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Asymptotic completeness for three-body Schrödinger equations with time-periodic potentials

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§1. Introduction.

The purpose of this paper is to study the asymptotic completeness for three-body Schrödinger equations with time-periodic potentials:

$$i\frac{\partial}{\partial t}\phi(t, x_1, x_2, x_3)=\left\{-\sum_{i=1}^{3}\frac{1}{2m_i}\Delta_{x_i}+\sum_{i<j}V_{ij}(t, x_i-x_j)\right\}\phi(t, x_1, x_2, x_3), \tag{1.1}$$

$$V_{ij}(t+\omega, x_i-x_j)=V_{ij}(t, x_i-x_j) \tag{1.2}$$

where $\phi(t, \cdot)\in L^2(\boldsymbol{R}^{3n})$, $x_i\in\boldsymbol{R}^n$ $(n>3)$, and ω is the period of the potentials. We assume that the potentials satisfy

ASSUMPTION (A). There exist constants p and q such that $1\leqq p<n/2<q\leqq\infty$ and for each $\alpha=(i, j)$, $t\rightarrow V_\alpha(t, \cdot)$ is an $(L^p(\boldsymbol{R}^n)\cap L^q(\boldsymbol{R}^n))$-valued absolutely continuous function. If $n=3$ and $q<2$, we assume further that they are continuously differentiable.

REMARK. If $n=3$, we suppose $q\geqq 2$ for the sake of simplicity.

Under Assumption (A), (1, 1) generates a family of evolution operators, and one can suitably define wave operators (see § 3 and § 4 for definitions). The central question in scattering theory is, then, to characterize the images of the wave operators, so called the problem of asymptotic completeness.

In this paper, we shall show that the completeness of wave operators holds for (1.1) if one of the following conditions is satisfied: (I) no two-body subsystems have bound states or "resonances" (§ 3); (II) two-body subsystems may have some bound states, but $n=3$ and the potentials decay exponentially (§ 4). In the second situation (II), the difficulty of embedded eigenvalues arises and we are forced to impose rather strong restrictions as they appear.

In the appendix, we shall give a new property of the eigenfunctions of two-body subsystems, which we need in § 4.

The asymptotic completeness for two-body Schrödinger equations with time-periodic perturbations has been studied by Schmidt [17], Yajima [19], Howland [9], Kitada-Yajima [13] and others ([2], [5]). Yajima and Howland employed a time-periodic analogue of the Howland stationary theory for time-dependent Hamiltonians ([8]), which we follow. On the other hand, the asymptotic completeness for three-body Schrödinger equations (with time-independent potentials) was proved first by Faddeev [6] by stationary method, which is subsequently generalized by Ginibre-Moulin [7] and Thomas [18], and by Enss by the time-dependent method ([3], [4]). We shall show that the synthesis of the Yajima-Howland method for the time-periodic system and the stationary theory for three-body system by Fadeev yields a proof of the asymptotic completeness for our system.

Notations. We shall use the following notations throughout the paper.

We denote the set of natural numbers by $\boldsymbol{N}$, integers by $\boldsymbol{Z}$, and reals by $\boldsymbol{R}$. We write $\boldsymbol{R}^m$ for the Euclidean m-space, and $\boldsymbol{T}$ for torus $\boldsymbol{R}/\omega\boldsymbol{Z}$.

For a Hilbert space $\mathscr{H}$, we write $L^p(\boldsymbol{T}, \mathscr{H})$ for the $\mathscr{H}$-valued L^p-space on $\boldsymbol{T}$, and $H^r(\boldsymbol{T}, \mathscr{H})$ for the H-valued Sobolev space of order r on $\boldsymbol{T}$. For a pair of Banach spaces X and Y, $B(X, Y)$ denotes the Banach space of all bounded operators from X to Y, and $B(X)=B(X, X)$. We write $B_c(X)$ for the space of all compact operators in $B(X)$. We write $L^2_\theta(\boldsymbol{R}^n)$ for the weighted L^2-space of order θ on $\boldsymbol{R}^n$: $\{\phi \in L^2_{\text{loc.}}(\boldsymbol{R}^n) : \phi(x) \times(1+|x|)^\theta \in L^2(\boldsymbol{R}^n)\}$.

For a function $F=F(x)$, we often denote the operator of multiplication by $F(x)$ with the same symbol F. We write $\langle x\rangle=(1+|x|^2)^{1/2}$ for $x\in\boldsymbol{R}^m$.

$\mathscr{F}_{x\to\xi}$ denote the Fourier transform from $\boldsymbol{R}^n_x$-space to $\boldsymbol{R}^n_\xi$-space:

$$(\mathscr{F}_{x\to\xi}\phi)(\xi)=(2\pi)^{-n/2}\int_{R^n} \exp(-ix\xi)\phi(x)dx\,. \tag{1.3}$$

$\mathscr{F}_{t\to\mu}\phi$ denotes the Fourier series expansion of ϕ on $\boldsymbol{T}\cong[0,\omega)$:

$$(\mathscr{F}_{t\to\mu}\phi)_\mu=\omega^{-1/2}\int_0^\omega \exp(-i2\pi\mu t/\omega)\phi(t)dt\,. \tag{1.4}$$

For an operator A, we denote the closure of A by $[A]$.

§ 2. Preliminaries.

2.1. Definition of Hamiltonians.

We assume that each particle is moving in $\boldsymbol{R}^n$ $(n \geqq 3)$, and its mass is m_i $(i=1,2,3)$. If they are interacting via two-body time-dependent potentials $V_{ij}(t, x_i - x_j)$, their motion is described by the Hamiltonian on $L^2(\boldsymbol{R}^{3n})$ given by

$$\tilde{H}(t) = -\sum_{i=1}^{3} \frac{1}{2m_i} \Delta_{x_i} + \sum_{\alpha=(i,j)} V_\alpha(t, x_i - x_j) \tag{2.1}$$

where $\boldsymbol{R}^{3n}$ is represented as $\{(x_1, x_2, x_3) : x_i \in \boldsymbol{R}^n,\ i=1,2,3\}$ and α runs over all pairs (i, j), $1 \leqq i < j \leqq 3$. We suppose time-periodicity of the potentials $\{V_\alpha\}$:

$$V_\alpha(t, x_i - x_j) = V_\alpha(t+\omega, x_i - x_j)\,. \tag{2.2}$$

Using the center of mass coordinate and the Jacobi coordinates for the pair $\alpha = (i, j)$:

$$\begin{cases} X = \sum_{i=1}^{3} m_i \cdot x_i \Big/ \sum_{i=1}^{3} m_i \\ x_\alpha = x_i - x_j \\ y_\alpha = x_k - (m_i x_i + m_j x_j)/(m_i + m_j)\,, \end{cases} \tag{2.3}$$

where k is determined by $\{i, j, k\} = \{1, 2, 3\}$, we represent $L^2(\boldsymbol{R}^{3n}) = L^2(\boldsymbol{R}^n\,;\, dX) \otimes L^2(\boldsymbol{R}^{2n}\,;\, dx_\alpha dy_\alpha)$ and

$$\begin{aligned} \tilde{H}(t) &= -\frac{1}{2M}\Delta_X - \frac{1}{2m_\alpha}\Delta_{x_\alpha} - \frac{1}{2n_\alpha}\Delta_{y_\alpha} + \sum_\beta V_\beta(t, x_\alpha) \\ &= \left(-\frac{1}{2M}\Delta_X \otimes 1\right) + 1 \otimes H(t) \end{aligned} \tag{2.4}$$

where $M = \sum_{i=1}^{3} m_i$, $m_\alpha^{-1} = m_i^{-1} + m_j^{-1}$ and $n_\alpha^{-1} = (m_i + m_j)^{-1} - m_k^{-1}$. Then separating out the trivial center of mass motion i.e. the motion due to $-(1/2M)\Delta_X$, we consider the equation

$$\begin{aligned} i\frac{\partial}{\partial t}\phi(t, x_\alpha, y_\alpha) &= H(t)\phi(t) \\ &= \left\{-\frac{1}{2m_\alpha}\Delta_{x_\alpha} - \frac{1}{2n_\alpha}\Delta_{y_\alpha} + \sum_\beta V_\beta(t, x_\beta)\right\}\phi(t, x_\alpha, y_\alpha) \end{aligned} \tag{2.5}$$

on $L^2(\boldsymbol{R}^{2n})=\mathcal{H}$ with $\boldsymbol{R}^{2n}=\{(x_\alpha, y_\alpha): x_\alpha, y_\alpha \in \boldsymbol{R}^n\}$. The free Hamiltonian H_0 is given as

$$H_0=-\frac{1}{2m_\alpha}\Delta_{x_\alpha}-\frac{1}{2n_\alpha}\Delta_{y_\alpha}. \tag{2.6}$$

We remark that these definitions are independent of the choice of the pair $\alpha=(i, j)$.

The Hamiltonian $h^\alpha(t)$ and the free Hamiltonian h_0^α for a two-body subsystems relative to α are defined by

$$h^\alpha(t)=-\frac{1}{2m_\alpha}\Delta_{x_\alpha}+V_\alpha(t, x_\alpha)=h_0^\alpha+V_\alpha(t, x_\alpha) \tag{2.7}$$

and the kinetic energy for the third particle is

$$l_0^\alpha=-\frac{1}{2n_\alpha}\Delta_{y_\alpha}. \tag{2.8}$$

They will be regarded as operators on $L^2(\boldsymbol{R}^n_{x_\alpha})=\mathfrak{h}^\alpha$ or $L^2(\boldsymbol{R}^n_{y_\alpha})$ as well as operators on $L^2(\boldsymbol{R}^{2n})=\mathcal{H}$.

For the details of these decompositions, we refer to Ginibre-Moulin [7] or Reed-Simon [16], Vol. III.

