$$K_i=\sum_{M\in\Gamma_n(\delta_i)\backslash\Gamma_n}\sum_{\lambda\in\boldsymbol{Z}^n} j(k, m\,; M, \lambda).$$

5.3 LEMMA. (i) *If $p\nmid m$, then*

$$K_i=p^{-ik+i(i+1)/2}g_p(n, i)E_{k,mp}.$$

(ii) *If $p|m$, then*

$$K_i(\tau, z)=p^{-ik+i(i+1)/2}\{g_p(n-1, n-i)E_{k,m/p}(\tau, pz)$$
$$+p^i g_p(n-i, n-i-1)E_{k,pm}(\tau, z)\}.$$

PROOF. We put $U=\left\{\begin{bmatrix}1_n & s\\ 0 & 1_n\end{bmatrix}; s={}^ts\in M_n(\boldsymbol{Z})\right\}$ and $\Gamma_n(\delta_i)_u=\Gamma_n(\delta_i)U$. Then it is easy to show that the set

$$\left\{\begin{bmatrix}1_n & x\\ 0 & 1_n\end{bmatrix};\ x=\begin{bmatrix}0 & 0\\ 0 & x_4\end{bmatrix}, x_4={}^tx_4\in M_i(\boldsymbol{Z}) \bmod p\right\}$$

is a set of representatives of $\Gamma_n(\delta_i)\backslash\Gamma_n(\delta_i)_u$. It follows from the definition that

$$K_i=p^{-ik}\sum_{M\in\Gamma_n(\delta_i)\backslash\Gamma_n}\sum_{\lambda\in\boldsymbol{Z}^n} j(k, pm\,; M, \lambda\delta_i^{-1}).$$

If we replace M by $\begin{bmatrix}1_n & x\\ 0 & 1_n\end{bmatrix}M$ with $x=\begin{bmatrix}0 & 0\\ 0 & x_4\end{bmatrix}$, then each term in the summation is multiplied by the factor $e^m(p\lambda\delta_i^{-1}x\delta_i^{-1}\,{}^t\lambda)$.

(i) The case $p\nmid m$. If we sum over $x_4 \bmod p$, we get

$$\sum_{x_4 \bmod p} e^m(p\lambda\delta_i^{-1}x\delta_i^{-1}\,{}^t\lambda)=\begin{cases}p^{i(i+1)/2} & \text{if } \lambda\in\boldsymbol{Z}^n\delta_i,\\ 0 & \text{otherwise.}\end{cases}$$

Hence we have

$$K_i = p^{-ik+i(i+1)/2} \sum_{M \in \Gamma_n(\delta_i)_u \backslash \Gamma_n} \sum_{\lambda \in Z^n} j(k, m ; M, \lambda) .$$

If $\{u_j\}_j$ is a set of representatives for $H_i \backslash H_n$, then $\left\{\begin{bmatrix} u_j & 0 \\ 0 & {}^t u_j^{-1} \end{bmatrix}\right\}_j$ is a set of representatives for $\Gamma_n(\delta_i)_u \backslash \Gamma_{n,0}$. Finally we get

$$\begin{aligned} K_i &= p^{-ik+i(i+1)/2} \sum_{u \in H_i \backslash H_n} \sum_{M \in \Gamma_{n,0} \backslash \Gamma_n} \sum_{\lambda \in Z^n} j(k, mp ; M, \lambda u) \\ &= p^{-ik+i(i+1)/2} [H_n : H_i] \sum_{M \in \Gamma_{n,0} \backslash \Gamma_n} \sum_{\lambda \in Z^n} j(k, mp ; M, \lambda) \\ &= p^{-ik+i(i+1)/2} [H_n : H_i] E_{k,mp} . \end{aligned}$$

(ii) The case $p|m$. Since $e^m(p\lambda\delta_i^{-1}x\delta_i^{-1}\,{}^t\lambda)=1$ for any x and λ, we have

$$\begin{aligned} K_i(\tau, z) &= p^{-ik+i(i+1)/2} \sum_{M \in \Gamma_n(\delta_i)_u \backslash \Gamma_n} \sum_{\lambda \in Z^n} j(k, m/p ; M, p\lambda\delta_i^{-1})(\tau, z) \\ &= p^{-ik+i(i+1)/2} \sum_{M \in \Gamma_n(\delta_i)_u \backslash \Gamma_n} \sum_{\lambda \in pZ^n\delta_i^{-1}} j(k, m/p ; M, \lambda)(\tau, pz) . \end{aligned}$$

Since H_i is the stabilizer of $p\boldsymbol{Z}^n\delta_i^{-1}$, we get

$$K_i(\tau, z) = p^{-ik+i(i+1)/2} \sum_{M \in \Gamma_{n,0} \backslash \Gamma_n} \sum_{L} \sum_{\lambda \in L} j(k, m/p ; M, \lambda)(\tau, pz) ,$$

in which the middle summation is taken over all lattices L in $\boldsymbol{Z}^n$ such that $\boldsymbol{Z}^n/L \cong (\boldsymbol{Z}/p\boldsymbol{Z})^{n-i}$. On the other hand it is easy to see that for any "good" function f on $\boldsymbol{Z}^n$, we have

$$\sum_{Z^n/L \cong (Z/pZ)^i} \sum_{\lambda \in L} f(\lambda) = g_p(n-1, i) \sum_{\lambda \in Z^n} f(\lambda) + p^{n-i} g_p(n-1, i-1) \sum_{\lambda \in Z^n} f(p\lambda) .$$

Observing that

$$j(k, m/p ; M, p\lambda)(\tau, pz) = j(k, m ; M, \lambda)(\tau, z) ,$$

we obtain the desired formula. Q. E. D.

