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Propagation of regularities of solutions to semilinear partial differential equations of quasi-homogeneous type

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0. Introduction.

The first work on microlocal analysis of nonlinear partial differential equations was done by Rauch [15]. Rauch used the microlocal estimate of the product of two functions and proved a theorem on propagation of regularity for semilinear wave equations.

Subsequently, Bony [3] dealt with the general nonlinear partial differential equations and considered the microlocal regularities of solutions at non-characteristic points and on real simple characteristics. In [3], Bony introduced the paradifferential operators. This machinery of paradifferential operators, which is based on the Littlewood-Paley decomposition of functions, turned out to be useful in the nonlinear microlocal analysis. Later, Meyer [13], [14] improved some of Bony's results, including remainder estimates in the paraproduct. See also Coifman-Meyer [7] and Bourdaud [4], where the Littlewood-Paley theory is applied to the problem on the boundedness of pseudo-differential operators on L^p spaces.

Partial differential equations discussed in this paper have mixed homogeneity in symbols in the sense that the order of differentiation may depend on variables. Typical examples are the heat equation and the Schrödinger equation. The pseudo-differential operators with such a homogeneity has been previously considered by Hörmander [9], Kumano-go (cf. [11]) and others. But microlocal analysis for operators with mixed homogeneity was first given by Lascar [12], and our terminology of quasi-homogeneous pseudo-differential operators comes from his work.

Recently, Yamazaki [18], [19], [20], [21], [22] extended the theory of paradifferential operators to the quasi-homogeneous case and derived boundedness of the operators in various spaces and introduced calculus of them. He applied the results to microlocal analysis of nonlinear partial differential equations of such a type and thus obtained the non-characteristic regularity theorem.

In this paper, we also consider microlocal structure of solutions to nonlinear partial differential equations of quasi-homogeneous type but with a particular emphasis on microlocal regularities of solutions on real simple characteristics. Actually we shall prove a propagation of regularity theorem for a class of semilinear equations having quasi-homogeneous leading terms. The main results of this paper will be stated precisely in Section 1, after describing some notation.

As in existing works toward similar directions (e. g., Hörmander [10]), our proof is based on a microlocal energy estimate along the real bicharacteristics. To this end, however, we have to prepare the following two things. The first one is to extend the sharp Gårding inequality to the case of quasi-homogeneous pseudo-differential operators. This will be done in Section 2. Incidentally, as an application of our version of the sharp Gårding inequality, we can improve Lascar's result on the propagation of regularities to some extent.

The other mean which is crucial to handle nonlinear terms is the use of the theory of quasi-homogeneous paradifferential operators due to Yamazaki [18], [21], [22]. As a matter of fact, the boundedness and the symbol calculus of these operators, which we need later, can be found in his papers [21], [22]. However, we only need to use the L^2-property of paradifferential operators as long as our interest is restricted to obtain a propagation regularity theorem. When we focus our attention to this limited target, many of the arguments of Yamazaki [21], [22] intended for extensive generality can be considerably simplified. This is actually carried out in Sections 3 and 4 in a manner close to Meyer [13], [14], which may be of some value by its own right and for the sake of self-containedness.

In Section 5, using the results of the preceding three sections we prove the microlocal energy estimate which yields our main results of this paper.

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1. Notation and statement of the results.

Here we recall shortly the definition of quasi-homogeneous pseudo-differential operators.

Let Ω denote an open set of $\boldsymbol{R}^n$. Let $M=(\mu_1, \cdots, \mu_n)$ be a weight vector on the dual space $\boldsymbol{R}_n$ satisfying $\inf\{\mu_j\}=1$. If $\xi\in\boldsymbol{R}_n$ and $t>0$ we set $t^M\xi=(t^{\mu_1}\xi_1,\cdots,t^{\mu_n}\xi_n)$. A function g on $\Omega\times(\boldsymbol{R}_n\backslash 0)$ is $(M\text{-})$quasi-homogeneous of degree m if $g(x,t^M\xi)=t^m g(x,\xi)$ for every $t>0$ and $(x,\xi)\in\Omega\times(\boldsymbol{R}\backslash 0)$. We also say that a subset Γ of $\Omega\times(\boldsymbol{R}_n\backslash 0)$ is an M-cone if $(x,\xi)\in\Gamma$ implies $(x,t^M\xi)\in\Gamma$ for every $t>0$.

We introduce the function ξ_M defined implicitly by $\sum_{j=1}^n \xi_j^2/[\xi]_M^{2\mu_j}=1$ if $\xi\neq 0$ and $(0)_M=0$. Then $[\xi]_M$ has the following properties:

(1) $[\xi]_M$ is M-quasi-homogeneous of degree 1,

(2) $[\xi]_M\in C^\infty(\boldsymbol{R}_n\backslash 0)$ and its derivatives satisfy the following estimate: for every $\alpha\in\boldsymbol{N}^n$

$$\partial_\xi^\alpha[\xi]_M=O([\xi]_M^{1-\langle\alpha,M\rangle}) \qquad \text{in } [\xi]_M\geqq 1\,,$$

where $\langle\alpha,M\rangle=\sum_{j=1}^n\alpha_j\mu_j$,

(3) $[\xi+\eta]_M\leqq[\xi]_M+[\eta]_M$ for every $\xi,\eta\in\boldsymbol{R}_n$,

(4) $(1+[\xi]_M)^{-s}\in L^2$ if and only if $s>|M|/2$, where $|M|=\sum_{j=1}^n\mu_j$.

(2) was proved by Fabes-Rivière [8]; the others can be seen easily.

Let $m\in\boldsymbol{R}$, $0\leqq\delta\leqq\rho\leqq 1$. We let $S_{\rho,\delta}^{M;m}(\Omega)$ denote the space of functions $p(x,\xi)$ of class C^∞ on $\Omega\times\boldsymbol{R}_n$ satisfying the following estimate: for every $\alpha,\beta\in\boldsymbol{N}^n$ and $K\Subset\Omega$ there exists a constant $C_{\alpha,\beta,K}$ such that

$$|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)|\leqq C_{\alpha,\beta,K}(1+[\xi]_M)^{m-\rho\langle\alpha,M\rangle+\delta\langle\beta,M\rangle},\quad x\in K\,. \tag{1.1}$$

We call elements of $S_{\rho,\delta}^{M;m}(\Omega)$ symbols of order m and type (ρ,δ). Also by $S_{\rho,\delta}^{M;m}$ we write the subclass of $S_{\rho,\delta}^{M;m}(\boldsymbol{R}^n)$ consisting of the symbols which satisfy (1.1) with the bound $C_{\alpha,\beta,K}$ independent of $K\Subset\boldsymbol{R}^n$.

We say that a symbol $p\in S_{1,0}^{M;m}(\Omega)$ is classical if p has an asymptotic expansion by quasi-homogeneous functions p_{m_j} of degree m_j:

$$p(x,\xi)\sim\sum_{j=0}^\infty p_{m_j}(x,\xi)\,, \tag{1.2}$$

where $m=m_0$ and $m^{-1}\geqq m_1>m_2>\cdots\to-\infty$. The precise meaning of (1.2) is that, for every integer k,

$$p(x,\xi)-\sum_{j=1}^{k-1}p_{m_j}(x,\xi)=O([\xi]_M^{m_k}) \qquad \text{in } [\xi]_M\geqq 1\,.$$

For a classical symbol p we call p_m the principal symbol.

Let p be an element of $S_{\rho,\delta}^{M;m}(\Omega)$. We set

$$p(x, D)u(x)=(2\pi)^{-n}\int e^{i<x,\xi>}p(x, \xi)\hat{u}(\xi)d\xi \qquad \text{for } u\in C_0^\infty(\Omega),$$

where $\hat{u}$ denotes the Fourier transform of u:

$$\hat{u}(\xi)=\int e^{-i<x,\xi>}u(x)dx .$$

$p(x, D)$ defines a continuous linear map $C_0^\infty(\Omega)\to C^\infty(\Omega)$, which can be extended with continuity to a map $\mathcal{E}'(\Omega)\to\mathcal{D}'(\Omega)$. We let $\mathrm{Op}\,S_{\rho,\delta}^{M;m}(\Omega)$ denote the space of all linear operators $P=p(x, D)$ with $p(x, \xi)\in S_{\rho,\delta}^{M;m}(\Omega)$. Elements of $\mathrm{Op}\,S_{\rho,\delta}^{M;m}(\Omega)$ are called M-pseudo-differential operators of order m and of type (ρ, δ).

Suppose $M=(\mu_1, \cdots, \mu_n)$. For real valued functions $p(x, \xi)$, $q(x, \xi)$ defined on $\Omega\times(\boldsymbol{R}_n\backslash 0)$ we let

$$\{p, q\}_M=\sum_{(j;\,\mu_j=1)}(\partial_{\xi_j}p\partial_{x_j}q-\partial_{x_j}p\partial_{\xi_j}q)$$

be the partial Poisson bracket of p and q, where the sum is taken over all j such that $\mu_j=1$. We can regard $\{p, q\}_M$ as a first order differential operator H_p^M acting on q, where H_p^M denote the M-Hamiltonian vector defined by

$$H_p^M=\sum_{(j;\,\mu_j=1)}(\partial_{\xi_j}p\partial_{x_j}-\partial_{x_j}p\partial_{\xi_j}) .$$

Since $H_p^M p=\{p, p\}_M=0$ it is clear that p is constant on integral curves of H_p^M; these on which $p=0$ are called (null) bicharacteristic strips for p.

We consider the regularities of solutions in the Sobolev space. We let H_M^s denotes the anisotropic Sobolev space with norm:

$$\|u\|_{M,s}=\left(\int(1+[\xi]_M)^{2s}|\hat{u}(\xi)|^2d\xi\right)^{1/2} .$$

We also define its microlocalization as follows:

DEFINITION 1.1. Let $u(x)$ be a distribution defined near $\mathring{x}\in\boldsymbol{R}^n$ and let $\mathring{\xi}\in\boldsymbol{R}_n\backslash 0$. We write $u\in H_M^s(\mathring{x}, \mathring{\xi})$ if there exist a classical symbol $a(x, \xi)$ of order 0 and a function $\chi\in C_0^\infty$ with $a(\mathring{x}, \mathring{\xi})\neq 0$, $\chi(\mathring{x})\neq 0$ such that $a(x, D)\chi u \in H_M^s$.

We then say that u belongs to H_M^s at $(\mathring{x}, \mathring{\xi})$. Moreover, let $\Gamma\subset\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)$ be an M-cone. Then we write $u\in H_M^s(\Gamma)$ if u belongs to H_M^s at all points of Γ. As usual, for $\Omega\subset\boldsymbol{R}^n$, $H_{M,\mathrm{loc}}^s(\Omega)$ denotes the space of $u\in\mathcal{D}'(\Omega)$ such that $\chi u\in H_M^s$ for every $\chi\in C_0^\infty(\Omega)$; this is equivalent to say that u belongs to H_M^s at every point of $\Omega\times(\boldsymbol{R}_n\backslash 0)$.

We are now ready to state the main results of this paper.