2.2. Time evolutions.

Under Assumption (A), $H(t)$ and $\{h^\alpha(t)\}$ generate unitary evolution operators:

PROPOSITION 2.1. *Suppose that* (A) *is satisfied. Then there exist sets of unitary operators* $\{U(t, s): t, s\in \boldsymbol{R}\}$ *on* $\mathcal{H}$ *and* $\{u^\alpha(t, s): t, s\in \boldsymbol{R}\}$ *on* $\mathfrak{h}^\alpha$ *such that*

(2.9) $(t, s) \to U(t, s)$ *is strongly continuous.*

(2.9)′ $(t, s) \to u^\alpha(t, s)$ *are strongly continuous.*

$$U(t, s)=U(t, r)U(r, s) \tag{2.10}$$

$$u^\alpha(t, s)=u^\alpha(t, r)u^\alpha(r, s) \tag*{(2.10)′}$$

$$U(t+\omega, s+\omega)=U(t, s) \tag{2.11}$$

$$u^\alpha(t+\omega, s+\omega)=u^\alpha(t, s) \tag*{(2.11)′}$$

$$U(t, s)H^2(\boldsymbol{R}^{2n})=H^2(\boldsymbol{R}^{2n}) \tag{2.12}$$

(2.12)′ $$u^{\alpha}(t, s)H^{2}(\boldsymbol{R}^{n})=H^{2}(\boldsymbol{R}^{n})$$

(2.13) $$\frac{d}{dt}U(t, s)\psi=-iH(t)U(t, s)\psi$$

$$\frac{d}{ds}U(t, s)\psi=iU(t, s)H(s)\psi$$

(2.13)′ $$\frac{d}{dt}u^{\alpha}(t, s)\phi=-ih^{\alpha}(t)u^{\alpha}(t, s)\phi$$

$$\frac{d}{ds}u^{\alpha}(t, s)\phi=iu^{\alpha}(t, s)h^{\alpha}(s)\phi$$

where $t, s, r\in\boldsymbol{R}$, $\alpha=(i, j)$, $\psi\in H^{2}(\boldsymbol{R}^{2n})$, $\phi\in H^{2}(\boldsymbol{R}^{n})$ *and the derivatives are taken in the strong sense in* $L^{2}(\boldsymbol{R}^{2n})$ *or* $L^{2}(\boldsymbol{R}^{n})$.

This is a simple consequence of a Kato's theorem [10] on the generation of evolution operators.

2.3. Howland-Yajima method.

Following Yajima [19] and Howland [8], [9], we introduce Hilbert spaces

(2.14)
$$\mathcal{K}=L^{2}(\boldsymbol{T}, \mathcal{H})\cong L^{2}(\boldsymbol{T})\otimes\mathcal{H}$$
$$k^{\alpha}=L^{2}(\boldsymbol{T}, \mathcal{H})\cong L^{2}(\boldsymbol{T})\otimes h^{\alpha}\,.$$

Define propagators $\{\mathcal{U}(\sigma):\sigma\in\boldsymbol{R}\}$ and $\{\mathcal{U}_{0}(\sigma)\}$ on $\mathcal{K}$, and $\{u^{\alpha}(\sigma)\}$ and $\{u_{0}^{\alpha}(\sigma)\}$ on K^{α} by

(2.15)
$$(\mathcal{U}(\sigma)\Psi)(t)=U(t, t-\sigma)\Psi(t-\sigma)$$
$$(\mathcal{U}_{0}(\sigma)\Psi)(t)=\exp(-i\sigma H_{0})\Psi(t-\sigma)$$

(2.15)′
$$(u^{\alpha}(\sigma)\Phi^{\alpha})(t)=u^{\alpha}(t, t-\sigma)\Phi^{\alpha}(t-\sigma)$$
$$(u_{0}^{\alpha}(\sigma)\Phi^{\alpha})(t)=\exp(-i\sigma h_{0}^{\alpha})\Phi^{\alpha}(t-\sigma)$$

where $\Psi=\{\Psi(t):t\in\boldsymbol{T},\ \Psi(t)\in\mathcal{H}$ a. e.$\}\in\mathcal{K}$ and $\Phi^{\alpha}=\{\Phi^{\alpha}(t):t\in\boldsymbol{T},\ \Phi^{\alpha}(t)\in h^{\alpha}$ a. e.$\}\in k^{\alpha}$. It is easy to see that these are one-parameter unitary groups on $\mathcal{K}$ or k^{α}. Hence there exist self-adjoint operators K, K_{0}, k^{α} and k_{0}^{α} such that $\mathcal{U}(\sigma)=\exp(-i\sigma K)$, $\mathcal{U}_{0}(\sigma)=\exp(-i\sigma K_{0})$, $u^{\alpha}(\sigma)=\exp(-i\sigma k^{\alpha})$ and $u_{0}^{\alpha}(\sigma)=\exp(-\sigma k_{0}^{\alpha})$. We let $K^{\alpha}=k^{\alpha}+l_{0}^{\alpha}=k^{\alpha}\otimes 1+1\otimes l_{0}^{\alpha}$ on $\mathcal{K}\cong h^{\alpha}\otimes L^{2}(\boldsymbol{R}_{y_{\alpha}}^{n})$, then K^{α} is also a self-adjoint operator and it satisfies

(2.16) $$(\exp(-i\sigma K^{\alpha})\Psi)(t)=(u^{\alpha}(t,t-\sigma)\otimes\exp(-i\sigma l_0^{\alpha}))\Psi(t-\sigma)$$

for $\Psi\in\mathcal{K}$. We denote their resolvents by

(2.17) $$\begin{cases} G(\zeta)=(\zeta-K)^{-1}, & G_0(\zeta)=(\zeta-K_0)^{-1} \\ g^{\alpha}(\zeta)=(\zeta-k^{\alpha})^{-1}, & g_0^{\alpha}(\zeta)=(\zeta-k_0^{\alpha})^{-1} \\ G_{\alpha}(\zeta)=(\zeta-K^{\alpha})^{-1} & \end{cases}$$

with $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$.

We denote by A_{α} (B_{α} resp.) the multiplication operator by $a_{\alpha}(t,x_{\alpha})=|V_{\alpha}(t,x_{\alpha})|^{1/2}\cdot\operatorname{sgn}V_{\alpha}(t,x_{\alpha})$ ($b_{\alpha}(t,x_{\alpha})=|V_{\alpha}(t,x_{\alpha})|^{1/2}$ resp.) on $\mathcal{K}$ or k^{α}.

For two-body subsystems, it is known that the boundary value of the (free) resolvent exists in the following sense:

PROPOSITION 2.2 ([15]). *For each* $\alpha, q_{\alpha}(\zeta)=[A_{\alpha}g_0^{\alpha}(\zeta)B_{\alpha}]$ *is a* $B_c(k^{\alpha})$-*-valued analytic function on* $\boldsymbol{C}^{\pm}$, *and it has continuous boundary value on* $\boldsymbol{R}\pm i0$. *Moreover,* $q_{\alpha}(\zeta)$ *is uniformly bounded, uniformly Hölder-continuous in* ζ *on the close upper (lower resp.) half plane, and* $\|q_{\alpha}(\zeta)\|$ *tends to zero if* $|\mathrm{Im}\,\zeta|\to\infty$.

For the proof, see Yajima [19] or Howland [9].

§ 3. Single-channel case.

3.1. Asymptotic completeness.

In this section, we assume (A) and

ASSUMPTION (B). For each $\alpha, q_{\alpha}(\lambda\pm i0)$ has no $(+1)$-eigenfunctions for any λ in $\boldsymbol{R}$.

Assumption (B) implies that k^{α} has no eigenvalues, and hence $u^{\alpha}(t+\omega,t)$ has no eigenvalues, for any α and any t. This fact can be verified by the standard method of scattering (see Kuroda [15], for example). Thus, here we consider the scattering with single channel.

THEOREM 1. *Under Assumptions* (A) *and* (B), *wave operators defined by*

(3.1) $$W_{\pm}(s)=\operatorname*{s-lim}_{t\to\pm\infty}U(s,t)\exp(-i(t-s)H_0)$$

exist and are complete for each $s\in\boldsymbol{R}$:

$$\text{Ran } W_{\pm}(s) = \mathscr{H}^{ac}(U(s+\omega, s)) . \tag{3.2}$$

The existence of $W_{\pm}(s)$ can be proved by Cook's method (see Reed-Simon [16], Vol. III) and we omit the proof. The completeness of $W_{\pm}(s)$, the equation (3.2), follows from the next proposition.

PROPOSITION 3.1. *Under Assumptions* (A) *and* (B), *wave operators defined by*

$$\mathscr{W}_{\pm} = \underset{\sigma \to \pm\infty}{\text{s-lim}} \exp(i\sigma K) \exp(-i\sigma K_0) \tag{3.3}$$

exist and are complete:

$$\text{Ran } \mathscr{W}_{\pm} = \mathscr{K}^{ac}(K) . \tag{3.4}$$

PROOF OF THEOREM 1. We follow the argument of [19]. Let $\mathscr{V}$ and $\mathscr{V}_0$ be operators on $\mathscr{K}$ defined by

$$(\mathscr{V}\Psi)(t) = U(t, s)\Psi(t) \qquad \Psi \in \mathscr{K},\ t \in [0, \omega) , \tag{3.5}$$

$$(\mathscr{V}_0\Psi)(t) = \exp(-i(t-s)H_0)\Psi(t) , \tag{3.6}$$

where we identified $\boldsymbol{T}$ with $[0, \omega)$. Then it is easily shown that $\exp(-i\omega K) = \mathscr{V}(1 \otimes U(s+\omega, s))\mathscr{V}^{-1}$. This implies

$$\mathscr{K}^{ac}(K) = \mathscr{V}(L^2(\boldsymbol{T}) \otimes \mathscr{H}^{ac}(U(s+\omega, s))) . \tag{3.7}$$

On the other hand, we see

$$\begin{aligned} \text{Ran}(\mathscr{W}_{\pm}) &= \text{Ran}(\mathscr{V}(1 \otimes W_{\pm}(s))\mathscr{V}_0^{-1}) \\ &= \mathscr{V}(L^2(\boldsymbol{T}) \otimes \text{Ran}(W_{\pm}(s))) . \end{aligned} \tag{3.8}$$

Theorem 1 follows from (3.7) and (3.8).

The existence of $\mathscr{W}_{\pm}$ follows from that of $W_{\pm}(s)$, and in the rest of this section we prove (3.4).

3.2. Faddeev matrix.

By analogy with Ginibre-Moulin [7], we consider Faddeev matrix $F_{\alpha\beta}(\zeta)$ defined by

$$F_{\alpha\beta}(\zeta) = A_{\alpha} G_{\alpha}(\zeta) B_{\beta}(\zeta) , \qquad (\alpha \neq \beta) \tag{3.9}$$

for $\zeta \in \boldsymbol{C} \backslash \boldsymbol{R}$ on $\mathscr{K}$.