5.4 THEOREM. *Let p be a prime number.*

(i) *If $p \nmid m$, then*

$$E_{k,m}|_{k,m} T(p) = \left\{ \prod_{1 \leq i \leq n} (1 + p^{k-i}) \right\} E_{k,pm} .$$

(ii) *If $p|m$, then*

$$(E_{k,m}|_{k,m} T(p))(\tau, z) = \left\{ \prod_{2 \leq i \leq n} (1 + p^{k-i}) \right\} \{E_{k,m/p}(\tau, pz) + p^{k-1} E_{k,pm}(\tau, z)\} .$$

PROOF. It follows from the definition that

$$E_{k,m}|_{k,m}T(p)=p^{nk-n(n+1)/2}\sum_{M\in\Gamma_n\backslash T(p)}\ \sum_{M'\in\Gamma_{n,0}\backslash\Gamma_n}\ \sum_{\lambda\in Z^n}1|_{k,m}[M'M, p^{-1}(\lambda,0)M, 0]$$
$$=p^{nk-n(n+1)/2}\sum_{M\in\Gamma_{n,0}\backslash T(p)}\ \sum_{\lambda\in Z^n}1|_{k,m}[1_{2n},(\lambda,0),0][M,0,0].$$

On the other hand, by Lemma 5.1 we can take

$$\{\delta_i'M\,;\ i=0,\cdots,n,\ M\in\Gamma_n(\delta_i)\backslash\Gamma_n\}$$

as a complete set of representatives for $\Gamma_{n,0}\backslash T(p)$. Hence we have

$$E_{k,m}|_{k,m}T(p)=p^{nk-n(n+1)/2}\sum_{0\le i\le n}\ \sum_{M\in\Gamma_n(\delta_i)\backslash\Gamma_n}\ \sum_{\lambda\in Z^n}1|_{k,m}[\delta_i'M, p^{-1}(\lambda,0)\delta_i'M, 0]$$
$$=p^{nk-n(n+1)/2}\sum_{0\le i\le n}K_i.$$

Therefore Theorem 5.4 follows from Lemmas 5.2 and 5.3. Q. E. D.

We expand the Siegel-Eisenstein series $E_k^{(n+1)}(\tau')$ into the Fourier-Jacobi series:

$$E_k^{(n+1)}(\tau')=\sum_{0\le m}e_{k,m}(\tau,z)e(mt).$$

As we showed before, $e_{k,m}(\tau,z)$ is a Jacobi form of weight k and index m for Γ_n. The Jacobi form $e_{k,m}$ is studied by Böcherer. A special case of Satz 7 of [B] may be formulated as follows.

5.5 THEOREM ([B]). *For any $m>0$, we have*

$$e_{k,m}(\tau,z)=\sum_{d^2|m,\,d>0}\sigma_{k-1}(md^{-2})\sum_{a|d,\,a>0}\mu(a)E_{k,ma^2/d^2}(\tau, da^{-1}z),$$

in which μ is the Möbius function and

$$\sigma_{k-1}(a)=\sum_{d|a,\,d>0}d^{k-1}.$$

5.6 COROLLARY. (i) *If $p\nmid m$, then*

$$e_{k,mp}=\left\{\prod_{2\le i\le n}(1+p^{k-i})^{-1}\right\}e_{k,m}|_{k,m}T(p);$$

(ii) *If $p|m$, then*

$$e_{k,mp}=\left\{\prod_{2\le i\le n}(1+p^{k-i})^{-1}\right\}e_{k,m}|_{k,m}T(p)-p^{(1-n)k+n(n+1)-1}e_{k,m/p}|_{k,m}T_{0,n}(p^2).$$

Since this is just a combination of Proposition 4.3, Theorems 5.4 and 5.5, we omit the proof.

For any natural number m, we define an element $D_n(m)$ in the Hecke algebra for Γ_n with similitude m by the following formal product:

$$\sum_{m \geq 0} D_n(m)m^{-s} = \prod_{p:\ \mathrm{prime}} \left\{1 - \prod_{1<i\leq n} (1+p^{k-i})^{-1}T(p)p^{-s} + T_{0,n}(p^2)p^{(1-n)k+n(n+1)-1-2s}\right\}^{-1}.$$

We can reformulate Corollary 5.6 in the following

5.7 THEOREM (Maass relation). *For the Fourier-Jacobi coefficients* $e_{k,m}(\tau, z)$ *of the Siegel-Eisenstein series, we have*

$$e_{k,m} = e_{k,1}|_{k,1}D_n(m).$$

5.8 REMARK. If $n=1$, then the above Euler factor is

$$1 - T(p)p^{-s} + T_{0,1}(p^2)p^{1-2s}.$$

This is the ordinary Euler factor for the modular group, hence we have $D_1(m) = T(m)$ for all m. Therefore in this case Theorem 5.7 is nothing but the Maass relation for the Siegel-Eisenstein series of degree two ([M]).

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