Let Ω be an open set of $\boldsymbol{R}^n$ and let $P=p(x, D)$ be a classical M-pseudo-differential operator on Ω of order m. Assume p has a real valued principal symbol and simple characteristics (i.e. $H_{p_m}^M \neq 0$ on $p_m^{-1}(0)$). Let $F=F(x; U, \cdots, U_\alpha, \cdots)$ be a C^∞-function defined on $\boldsymbol{R}^n \times \boldsymbol{C}^N$, where N is the number of multi-indices α satisfying $\langle \alpha, M\rangle \leqq m-1$, and assumed further to be holomorphic with respect to $U, \cdots, U_\alpha, \cdots$. Let $u=u(x)$ be a function defined on Ω; we shall use the abbreviation

$$F(D^\alpha u)=F(x; u(x), \cdots, D^\alpha u(x), \cdots), \qquad \langle \alpha, M\rangle \leqq m-1 .$$

We then consider the following semilinear equation

$$Pu+F(D^\alpha u)=0 \qquad \text{in } \Omega . \tag{1.3}$$

THEOREM A. *Let $s>(m-1)+|M|/2$ and let $\sigma \leqq s-(m-1)-|M|/2$. Suppose u is a solution of* (1.3) *which belongs to $H_{M,\mathrm{loc}}^s(\Omega)$. If u belongs to $H_M^{s+\sigma}$ at some point $(\mathring{x}, \mathring{\xi})$ in $p_m^{-1}(0)$ then u belongs to $H_M^{s+\sigma}$ on the null bicharacteristic strip through $(\mathring{x}, \mathring{\xi})$.*

If F is free from the higher derivatives of u then we can weaken the assumption on s and on σ accordingly.

THEOREM A′. *Suppose F is independent of $D^\alpha u$ for α such that $\langle \alpha, M\rangle > m-h$, where h is real and $h>1$. Let $s>(m-h)+|M|/2$ and let $\sigma \leqq s-m+2h-1-|M|/2$. If $u \in H_{M,\mathrm{loc}}^s(\Omega)$ is a solution of* (1.3) *and belongs to $H_M^{s+\sigma}$ at $(\mathring{x}, \mathring{\xi}) \in p_m^{-1}(0)$ then u belongs to $H_M^{s+\sigma}$ on the whole bicharacteristic strip through $(\mathring{x}, \mathring{\xi})$.*

The semilinear Schrödinger equation is a typical example; however, from the additional fact that the anti-podal image of the bicharacteristic strip consists only of non-characteristic points, we allow the nonlinear term F of the equation to be a holomorphic function of u and $\bar{u}$ (the complex conjugate of u). This was considered in Sakurai [16]. Here we extend the result for more general equations.

Let P be a classical M-pseudo-differential operator on $\Omega \subset \boldsymbol{R}^n$, which satisfies the same assumption as in Theorem A. Let $F(D^\alpha u, D^\alpha \bar{u})=F(x; u, \bar{u}, \cdots, D^\alpha u, D^\alpha \bar{u}, \cdots)$, $\langle \alpha, M\rangle \leqq M-h$ with $h \geqq 1$, be a smooth function holomorphic with respect to $D^\alpha u$, $D^\alpha \bar{u}$ where $\bar{u}$ denotes the complex conjugate of u. Now consider the equation:

$$Pu+F(D^\alpha u, D^\alpha \bar{u})=0 \qquad \text{in } \Omega . \tag{1.4}$$

Then we have

THEOREM B. *Let* $s>(m-h)+|M|/2$ *and let* $\sigma\leqq s-m+2h-1-|M|/2$. *Let* $u\in H_{M,\mathrm{loc}}^{s}(\Omega)$ *be a solution of* (1.4). *Consider a null bicharacteristic strip* γ *for* P *such that the anti-podal image of which consists only of non-characteristic points. Then* u *belongs to* $H_{M}^{s+\sigma}$ *on* γ *if this is true at some point on* γ.

2. Quasi-homogeneous pseudo-differential operators.

In this section we list the facts on the quasi-homogeneous pseudo-differential operators, which will be used in the proof of the main results.

The classical results on the continuity of pseudo-differential operators is the following.

THEOREM 2.1. *Let* $m\in \boldsymbol{R}$ *and let* $0\leqq\delta\leqq\rho\leqq 1$, $\delta<1$. *Let* P *be an element of* $\mathrm{Op}S_{\rho,\delta}^{M,m}(\Omega)$. *Then* P *maps* $\mathcal{E}'(\Omega)\cap H_{M}^{s}$ *continuously to* $H_{M,\mathrm{loc}}^{s-m}(\Omega)$.

When $M=(1,\cdots,1)$ Theorem 2.1 was proved by Hörmander [9] if $0\leqq\delta<\rho\leqq 1$ and by Calderón-Vaillancourt [5] if $0\leqq\delta=\rho<1$. See also Kumano-go [11], where the pseudo-differential operators with weight functions including our $(1+[\xi]_{M})$ are considered.

Symbol calculus of M-pseudo-differential operators is the same as the homogeneous one. The calculus is well known; see for example Kumano-go [11].

PROPOSITION 2.2. *Suppose that* $0\leqq\delta<\rho\leqq 1$, $\delta<1$.

(1) *For* $p\in S_{\rho,\delta}^{M,m}$ *and* $p'\in S_{\rho,\delta}^{M,m'}$ *define* $q\in S_{\rho,\delta}^{M,m+m'}$ *by*

$$q(x,\xi)=(2\pi)^{-n}\iint e^{-i<y,\eta>}p(x,\xi+\eta)p'(x+y,\xi)dyd\eta\,.$$

Then $p(x,D)p'(x,D)=q(x,D)$. *Moreover,* q *has the asymptotic expansion*

$$q(x,\xi)\sim\sum_{\alpha}\frac{(-i)^{|\alpha|}}{\alpha!}\partial_{\xi}^{\alpha}p(x,\xi)\partial_{x}^{\alpha}p'(x,\xi)\,.$$

(2) *For* $P=p(x,D)\in\mathrm{Op}\,S_{\rho,\delta}^{M,m}$ *define the formal adjoint* P^{*} *by* $(Pu,v)=(u,P^{*}v)$ *for* $u,v\in C_{0}^{\infty}(\boldsymbol{R}^{n})$, *where* (,) *denotes the usual sesquilinear product, and define* $p^{*}(x,\xi)\in S_{\rho,\delta}^{M,m}$ *by*

$$p^{*}(x,\xi)=(2\pi)^{-n}\iint e^{-i<y,\eta>}\overline{p(x+y,\xi+\eta)}dyd\eta\,.$$

Then $P^*=p^*(x, D)$. *Moreover*, p^* *has the asymptotic expansion*

$$p^*(x, \xi) \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{p(x, \xi)}\,.$$

For operators P and Q we define the commutator $[P, Q]$ by $[P, Q]=PQ-QP$. Then we have

COROLLARY 2.3. *For* $M=(\mu_1, \cdots, \mu_n)$ *we set* $\nu=\inf(\{\mu_j-1\,;\ \mu_j>1\}\cup\{1\})$. *Let* $p\in S_{1,0}^{M,m}$ *and* $p'\in S_{1,0}^{M,m'}$ *be two classical M-pseudo-differential operators with the principal symbols* p_m *and* $p'_{m'}$ *respectively. Then* $[p(x, D), p'(x, D)]\in \mathrm{Op}\, S_{1,0}^{M,m+m'-1}$. *Moreover, we have*

$$[p(x, D), p'(x, D)]=-i\{p_m, p'_{m'}\}_M(x, D)+\mathrm{Op}\, S_{1,0}^{M,m+m'-1-\nu}\,.$$

The starting point of microlocal analysis is the following non-characteristic regularity theorem, which was proved initially by Lascar [12].

THEOREM 2.4. *Let* P *be a classical M-pseudo-differential operator on* $\Omega\subset \boldsymbol{R}^n$ *of order* m; *let* p_m *be the principal symbol of* P. *Let* $(\mathring{x}, \mathring{\xi})\in\Omega\times(\boldsymbol{R}_n\backslash 0)$ *and let* u *be a distribution defined near* $\mathring{x}$. *Suppose that* $p(\mathring{x}, \mathring{\xi})\neq 0$. *Then* $Pu\in H_M^s(\mathring{x}, \mathring{\xi})$ *implies* $u\in H_M^{s+m}(\mathring{x}, \mathring{\xi})$.

This theorem means that if Pu is smooth then the singularities of u must be included in the characteristic set $p_m^{-1}(0)$.

Now we generalize the sharp Gårding inequality to quasi-homogeneous pseudo-differential operators. This inequality is essential for the energy estimate, which yields the propagation of regularity theorem (Proposition 2.7).

PROPOSITION 2.5. *Let* P *be a classical M-pseudo-differential operator of order* m *defined on* $\Omega\subset\boldsymbol{R}^n$ *and let* p_m *be the principal symbol of* P. *Assume that*

$$\mathrm{Re}\, p_m(x, \xi)\geqq 0\,.$$

Then, for every $K\Subset\Omega$, *there exists a constant* C_K *such that*

$$\mathrm{Re}\,(Pu, u)\geqq C_K\|u\|^2_{M,(m-1)/2} \qquad \textit{for } u\in C_0^\infty(K)\,. \tag{2.1}$$

PROOF. Let u be an element of $C_0^\infty(K)$. Take a real valued function $\chi\in C_0^\infty(\Omega)$ satisfying $\chi=1$ in a neighborhood of K. Then $(Pu, u)=(\chi Pu, u)$. Hence, we can assume that the symbol $p(x, \xi)$ of P is defined on the whole of $\boldsymbol{R}^n\times\boldsymbol{R}_n$. More precisely, we can assume $p(x, \xi)\in S_{1,0}^{M,m}$ without loss of generality.

In order to prove the proposition we shall make use of the "wave packet transform" introduced by Cordoba-Fefferman [6]. Let us now define the operator $W: L^2(\boldsymbol{R}^n)\to L^2(\boldsymbol{R}^n\times\boldsymbol{R}_n)$ by

$$Wu(y,\xi)=c_n[\xi]_M^{n/4}\int e^{i<y-x,\xi>-[\xi]_M|y-x|^2/2}u(x)dx\,,$$

where $c_n=(2\pi)^{-3n/4}$ and let W^* be its adjoint:

$$W^*F(x)=c_n\iint e^{i<x-y,\xi>-[\xi]_M|y-x|^2/2}[\xi]_M^{n/4}F(y,\xi)dyd\xi\,.$$

Then we have

LEMMA 2.6. *Let p be an element of $S_{1,0}^{M,m}$. Then*

$$W^*pW=p(x,D)+R\,,$$

where $R\in\mathrm{Op}\,S_{1,1/2}^{M,m-1}$.

PROOF OF THE LEMMA. The distribution kernel of W^*pW is

$$W^*pW(z,x)=(2\pi)^{-n}\int e^{i<z-x,\xi>}t(z,\xi,x)d\xi\,,$$

where

$$t(z,\xi,x)=(2\pi)^{-n/2}[\xi]_M^{n/2}\int e^{-[\xi]_M(|z-y|^2+|x-y|^2)/2}p(y,\xi)dy\,.$$

Observing that

$$\begin{aligned}&\left|\partial_{\xi_j}\int e^{-[\xi]_M(|z-y|^2+|x-y|^2)/2}dy\right|\\ &\leqq\frac{1}{2}\left|\partial_{\xi_j}[\xi]_M\right|\left|\int(|z-y|^2+|x-y|^2)e^{-[\xi]_M(|z-y|^2+|x-y|^2)/2}dy\right.\\ &\leqq C[\xi]_M^{-\mu_j-n/2}\end{aligned}$$

and that

$$\begin{aligned}&\left|\partial_{x_j}\int e^{-[\xi]_M(|z-y|^2+|x-y|^2)/2}dy\right|\\ &\leqq\int|x_j-y_j|[\xi]_M e^{-[\xi]_M(|z-y|^2+|x-y|^2)/2}dy\\ &\leqq[\xi]_M^{1/2}\int((|z-y|^2+|x-y|^2)[\xi]_M)^{1/2}e^{-[\xi]_M(|z-y|^2+|x-y|^2)/2}dy\\ &\leqq C[\xi]_M^{1/2-n/2}\end{aligned}$$

$\left(\text{and the same estimate for } \partial_{z_j}\int e^{-[\xi]_M(|z-y|^2+|x-y|^2)/2}dy\right)$, we can easily verify

$$|\partial_\xi^\alpha \partial_z^\beta \partial_x^\gamma t(z, \xi, x)| \leqq C_{\alpha,\beta,\gamma}(1+[\xi]_M)^{m-<\alpha, M>+(|\beta|+|\gamma|)/2}$$

for every $\alpha, \beta, \gamma \in \boldsymbol{N}^n$. Then, from the calculus of multiple symbols given by Kumano-go (cf. [11]), we can see that $W^*pW \in \mathrm{Op}\, S^{M,m}_{1,1/2}$ and that the symbol of W^*pW is

$$t(x, \xi, x) + \sum_{j=1}^n \partial_{\xi_j}\partial_{x_j} t(z, \xi, x)|_{z=x},$$

modulo a function which belongs to $S^{M,m-1}_{1,1/2}$.