PROPOSITION 3.2. *As a $B(\mathcal{K})$-valued function of ζ on $\boldsymbol{C}\backslash\boldsymbol{R}$, $F_{\alpha\beta}(\zeta)$ $(\alpha\neq\beta)$ satisfies the following properties*:

(i) *$F_{\alpha\beta}(\zeta)$ is uniformly bounded.*
(ii) *$F_{\alpha\beta}(\zeta)$ is analytic.*
(iii) *$F_{\alpha\beta}(\zeta)$ is uniformly Hölder-continuous.*
(iv) *$\|F_{\alpha\beta}(\zeta)\|$ tends to zero if $|\mathrm{Im}\,\zeta|\to\infty$.*
(v) *$F_{\alpha\beta}(\zeta)\in B_c(\mathcal{K})$ for each $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$.*

The proof will be given in the next subsection. Proposition 3.2 implies that $F_{\alpha\beta}(\zeta)$ has a continuous boundary value on $\boldsymbol{R}\pm i0$ as a compact-operator valued function. Let $\tilde{\mathcal{K}}$ be the direct sum of three copies of $\mathcal{K}$:

$$\tilde{\mathcal{K}}=\bigoplus_{\alpha}\mathcal{K}^{\alpha}\,,\qquad \mathcal{K}^{\alpha}=\mathcal{K}\,. \tag{3.10}$$

We define an operator $F(\zeta)$ on $\tilde{\mathcal{K}}$ by

$$(F(\zeta)\phi)_{\alpha}=\sum_{\alpha\neq\beta}F_{\alpha\beta}(\zeta)\phi_{\beta}\,,\qquad \phi=(\phi_{\alpha})\in\tilde{\mathcal{K}}\,. \tag{3.11}$$

LEMMA 3.1. (i) *$F(\zeta)$ is a $B(\tilde{\mathcal{K}})$-valued function of $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$, and is uniformly bounded, analytic and uniformly Hölder continuous in ζ.*

(ii) *$(1-F(\zeta))^{-1}$ exists for $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$.*

(iii) *There exists a closed null set $\mathcal{E}\subset\boldsymbol{R}$, such that $(1-F(\zeta))^{-1}$ can be extended continuously on $(\boldsymbol{R}\pm i0)\backslash\mathcal{E}$ as a $B(\tilde{\mathcal{K}})$-valued function of ζ.*

Lemma 3.1 is a direct consequence of Proposition 3.2 and a Kuroda's theorem (Theorem 3.10 of Kuroda [14]) on the stationary scattering theory.

LEMMA 3.2. *Under Assumption* (A), *we have*

$$G(\zeta)=G_0(\zeta)+\sum_{\alpha,\beta}[B_{\alpha}G_0(\bar{\zeta})]^*(1-F(\zeta))^{-1}_{\alpha\beta}[A_{\beta}G_{\beta}(\zeta)] \tag{3.12}$$

for $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$.

PROOF. By the argument of Lemma 3.3 of Yajima [19], we have the "second resolvent equations":

$$G(\zeta)=G_0(\zeta)+\sum_{\alpha}[B_{\alpha}G_0(\bar{\zeta})]^*[A_{\alpha}G(\zeta)]\,, \tag{3.13}$$

$$G(\zeta)=G_{\alpha}(\zeta)+\sum_{\alpha\neq\beta}[B_{\beta}G_{\alpha}(\bar{\zeta})]^*[A_{\beta}G(\zeta)]\,. \tag{3.14}$$

We substitute (3.13) to (3.14), and by iterations we have (3.12) if $|\mathrm{Im}\,\zeta|$ is sufficiently large. By analyticity, (3.12) holds for any $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$.

3.3. Proof of Propositions 3.1 and 3.2.

At first, by the "second resolvent equations" (see (3.13) and (3.14)), we obtain

$$F_{\alpha\beta}(\zeta)=(1+[A_\alpha G_\alpha(\zeta)B_\alpha])(A_\alpha G_0(\zeta)B_\beta)\,. \tag{3.15}$$

LEMMA 3.3. *$[A_\alpha G_\alpha(\zeta)B_\alpha]$ and $[A_\alpha G_0(\zeta)B_\alpha]$ satisfy the assertions* (i), (ii) *and* (iii) *of Proposition* 3.2.

PROOF. We see

$$\mathscr{F}_{y_\alpha\to\eta}G_\alpha(\zeta)\mathscr{F}^{-1}_{y_\alpha\to\eta}=g^\alpha\Big(\zeta-\frac{1}{2n_\alpha}\eta^2\Big), \tag{3.16}$$

and

$$A_\alpha g^\alpha(\zeta)B_\alpha=A_\alpha g_0^\alpha(\zeta)B_\alpha+[A_\alpha g_0^\alpha(\zeta)B_\alpha]\cdot(1-q_\alpha(\zeta))^{-1}\cdot[A_\alpha g_0^\alpha(\zeta)B_\alpha]\,. \tag{3.17}$$

Then Proposition 2.2 and Assumption (B) implies the lemma.

Thus it is sufficient to prove the assertions for $\tilde F_{\alpha\beta}(\zeta)=A_\alpha G_0(\zeta)B_\beta$. We may suppose $\alpha=(1,2)$, $\beta=(2,3)$ without loss of generality.

LEMMA 3.4. *$\tilde F_{\alpha\beta}(\zeta)$ satisfies the assertions* (i)-(iv) *of Proposition* 3.2 *replacing $F_{\alpha\beta}$ by $\tilde F_{\alpha\beta}$.*

PROOF. We consider the system without separating the center of mass motion. Let $\mathcal{K}'$ be $L^2(\boldsymbol{T},L^2(\boldsymbol{R}^{3n}))$ and K_0' be the operator defined by

$$(\exp(-i\sigma K_0')f)(t)=\exp(-i\sigma\tilde H_0)f(t-\sigma) \tag{3.18}$$

where $f\in\mathcal{K}'$ and $\tilde H_0=-\sum_{i=1}^3(1/2m_i)\Delta_{x_i}$. $G_0'(\zeta)$ denotes its resolvent $(\zeta-K_0')^{-1}$. Clearly, the assertions (i)-(iv) for $\tilde F'_{\alpha\beta}(\zeta)=A_\alpha G_0'(\zeta)B_\beta$ are equivalent to those for $\tilde F_{\alpha\beta}(\zeta)$.

Mimicking the proof of Proposition 5.1 of Ginibre-Moulin [7], we see

$$\begin{aligned}&\|(A_{12}\exp(-itK_0')B_{23}f)(s)\|_{L^2(R^{3n})}\\&\quad\leqq C\|a_{12}(s)\|_{L^{2r}(R^n)}\|b_{23}(s-t)\|_{L^{2r}(R^n)}|t|^{-n/2r}\|f(s-t)\|_{L^2(R^{3n})}\end{aligned} \tag{3.19}$$

where $f\in\mathcal{K}'$ and $r=p$ or q. Using Laplace transform, Young's inequality and Hölder's inequality, we obtain

$$(3.20)\qquad \|A_{12}G_0'(\zeta)B_{23}\|_{B(\mathcal{K}')}$$
$$\leqq C\{\|e^{-|\mathrm{Im}\zeta|\cdot|t|}\cdot|t|\|_{L_t^2(0,2)}\|a_{12}\|_{L^2(T,L^{2q})}\cdot\|b_{23}\|_{L^{\alpha/(\alpha-1)}(T,L^{2q})}$$
$$+e^{-|\mathrm{Im}\zeta|}\cdot\|a_{12}\|_{L^2(T,L^{2p})}\cdot\|b_{23}\|_{L^2(T,L^{2p})}\}$$

with $1\leqq\alpha\leqq 2q/n$ (see the proof for Lemma 3.1 of [19]). (i) and (iv) follow immediately from (3.20).

From (3.19), we also obtain

$$(3.21)\qquad \|A_{12}G_0'(\zeta)B_{23}-A_{12}G_0'(\xi)B_{23}\|_{B(\mathcal{K}')}$$
$$\leqq C|\zeta-\xi|^{\beta}\cdot\{\|a_{12}\|_{L^2(T,L^{2q})}\cdot\|b_{23}\|_{L^{\alpha/(\alpha-1)}(T,L^{2q})}$$
$$+\|a_{12}\|_{L^2(T,L^{2p})}\cdot\|b_{23}\|_{L^2(T,L^{2p})}\}$$

for any α and β satisfying $1<\alpha<2q/n$, $0<\beta<-1+n/2p$. This implies the assertion (iii).

(ii) follows since $\tilde{F}'_{\alpha\beta}(\zeta)$ is weakly analytic on $D(A_{12})\times D(B_{23})$ and is uniformly bounded.

LEMMA 3.5. *For each* $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$, $\tilde{F}_{\alpha\beta}(\zeta)\in B_c(\mathcal{K})$ $(\alpha\neq\beta)$.

PROOF. By virtue of (3.20), it is sufficient to prove it for a_{12}, $b_{23}\in C^\infty(\boldsymbol{T}, C_0^\infty(\boldsymbol{R}^n))$ which is dense in $L^2(\boldsymbol{T}, L^{2p}\cap L^{2q})$ or $L^2(\boldsymbol{T}, L^{2p})\cap L^{\alpha/(\alpha-1)}(\boldsymbol{T}, L^{2q})$. We suppose $a_{12}, b_{23}\in C^\infty(\boldsymbol{T}, C_0^\infty(\boldsymbol{R}^n))$.

Let ϕ be a C_0^∞-function such that $\phi(x)=1$ ($=0$ resp.) if $|x|\leqq 1$ ($|x|\geqq 2$ resp.), and set $\phi_R(x)=\phi(x/R)$. Take $R_0>0$ large enough so that for any $R>R_0$,

$$(3.22)\qquad \begin{cases}\{1-\phi_R(x)\}a_\alpha(x)=0\,,\\ \{1-\phi_R(x)\}b_\beta(x)=0\,.\end{cases}$$

Then we see for $R>R_0$,

$$(3.23)\qquad A_\alpha G_0(\zeta)B_\beta=A_\alpha\phi_{R_0}(x_\alpha)\phi_R(x_\beta)G_0(\zeta)\phi_R(x_\alpha)\phi_{R_0}(x_\beta)B_\beta$$
$$+A_\alpha\phi_{R_0}(x_\alpha)[G_0(\zeta),\phi_R(x_\beta)]\phi_R(x_\alpha)\phi_{R_0}(x_\beta)B_\beta$$
$$+A_\alpha\phi_{R_0}(x_\alpha)[\phi_R(x_\alpha),G_0(\zeta)]\phi_R(x_\beta)\phi_{R_0}(x_\beta)B_\beta\,.$$

The first term in the right hand side is compact because $\phi_{R_0}(x_\alpha)\cdot\phi_R(x_\beta)$ and $\phi_R(x_\alpha)\cdot\phi_{R_0}(x_\beta)$ have compact supports in $\boldsymbol{R}^{2n}$ (Lemma 3.1 of [19]).