Using Taylor's formula we obtain

$$\begin{aligned} t(x, \xi, x) &= \left(\frac{[\xi]_M}{2\pi}\right)^{n/2}\int e^{-[\xi]_M|x-y|^2/2} p(y, \xi)dy \\ &= p(x, \xi) + \sum_{j=1}^n \partial_{x_j} p(x, \xi)\left(\frac{[\xi]_M}{2\pi}\right)^{n/2}\int (y_j - x_j) e^{-[\xi]_M|x-y|^2}dy \\ &\quad + [\xi]_M^{m-1}\int O([\xi]_M|x-y|^2)[\xi]_M^{n/2} e^{-[\xi]_M|x-y|^2}dy \\ &= p(x, \xi) + 0 + \text{symbol in } S^{M,m-1}_{1,1/2}. \end{aligned}$$

In the same way we can show that

$$\sum_{j=1}^n \partial_{\xi_j}\partial_{x_j} t|_{z=x} \in S^{M,m-1}_{1,1/2}.$$

The lemma is now proved.

End of the Proof of Proposition 2.5. In view of the preceding lemma we can write

$$(Pu, u) = (W^*p_m Wu, u) + (Ru, u)$$

with $R \in \mathrm{Op}\, S^{M,m-1}_{1,1/2}$. Note that

$$\mathrm{Re}(W^*p_m Wu, u) \geqq 0 \qquad \text{if } \mathrm{Re}\, p_m \geqq 0, \tag{2.2}$$

which comes from

$$\int W^*p_m Wu(x)\bar{u}(x)dx = \iint p_m(y, \xi)|Wu(y, \xi)|^2 dyd\xi$$

and that if $R \in \mathrm{Op}\, S^{M,m-1}_{1,1/2}$ then

$$(2.3)\qquad |(Ru, u)| \leqq \|Ru\|_{M,-(m-1)/2}\|u\|_{M,(m-1)/2} \leqq C\|u\|^2_{M,(m-1)/2}\,.$$

From (2.2) and (2.3), (2.1) follows immediately.

Using Proposition 2.5 one can prove the following result along the same line as in the proof of Proposition 3.5.1 of Hörmander (10). See also the proof of Proposition 5.1 below, where the same argument will be used for a more complicated operator.

THEOREM 2.7. *Let Ω be an open set of $\boldsymbol{R}^n$. Let P be a classical M-pseudo-differential operator on Ω of order m and let p_m be the principal symbol of P. Consider a null bicharacteristic strip for* $\mathrm{Re}\, p_m : I \ni t \to \gamma(t) \in \Omega \times (\boldsymbol{R}_n \backslash 0)$, *where $I=\{t\in \boldsymbol{R}\,;\, t_1 \leqq t \leqq t_2\}$. Assume that* $\mathrm{Im}\, p_m \geqq 0$ *in a neighborhood of $\gamma(I)$. If $u \in \mathcal{D}'(\Omega)$ satisfies that $Pu \in H^s_M(\gamma(I))$ and that $u \in H^{s+m-1}_M(\gamma(t_2))$, then $u \in H^{s+m-1}_M(\gamma(I))$.*

If p_m is real valued then, by applying Theorem 2.7 for both P and $-P$, we obtain the following result of Lascar (Theorem 4.1 in (12)).

COROLLARY 2.8. *Let Ω be an open set of $\boldsymbol{R}^n$ and let $u \in \mathcal{D}'(\Omega)$. Let P be a classical M-pseudo-differential operator on Ω of order m with real principal symbol. Assume γ is an interval on a null bicharacteristic strip where Pu belongs to H^s_M. Then u belongs to H^{s+m-1}_M on γ if this is true at some point on γ.*

3. Paraproducts.

We first define the Littlewood-Paley decomposition adapted to the anisotropic function spaces.

Let ψ be a C^∞-function of $t \in \overline{\boldsymbol{R}}^+ = \{t \in \boldsymbol{R}\,;\, t \geqq 0\}$ with the value in $[0, 1]$, which satisfies $\psi(t)=1$ if $t \leqq 1/2$ and $\psi(t)=0$ if $t \geqq 1$. We set

$$\psi_k^M(\xi) = \psi([\xi]_M/2^k)\,, \qquad k=0, 1, 2, \cdots,$$

$$\varphi_k^M(\xi) = \psi([\xi]_M/2^k) - \psi([\xi]_M/2^{k-1})\,, \qquad k=1, 2, \cdots$$

and define the operators $S_k : L^p \to L^p$ and $\Delta_k : L^p \to L^p$ by

$$S_k(f)^\wedge(\xi) = \psi_k^M(\xi)\hat{f}(\xi)\,, \qquad k=0, 1, 2, \cdots,$$

$$\Delta_k(f)^\wedge(\xi) = \varphi_k^M(\xi)\hat{f}(\xi)\,, \qquad k=1, 2, \cdots.$$

By convention we set $\phi_0^M(\xi)=\psi_0^M(\xi)$ and $\Delta_0=S_0$. We then define the Littlewood-Paley decomposition of $f\in S'$:

$$f(x)=\sum_{k=0}^{\infty}\Delta_k(f)(x)\,.$$

Let M_p denote the linear space of all Fourier multiplier on L^p $(1\leqq p\leqq\infty)$; see Bergh-Löfström [2], Chapter 6. Note that

$$\psi_k^M(\xi)=\psi_0^M(2^{-kM}\xi)$$

$$\phi_k^M(\xi)=\phi_1^M(2^{-(k-1)M}\xi)$$

for $k=1,2,3,\cdots$. Hence, by Lemma 6.1.3 of Bergh-Löfstöm [2], each ψ_k^M (resp. ϕ_k^M) has the same norm in M_p as ψ_0^M (resp. ϕ_1^M). Since the Fourier inverse images of ψ_0^M, ϕ_1^M are in $\mathcal{S}$ $(\subset L^1)$, it is evident that ψ_0^M and ϕ_1^M belong to M_p for $1\leqq p\leqq\infty$.

The functions $S_k(f)$ and $\Delta_k(f)$ satisfy the following estimate for their derivatives.

LEMMA 3.1. *Let $f\in S'$, and assume that $\Delta_k(f)$, $S_k(f)\in L^p$, $1\leqq p\leqq\infty$. Then for every α*

$$\|\partial_x^\alpha\Delta_k(f)\|_{L^p}\leqq C_\alpha 2^{k<\alpha,M>}\|\Delta_k(f)\|_{L^p} \tag{3.1}$$

$$\|\partial_x^\alpha S_k(f)\|_{L^p}\leqq C_\alpha 2^{k<\alpha,M>}\|S_k(f)\|_{L^p}\,. \tag{3.2}$$

PROOF. Note that

$$\|\partial_x^\alpha\Delta_k(f)\|_{L^p}\in\|\partial_x^\alpha S_{k+1}\Delta_k(f))\|_{L^p}$$

$$\|\partial_x^\alpha S_k(f)\|_{L^p}\in\|\partial_x^\alpha S_{k+1}(S_k(f))\|_{L^p}\,.$$

Then $\partial_x^\alpha S_{k+1}$ has the symbol

$$\xi^\alpha\psi_{k+1}^M(\xi)=2^{k<\alpha,M>}(i2^{-kM}\xi)^\alpha\psi_1^M(2^{-kM}\xi)\,.$$

Hence, by Lemma 6.1.3 of [2], we obtain (3.1), (3.2) with $C_\alpha=\|(i\xi)^\alpha\psi_1^M(\xi)\|_{M_p}<\infty$.

It is convenient to introduce the anisotropic Besov space though our results only concern with the regularity in the Sobolev space. Now suppose $1\leqq p$, $q\leqq\infty$ and let $s\in R$. We define the anisotropic Besov space $B_{p,q}^{M,s}$ as follows:

DEFINITION 3.2. A tempered distribution u belongs to $B_{p,q}^{M,s}$ if there exists a sequence $\{\varepsilon_k\}\in l^q$ such that

$$\|\Delta_k u\|_{L^p}\leqq\varepsilon_k 2^{-ks}.$$

$B_{p,q}^{M,s}$ becomes a Banach space endowed with the norm

$$\|u\|_{B_{p,q}^{M,s}}=\|\{2^{ks}\|\Delta_k(u)\|_{L^p}\}\|_{l^q}.$$

REMARK 3.3. As in the isotropic case we can show that

(1) $B_{2,2}^{M,s}=H_M^s$,

(2) if $q_1<q_2$ then $B_{p,q_1}^{M,s}\subset B_{p,q_2}^{M,s}$,

(3) (Sobolev embedding theorem) if $p<r$ and $s-|M|/p=t-|M|/r$ then $B_{p,q}^{M,s}\subset B_{r,q}^{M,t}$.

Here, in (2) and in (3), the inclusion stands for a continuous embedding between two Banach spaces. These are proved in quite the same way as in the isotropic case; see Yamazaki [21] for complete proofs of them.

From (3) above it follows especially that if $s>|M|/2$ then

$$H_M^s(=B_{2,2}^{M,s})\subset B_{\infty,2}^{M,s-|M|/2}\subset L^\infty.$$

In this paper, we shall only use the Besov space of a positive order. When $s>0$, $B_{p,q}^{M,s}$ is characterized by the following lemma.

LEMMA 3.4. *Let $1\leqq p,\ q\leqq\infty$, and assume that $s>0$. Let m be an even integer greater than* $\mathrm{Max}\{s, n/2\}$. *Then a function $f\in L^p$ belongs to $B_{p,q}^{M,s}$ if and only if there exist a sequence $\{\varepsilon_k\}\in l^q$ and a sequence $\{f_k\}$ consisting of smooth functions such that, for every integer $k\geqq 0$,*

$$\|f-f_k\|_{L^p}\leqq\varepsilon_k 2^{-ks}, \tag{3.3}$$

and that

$$\|\partial_x^\alpha f_k\|_{L^p}\leqq\varepsilon_k 2^{k\langle\alpha,M\rangle-ks} \qquad \text{for } |\alpha|=m. \tag{3.4}$$

PROOF. First we prove the "if" part of the lemma. Let $g_k=S_k(f_k)$. Then

$$\begin{aligned}\|f-g\|_{L^p}&\leqq\|f-f_k\|_{L^p}+\|(1-S_k)f_k\|_{L^p}\\&\leqq\varepsilon_k 2^{-ks}+\left\|\left(\frac{1-\phi_k^M(D)}{|2^{-kM}D|^m}\right)(|2^{-kM}D|^m f_k)\right\|_{L^p}\\&\leqq\varepsilon_k 2^{-ks}+C\left\|\frac{1-\phi_0^M(2^{-kM}\xi)}{|2^{-kM}\xi|^m}\right\|_{M_p}\sup_{|\alpha|=m}\{2^{-k\langle\alpha,M\rangle}\|D^\alpha f_k\|_{L^p}\}\end{aligned}$$

where $\|\cdot\|_{M_p}$ denotes the norm of M_p. Note that the function

$$(1-\phi_0^M(2^{-kM}\xi))/|2^{-kM}\xi|^m$$

has the same M_p norm as the function

$$(1-\phi_0^M(\xi))/|\xi|^m$$

by Lemma 6.1.3 of [2]. Using Lemma 6.1.5 of [2] we see that the latter function in fact belongs to M_p $(1\leqq p\leqq\infty)$, because $m>n/2$. Therefore

$$\|f-g_k\|_{L^p}\leqq C\varepsilon_k 2^{-ks}$$

with C independent of k; in particular we have $g_k\to f$ in L^p.

Now let $u_k=g_k-g_{k-1}$, $k=0, 1, 2, \cdots$ $(g_{-1}=0)$ so that $f=\sum_{k=0}^{\infty} u_k$ in L^p. Then

$$\operatorname{supp}\hat{u}_k\subset\{\xi\in\boldsymbol{R}^n\,;\,(\xi)_M\leqq 2^k\}$$

$$\begin{aligned}\|u_k\|_{L^p}&\leqq\|f-g_k\|_{L^p}+\|f-g_{k-1}\|_{L^p}\\&\leqq\varepsilon_k' 2^{-ks},\end{aligned}$$

with another $\{\varepsilon_k'\}\in l^q$. Applying Theorem 3.8 of Yamazaki [21] we can then conclude that $f\in B_{p,q}^{M,s}$.

In order to prove the "only if" part of the lemma we set $f_k=S_k(f)$. Then

$$\begin{aligned}\|f-f_k\|_{L^p}&\leqq\sum_{j=k}^{\infty}\|\Delta_j f\|_{L^p}\\&\leqq\sum_{j=k}^{\infty}\varepsilon_j 2^{-js}\\&\leqq\eta_k 2^{-ks},\end{aligned}$$

where $\eta_k=\sum_{j=k}^{\infty}\varepsilon_j 2^{-(j-k)s}$. Also, by Lemma 3.1,

$$\begin{aligned}\|\partial_x^\alpha f_k\|_{L^p}&\leqq\sum_{j=0}^{k}\|\partial_x^\alpha\Delta_j(f)\|_{L^p}\\&\leqq C_\alpha\sum_{j=0}^{k}\varepsilon_j 2^{j<\alpha,M>-js}\\&\leqq\eta_k' 2^{k<\alpha,M>-ks},\end{aligned}$$

where $\eta_k'=\sum_{j=0}^{\infty}\varepsilon_j 2^{-(k-j)(<\alpha,M>-s)}$.

Since $s>0$ (resp. $\langle\alpha, M\rangle\geqq|\alpha|=m>s$) we see that $\{\eta_k\}$ (resp. $\{\eta_k'\}$) belongs to l^q by means of the following sublemma.

SUBLEMMA 3.5. *Suppose r is real, $0<r<1$ and let $\{\varepsilon_j\}\in l^q$ with $1\leqq q \leqq\infty$. It follows that*

(1) *if* $\eta_k=\sum_{j=k}^{\infty}\varepsilon_j r^{(j-k)}$ *then* $\{\eta_k\}\in l^q$,

(2) *if* $\eta'_k=\sum_{j=0}^{k}\varepsilon_j r^{(k-j)}$ *then* $\{\eta'_k\}\in l^q$.

PROOF. If $q=1$ or ∞ then the assertions of the sublemma are evident. Hence, by the Riesz-Thorin theorem, we get the sublemma for $1\leqq q\leqq\infty$.

REMARK 3.6. We intend to apply Lemma 3.4 to a series $f=\sum_{j=0}^{\infty}u_j$. To verify (3.3), (3.4) with $f_k=\sum_{j=0}^{k}u_j$ it is enough to show that there exists a sequence $\{\varepsilon_j\}\in l^q$ such that

$$\|u_j\|_{L^p}\leqq\varepsilon_j 2^{-js} \tag{3.5}$$

and that

$$\|\partial_x^\alpha u_j\|_{L^p}\leqq\varepsilon_j 2^{j\langle\alpha, M\rangle-js} \qquad \text{for } |\alpha|=m>s\,. \tag{3.6}$$

In fact, arguing as in the latter part of the proof of Lemma 3.4, we can easily show that $\{f_k\}$ satisfies (3.3), (3.4).

Employing the Littlewood-Paley decomposition and using Lemma 3.4 we obtain

PROPOSITION 3.7. *Let $s>|M|/2$ and let $u\in H_M^s$. Assume that $F=F(U)$ is a holomorphic function of $U\in\boldsymbol{C}$, vanishing at 0. Then $F(u(x))\in H_M^s$.*

PROOF. First note that u belongs to L^∞ by the Sobolev embedding theorem. Thus $F(u(x))$ becomes a well defined element of L^∞.

Now set $u_k=S_k(u)$ and $v_k=\Delta_k(u)=u_k-u_{k-1}$. We have

$$F(u)=F(u_0)+(F(u_1)-F(u_0))+\cdots+(F(u_k)-F(u_{k-1}))+\cdots$$

and

$$F(u_k)-F(u_{k-1})=m_k v_k\,,$$

where

$$m_k=\int_0^1 F'(u_{k-1}+tv_k)dt\,.$$

Then m_k satisfies

$$|\partial_x^\alpha m_k(x)| \leqq C_\alpha 2^{k<\alpha, M>}. \tag{3.7}$$

Admitting this for the moment we shall complete the proof of the proposition.

Since $\partial_x^\alpha u \in H_M^{s-<\alpha,M>}$ $(=B_{2,2}^{M,s-<\alpha,M>})$, for $|\alpha| \leqq m$,

$$\|\partial_x^\alpha v_k\|_{L^2} \leqq \varepsilon_k 2^{k<\alpha,M>-ks}, \qquad \{\varepsilon_k\} \in l^2.$$

Then, from (3.7) and the above estimate, it follows that

$$\begin{aligned}\|\partial_x^\alpha m_k v_k\|_{L^2} &\leqq \sum_{\alpha'+\alpha''=\alpha} C_{\alpha',\alpha''} \|\partial_x^\alpha m_k\|_{L^\infty} \|\partial_x^{\alpha''} v_k\|_{L^2} \\ &\leqq \varepsilon_k 2^{k<\alpha,M>-ks},\end{aligned}$$

where $\{\varepsilon_k\}$ is another sequence belonging to l^2. Applying Lemma 3.4 or rather Remark 3.6 to the series $\sum_{j=0}^{\infty} m_k v_k$, we conclude that $F(u(x)) \in H_M^s$.

Now we shall show (3.7). Using the Sobolev inequality and Lemma 3.1, we obtain

$$\begin{aligned}|\partial_x^\alpha(u_{k-1}(x)+tv_k(x))| &\leqq C(\|(1+(D)_M)^s \partial_x^\alpha \hat{u}_{k-1}(\xi)\|_{L^2} + \|(1+(D_M))^s \partial_x^\alpha v_k(\xi)\|_{L^2}) \\ &\leqq C_\alpha 2^{k<\alpha,M>}(\|u_{k-1}\|_{H_M^s} + \|v_k\|_{H_M^s}) \\ &\leqq C'_\alpha 2^{k<\alpha,M>}.\end{aligned}$$

Hence (3.7) follows from

LEMMA 3.8. *Let F be a holomorphic function. Suppose that a sequence $g_k \in C^\infty(\boldsymbol{R}^n)$ satisfies $|\partial_x^\alpha g_k(x)| \leqq C_\alpha 2^{k<\alpha,M>}$ for every α. Then $|\partial_x^\alpha F(g_k(x))| \leqq C'_\alpha 2^{k<\alpha,M>}$.*

PROOF OF LEMMA 3.8. By using the chain rule, we can write

$$\partial_x^\alpha F(g_k) = \sum_{\alpha=\alpha^{(1)}+\cdots+\alpha^{(q)}} C_{\alpha^{(1)},\ldots,\alpha^{(q)}} D^q F(g_k) \partial^{\alpha^{(1)}} g_k \cdots \partial_x^{\alpha^{(q)}} g_k,$$

where the sum is taken over all the decompositions of α, and $1 \leqq q \leqq |\alpha|$. Then

$$|\partial_x^\alpha F(g_k(x))| \leqq C_\alpha \sum |\partial_x^{\alpha^{(1)}} g_k(x)| \cdots |\partial_x^{\alpha^{(q)}} g_k(x)| \leqq C'_\alpha 2^{k<\alpha,M>},$$

which proves the lemma.

Let us now recall the definition of the paraproduct $\pi: \mathcal{S}' \times \mathcal{S}' \to \mathcal{S}'$; which was introduced by Bony [3] in the homogeneous case and extended by Yamazaki [18] to quasi-homogeneous case.

DEFINITION 3.9. Let u and v be two tempered distributions. We define $\pi(u, v)$ by the following:

$$\pi(u, v)=\sum_{k=6}^{\infty} S_{k-6}(u)\Delta_{k}(v) .$$

If $u\in L^{\infty}$, we can easily see that the map $\pi(u, \cdot)$, which associates $\pi(u, v)$ to v, defines a continuous map on L^{p} for every p with $1<p<\infty$. The linear operator $\pi(u, \cdot)$ can be regarded as a pseudo-differential operator with the symbol

$$(2\pi)^{-n}\int e^{i<x,\eta>}\theta(\eta, \xi)\hat{u}(\eta)d\eta$$

of type $(1, 1)$, where $\theta(\eta, \xi)=\sum_{k=6}^{\infty}\psi_{k-6}^{M}(\eta)\phi_{k}^{M}(\xi)$ is supported by $\{(\eta, \xi)\in \boldsymbol{R}_{n}\times\boldsymbol{R}_{n}; (\eta)_{M}\leqq(1/16)[\xi]_{M}\}$. Such an operator will be considered in detail in Section 4 as a paradifferential operator.

This notion of paraproduct brings us a new method of linearizing nonlinear equations. Namely, we have

THEOREM 3.10. *Suppose that u and F satisfy the same assumptions as in Proposition* 3.7. *Then we have*

$$F(u)=\pi(F'(u), u)+w$$

with $w\in B_{1,1}^{M,2s}$ $(\subset H_{M}^{2s-|M|/2})$.

A similar result with $w\in H_{M}^{2s-|M|/2}$ has been proved originally by Yamazaki [18]. Thus, the above-stated result is a little improvement of his result[1], while our proof is based on the argument of Meyer [14], as is carried out below for the sake of completeness.

PROOF. As in the proof of Proposition 3.6 we set $u_{k}=S_{k}(u)$, $v_{k}=\Delta_{k}(u)=u_{k}-u_{k-1}$ and write

$$F(u)=\sum_{k=0}^{\infty} m_{k}v_{k} ,$$

where

$$m_{k}=\int_{0}^{1}F'(u_{k-1}+tv_{k})dt .$$

Then

$$w=F(u)-\pi(F'(u), u)=(m_{k}-S_{k-6}(u))v_{k} .$$

1) After completion of this work, the author was informed that the same result as our theorem has been proved among others in a recent preprint by Yamazaki.

Let m be an integer fixed as $m>s$. We shall show that there exists a sequence $\{\varepsilon_k\}\in l^2$ such that the estimate

$$\|\partial_x^\alpha(m_k-S_{k-6}(F'(u))\|_{L^2}\leqq\varepsilon_k 2^{k<\alpha,M>-ks} \tag{3.8}$$

holds for $0\leqq|\alpha|\leqq m$.