Thus it is sufficient to prove

$$(3.24)\qquad \|[G_0(\zeta),\phi_R(x_\alpha)]\phi_{R_0}(x_\alpha)\|_{B(\mathcal{K})}\longrightarrow 0\qquad(R\to\infty)$$

for each $\zeta\in \boldsymbol{C}\backslash\boldsymbol{R}$. By simple computations,

$$[G_0(\zeta), \phi_R(x_\alpha)]=G_0(\zeta)\left[-\frac{1}{2m_\alpha}\Delta_{x_\alpha}, \phi_R(x_\alpha)\right]G_0(\zeta) \tag{3.25}$$

$$=\left(-\frac{1}{2m_\alpha}\right)G_0(\zeta)\left\{\frac{(\Delta\phi)(x_\alpha/R)}{R^2}+2\frac{(\nabla\phi)(x_\alpha/R)}{R}\nabla_{x_\alpha}\right\}\cdot G_0(\zeta).$$

It is clear that $R^{-2}\|G_0(\zeta)\cdot(\Delta\phi)(x_\alpha/R)\cdot G_0(\zeta)\|\to 0$ as $R\to\infty$. We prove

$$R^{-1}\cdot\|(\nabla\phi)(x_\alpha/R)\cdot\nabla_{x_\alpha}\cdot G_0(\zeta)\cdot\phi_{R_0}(x_\alpha)\|\longrightarrow 0 \qquad (R\to\infty). \tag{3.26}$$

LEMMA 3.6. *Let $g_0(\zeta)=(\zeta-h_0)^{-1}$ with $h_0=-\Delta$ on $L^2(\boldsymbol{R}^n)$. Then for any $\delta>1/2$, $\langle x\rangle^{-\delta}\nabla_x g_0(\zeta)\langle x\rangle^{-\delta}$ is uniformly bounded in $\zeta\in\boldsymbol{C}\backslash\boldsymbol{R}$.*

Granting Lemma 3.6 for the moment, we proceed with the proof of (3.26). Performing Fourier transforms in t and y_α, we see

$$\mathscr{F}_{y_\alpha\to\eta}\mathscr{F}_{t\to\mu}\{R^{-1}\cdot(\nabla\phi)(x_\alpha/R)\nabla_{x_\alpha}G_0(\zeta)\phi_{R_0}(x_\alpha)\}\cdot\mathscr{F}_{t\to\mu}^{-1}\mathscr{F}_{y_\alpha\to\eta}^{-1} \tag{3.27}$$

$$=R^{-1}(\nabla\phi)(x_\alpha/R)\nabla_{x_\alpha}g_0^\alpha\left(\zeta-\frac{2\pi}{\omega}\mu-\frac{1}{2n_\alpha}\eta^2\right)\phi_{R_0}(x_\alpha).$$

Let δ be a constant such that $1/2<\delta<1$, then obviously $|\phi_{R_0}(x_\alpha)|\leqq C\langle x_\alpha\rangle^{-\delta}$ and $|R^{-\delta}(\nabla\phi)(x_\alpha/R)|\leqq C\langle x_\alpha\rangle^{-\delta}$ with some R-independent constant $C>0$. Hence, by Lemma 3.6 we obtain

$$R^{-1}\left\|(\nabla\phi)(x_\alpha/R)\nabla_{x_\alpha}g_0^\alpha\left(\zeta-\frac{2\pi}{\omega}\mu-\frac{1}{2n_\alpha}\eta^2\right)\phi_{R_0}(x_\alpha)\right\|\leqq CR^{-(1-\delta)} \tag{3.28}$$

with some C independent of ζ, μ and η. This proves (3.26).

PROOF OF LEMMA 3.6. In the case $n=1$, $\nabla_x g_0(\zeta)$ is represented by the integral kernel $((x-x')/2|x-x'|)\exp(ik|x-x'|)$ where $k=\zeta^{1/2}$, $\operatorname{Im}k\geqq 0$. Thus the Hilbert-Schmidt norm of $\langle x\rangle^{-\delta}\nabla_x g_0(\zeta)\langle x\rangle^{-\delta}$ is bounded by $\|\langle x\rangle^{-\delta}\|^2/2$ which is obviously uniformly bounded in ζ.

In the case $n>1$, we write $x\in\boldsymbol{R}^n$ as $x=(x_1, x')\in\boldsymbol{R}^1\times\boldsymbol{R}^{(n-1)}$. Performing a Fourier transform in x', we see

$$\mathscr{F}_{x'\to\xi'}\frac{\partial}{\partial x_1}g_0(\zeta)\mathscr{F}_{x'\to\xi'}^{-1}=\frac{\partial}{\partial x_1}g_0^{(1)}(\zeta-\xi'^2) \tag{3.29}$$

where $g_0^{(1)}(\zeta)=(\zeta+\partial^2/\partial x_1^2)^{-1}$. Hence $\langle x_1\rangle^{-\delta}((\partial/\partial x_1)g_0(\zeta))\langle x_1\rangle^{-\delta}$ is uniformly bounded in ζ and the lemma follows since $\langle x\rangle^{-\delta}\leqq\langle x_1\rangle^{-\delta}$.

(3.15) and Lemmata 3.3-3.5 imply Proposition 3.2.

PROOF OF PROPOSITION 3.1. It is easily seen from Lemmata 3.3-3.5 that $A_\alpha G_0(\zeta)B_\beta$ has a continuous boundary value on $\boldsymbol{R}\pm i0$. On the other hand, Lemma 3.2, Proposition 3.2 and Lemmata 3.3-3.5 follow that $A_\alpha G(\zeta)B_\beta$ has a continuous boundary value on $(\boldsymbol{R}\pm i0)\backslash\mathcal{E}$. Then Proposition 3.1 follows from a abstract theorem of scattering (Theorem 5.2 of [14]).

§ 4. Multi-channel case.

4.1. Asymptotic completeness.

In this section, we suppose $n=3$, and instead of (A) we suppose

ASSUMPTION (C). There exists a constant $\delta>0$ such that $t\to e^{2\delta|x|}\times V_\alpha(t,x)$ is an $L^\infty(\boldsymbol{R}^3)$-valued continuously differentiable function for each α.

Of course, propositions in § 2 remain valid under (C). Two-body subsystems may have some bound states with non-threshold energies, but we assume for simplicity that each two-body subsystem has exactly one bound state:

ASSUMPTION (D). Each $q_\alpha(\zeta)$ has no $(+1)$-eigenfunctions for any $\zeta\in[0,2\pi/\omega)\pm i0$ except for $\lambda_\alpha\pm i0$ with $\lambda_\alpha\in(0,2\pi/\omega)$. The $(+1)$-eigenspace of $q_\alpha(\lambda_\alpha\pm i0)$ is one-dimensional.

Assumption (D) implies that k^α has exactly one eigenvalue λ_α in $[0,2\pi/\omega)$ and it is simple. We denote by $\phi_\alpha\in\mathscr{k}^\alpha$ the λ_α-eigenfunction of k^α with normalization $\|\phi_\alpha\|=\omega^{1/2}$. Then by the definition of k^α, we see

$$u^\alpha(s,t)\phi_\alpha(t)=\exp(-i(s-t)\lambda_\alpha)\phi_\alpha(s) \qquad (\text{a. e. } (s,t)) \tag{4.1}$$

where $\phi_\alpha=\{\phi_\alpha(t): t\in\boldsymbol{T}, \phi_\alpha(t)\in\mathscr{h}^\alpha\}\in\mathscr{k}^\alpha$. Thus we may assume that $\phi_\alpha(t)$ is continuous in t as a $\mathscr{h}^\alpha$-valued function, and (4.1) holds for all s and t.

Under these assumptions, we can prove the asymptotic completeness. To state our result explicitly, we introduce some spaces and operators. Let $\mathscr{H}_i$ $(i=1,2)$ be

$$\begin{cases} \mathscr{H}_1=L^2(\boldsymbol{R}^6_{(x_\alpha,y_\alpha)})\oplus\bigoplus_\beta L^2(\boldsymbol{R}^3_{y_\beta}) \\ \qquad =\mathscr{H}_1^{(0)}\oplus\bigoplus_\alpha\mathscr{H}_1^{(\alpha)} \\ \mathscr{H}_2=L^2(\boldsymbol{R}^6_{(x_\alpha,y_\alpha)}). \end{cases} \tag{4.2}$$

We define $J(t)\in B(\mathscr{H}_1,\mathscr{H}_2)$ as

$$(4.3)\qquad \begin{cases} J(t)f=J_0f_0+\sum_\alpha J_\alpha(t)f_\alpha\,, & f=f_0\oplus\left[\bigoplus_\alpha f_\alpha\right]\in\mathcal{H}_1 \\ J_0f_0=f_0\,, \quad f_0\in\mathcal{H}_1^{(0)} & \\ (J_\alpha(t)f_\alpha)(x_\alpha,y_\alpha)=\phi_\alpha(t,x_\alpha)f_\alpha(y_\alpha)\,, & f_\alpha\in\mathcal{H}_1^{(0)}\,, \end{cases}$$

and time evolutions U_i on $\mathcal{H}_i$ $(i=1,2)$ as

$$(4.4)\qquad \begin{cases} U_1(t,s)f=[\exp(-i(t-s)H_0)f_0]\oplus\left[\bigoplus_\alpha \exp(-i(t-s)\tilde{l}_0^\alpha)f_\alpha\right], & f\in\mathcal{H}_1 \\ U_2(t,s)f=U(t,s)f\,, \quad f\in\mathcal{H}_2 & \end{cases}$$

where $\tilde{l}_0^\alpha=l_0^\alpha+\lambda_\alpha=-(1/2n_\alpha)\,\Delta_{y_\alpha}+\lambda_\alpha$.