In order to show (3.8) we set $F'(u)=F'(0)+G(u)$, where G is holomorphic satisfying $G(0)=0$. Noticing that

$$m_k-S_{k-6}(F'(u))=\int_0^1 G(u_{k-1}+tv_k)dt-S_{k-6}(G(u))$$

we can write

$$m_k-S_{k-6}(F'(u))=G(u_k)-S_k(G(u))+\rho_k+\gamma_k\,,$$

where

$$\rho_k=S_k(G(u))-S_{k-6}(G(u))\,,$$

$$\gamma_k=\left(\int_0^1(1-t)G'(u_{k-1}+tv_k)dt\right)v_k\,.$$

In Proposition 3.7 we have proved that $G(u)\in H_M^s$. Hence

$$\|\partial_x^\alpha\rho_k\|_{L^2}\leqq\sum_0^5\|\partial_x^\alpha\Delta_{k-j}(G(u))\|_{L^2}\leqq\varepsilon_k 2^{k<\alpha,M>-ks}\,. \tag{3.9}$$

Arguing as in the proof of Proposition 3.7 we also have

$$\|\partial_x^\alpha\gamma_k\|_{L^2}\leqq\varepsilon_k 2^{k<\alpha,M>-ks}\,,\qquad\{\varepsilon_k\}\in l^2\,. \tag{3.10}$$

Our problem is thus to examine in what sense the linear operator S_k and the nonlinear operator G commute. We shall show the following:

LEMMA 3.11. *Assume G is a holomorphic function vanishing at 0 and let u be an element of H_M^s with $s>|M|/2$. Let m be an integer, $m>s$. Then there exists a sequence $\{\varepsilon_k\}\in l^2$ such that, for $0\leqq|\alpha|\leqq m$,*

$$\|\partial_x^\alpha(G(S_k(u))-S_k(G(u)))\|_{L^2}\leqq\varepsilon_k 2^{k<\alpha,M>-ks}\,. \tag{3.11}$$

This lemma, together with (3.9) and (3.10), will prove (3.8). Before proving the lemma we shall now complete the proof of Theorem 3.10.

PROOF OF (3.8)$\Rightarrow$THEOREM 3.10. It follows from $u\in H_M^s$ $(=B_{2,2}^{M,s})$ that $\|\partial_x^\alpha v_k\|_{L^2}\leqq\varepsilon'_k 2^{k<\alpha,M>-ks}$ with a sequence $\{\varepsilon'_k\}\in l^2$. By using the Schwarz inequality, this and the estimate (3.8) yield

$$\|\partial_x^\alpha((m_k-S_{k-6}(u))v_k)\|_{L^1}\leqq\eta_k 2^{k<\alpha,M>-2ks}$$

for $0\leqq|\alpha|\leqq m$, where $\gamma_k=C\varepsilon_k\varepsilon'_k$ with the constant C depending only on m, thus $\{\eta_k\}\in l^1$. From Remark 3.4 we can then conclude that $w\in B^{M;2s}_{1,1}$.

PROOF OF LEMMA 3.11. First we consider the case $\alpha=0$. Since G is holomorphic, the following estimate holds with $\{\varepsilon_k\}\in l^2$:

$$\|G(S_k(u))-G(u)\|_{L^2}\leqq C\|S_k(u)-u\|_{L^2}\leqq\varepsilon_k 2^{-ks}.$$

Here the second inequality is a consequence of Sublemma 3.5. Also, because $G(u)\in H^s_M$,

$$\|S_k(G(u))-G(u)\|_{L^2}\leqq\varepsilon_k 2^{-ks}$$

with another sequence $\varepsilon_k\in l^2$. By the triangle inequality, these estimate yield (3.11) for $\alpha=0$.

Next we suppose $|\alpha|=m$. To prove (3.11) it suffices to show

$$\|\partial_x^\alpha S_k(G(u))\|_{L^2}\leqq\varepsilon_k 2^{k<\alpha,M>-ks}, \tag{3.12}$$

$$\|\partial_x^\alpha G(S_k(u))\|_{L^2}\leqq\varepsilon_k 2^{k<\alpha,M>-ks} \tag{3.13}$$

with $\varepsilon_k\in l^2$. Since $G(u)\in H^s_M$, (3.12) follows by Lemma 3.1. The crucial part of the proof is thus to show (3.13).

PROOF OF (3.13). We write, as in the proof of Lemma 3.8, (setting $u_k=S_k(u)$)

$$G(u_k)=\sum_{\alpha=\alpha^{(1)}+\cdots+\alpha^{(q)}}C_{\alpha^{(1)},\dots,\alpha^{(q)}}D^qG(u_k)\partial_x^{\alpha^{(1)}}u_k\cdots\partial_x^{\alpha^{(q)}}u_k, \tag{3.14}$$

where the sum is taken over all the decomposition of α. Choose p_j so that $2/p_j=\langle\alpha^{(j)},M\rangle/\langle\alpha,M\rangle$ and take the L^2 norm of the each term on the right hand side of (3.14). Using the Hölder inequality we obtain

$$\|D^qG(u_k)\partial_x^{\alpha^{(1)}}u_k\cdots\partial_x^{\alpha^{(q)}}u_k\|_{L^2}\leqq\|D^qG(u_k)\|_{L^\infty}\|\partial_x^{\alpha^{(1)}}u_k\|_{L^{p_1}}\cdots\|\partial_x^{\alpha^{(q)}}u_q\|_{L^{p_q}}. \tag{3.15}$$

Now, we assert that there exists a sequence $\varepsilon_k\in l^2$ such that the estimate

$$\|\partial_x^{\alpha^{(j)}}u_k\|_{L^{p_j}}\leqq\varepsilon_k 2^{k<\alpha^{(j)},M>(1-s/<\alpha,M>)} \tag{3.16}$$

holds for each j.

In order to show (3.16) we distinguish three cases as follows:

$$\text{(I)}\qquad \langle\alpha^{(j)}, M\rangle+|M|\left(\frac{1}{2}-\frac{1}{p_j}\right)>s\,,$$

$$\text{(II)}\qquad \langle\alpha^{(j)}, M\rangle+|M|\left(\frac{1}{2}-\frac{1}{p_j}\right)=s\,,$$

$$\text{(III)}\qquad \langle\alpha^{(j)}, M\rangle+|M|\left(\frac{1}{2}-\frac{1}{p_j}\right)<s\,.$$

In (I) we set $u_k=v_0+v_1+\cdots+v_k$, where $v_0=u_0=S_0(u)$ and $v_j=\Delta_j(u)$, $j=1,2,\cdots$. The Sobolev inequality gives

$$\begin{aligned}\|\partial_x^{\alpha^{(j)}}v_k\|_{L^{p_j}}&\leqq C\|\partial_x^{\alpha^{(j)}}v_k\|_{H_M^{|M|(1/2-1/p_j)}}\\&\leqq \varepsilon_k 2^{k(<\alpha^{(j)},M>+|M|(1/2-1/p_j)-s)}\,.\end{aligned}$$

Hence

$$\begin{aligned}\|\partial_x^{\alpha^{(j)}}u_k\|_{L^{p_j}}&\leqq C\sum_{i=0}^{k}\varepsilon_i 2^{i(<\alpha^{(j)},M>+|M|(1/2-1/p_j)-s)}\\&\leqq \eta_k 2^{k(<\alpha^{(j)},M>+|M|(1/2-1/p_j)-s)}\,,\end{aligned}\tag{3.17}$$

where

$$\eta_k=C\sum_{i=0}^{k}\varepsilon_i 2^{(k-i)(<\alpha^{(j)},M>+|M|(1/2-1/p_j)-s)}\in l^2\,,$$

by Sublemma 3.5. Since $s>|M|/2$ and $1/2-1/p_j>0$, (recalling that $2/p_j=\langle\alpha^{(j)}, M\rangle/\langle\alpha, M\rangle$)

$$\begin{aligned}\langle\alpha^{(j)}, M\rangle+|M|\left(\frac{1}{2}-\frac{1}{p_j}\right)-s&\leqq\langle\alpha^{(j)}, M\rangle+2s\left(\frac{1}{2}-\frac{1}{p_j}\right)-s\\&\leqq\langle\alpha^{(j)}, M\rangle\left(1-\frac{s}{\langle\alpha, M\rangle}\right).\end{aligned}$$

Thus (3.17) yields (3.16).

In the same way we can show that

$$\|\partial_x^{\alpha^{(j)}}u_k\|_{L^{p_j}}\leqq O(k)$$

in (II) and that

$$\|\partial_x^{\alpha^{(j)}}u_k\|_{L^{p_j}}\leqq O(1)$$

in (III). In both cases (3.16) is clearly satisfied because $\langle\alpha, M\rangle\geqq|\alpha|=m>s$. Hence (3.16) is established.

Now combining (3.16) with (3.15) we have

$$\|D^qG(u_k)\partial_x^{\alpha^{(1)}}u_k\cdots\partial_x^{\alpha^{(q)}}u_k\|_{L^2}\leqq\prod_{j=1}^{q}\varepsilon_k 2^{k<\alpha^{(j)},M>(1-s/<\alpha,M>)}$$

$$\leqq \varepsilon_k' 2^{k<\alpha, M>-ks},$$

where ε_k' is another sequence belonging to l^2. This proves (3.13) in view of (3.14).

END OF THE PROOF OF LEMMA 3.11. We have proved (3.11) for $\alpha=0$ and $|\alpha|=m$. It only remains to prove (3.11) for $1\leqq|\alpha|\leqq m-1$. Now let $1\leqq|\alpha|\leqq m-1$. We can then prove (3.11) by an interpolation method; see for example Bergh-Löfström [2]. Now the proof of Lemma 3.10, and therefore, the proof of Theorem 3.9 are complete.

The following theorem is proved just in the same way as in the proofs of Proposition 3.6 and Theorem 3.9.

THEOREM 3.12. *Let $F=F(x; U_1, \cdots, U_N)$ be a C^∞-function defined on $\boldsymbol{R}^n\times C^N$ which is holomorphic with respect to $U_1, \cdots, U_N$. Let $s>|M|/2$ and let $u_1(x), \cdots, u_N(x)$ be elements of H_M^s. Then $F(x; u_1(x), \cdots, u_N(x))$ belongs to H_M^s locally. Moreover, we have*

$$F(x; u_1(x), \cdots, u_N(x))=\sum_{j=1}^N \pi(\partial_{U_j}F(x; u_1(x), \cdots, u_N(x)), u_j(x))+G(x),$$

where G belongs to $B_{1,1}^{M,2s}$ locally.

REMARK 3.13. (1) We note that $B_{1,1}^{M,2s}\subset B_{2,1}^{M,2s-|M|/2}\subset B_{2,2}^{M,2s-|M|/2}=H_M^{2s-|M|/2}$ (see Remark 3.3). Thus the theorem asserts that the remainder function G is more regular than $F(x; u_1(x), \cdots, u_N(x))$ by degree $s-|M|/2$.

(2) Suppose further that $F(x; 0, \cdots, 0)$ belongs to S. Then under the assumptions of Theorem 3.11 we can show that $F(x; u_1, \cdots, u_n)\in H_M^{2s}$ and that $G\in B_{1,1}^{M,2s}\subset H_M^{2s-|M|/2}$.

4. Paradifferential operators.

In this section, we define our symbol class of paradifferential operators and describe the symbol calculus for them. Partly we use results of Yamazaki [21], [22]. But partly we follow a line adapted from Meyer [13], [14]. This line is relatively simpler than that in [21], [22] and sufficient for our purpose.

The following definition of a symbol class of paradifferential operators is due to Meyer [14].

DEFINITION 4.1. Let r be real, $r>0$. We let $\Sigma_r^{M,m}$ denote the set of all functions $\sigma\in C^{\infty}(\boldsymbol{R}^{n}\times\boldsymbol{R}_{n})$ such that, for every $\alpha\in N^{n}$,

$$\|\partial_{\xi}^{\alpha}\sigma(x,\xi)\|_{H_{M}^{r+|M|/2}(dx)}\leqq C_{\alpha}(1+[\xi]_{M})^{m-<\alpha,M>} \tag{4.1}$$

and that the spectrum of $x\rightarrow\sigma(x,\xi)$ is supported by $\{(\xi,\eta)\in\boldsymbol{R}_{n}\times\boldsymbol{R}_{n}\,;\,(\eta)_{M}\leqq (1/10)[\xi]_{M}\}$, that is, $\hat{\sigma}(\eta,\xi)=0$ when $[\eta]_{M}\geqq(1/10)[\xi]_{M}$, where

$$\hat{\sigma}(\eta,\xi)=\int e^{-i<x,\eta>}\sigma(x,\xi)dx\;.$$

Moreover, we let $\mathrm{Op}\,\Sigma_{r}^{M,m}$ denote the space of pseudo-differential operators with symbols in $\Sigma_{r}^{M,m}$ and call elements of $\Sigma_{r}^{M,m}$ (quasi-homogeneous) paradifferential operators.

REMARK 4.2. Let σ be an element of $\Sigma_{r}^{M,m}$. Then, using the Sobolev inequality, we have

$$|\partial_{\xi}^{\alpha}\partial_{x}^{\beta}\sigma(x,\xi)|\leqq C_{\alpha,\beta}(1+[\xi]_{M})^{m+(<\beta,M>-r)_{+}-<\alpha,M>}$$

provided $\langle\beta,M\rangle\neq r$, where $(X)_{+}=\max(X,0)$ and

$$|\partial_{\xi}^{\alpha}\partial_{x}^{\beta}\sigma(x,\xi)|\leqq C_{\alpha,\beta}\log(2+[\xi]_{M})\cdot(1+[\xi]_{M})^{m-<\alpha,M>}$$

when $\langle\beta,M\rangle=r$. From this it follows especially that $\Sigma_{r}^{M,m}\subset S_{1,1}^{M,m}$.

In view of the above remark we can consider the paradifferential operator as a subclass of M-pseudo-differential operators of type (1, 1). On the continuity of $S_{1,1}^{M,m}$ the following result was given by Yamazaki [21], Theorem D (see the remark after it).

THEOREM 4.3. *Let $s>0$ and $\sigma\in S_{1,1}^{M,m}$. Then $\sigma(x,D)$ maps H_{M}^{s+m} continuously to H_{M}^{s}.*

We shall now give a symbol calculus for $\Sigma_{r}^{M,m}$. Symbol calculus of paradifferential operators is also given in [22] for more general function spaces. But here we present a somewhat simpler version sufficient for our purpose.

PROPOSITION 4.4. *Let $\sigma\in\Sigma_{r}^{M,0}$ and let $\tau\in S_{1,1}^{M,0}$. Then*

$$\tau(x,D)\sigma(x,D)=\omega(x,D)+\rho(x,D)\,,$$

where

$$\omega(x,\xi)=\sum_{<\alpha,M>\leqq r}\frac{(-i)^{|\alpha|}}{\alpha!}\partial_{\xi}^{\alpha}\tau(x,\xi)\partial_{x}^{\alpha}\sigma(x,\xi) \tag{4.2}$$

and $\rho\in S_{1,1}^{M;-r}$.

PROOF. Let us define ω by (4.2). Then, by the explicit formula of the composition, we have

$$\rho(x,\xi)=(2\pi)^{-n}\int\Big\{\tau(x,\xi+\eta)-\sum_{\langle\alpha,M\rangle\leq r}\frac{\eta^\alpha}{\alpha!}\partial_\xi^\alpha\tau(x,\xi)\Big\}e^{i\langle x,\eta\rangle}\hat\sigma(\eta,\xi)d\eta\,.$$

By using the Taylor formula we can rewrite this

$$\rho(x,\xi)=\sum_{\substack{\langle\alpha,M\rangle\geq r\\|\alpha|<N}}\int\frac{\eta^\alpha}{\alpha!}\partial_\xi^\alpha\tau(x,\xi)e^{i\langle x,\eta\rangle}\hat\sigma(\eta,\xi)d\eta$$

$$+\sum_{|\alpha|=N+1}\frac{N+1}{\alpha!}\int\Big(\int_0^1(1-t)^N\partial_\xi^\alpha\tau(x,\xi+t\eta)dt\Big)\eta^\alpha e^{i\langle x,\xi\rangle}\hat\sigma(\eta,\xi)d\eta\,,$$

where $N=[r]$.

Now we set, for $\langle\alpha,M\rangle>r$ and $|\alpha|\leq N$,

$$\rho^\alpha(x,\xi)=\int\eta^\alpha\mu_\xi^\alpha(x,\xi)e^{i\langle x,\eta\rangle}\hat\sigma(\eta,\xi)d\eta$$

and, for $|\alpha|=N+1$,

$$\rho^\alpha(x,\xi)=\int\Big(\int_0^1(1-t)^N\partial_\xi^\alpha\tau(x,\xi+t\eta)dt\Big)\eta^\alpha e^{i\langle x,\eta\rangle}\hat\sigma(\eta,\xi)d\eta\,.$$

It suffices to show that ρ^α belongs to $S_{1,1}^{M;-r}$ for each α.

First we consider ρ^α for $\langle\alpha,M\rangle>r$ and $|\alpha|\leq N$. Differentiating under the integral sign we have

$$\begin{aligned}(4.3)\qquad &\partial_x^\beta\partial_\xi^\gamma\rho^\alpha(x,\xi)\\ &=\sum_{\substack{\beta'+\beta''=\beta\\\gamma'+\gamma''=\gamma}}C_{\beta'\beta''\gamma'\gamma''}\int\partial_x^{\beta'}\partial_\xi^{\gamma'}\tau(x,\xi)(-i)^{|\beta''|}\eta^{\alpha+\beta''}e^{i\langle x,\eta\rangle}\partial_\xi^{\gamma''}\hat\sigma(\eta,\xi)d\eta\,.\end{aligned}$$

Since $\tau\in S_{1,1}^{M;0}$ we have

$$|\partial_x^{\beta'}\partial_\xi^{\alpha+\gamma'}\tau(x,\xi)|\leq C(1+[\xi]_M)^{\langle\beta',M\rangle-\langle\alpha+\gamma',M\rangle}\,.$$

Also, by the Schwarz inequality, it follows that

$$\begin{aligned}&\int|\eta^{\alpha+\beta''}\partial_\xi^{\gamma''}\hat\sigma(\eta,\xi)|d\eta\\ &\leq\Big(\int_{[\eta]_M\leq(1/10)[\xi]_M}|\eta^{\alpha+\beta''}(1+[\eta]_M)^{-r-|M|/2}|^2d\eta\Big)^{1/2}\|\partial_\xi^{\gamma''}\sigma(\eta,\xi)\|_{H_M^{r+|M|/2}}\\ &\leq C(1+[\xi]_M)^{\langle\alpha+\beta'',M\rangle-r-\langle\gamma'',M\rangle}\,.\end{aligned}$$

Here we have used the fact that $\langle\alpha, M\rangle > r$. Hence

$$|\partial_x^\beta \partial_\xi^\gamma \rho^\alpha(x, \xi)| \leqq C(1+[\xi]_M)^{-r+<\beta, M>-<\gamma, M>},$$

which implies $\rho^\alpha \in S_{1,1}^{M;-r}$ for $\langle\alpha, M\rangle > r$, $|\alpha| \leqq N$.

Next we consider the estimate of ρ^α for $|\alpha| = N+1$. Since $(1/2)[\xi]_M \leqq [\xi+t\eta]_M \leqq 2[\xi]_M$ on the support of $\hat{\sigma}$ the following estimate holds for $(\eta, \xi) \in \operatorname{supp} \hat{\sigma}$:

$$\left|\partial_x^\beta \partial_\xi^\gamma \left(\int (1-t)^N \partial_\xi^\alpha \tau(x, \xi+t\eta) dt\right)\right| \leqq C(1+[\xi]_M)^{<\beta, M>-<\alpha+\gamma, M>},$$

where C denotes a constant independent of η. Accordingly, the proof for $\langle\alpha, M\rangle > r$, $|\alpha| \leqq N$ can be applied without any change; hence we have $\rho^\alpha \in S_{1,1}^{M;-r}$ for $|\alpha| = N+1$.

Now all terms in the right-hand side of (4.3) are in $S_{1,1}^{M;-r}$. Thus ω belongs to $S_{1,1}^{M;-r}$ as desired.

REMARK 4.5. Let σ and p be as in Proposition 4.4. Then, by Remark 4.2, we have

$$\partial_\xi^\alpha p(x, \xi) \partial_x^\alpha \sigma(x, \xi) \in S_{1,1}^{M;-r}$$

provided $\langle\alpha, M\rangle \neq r$, and also

$$\partial_\xi^\alpha p(x, \xi) \partial_x^\alpha \sigma(x, \xi) \in S_{1,1}^{M;-r+\varepsilon} \qquad \text{for every } \varepsilon > 0,$$

when $\langle\alpha, M\rangle = r$. Hence $\omega \in S_{1,1}^{M;0}$ for every $r > 0$.

Moreover, for the commutator with an element of $S_{1,0}^{M;0}$, we can easily see the following.

COROLLARY 4.6. *Let $\sigma(s, D) \in \operatorname{Op} \Sigma_r^{M,0}$ and let $p(x, D) \in \operatorname{Op} S_{1,0}^{M;0}$. Then, modulo an operator belonging to $\operatorname{Op} S_{1,1}^{M;-r}$, it follows that*

(1) *$[p(x, D), \sigma(x, D)] \in \operatorname{Op} S_{1,1}^{M;-1}$ if $r > 1$, with the symbol supported by those of σ and p,*

(2) *$[p(x, D), \sigma(x, D)] \in \operatorname{Op} S_{1,1}^{M;-1+\varepsilon}$ for every $\varepsilon > 0$, if $r = 1$, with the symbol supported by those of σ and p, or*

(3) *$[p(x, D), \sigma(x, D)] = 0$ if $0 < r < 1$.*

Since this is a direct consequence of Proposition 4.4 and Remark 4.5, we omit the proof.

Proposition 4.4 allows one to study the microlocal regularities of $\sigma(x, D)u$, with σ in $\Sigma_r^{M,0}$ up to the finite order, say r. Indeed, we have

PROPOSITION 4.6. *Let* $s>0$, $0<\rho\leqq r$ *and let* $\sigma(x,D)\in\Sigma_{r}^{M,0}$. *If* u *is an element in* $H_{M}^{s+\rho}(\mathring{x},\mathring{\xi})\cup H_{M}^{s}$, *then so is* $\sigma(x,D)u$.

PROOF. Choose a closed M-conic neighborhood V of $(\mathring{x},\mathring{\xi})$ where u belongs to $H_{M}^{s+\rho}$ and a classical symbol $a\in S_{1,0}^{M;0}$ supported by V such that $a_{0}(\mathring{x},\mathring{\xi})\neq 0$. Then Proposition 4.5 gives

$$a(x,D)\sigma(x,D)=\tau(x,D)+\omega(x,D)\,,$$

where $\tau\in S_{1,1}^{M;0}$ supported by V and $\omega\in S_{1,1}^{M;-r}$. Since $s+\sigma>0$ we can apply Theorem 4.1 to obtain

$$\|a(x,D)\sigma(x,D)u\|_{H_{M}^{s+\rho}}\leqq C(\|u\|_{H_{M}^{s+\rho}(V)}+\|u\|_{H_{M}^{s+\rho-r}})\,,$$

where $\|u\|_{H_{M}^{s+\rho}(V)}$ stands for a fixed seminorm induced from the definition of $u\in H_{M}^{s+\rho}(V)$.