We define wave operators $W_\pm(s)=W_\pm(s\,;\,U_2,U_1,J)$ by

$$(4.5)\qquad W_\pm(s\,;\,U_2,U_1,J)=\operatorname*{s-lim}_{t\to\pm\infty} U_2(s,t)J(t)U_1(t,s)\,.$$

PROPOSITION 4.1. *$W_\pm(s)$ exist and are isometric operators from $\mathcal{H}_1$ into $\mathcal{H}_2$.*

Since $\{\phi_\alpha(t): t\in \boldsymbol{T}\}$ forms a compact set in $L^2(\boldsymbol{R}^3)$, the proof for Proposition 4.1 can be carried out in a way almost identical with the standard one for time-independent Hamiltonians (see Reed-Simon [16], § XI. 5), and we omit its proof.

Now, we can state our goal of this section.

THEOREM 2. *Under Assumptions* (C) *and* (D), *$W_\pm(s)$ are complete*:

$$(4.6)\qquad \operatorname{Ran} W_\pm(s)=\mathcal{H}_2^{ac}(U_2(s+\omega,s))\,.$$

By analogy with the last section, we reduce the problem to time-independent one by the Howland-Yajima method. Let $\mathcal{K}_i$ $(i=1,2)$ be

$$(4.7)\qquad \begin{cases} \mathcal{K}_1=L^2(T,\mathcal{H}_1) \\ \qquad \cong L^2(T,\mathcal{H}_1^{(0)})\oplus\bigoplus_\alpha L^2(T,\mathcal{H}_1^{(\alpha)}) \\ \qquad =\mathcal{K}_1^{(0)}\oplus\bigoplus_\alpha \mathcal{K}_1^{(\alpha)} \\ \mathcal{K}_2=L^2(T,\mathcal{H}_2)\,. \end{cases}$$

Self-adjoint operators K_i are defined on $\mathcal{K}_i$ by

$$(4.8)\qquad (\exp(-i\sigma K_i)f)(t)=U_i(t,t-\sigma)f(t-\sigma)\,,\qquad f\in\mathcal{K}_i\,.$$

We define $\mathcal{J}\in B(\mathcal{K}_1,\mathcal{K}_2)$ by

$$(\mathcal{J}f)(t)=J(t)f(t)\,, \qquad f\in\mathcal{K}_1\,. \tag{4.9}$$

Then wave operators $\mathcal{W}_\pm=\mathcal{W}_\pm(K_2, K_1, \mathcal{J})$ are defined by

$$\mathcal{W}_\pm(K_2, K_1, \mathcal{J})=\operatorname*{s-lim}_{\sigma\to\pm\infty}\exp(i\sigma K_2)\mathcal{J}\exp(-i\sigma K_1)\,. \tag{4.10}$$

It is easy to see that the existence of $\mathcal{W}_\pm$ follows from that of $W_\pm(s)$. It is not difficult to prove that $\mathcal{W}_\pm$ are isometric operators.

PROPOSITION 4.2. *Under Assumptions* (C) *and* (D), $\mathcal{W}_\pm(K_2, K_1, \mathcal{J})$ *are complete*:

$$\operatorname{Ran}\mathcal{W}_\pm(K_2, K_1, J)=\mathcal{K}_2^{ac}(K_2)\,. \tag{4.11}$$

As was in § 3, Theorem 2 follows from Proposition 4.2, and we devote the rest of this section to the proof of Proposition 4.2.

4.2. Faddeev matrix.

Following Ginibre-Moulin [7], we introduce the (modified) Faddeev matrix as follows. We define $G'_\alpha(\zeta), L_0^\alpha, r_0^\alpha(\zeta), C_\alpha$ and $\mathcal{J}_\alpha$ as:

$$G'_\alpha(\zeta)=G_\alpha(\zeta)(1-P^{pp}(k^\alpha)) \tag{4.12}$$

$$(\exp(-i\sigma L_0^\alpha)f)(t)=\exp(-i\sigma\tilde{l}_0^\alpha)f(t-\sigma)\,, \qquad f\in L^2(T, L^2(\boldsymbol{R}^3)) \tag{4.13}$$

$$r_0^\alpha(\zeta)=(\zeta-L_0^\alpha)^{-1} \tag{4.14}$$

$$(C_\alpha f)(y_\alpha)=\langle y_\alpha\rangle^{-\theta}f(y_\alpha)\,, \qquad f\in L^2(\boldsymbol{T}, L^2(\boldsymbol{R}^3)) \tag{4.15}$$

$$(\mathcal{J}_\alpha f)(t)=J_\alpha(t)f(t)\,, \qquad f\in L^2(\boldsymbol{T}, L^2(\boldsymbol{R}^3)) \tag{4.16}$$

with a fixed constant $\theta>1$. $\mathcal{J}_\alpha$ is an operator from $L^2(\boldsymbol{T}, L^2(\boldsymbol{R}^3))$ to $L^2(\boldsymbol{T}, L^2(\boldsymbol{R}^6))$. Let $\tilde{\mathcal{K}}$ be the direct sum of three copies of $\mathcal{K}$ and $L^2(\boldsymbol{T}, L^2(\boldsymbol{R}^3_{y_\alpha}))$'s:

$$\tilde{\mathcal{K}}=\bigoplus_\alpha\tilde{\mathcal{K}}_0^{(\alpha)}\oplus\bigoplus_\alpha\tilde{\mathcal{K}}_1^{(\alpha)}\,, \tag{4.17}$$

$$\tilde{\mathcal{K}}_0^{(\alpha)}=\mathcal{K}\,, \qquad \tilde{\mathcal{K}}_1^{(\alpha)}=L^2(\boldsymbol{T}, L^2(\boldsymbol{R}^3_{y_\alpha}))\,.$$

Then the (modified) Faddeev matrix $F(\zeta)$ is defined on $\tilde{\mathcal{K}}$ by the following equations:

$$(F(\zeta)f)=\sum_{\substack{\beta\neq\alpha\\ j=1,2}}F_{\alpha i\beta j}(\zeta)f_{\beta j}\,, \qquad f=(f_{\alpha i})\in\tilde{\mathcal{K}} \tag{4.18}$$

$$F_{\alpha 0 \beta 0}(\zeta)=A_\alpha G'_\alpha(\zeta)B_\beta \tag{4.19}$$

$$F_{\alpha 0 \beta 1}(\zeta)=A_\alpha G'_\alpha(\zeta)B_\beta A_\beta \mathcal{J}_\beta r_0^\beta(\zeta)C_\beta \tag{4.20}$$

$$F_{\alpha 1 \beta 0}(\zeta)=C_\alpha^{-1}\mathcal{J}_\alpha^* B_\beta \tag{4.21}$$

$$F_{\alpha 1 \beta 1}(\zeta)=C_\alpha^{-1}\mathcal{J}_\alpha^* B_\beta A_\beta \mathcal{J}_\beta r_0^\beta(\zeta)C_\beta\,. \tag{4.22}$$

PROPOSITION 4.3. *For $\zeta\in \boldsymbol{C}\setminus\boldsymbol{R}$, $F_{\alpha 0\beta 0}(\zeta)\in B(\tilde{\mathcal{K}}_0^{(\alpha)}, \tilde{\mathcal{K}}_0^{(\beta)})$, and as a $B(\tilde{\mathcal{K}}_0^{(\alpha)}, \tilde{\mathcal{K}}_0^{(\beta)})$-valued function of $\zeta\in\boldsymbol{C}\setminus\boldsymbol{R}$, $F_{\alpha 0\beta 0}(\zeta)$ satisfies the following properties:*

(i) *$F_{\alpha 0\beta 0}(\zeta)$ is uniformly bounded.*
(ii) *$F_{\alpha 0\beta 0}(\zeta)$ is analytic.*
(iii) *$F_{\alpha 0\beta 0}(\zeta)$ is uniformly Hölder-continuous.*
(iv) *$\|F_{\alpha 0\beta 0}(\zeta)\|$ tends to zero if $|\mathrm{Im}\,\zeta|\to\infty$.*
(v) *$F_{\alpha 0\beta 0}(\zeta)\in B_c(\tilde{\mathcal{K}}^{(\alpha)}, \tilde{\mathcal{K}}^{(\beta)})$ for each ζ.*

PROPOSITION 4.4. *As a $B(\tilde{\mathcal{K}}_0^{(\alpha)}, \tilde{\mathcal{K}}_1^{(\beta)})$-valued function of $\zeta, F_{\alpha 0\beta 1}(\zeta)$ $(\alpha\neq\beta)$ satisfies the properties* (i)-(v) *of Proposition* 4.3, *if we replace $F_{\alpha 0\beta 0}(\zeta)$ by $F_{\alpha 0\beta 1}(\zeta)$.*

PROPOSITION 4.5. *$F_{\alpha 1\beta 0}(\zeta)=F_{\alpha 1\beta 0}$ $(\alpha\neq\beta)$ is a bounded operator from $\tilde{\mathcal{K}}_1^{(\alpha)}$ to $\tilde{\mathcal{K}}_0^{(\beta)}$.*

PROPOSITION 4.6. *As a $B(\tilde{\mathcal{K}}_1^{(\alpha)}, \tilde{\mathcal{K}}_1^{(\beta)})$-valued function of $\zeta, F_{\alpha 1\beta 1}(\zeta)$ $(\alpha\neq\beta)$ satisfies the properties* (i)-(v) *of Proposition* 4.3, *if we replace $F_{\alpha 0\beta 0}(\zeta)$ by $F_{\alpha 1\beta 1}(\zeta)$.*

We postpone the proof of Propositions 4.3-4.6 till subsection 4.4 and proceed with the proof of Proposition 4.2.

LEMMA 4.1. *$(1-F(\zeta))^{-1}$ exists for $\zeta\in\boldsymbol{C}\setminus\boldsymbol{R}$. Moreover except for a closed null set $\mathcal{E}\subset\boldsymbol{R}$, $(1-F(\zeta))^{-1}$ has a continuous boundary value on $\boldsymbol{R}\pm i0$.*

PROOF. Propositions 4.3-4.6 imply that $F(\zeta)^2$ is compact on $\mathcal{K}$. Hence the lemma follows from Theorem 3.10 of Kuroda [14].

PROOF OF PROPOSITION 4.2. We apply the abstract theory of two-space scattering:

THEOREM 4.1 ([11]). *For $i=1,2$, let K_i be a self-adjoint operator in a separable Hilbert space $\mathcal{K}_i$, with the spectral family $\{E_i(\lambda)\}$ and the* *resolvent* $G_i(\zeta)=(\zeta-K_i)^{-1}$. *Let* $\mathcal{J}\in B(\mathcal{K}_1,\mathcal{K}_2)$. *Let* Γ *be an open subset of* $\boldsymbol{R}$. *Suppose* $\mathcal{X}_i$ *is a Banach space densely embedded in* $\mathcal{K}_i$ $(i=1,2)$. *Suppose, further, that the following conditions are satisfied.*

(a) $\lim_{\sigma\to\pm\infty}\|\mathcal{J}\exp(-i\sigma K_1)u\|=\|u\|$ *for* $u\in\mathcal{K}_1$.