Recalling that $\rho\leqq r$ and that $u\in H_{M}^{s+\rho}(V)$, by the hypothesis, we can conclude that $a(x,D)\sigma(x,D)u\in H_{M}^{s+\rho}$. Namely, $\sigma(x,D)u$ belongs to $H_{M}^{s+\sigma}$ at $(\mathring{x},\mathring{\xi})$.

5. Proof of the main results.

In this section we shall prove the main results of this paper.

PROOF OF THEOREM A. Let u be a solution of

$$Pu+F(D^{\alpha}u)=0 \qquad \text{in } \Omega\subset\boldsymbol{R}^{n}\,, \tag{5.1}$$

where $u\in H_{M,\mathrm{loc}}^{s}(\Omega)$ for $s>(m-1)+|M|/2$ and belongs to $H_{M}^{s+\sigma}$ at $(\mathring{x},\mathring{\xi})$. To prove Theorem A, it suffices to show that u belongs to $H_{M}^{s+\sigma}$ on each closed interval γ on the null bicharacteristic strip through $(\mathring{x},\mathring{\xi})$. Now let γ be a closed interval on the null bicharacteristic strips and consider (5.1) in a neighborhood $U\subset\Omega$ of $\pi(\gamma)$, where $\pi:\Omega\times(\boldsymbol{R}_{n}\backslash 0)\rightarrow\Omega$ is the natural projection. Multiplying (5.1) by a real valued function $\chi\in C_{0}^{\infty}(\Omega)$ which is equal to 1 on U, we may assume that the symbol of P belongs to $S_{1,0}^{M;m}$ and that $\operatorname{supp}_{x} F$ is contained in a fixed compact set of Ω.

We let $r=s-(m-1)-|M|/2$ and use the abbreviation

$$F_{\alpha}'(D^{\alpha}u)=\frac{\partial F}{\partial(D^{\alpha}u)}(x\,;u,\cdots,D^{\alpha}u,\cdots)\,.$$

Since $D^{\alpha}u\in H_{M}^{s-(m-1)}$ $(=H_{M}^{r+|M|/2})$ on the support of $F(D^{\alpha}u)$ it follows from Remark 3.13 that

$$F(D^\alpha u)=\sum_\alpha \pi(F'_\alpha(D^\alpha u), D^\alpha u)-g\,, \tag{5.2}$$

where g belongs to $H_M^{2(s-m+1)-|M|/2}(=H_M^{s-(m-1)+r})$ and that

$$F'_\alpha(D^\alpha u)\in H_M^{r+|M|/2}\cap \mathcal{E}'(\Omega)\,. \tag{5.3}$$

By the definition of paraproducts the symbol of $\sum_\alpha \pi(F'_\alpha, D^\alpha\cdot)$ is

$$\sigma(x,\xi)=(2\pi)^{-n}\sum_\alpha\int F'_\alpha(D^\alpha u)\hat{\ }(\eta)\theta(\eta,\xi)e^{i<x,\eta>}d\eta\cdot\xi^\alpha\,,$$

where $\theta(x,\xi)=\sum_{k=6}^{\infty}\phi_{k-6}^M(\eta)\phi_k^M(\xi)\in C^\infty(\boldsymbol{R}_n\times\boldsymbol{R}_n)$ introduced in Section 3. Recalling that θ is supported by $\{(\eta,\xi)\in\boldsymbol{R}_n\times\boldsymbol{R}_n\,;\,[\eta]_M\leqq(1/16)[\xi]_M\}$ we see, in view of (5.3), that $\sigma\in\Sigma_r^{M,m-1}$.

Now, writing $\mathcal{L}=\sigma(x,D)\in\Sigma_r^{M,0}$, we need to consider the following linearized equation:

$$(P+\mathcal{L})u=g\in H_M^{s-(m-1)+r}\,.$$

PROPOSITION 5.1. *Let $P\in\mathrm{Op}\,S_{1,0}^{M;m}$ be a classical M-pseudodifferential operator with principal symbol p_m and let $\mathcal{L}$ be an element of $\mathrm{Op}\,\Sigma_r^{M,m-1}$. Let $s>(m-1)+|M|/2$, $0<\sigma\leqq r$. Consider a null bicharacteristic strip $I\ni t\to\gamma(t)\in\boldsymbol{R}^n\times(\boldsymbol{R}_n\backslash 0)$ for $\mathrm{Re}\,p_m$, where $I=\{t\in R\,;\,t_1\leqq t\leqq t_2\}$. Assume $\mathrm{Im}\,p_m\geqq 0$ on a neighborhood of $\gamma(I)$. Then if $u\in H_M^s$ satisfies $(P+\mathcal{L})u=g\in H_M^{s-(m-1)+\sigma}(\gamma(I))$ and $u\in H_M^{s+\sigma}(\gamma(t_2))$ it follows that $u\in H_M^{s+\sigma}(\gamma(I))$.*

PROOF. First of all, we set $s'=s-(m-1)$ and

$$v=(1+[D]_M)^{m-1}u\,,$$
$$P_1=P(1+[D]_M)^{1-m}\,,$$
$$\mathcal{L}_0=\mathcal{L}(1+[D]_M)^{1-m}\,.$$

Replacing s through $\mathcal{L}$ by s' through $\mathcal{L}_0$ we can assume $m=1$ without loss of generality.

Now set $\nu=\inf(\{\mu_j-1\,;\,\mu_j>1\}\cup\{1\})$ for $M=(\mu_1,\cdots,\mu_n)$ and let $\rho=s+\sigma$. We prove the proposition by increasing the regularity of u step by step. Each step shows an incriment of $\nu/2$ regularity on $\gamma(I)$. Thus by replacing ρ we may assume that u belongs to $H_M^{\rho-\nu/2}$ on $\gamma(I)$.

Choose a closed M-conic neighborhood Γ of $\gamma(I)$ such that $g\in H_M^\rho(\Gamma)$, $u\in H_M^{\rho-\nu/2}(\Gamma)$ and that $\mathrm{Im}\,p_1\geqq 0$ on Γ. We let $\mathcal{M}$ be a bounded subset of $S_{1,0}^{M;\rho}$ which consists only of real valued symbols $c\in S_{1,0}^{M;\rho-\nu/2}$ supported by Γ. With $c\in\mathcal{M}$ we put $C=c(x,D)$ and consider the form:

$$(5.4)\qquad (Cg, Cu)=(CPu, Cu)+C\mathcal{L}u, Cu)$$
$$=(PCu, Cu)+([C, P]u, Cu)+(C\mathcal{L}u, Cu)\ .$$

Let us write $p=A+iB$ with A and B self-adjoint, that is, $A=(P+P^{*})/2$ and $B=(P-P^{*})/2i$. Taking the imaginary part of (5.4) we obtain

$$\mathrm{Im}(Cg, Cu)=(BCu, Cu)+\mathrm{Re}([C, B]u, Cu)$$
$$+\mathrm{Im}([C, A]u, Cu)+\mathrm{Im}(C\mathcal{L}u, Cu)\ .$$

We can write $B=B'+B''$ where the principal symbol of B' is non-negative everywhere and the support of the symbol of B'' does not meet Γ. Applying the sharp Gårding inequality (Proposition 2.5) we have

$$(5.5)\qquad \mathrm{Re}(B'Cu, Cu)\geqq -C_{1}\|Cu\|_{L^{2}}^{2}.$$

Also, by noting that $B''C$ is of order $-\infty$, we have $|(B''Cu, Cu)|\leqq C_{1}'$, where C_{1}' is a constant depending on u but not on $c\in\mathcal{M}$. Hence

$$(5.6)\qquad (BCu, Cu)\geqq -C_{1}\|Cu\|_{L^{2}}^{2}-C_{1}'.$$

Next we note that the symbol of $C^{*}[C, B]$ is $ic\{b_{1}, c\}_{M}=(i/2)\{b_{1}, c^{2}\}_{M}$ apart from an error which belongs to a bounded set of $S_{1,0}^{M;2\rho-\nu}(\Omega)$, where b_{1} denotes the principal symbol of B and $\{\cdot, \cdot\}_{M}$ denotes the partial Poisson bracket defined in Section 1. Since $\{b_{1}, c^{2}\}_{M}$ is real valued the sum of $C^{*}[C, B]$ and its adjoint belongs to a bounded set of $S_{1,0}^{M;2\rho-\nu}$. This yields, with a constant C_{2}' depending on u,

$$(5.7)\qquad \mathrm{Re}([C, B]u, Cu)\geqq -C_{2}'.$$

In the same way we can show the estimate:

$$(5.8)\qquad \mathrm{Im}([C, A]u, Cu)\geqq\frac{1}{2}\mathrm{Re}(\{a_{1}, c^{2}\}_{M}(x, D)u, u)-C_{3}',$$

where a_{1} denotes the principal symbol of A.

Now we turn to the estimate for $\mathrm{Im}(C\mathcal{L}u, Cu)$. Let $\tilde{C}=(1+[D]_{M})^{-\rho}C$, which is in a bounded set of $S_{1,0}^{M;0}$. Then we have

$$(5.9)\qquad \|C\mathcal{L}u\|_{L^{2}}^{2}=\|\tilde{C}\mathcal{L}u\|_{H_{M}^{\rho}}^{2}$$
$$\leqq 2(\|\mathcal{L}\tilde{C}u\|_{H_{M}^{\rho}}^{2}+\|[\tilde{C}, \mathcal{L}]u\|_{H_{M}^{\rho}}^{2})\ .$$

The boundedness of paradifferential operator as a pseudo-differential operator with a symbol in $S_{1,1}^{M;0}$ gives

$$\|\mathcal{L}\tilde{C}u\|^2_{H^\rho_M} \leqq C_4\|\tilde{C}u\|^2_{H^\rho_M} = C_4\|Cu\|^2_{L^2}. \tag{5.10}$$

On the other hand, by Corollary 4.6, we can write

$$[\tilde{C}, \mathcal{L}] = \tau(x, D) + \omega(x, D),$$

where $\omega \in S^{M;-r}_{1,1}$ and the symbol $\tau(x, \xi)$ is supported by Γ such that

$$\tau \in S^{M;-1}_{1,1} \quad \text{if } r > 1,$$

$$\tau \in S^{M;-r+\varepsilon}_{1,1} \quad \text{for every } \varepsilon > 0 \text{ if } r = 1 \text{ or}$$

$$\tau = 0 \quad \text{if } 0 < r < 1.$$

Even when $r=1$ the symbol $\tau(x, \xi)$ belongs to $S^{M;-1/2}_{1,1}$. Thus, by Theorem 4.3, we have

$$\begin{aligned} \|[\tilde{C}, \mathcal{L}]u\|_{H^\rho_M} &\leqq \|\tau u\|_{H^\rho_M} + \|\omega u\|_{H^\rho_M} \\ &\leqq C_5(\|u\|_{H^{\rho-1/2}_M(\Gamma)} + \|u\|_{H^{\rho-r}_M}) \\ &\leqq C_5, \end{aligned} \tag{5.11}$$

where $\|u\|_{H^{\rho-1/2}_M(\Gamma)}$ denotes a fixed seminorm induced naturally from the definition of $u \in H^{\rho-1/2}_M(\Gamma)$. Combining (5.9) and (5.10), (5.11) we have

$$\|C\mathcal{L}u\|^2_{L^2} \leqq 2C_4\|Cu\|^2_{L^2} + C'_5.$$

Hence

$$\begin{aligned} \operatorname{Im}(C\mathcal{L}u, Cu) &\geqq -\frac{1}{2}(\|C\mathcal{L}u\|^2_{L^2} + \|Cu\|^2_{L^2}) \\ &\geqq -\left(C_4 + \frac{1}{2}\right)\|Cu\|^2_{L^2} - C'_5. \end{aligned} \tag{5.12}$$

Summing up (5.6)-(5.8) and (5.12) we obtain, with another constant C'_6,

$$\operatorname{Re}(e(x, D)u, u) \leqq \|Cg\|^2_{L^2} + C'_6, \tag{5.13}$$

where

$$e(x, \xi) = \{a_1, c^2\}_M(x, \xi) - (2C_1 + 2C_4 + 2)c^2(x, \xi). \tag{5.14}$$

Note that while C'_6, which comes from C'_1 to C'_5, may depend on $\mathcal{M}$, the constants C_1 and C_4 are completely independent of the choice of $\mathcal{M}$.

We may assume that the map γ is injective. Let V_0 be an open M-conic neighborhood of $\gamma(t_2)$ where $u \in H^\rho_M$ and choose a non-negative C^∞

function c which is M-quasi-homogeneous of degree ρ and supported by Γ such that $\{a_1, c^2\}_M = H^M_{\mathrm{Re}\, p_1}(c^2) \geqq 0$ in $\Gamma \setminus V_0$ with strict inequality on $\gamma(I)\setminus V_0$. Also choose C^∞ functions d_0 and d_1 M-quasi-homogeneous of degree 0 and 1 respectively so that $H^M_{\mathrm{Re}\, p_1}(d_0)=1$, $H^M_{\mathrm{Re}\, p_1}(d_1)=0$ and d_1 is different from 0 on the support of c. Now we let $\mathcal{M}$ consist of the functions

$$e_{\lambda,\varepsilon} = c\, e^{\lambda d_0}(1+\varepsilon^2 d_1^2)^{-\nu/2}, \qquad 0<\varepsilon\leqq 1\,,$$

where λ is a fixed real number satisfying $\lambda \geqq C_1+C_4+2$.

If c is replaced by $c_{\lambda,\varepsilon}$ the function e in (5.14) becomes

$$e_{\lambda,\varepsilon} = (\{a_1, c^2\}_M + (2\lambda - 2C_1 - 2C_4 - 2)c^2)e^{2\lambda d_0}(1+\varepsilon^2 d_1^2)^{-\nu}\,.$$

Since $e_{\lambda,\varepsilon}\geqq 0$ outside of V_0 with strict inequality on $\gamma(I)\setminus V_0$ we can choose a non-negative function $f\in S^{M;\rho}_{1,0}$ which is positive on $\gamma(I)$ and a real valued function $q\in S^{M;\rho}_{1,0}$ supported by V_0 so that

$$f^2 \leqq (\{a_1, c^2\}_M + (2\lambda - 2C_1 - 2C_4 - 2)c^2)e^{2\lambda d_0} + q^2\,. \tag{5.15}$$

Let $f_\varepsilon = f(1+\varepsilon^2 d_1^2)^{-\nu/2}$, $q_\varepsilon = q(1+\varepsilon^2 d_1^2)^{-\nu/2}$. An application of the sharp Gårding inequality (Proposition 2.5) to the difference of the two sides in (5.15), multiplied by $(1+\varepsilon^2 d_1^2)^{-\nu}$ leads to the estimate

$$\|f_\varepsilon(x, D)u\|^2_{L^2} \leqq \mathrm{Re}\,(e_{\lambda,\varepsilon}(x, D)u, u) + \|q_\varepsilon(x, D)u\|^2_{L^2} + C'\,.$$

Then it follows from (5.13) and the fact $g\in H^\rho_M(\Gamma)$ that $\|f_\varepsilon(x, D)u\|^2_{L^2}$ is bounded when $\varepsilon \downarrow 0$. This proves that $f(x, D)u$: the limit of $F(x, D)u$ in D', must belong to L^2. Hence $u\in H^\rho_M$ on $\gamma(I)$, and in view of the induction mentioned at the beginning this proves the proposition.

Now we shall complete the proof of Theorem A. Apply the preceding proposition for both $P+\mathcal{L}$ and $-(P+\mathcal{L})$. Then we can conclude that u belongs to $H^{s+\sigma}_M$ on bicharacteristic strip γ passing through $(\mathring{x}, \mathring{\xi})$ in both direction. This proves Theorem A.

To prove Theorem A$'$ we prepare the following lemma.

LEMMA 5.2. *Let $F(x, U_1, \cdots, U_N)\in C^\infty(\boldsymbol{R}^n\times \boldsymbol{C}^N)$ be holomorphic with respect to $U_1, \cdots, U_N$. Let $s>|M|/2$ and $\sigma\leqq s-|M|/2$. Assume $u_1(x), \cdots, u_N(x)$ are elements of $H^s_{M,\mathrm{loc}}(\Omega)$ which belong to $H^{s+\sigma}_M$ at $(\mathring{x}, \mathring{\xi})$. Then $F(x\,; u_1(x), \cdots, u_N(x))$ is an element of $H^s_{M,\mathrm{loc}}(\Omega)$ and belongs to $H^{s+\sigma}_M$ at $(\mathring{x}, \mathring{\xi})$.*

PROOF. Since the statement is local we may assume that $\operatorname{supp}_x F$ is contained in a fixed compact set in Ω and that $u_j\in H^s_M$ for $j=1, \cdots, N$. By Remark 3.13 we have $F(x\,; u_1(x), \cdots, u_N(x))\in H^s_M$ and

$$F(x\,;u_1,\cdots,u_n)=\sum_{j=1}^{N}\pi(\partial_{u_j}F,u_j)+G\,,$$

where $G\in H_M^{2s-|M|/2}$. From the definition of paraproducts and from the fact that $\partial_{u_j}F(x\,;u_1,\cdots,u_N)\in H_M^s$ for $j=1,\cdots,N$, which is also given by Remark 3.13, it follows that

$$\sigma_j(x,D)=\pi(\partial_{u_j}F,\cdot)\in \mathrm{Op}\,\Sigma_r^{M,0}\,,$$

where $r=s-|M|/2$.

Then, by Proposition 4.6, $\sigma_j u_j$ belongs to $H_M^{s+\sigma}$ at $(\mathring{x},\mathring{\xi})$ provided $u_j\in H_M^{s+\sigma}(\mathring{x},\mathring{\xi})$. Since this is true for every j and since $G\in H_M^{2s-|M|/2}\subset H_M^{s+\sigma}$ we have proved the lemma.

PROOF OF THEOREM A$'$. Arguments of F consist of x and $D^\alpha u$, $\langle\alpha,M\rangle\leqq m-h$. Clearly we may assume $m\geqq h$.

An application of Lemma 5.2 with no microlocal assumption gives

$$Pu=-F(D^\alpha u)\in H_{M,\mathrm{loc}}^{s-(m-h)}(\Omega)\,. \tag{5.16}$$

Now let γ be the bicharacteristic strip through $(\mathring{x},\mathring{\xi})$. Then it follows from (5.16) and from the assumption that $u\in H_M^{s+\sigma}(\mathring{x},\mathring{\xi})$ that

$$u\in H_{M,\mathrm{loc}}^s(\Omega)\cap H_M^{\min(s-(m-h)+(m-1),s+\sigma)}(\gamma)\,,$$

by Corollary 2.8. Again Lemma 5.2 implies

$$F(D^\alpha u)\in H_M^{\min(s+(h-1),s+\sigma)-(m-h)}(\gamma)\,.$$

Then, by Corollary 2.8, it follows that

$$u\in H_M^{\min(s+2(h-1),s+\sigma)}(\gamma)\,.$$

If $2(h-1)\geqq\sigma$ we have done. If not, we can continue this process and obtain

$$u\in H_M^{\min(s+k(h-1),s+\sigma)}(\gamma)\,,$$

after k-times use of Corollary 2.8 together with Lemma 5.2. Since k can be arbitrarily large we can conclude that

$$u\in H_M^{s+\sigma}(\gamma)\,,$$

which proves Theorem A$'$.

Before proving Theorem B we give a remark as follows: Let $(x,\xi)^\vee=(x,-\xi)$ and $\gamma^\vee=\{(x,\xi)\,;(x,-\xi)\in\gamma\}$. Then, for the complex conjugate $\bar{u}$ of u, $\bar{u}\in H_M^s((x,\xi))$ $(\in H_M^s(\gamma))$ if and only if $u\in H_M^s((x,\xi)^\vee)$ $(\in H_M^s(\gamma^\vee))$.

PROOF OF THEOREM B. First we suppose $h=1$. As in the proof of Theorem A we can assume that $P\in \mathrm{Op}\, S_{1,0}^{M;m}$ and that $\underset{x}{\mathrm{supp}}\, F\Subset\Omega$, $u\in H_M^s\cap\mathcal{E}'(\Omega)$.

Now let γ be a null bicharacteristic strip such that the anti-podal image of which consists only of non-characteristic points. Let us further make an inductive assumption that u belongs to $H_M^{s+\sigma-1}$ both on γ and $\gamma^{\vee}$. We shall then prove that u belongs to $H_M^{s+\sigma}$ on γ and $\gamma^{\vee}$.

An application of Lemma 5.2 yields

$$Pu=-F(D^\alpha u, D^\alpha \bar{u})\in H_M^{s+\sigma-m}(\gamma)\,.$$

Then, by Theorem 2.4, we have

$$u\in H_M^{s+\sigma}(\gamma^{\vee})\,. \tag{5.17}$$

Using Theorem 3.12 with Remark 3.13 we can write

$$Pu+\mathcal{L}u=G-\tilde{\mathcal{L}}\bar{u}\,,$$

where, with $r=s-(m-1)-|M|/2$

$$\mathcal{L}=\sum_{<\alpha,M>\leq m-1}\pi\Big(\frac{\partial F}{\partial(D^\alpha u)}, D^\alpha\cdot\Big)\in \mathrm{Op}\,\Sigma_r^{M,m-1}\,.$$

$$\tilde{\mathcal{L}}=\sum_{<\alpha,M>\leq m-1}\pi\Big(\frac{\partial F}{\partial(D^\alpha \bar{u})}, D^\alpha\cdot\Big)\in \mathrm{Op}\,\Sigma_r^{M,m-1}\,,$$

and

$$G\in H_M^{s-(m-1)+r}\subset H_M^{s-(m-1)+\sigma}\,.$$

Then from (5.17), by Proposition 4.8, it follows that $\tilde{\mathcal{L}}\bar{u}\in H_M^{s+\sigma-(m-1)}(\gamma)$, and therefore, $G-\tilde{\mathcal{L}}\bar{u}\in H_M^{s+\sigma-(m-1)}(\gamma)$. Now applying Proposition 5.1 we conclude that

$$u\in H_M^{s+\sigma}(\gamma)\,.$$

This together with (5.17) will prove Theorem B for $h=1$, because of the induction mentioned at the beginning.

The proof for $h>1$ is quite similar to that of Theorem A′, except that, here in Theorem B, we need to work on γ and $\gamma^{\vee}$ simultaneously. Apply Corollary 2.8 on γ and apply Theorem 2.4 on $\gamma^{\vee}$. Then we can prove Theorem B in the same way as in Theorem A′.

All the results of the present paper are now established.

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