(b) $\mathcal{W}_\pm(\Gamma)=\text{s-lim}_{s\to\pm\infty}\exp(i\sigma K_2)\mathcal{J}\exp(-i\sigma K_1)E_1(\Gamma)$ *exist.*

(c) $f(\zeta;\phi,\psi)=\pi^{-1}|\mathrm{Im}\,\zeta|(G_1(\zeta)\phi,G_1(\zeta)\psi)$ *has a continuous boundary value on* $\Gamma\pm i0$ *if* $\phi,\psi\in\mathcal{X}_1$.

(d) *There is a* $B(\mathcal{X}_2,\mathcal{X}_1)$*-valued strongly continuous function* $N(\zeta)$ *on* $\Gamma^\pm=\{x+iy: x\in\Gamma,\ \pm iy\geqq 0\}$, *such that* $G_2(\zeta)\phi=\mathcal{J}G_1(\zeta)N(\zeta)\phi$ *for* $\phi\in\mathcal{X}_2$.

Then K_2 *is absolutely continuous on* Γ, *and there exists* $\mathcal{Z}_\pm\in B(\mathcal{K}_2,\mathcal{K}_1)$ *such that* $\mathcal{W}_\pm(\Gamma)\mathcal{Z}_\pm=E_2(\Gamma)$.

In particular, if Γ *is dense in* $\boldsymbol{R}$, *the wave operators* $\mathcal{W}_\pm$ *are complete, i. e.* $\mathrm{Ran}\,\mathcal{W}_\pm=\mathcal{K}_2^{ac}(K_2)$.

Let $\Gamma=\boldsymbol{R}\backslash\mathcal{E}$. We define $\mathcal{X}_i$ $(i=1,2)$ as:

$$\left\{\begin{array}{l}\mathcal{X}_1=\mathcal{X}_1^{(0)}\oplus\bigoplus_\alpha\mathcal{X}_1^{(\alpha)}\subset\mathcal{K}_1\\ \mathcal{X}_1^{(0)}=\left\{\sum_\alpha\phi_\alpha\in\mathcal{K}_1^{(0)}:\phi_\alpha\in L^2(\boldsymbol{T},L_\theta^2(\boldsymbol{R}_{x_\alpha}^3))\otimes L^2(\boldsymbol{R}_{y_\alpha}^3)\right\}\\ \mathcal{X}_1^{(\alpha)}=L^2(\boldsymbol{T},L_\theta^2(\boldsymbol{R}_{y_\alpha}^3))\subset\mathcal{K}_1^{(\alpha)},\end{array}\right.\tag{4.23}$$

$$\mathcal{X}_2=\{\phi\in\mathcal{K}_2:\exp\delta(|x_\alpha|+|y_\alpha|)\cdot\phi\in\mathcal{K}_2\}.\tag{4.24}$$

$D(\zeta)\in B(\mathcal{X}_2,\tilde{\mathcal{K}})$ is defined by

$$\left\{\begin{array}{l}(D(\zeta)\phi)_{\alpha 0}=A_\alpha G'_\alpha(\zeta)\phi\\ (D(\zeta)\phi)_{\alpha 1}=C_\alpha^{-1}\mathcal{J}_\alpha^*\phi.\end{array}\right.\tag{4.25}$$

Note that $D(\zeta)$ has a continuous boundary value on $\boldsymbol{R}\pm i0$ by Proposition 4.3. By virtue of Lemma 4.1, we can define $N(\zeta)\in B(\mathcal{X}_2,\mathcal{X}_1)$by

$$\left\{\begin{array}{l}(N(\zeta)\phi)_0=\phi+\sum_\alpha B_\alpha((1-F(\zeta))^{-1}D(\zeta)\phi)_{\alpha 0}-\sum_\alpha C_\alpha\mathcal{J}_\alpha((1-F(\zeta))^{-1}D(\zeta)\phi)_{\alpha 1}\\ (N(\zeta)\phi)_\alpha=C_\alpha((1-F(\zeta))^{-1}D(\zeta)\phi)_{\alpha 1}\end{array}\right.\tag{4.26}$$

for $\zeta\in\Gamma^\pm$, $\phi\in\mathcal{X}_2$.

Now, we have remarked that (a) and (b) hold, and it is well-known that (c) holds. Following the computations of Ginibre-Moulin [7], § 6 and § 8, we obtain (d). Then we apply Theorem 4.1, and Theorem 2 is proved.

4.3. Two-body subsystems.

To prove Propositions 4.3–4.6, we need some facts on the resolvents of two-body subsystems, and this is the reason we impose Assumption (C).

Let $h(t)=h_0+V(t,x)$, $h_0=-\Delta$ on $L^2(\boldsymbol{R}^3)$ and let V satisfy (C). $\mathcal{X}_\beta$ denotes the exponentially weighted L^2-space of order β:

$$\mathcal{X}_\beta=\{\phi\in L^2_{\mathrm{loc.}}(\boldsymbol{R}^3):\exp(\beta|x|)\cdot\phi\in L^2(\boldsymbol{R}^3)\} \tag{4.28}$$

and we set $\mathcal{Y}_\beta=L^2(\boldsymbol{T},\mathcal{X}_\beta)$. For $r\in\boldsymbol{R}$ and $\varepsilon>0$, we set

$$\Gamma_\varepsilon^{r,\pm}=\left\{\zeta\in\boldsymbol{C}:\varepsilon<\mathrm{Re}\,\zeta<\frac{2\pi}{\omega}-\varepsilon,\ \pm\mathrm{Im}\,\zeta>r\right\}. \tag{4.29}$$

PROPOSITION 4.7. *For any $\varepsilon>0$, $g_0(\zeta)=(\zeta-k_0)^{-1}$ has an analytic continuation from $\Gamma_\varepsilon^{0,\pm}$ to $\Gamma_\varepsilon^{-\gamma,\pm}$ as a $B(\mathcal{Y}_\delta,\mathcal{Y}_{-\delta})$-valued function for some $\gamma>0$. Further, it is norm-continuous on $\Gamma_\varepsilon^{-\gamma,\pm}$ and is compact for each $\zeta\in\Gamma_\varepsilon^{-\gamma,\pm}$.*

PROOF. Take $\gamma>0$ so small that $-\delta<\mathrm{Im}\sqrt{\varepsilon-i\gamma}$, where we take a branch of $\sqrt{}$ such that $\mathrm{Im}\sqrt{\zeta}>0$ for $\zeta\in\Gamma_\varepsilon^{0,\pm}$.

Performing a Fourier transform in t, we see

$$\mathcal{F}_{t\to\mu}(\zeta-k_0)^{-1}\mathcal{F}_{t\to\mu}^{-1}=\left(\zeta-\frac{2\pi}{\omega}\mu-h_0\right)^{-1}. \tag{4.30}$$

Since $(\zeta-h_0)^{-1}$ is an integral operator with the kernel $-(4\pi)^{-1}|x-x'|\times\exp(i\sqrt{\zeta}\cdot|x-x'|)$, it is easy to see that $(\zeta-(2\pi/\omega)\mu-h_0)^{-1}$ has an analytic continuation from $\Gamma_\varepsilon^{0,\pm}$ to $\Gamma_\varepsilon^{-\gamma,\pm}$ as a $B(\mathcal{X}_\delta,\mathcal{X}_{-\delta})$-valued function for each μ. Further it is uniformly bounded and uniformly continuous in μ and ζ with fixed γ. Thus (4.30) implies that $(\zeta-k_0)^{-1}$ can be analytically extended to $\Gamma_\varepsilon^{-\gamma,\pm}$ as a $B(\mathcal{Y}_\delta,\mathcal{Y}_{-\delta})$-valued function.

For proving that $(\zeta-k_0)^{-1}$ is compact, we use that $(\zeta-(2\pi/\omega)\mu-h_0)^{-1}$ is of Hilbert-Schmidt from $\mathcal{X}_\delta$ to $\mathcal{X}_{-\delta}$ for each μ, and that $\|(\zeta-(2\pi/\omega)\mu-h_0)^{-1}\|\to 0$ $(\mu\to\infty)$, which can be proved by using integrations by parts as in the proof of the Riemann-Lebesgue theorem (cf. Reed-Simon [16], Chap. XI, Problem 60).

The next corollary follows from the proposition above and the analytic Fredholm theorem.

COROLLARY 4.1. *Suppose Assumption* (C), *and $\varepsilon>0$. Then, as a*

$B(\mathcal{Y}_\delta, \mathcal{Y}_{-\delta})$*-valued function,* $g(\zeta)=(\zeta-k)^{-1}$ *has a meromorphic continuation from* $\Gamma_\varepsilon^{0,\pm}$ *to* $\Gamma_\varepsilon^{-\gamma,\pm}$ *for some* $\gamma>0$. *Moreover the residues of* $g(\zeta)$ *at the poles are finite rank operators.*

The next proposition gives a relation between the poles of $g(\zeta)$ and the eigenvalues of k.

PROPOSITION 4.8. *Suppose Assumption* (C). *Let* $\lambda\in(0, 2\pi/\omega)$ *be a simple eigenvalue of* k, *and* P *be the orthogonal projection onto the* λ-*eigenspace. Then*

$$g'_\lambda(\zeta)=g(\zeta)-(\zeta-\lambda)^{-1}P \tag{4.31}$$

has an analytic continuation from $\boldsymbol{C}^\pm=\{\zeta\in\boldsymbol{C}:\operatorname{Im}\zeta\gtrless 0\}$ *to a neighborhood of* λ *as a* $B(\mathcal{Y}_\delta, \mathcal{Y}_{-\delta})$*-valued function.*

PROOF. We employ an argument similar to the proof of Theorem III-1 of Aguiler-Combes [1]. We know that $g(\zeta)$ has a meromorphic continuation by Corollary 4.1. We show that λ is a simple pole, and its residue is given by $-P$.

We let ϕ and ψ be elements of $\mathcal{Y}_\delta$. Then $(\phi, g(\zeta)\psi)$ is meromorphic and has no poles except for λ in a neighborhood of λ. Let $\boldsymbol{C}_d^\pm=\{\zeta\in\boldsymbol{C}: \pi/2-d<\pm\operatorname{Arg}(\zeta-\lambda)<\pi/2+d\}$ $(0<d<\pi/2)$ be a conic neighborhood of λ. The pole of $(\phi, g(\zeta)\psi)$ at λ is at most of order one because

$$(\zeta-\lambda)^2(\phi, g(\zeta)\psi)=\int\left(\frac{(\zeta-\lambda)^2}{\zeta-\rho}\right)d(\phi, E(\rho)\psi) \longrightarrow 0 \quad (\zeta\in\boldsymbol{C}_d^\pm, \zeta\to\lambda) \tag{4.32}$$

by Lebesgue's theorem, where $\{E(\cdot)\}$ denotes the spectral measure of k. Further, the residue of $(\phi, g(\zeta)\psi)$ coincides with $-(\phi, P\psi)$ since

$$(\zeta-\lambda)(\phi, g(\zeta)\psi)=\int\left(\frac{\lambda-\zeta}{\zeta-\rho}\right)d(\phi, E(\rho)\psi) \longrightarrow -(\phi, P\psi) \quad (\zeta\in\boldsymbol{C}_d^\pm, \zeta\to\lambda) \tag{4.33}$$

again by Lebesgue's theorem. Hence $(\phi, g'_\lambda(\zeta)\psi)=(\phi, g(\zeta)\psi)-(\zeta-\lambda)^{-1}\cdot(\phi, P\psi)$ is holomorphic near λ.

COROLLARY 4.2. *Suppose that* (C) *and* (D) *are satisfied. For any* $\varepsilon>0$, *there exists* $\gamma>0$ *such that*

$$g'(\zeta)=g(\zeta)(1-P^{pp}(k)) \tag{4.34}$$

has an analytic continuation from $\Gamma_\varepsilon^{0,\pm}$ *to* $\Gamma_\varepsilon^{-\gamma,\pm}$ *as a* $B(\mathcal{Y}_\delta, \mathcal{Y}_{-\delta})$*-valued function, and it is uniformly bounded, uniformly continuous on* $\Gamma_\varepsilon^{-\gamma,\pm}\cup\boldsymbol{C}^\pm$.

We prepare two more lemmata on subsystems. We assume (C) and (D), and let ϕ_α be the λ_α-eigenfunction of k^α.

LEMMA 4.2. *For any* μ, $\nu \in \boldsymbol{R}$,

$$\phi_\alpha \in H^1(T, L^2_\mu(\boldsymbol{R}^3)), \tag{4.35}$$

$$\sup_t \int |\phi_\alpha(t, x_\alpha)|^2 \langle x_\alpha \rangle^\mu \sum_{\beta \neq \alpha} |V_\beta(t, x_\beta)| dx_\alpha \leqq C_{\mu\nu} \langle y_\alpha \rangle^{-\nu}. \tag{4.36}$$

The proof of (4.35) will be given in the appendix. (4.36) can be proved analogously to the time-independent case (see Ginibre-Moulin [7]).

LEMMA 4.3. *Let* $f \in k^\alpha$, *then*

$$(P^{pp}(k^\alpha)f)(t, x_\alpha) = \phi_\alpha(t, x_\alpha) \int \overline{\phi_\alpha(t, x'_\alpha)} f(t, x'_\alpha) dx'_\alpha. \tag{4.37}$$

PROOF. Under the assumption (D), the set of all eigenvalues of k^α is $\{\lambda_\alpha + (2\pi/\omega)\mu : \mu \in \boldsymbol{Z}\}$ and the $(\lambda_\alpha + (2\pi/\omega)\mu)$-eigenfunction is $\exp(i\mu(2\pi/\omega)t)\phi_\alpha$. Hence, for each $f, g \in C^\infty(\boldsymbol{T}, C_0^\infty(\boldsymbol{R}^3))$ we obtain

$$\begin{aligned}(f, P^{pp}(k^\alpha)g) = \sum_\mu \omega^{-1} \int \exp\Big(i\mu \frac{2\pi}{\omega} t\Big) \phi_\alpha(t, x_\alpha) \overline{f(t, x_\alpha)} dt dx_\alpha \\ \times \int \exp\Big(-i\mu \frac{2\pi}{\omega} u\Big) \overline{\phi_\alpha(u, x'_\alpha)} g(u, x'_\alpha) du dx'_\alpha.\end{aligned} \tag{4.38}$$

Remark that $\int \phi_\alpha(t, x_\alpha) \overline{f(t, x_\alpha)} dx_\alpha \Big(\int \overline{\phi_\alpha(u, x'_\alpha)} g(u, x'_\alpha) dx'_\alpha \text{ resp.} \Big)$ is a H^1-function of t (u resp.). Since $\sum_{|\mu|<N} \exp(i\mu(2\pi/\omega)(t-u))$ converges to $\omega \cdot \delta(t-u)$ in $H^{-1}(\boldsymbol{T} \times \boldsymbol{T})$ as $N \to \infty$, we obtain

$$(f, P^{pp}(k^\alpha)g) = \int \phi_\alpha(t, x_\alpha) \overline{f(t, x_\alpha)} \overline{\phi_\alpha(t, x'_\alpha)} g(t, x'_\alpha) dt dx_\alpha dx'_\alpha. \tag{4.39}$$

4.4. Proof of Propositions 4.3-3.6.

Now, we can prove Propositions 4.3-4.6 using the results of the last subsection.

PROOF OF PROPOSITION 4.3. By the second resolvent equation, we see

$$F_{\alpha 0 \beta 0}(\zeta) = (1 + A_\alpha G'_\alpha(\zeta) B_\alpha) A_\alpha G_0(\zeta) B_\beta - A_\alpha P^{pp}(k^\alpha) G_0(\zeta) B_\beta. \tag{4.40}$$

If we perform a Fourier transform with respect to y_α, we obtain

$$\mathscr{F}_{y_\alpha\to\eta} A_\alpha G'_\alpha(\zeta) B_\alpha \mathscr{F}^{-1}_{y_\alpha\to\eta} = A_\alpha g^\alpha\Big(\zeta - \frac{1}{2n_\alpha}\eta^2\Big)(1-P^{pp}(k^\alpha))B_\alpha . \tag{4.41}$$

Then by Corollary 4.2, we can conclude that $(1+A_\alpha G'_\alpha(\zeta)B_\alpha)$ enjoys the properties (i), (ii) and (iii) of Proposition 4.3. Thus it is sufficient to prove the properties (i)-(v) for $A_\alpha G_0(\zeta)B_\beta$ and $A_\alpha P^{pp}(k^\alpha)G_0(\zeta)B_\beta$. Those for $A_\alpha G_0(\zeta)B_\beta$ have been proved in Lemmata 3.4, 3.5. On the other hand, since $A_\alpha P^{pp}(k^\alpha)\langle x_\alpha\rangle^\theta \in B(\tilde{\mathscr{K}}_0^{(\alpha)})$ by (4.35) and (4.37), and $\langle x_\alpha\rangle^{-\theta}G_0(\zeta)B_\beta$ satisfies the properties $(\theta>1)$ by the proof of Lemmata 3.4, 3.5, we obtain those for $A_\alpha P^{pp}(k^\alpha)G_0(\zeta)B_\beta$.

PROOF OF PROPOSITION 4.4. Using the second resolvent equation and the formula $\mathscr{J}_\beta r_0^\beta(\zeta) = G_\beta(\zeta)\mathscr{J}_\beta$, we obtain after some computations

$$\begin{aligned} F_{\alpha 0\beta 1}(\zeta) &= (1+A_\alpha G'_\alpha(\zeta)B_\alpha)(A_\alpha \mathscr{J}_\beta r_0^\beta(\zeta)C_\beta - A_\alpha G_0(\zeta)\mathscr{J}_\beta C_\beta) \\ &\quad - A_\alpha P^{pp}(k^\alpha)\mathscr{J}_\beta r_0^\beta(\zeta)C_\beta + A_\alpha P^{pp}(k^\alpha)G_0(\zeta)\mathscr{J}_\beta C_\beta \\ &= I_1 + I_2 + I_3 . \end{aligned} \tag{4.42}$$

$A_\alpha \mathscr{J}_\beta r_0^\beta(\zeta)C_\beta = (A_\alpha \mathscr{J}_\beta C_\beta^{-1})(C_\beta r_0^\beta(\zeta)C_\beta)$ satisfies the properties (i)-(v), because $A_\alpha \mathscr{J}_\beta C_\beta^{-1} \in B(\tilde{\mathscr{K}}_1^{(\beta)}, \tilde{\mathscr{K}}_0^{(\alpha)})$ by Lemma 4.2, and $C_\beta r_0^\beta(\zeta)C_\beta$ satisfies Proposition 2.2. $A_\beta G_0(\zeta)\mathscr{J}_\beta C_\beta = (A_\beta G_0(\zeta)C_\beta)\mathscr{J}_\beta$ satisfies them also. Since $(1+A_\alpha G'_\alpha(\zeta)B_\alpha)$ enjoys the assertion (i)-(iii) by Proposition 4.3, the proof for I_1 is completed.

$I_2 = (A_\alpha P^{pp}(k^\alpha)\mathscr{J} C_\beta^{-1})(C_\beta r_0^\beta(\zeta)C_\beta)$ has the properties if $A_\alpha P^{pp}(k^\alpha)\mathscr{J}_\beta C_\beta^{-1} \in B(\tilde{\mathscr{K}}_1^{(\beta)}, \tilde{\mathscr{K}}_0^{(\alpha)})$. Using Lemmata 4.2 and 4.3, one obtains an estimate:

$$\begin{aligned} \|A_\alpha P^{pp}(k^\alpha)\mathscr{J}_\beta C_\beta^{-1} f\|_{\tilde{\mathscr{K}}_0^{(\alpha)}} &\leqq C\|A_\alpha\|_{B(\tilde{\mathscr{K}}_0^{(\alpha)})}\|\phi_\alpha\|_{L^\infty(T, L^2(R^3))} \\ &\quad \times \|\phi_\alpha\|_{L^\infty(T, L^2_\theta(R^3))}\|\phi_\beta\|_{L^\infty(T, L^2_\theta(R^3))} \\ &\quad \times \|f\|_{\tilde{\mathscr{K}}_1^{(\beta)}}, \qquad (f\in \tilde{\mathscr{K}}_1^{(\beta)}) . \end{aligned} \tag{4.43}$$

Proof for $I_3 = (A_\alpha P^{pp}(k^\alpha)\langle x_\alpha\rangle^\theta)(\langle x_\alpha\rangle^{-\theta}G_0(\zeta)C_\beta)\mathscr{J}_\beta$ can be done analogously.

PROOF OF PROPOSITION 4.5. By definitions, we see

$$(F_{\alpha 1\beta 0}f)(y_\alpha) = \langle y_\alpha\rangle^\theta \int \phi_\alpha(t, x_\alpha) b_\beta(t, x_\beta) f(t, x_\beta, y_\beta) dx_\alpha \tag{4.44}$$

for $f\in \tilde{\mathscr{K}}^{(\beta)}$. Then by the Schwartz inequality, Lemma 4.2 and (4.36), we obtain

$$\text{(4.45)} \qquad \|F_{\alpha 1\beta 0}f\|^2 \leqq \iint \langle y_\alpha\rangle^{2\theta} \left|\int \phi_\alpha(t, x_\alpha) b_\beta(t, x_\beta) f(t, x_\beta, y_\beta) dx_\alpha\right|^2 dy_\alpha dt$$

$$\leqq \iint \langle y_\alpha\rangle^{2\theta} \left(\int |\phi_\alpha(t, x_\alpha)|^2 |V_\beta(t, x_\beta)| dx_\alpha\right)$$

$$\times \left(\int |f(t, x_\beta, y_\beta)|^2 dx_\alpha\right) dy_\beta dt$$

$$\leqq C_{0,2\theta} \|f\|^2 .$$

PROOF OF PROPOSITION 4.6. Since

$$\text{(4.46)} \qquad F_{\alpha 1\beta 1}(\zeta) = (C_\alpha^{-1} \mathcal{J}_\alpha^* B_\beta A_\beta \mathcal{J}_\beta C_\beta^{-1})(C_\beta r_0^\beta(\zeta) C_\beta) ,$$

it is sufficient to prove that $C_\alpha^{-1} \mathcal{J}_\alpha^* B_\beta A_\beta \mathcal{J}_\beta C_\beta^{-1} \in B(\tilde{\mathcal{K}}_1^{(\beta)}, \tilde{\mathcal{K}}_1^{(\alpha)})$. This can be estimated analogously to (4.43).

Appendix. Property of eigenfunctions.

Here we prove a property of eigenfunctions of two-body subsystems. Let $h(t) = h_0 + V(t, x)$ ($h_0 = -\Delta$ resp.) on $L^2(\boldsymbol{R}^n)$ be the Hamiltonian (free Hamiltonian resp.) of our system. $U(t, s)$ denotes the evolution operator generated by $h(t)$, and k (k_0 resp.) denotes the self-adjoint operator defined by $(\exp(-i\sigma k)\phi)(t) = U(t, t-\sigma)\phi(t-\sigma)$ $((\exp(-i\sigma k_0)\phi)(t) = \exp(-i\sigma h_0)\phi(t-\sigma)$ resp.) on $k = L^2(\boldsymbol{T}, L^2(\boldsymbol{R}^n))$. We suppose

ASSUMPTION (F). There exists a $\delta > 1/2$ such that $t \to \langle x\rangle^{2\delta} V(t, x)$ is an $L^\infty(\boldsymbol{R}^n)$-valued continuously differentiable function.

The purpose of this appendix is to prove the next Theorem.

THEOREM A.1. *Let ϕ be an eigenfunction of k with an eigenvalue $\lambda \in \boldsymbol{R} \backslash (2\pi/\omega)\boldsymbol{Z}$. Then for any $\mu \in \boldsymbol{R}$,*

$$\text{(A.1)} \qquad \phi \in H^1(\boldsymbol{T}, L^2_\mu(\boldsymbol{R}^n)) .$$

Now, we set $a(t, x) = \langle x\rangle^\delta V(t, x)$ ($b(t, x) = \langle x\rangle^{-\delta}$ resp.) and a (b resp.) be the multiplication operator by $a(t, x)$ ($b(t, x)$ resp.) on k. We write $q(\zeta) = a(\zeta - k_0)^{-1} b$. Then we have (cf. Proposition 2.2)

PROPOSITION A.1 (Yajima [19]). *$q(\zeta)$ has a continuous boundary value on $(\boldsymbol{R} \backslash (2\pi/\omega)\boldsymbol{Z}) \pm i0$ as a compact-operator valued analytic function in k.*

If we remark

$$\left[\frac{\partial}{\partial t}, q(\zeta)\right]=\left(\frac{\partial a}{\partial t}\right)(\zeta-k_0)^{-1}b\,, \tag{A.2}$$

$q(\zeta)$ can be regarded as an operator in $H^1(\boldsymbol{T}, L^2(\boldsymbol{R}^n))$ and moreover,

LEMMA A.1. *$q(\zeta)$ satisfies the property stated in Proposition A.1 regarded as an operator-valued function in $H^1(\boldsymbol{T}, L^2(\boldsymbol{R}^n))$ also.*

LEMMA A.2. *Let X be a Hilbert space and Y be a Banach space densely embedded in X. Suppose that T is a compact operator in X and that its restriction $T|_Y$ on Y is also a compact operator in Y. Then, if $f\in X$ is an eigenfunction of T: $Tf=\lambda f$, $f\in Y$.*

PROOF. Let m be the multiplicity of the eigenvalue λ of T on X. It follows from the Riesz-Schauder theorem that T^* has eigenvalue $\bar{\lambda}$ of the same multiplicity m on X. Since $T^*=(T|_Y)^*|_X$, T^* has eigenvalue $\bar{\lambda}$ with multiplicity $\geqq m$ on Y^*. Hence we conclude that $T|_Y$ has an eigenvalue λ of a multiplicity $\geqq m$ by using the Riesz-Schauder theorem again and the λ-eigenspace of T in X must coincide with that of $T|_Y$.

Lemma A.1 and Lemma A.2 yield

PROPOSITION A.2. *Let $\phi\in$ be a $(+1)$-eigenfunction of $q(\lambda+i0)$: $q(\lambda+i0)\phi=\phi$, then $\phi\in H^1(\boldsymbol{T}, L^2(\boldsymbol{R}^n))$.*

To prove Theorem A.1, we need one more lemma.

LEMMA A.3. *Let $\lambda\in\boldsymbol{R}\setminus(2\pi/\omega)\boldsymbol{Z}$. Suppose that $\phi\in H^1(\boldsymbol{T}, L^2_\gamma(\boldsymbol{R}^n))$ satisfies*

$$\lim_{\eta\downarrow 0}\eta\|(\lambda+i\eta-k_0)^{-1}\phi\|_k^2=0\,. \tag{A.3}$$

Then $(\lambda+i0-k_0)^{-1}\phi\in H^1(\boldsymbol{T}, L^2_{\gamma-1}(\boldsymbol{R}^n))$ and

$$\|(\lambda+i0-k_0)^{-1}\phi\|_{H^1(\boldsymbol{T}, L^2_{\gamma-1}(\boldsymbol{R}^n))}\leqq C_{\lambda\gamma}\|\phi\|_{H^1(\boldsymbol{T}, L^2_\gamma(\boldsymbol{R}^n))}\,. \tag{A.4}$$

PROOF. Using a Fourier transform in t, we obtain from (A.3),

$$\lim_{\eta\downarrow 0}\eta\left\|\left(\lambda-\frac{2\pi}{\omega}\mu+i\eta-h_0\right)^{-1}(\mathscr{F}_{t\to\mu}\phi)_\mu\right\|^2=0\,. \tag{A.5}$$

Hence we can apply Proposition (2.6.1) of Ginibre-Moulin [7], and see

$$\text{(A.6)} \qquad \left\| \left(\lambda - \frac{2\pi}{\omega}\mu + i0 - h_0\right)^{-1} (\mathscr{F}_{t\to\mu}\phi)_\mu \right\|_{L^2_{\gamma-1}(R^n)} \leqq C'_{\lambda\gamma} \|(\mathscr{F}_{t\to\mu}\phi)_\mu\|_{L^2_\gamma(R^n)},$$

$$\text{(A.7)} \qquad \sum_\mu \langle\mu\rangle^2 \left\| \left(\lambda - \frac{2\pi}{\omega}\mu + i0 - h_0\right)^{-1} (\mathscr{F}_{t\to\mu}\phi)_\mu \right\|^2_{L^2_{\gamma-1}(R^n)}$$

$$\leqq C'^2_{\lambda\gamma} \sum_\mu \langle\mu\rangle^2 \|(\mathscr{F}_{t\to\mu}\phi)_\mu\|^2_{L^2_\gamma(R^n)}.$$

This implies (A.4).

PROOF OF THEOREM A.1. Let $\psi \in \mathcal{k}$ be the λ-eigenfunction of k. Then it is easily seen that $\phi = a\psi$ is a $(+1)$-eigenfunction of $q(\lambda+i0)$: $\phi = q(\lambda+i0)\phi$. Hence Proposition A.2 yields $\phi \in H^1(\boldsymbol{T}, L^2(\boldsymbol{R}^n))$. Moreover, because $b\phi \in H^1(\boldsymbol{T}, L^2_\delta(\boldsymbol{R}^n))$ and

$$\text{(A.8)} \qquad \begin{aligned} \eta\|(\lambda+i\eta-k_0)^{-1}b\phi\|^2 &= -\mathrm{Im}(b\phi, (\lambda+i\eta-k_0)^{-1}b\phi) \\ &= -\mathrm{Im}(V(\lambda+i\eta-k_0)^{-1}b\phi, (\lambda+i\eta-k_0)^{-1}b\phi) \\ &= 0, \end{aligned}$$

it follows from Lemma A.3 that $(\lambda+i\eta-k_0)^{-1}b\phi = \psi \in H^1(\boldsymbol{T}, L^2_{\delta-1}(\boldsymbol{R}^n))$, and that $\phi = a\psi \in H^1(\boldsymbol{T}, L^2_{2\delta-1}(\boldsymbol{R}^n))$. Repeating this process ν-times, we obtain

$$\text{(A.9)} \qquad \phi \in H^1(\boldsymbol{T}, L^2_{\nu(2\delta-1)}(\boldsymbol{R}^n)).$$

Thus $\psi = (\lambda+i0-k_0)^{-1}b\phi \in H^1(\boldsymbol{T}, L^2_\mu(\boldsymbol{R}^n))$ for any $\mu \in \boldsymbol{R}$.

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