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# *Removable singularities of solutions of linear partial differential equations —Systems and Fuchsian equations—*

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## Introduction.

A. Kaneko has studied continuation problem for solutions of linear partial differential equations (with real analytic coefficients) systematically by using the theory of non-characteristic boundary value problem for hyperfunctions. In particular, he has proved in [2] and [3] fundamental results on continuation of real analytic (and hyperfunction) solutions whose singularities (i. e. points where the solution is not defined) are contained in a real analytic and non-characteristic hypersurface with respect to a single equation.

The purpose of this paper is to extend main results of Kaneko in [2] and [3] in two directions: on the one hand, to general systems of linear partial differential equations with a real analytic and non-characteristic hypersurface, and on the other hand, to (single) Fuchsian equations (in the sense of Baouendi-Goulaouic [1]) with respect to a real analytic hypersurface (which is characteristic for the equation).

Let $M$ be an open set in $\boldsymbol{R}^n$ and $N$ be a real analytic hypersurface in $M$. Since the problems are of local character, we may assume that $N=\{x\in M;\ x_1=0\}$ with the notation $x=(x_1, x')$, $x'=(x_2, \cdots, x_n)$. Let

$$\mathcal{M}:\ \sum_{j=1}^{\nu} P_{ij}u_j=0 \qquad (i=1, \cdots, \mu)$$

be a system of linear partial differential equations with analytic coefficients defined on $M$. We assume that $\mathcal{M}$ satisfies either one of the following two conditions (we use the notation $D'=(D_2, \cdots, D_n)$, $D_j=\partial/\partial x_j$):

(NC) $N$ is non-characteristic with respect to $\mathcal{M}$.

(F) $\mathcal{M}$ is a single equation $Pu=0$ with a Fuchsian operator $P$ of weight $m-k$ with respect to $N$ in the sense of [1]; i. e.,

$$P=x_1^kD_1^m+A_1(x, D')x_1^{k-1}D_1^{m-1}+\cdots+A_k(x, D')D_1^{m-k}+\cdots+A_m(x, D')\ ,$$

where $0\leqq k\leqq m$, $A_j(x, D')$ is of order $\leqq j$ for $j=1, \cdots, m$, and $A_j(0, x', D')$ is of order 0 for $j=1, \cdots, k$. In addition, none of the non-trivial characteristic exponents is an integer, and no pair of non-trivial characteristic exponents differ by nonzero integers.

For such a system $\mathcal{M}$, we define a closed subset $V_{N,A}(\mathcal{M})$ of the purely imaginary cosphere bundle $\sqrt{-1}S^*N$ of $N$, which is a generalization of the set of $A$-boundary characteristic points defined by Kaneko [2] when both (NC) and (F) are satisfied (i.e. $k=0$ in (F)). Let $\varphi(x')$ be a real valued $C^1$ function on $N$ such that $\varphi(\mathring{x})=0$, $d\varphi(\mathring{x})\neq 0$ with a point $\mathring{x}$ of $N$ and assume that not both of $(\mathring{x}, \pm\sqrt{-1}d\varphi(\mathring{x})\infty)\in\sqrt{-1}S^*N$ are contained in $V_{N,A}(\mathcal{M})$. Then our first result is as follows: any real analytic solution of $\mathcal{M}$ defined on $M\backslash\{(0, x')\in N;\ \varphi(x')\leqq 0\}$ is continued uniquely to a neighborhood of $\mathring{x}$ as a hyperfunction solution (Theorems 1.1 and 3.1). We also give results on continuation of hyperfunction solutions under a stronger condition.

In the proof of the above result, we use the theory of boundary value problems for non-characteristic or Fuchsian systems developed in Ôaku [11]. The most essential tool is the theorem on propagation of micro-analyticity up to the boundary for micro-hyperbolic systems in a weak sense (Theorems 3.2 and 3.3 of [11]), which generalizes Theorem 2.1 of Kaneko [2].

Next we consider real analytic continuation of real analytic solutions. Since we have continuation theorems as hyperfunction solutions as above, this reduces to a problem of propagation of (micro-) analyticity (interior problem), which requires further assumption on $\mathcal{M}$. In the case (NC), if we impose on $\mathcal{M}$, e.g., the condition of micro-hyperbolicity in the sense of Kashiwara-Schapira [7], then we obtain this propagation of analyticity as an immediate consequence of their results. In the case (F), however, we have to prove in the first place that the continued solution has $x_1$ as a real analytic parameter (this fact is trivial for (NC)). For this purpose we use the theory of F-mild hyperfunctions developed in Ôaku [10]. Theorem 3.2 and Corollary 3.2 in § 3 are the main results on real analytic continuation in case (F).

Note that continuation problem for systems has been also studied by Kawai [8] (both for real analytic and hyperfunction solutions) from a different viewpoint.

In § 1, we study the continuation problem in the case (NC). In § 2, we develop the theory of boundary value problem for Fuchsian equations. In § 3, we study the continuation problem in the case (F) by using the results in § 2.

I would like to thank Professor A. Kaneko for suggesting these problems.

## § 1. Systems with non-characteristic hypersurface.

First let us recall the formulation of non-characteristic boundary value problem for systems developed in Ôaku [11]. Let $M$ be an $n$-dimensional paracompact real analytic manifold and assume that there is a real valued real analytic function $f$ on $M$ whose differential $df$ does not vanish on $M$. We put $M_{\pm}=\{x\in M;\ \pm f(x)>0\}$, $N=\{x\in M;\ f(x)=0\}$ and set

$$\mathcal{B}_{N|M_{\pm}}=(\iota_{\pm})_{*}(\iota_{\pm})^{-1}\mathcal{B}_{M}|_{N},$$

where $\iota_{\pm}: M_{\pm}\to M$ are the embeddings and $\mathcal{B}_{M}$ is the sheaf of hyperfunctions on $M$. Sections of $\mathcal{B}_{N|M_{\pm}}$ are represented by hyperfunctions on the intersection of $M_{\pm}$ and of a neighborhood in $M$ of a point of $N$.

There exists a complex neighborhood $X$ of $M$ such that $f$ is extended to $X$ as a holomorphic function and that $df\neq 0$ on $X$. We put $Y=\{z\in X;\ f(z)=0\}$ and

$$\tilde{M}=\{z\in X;\ f(z)\in \boldsymbol{R}\}, \qquad \tilde{M}_{\pm}=\{z\in\tilde{M};\ \pm f(z)>0\}.$$

We use a local coordinate system $z=(z_1, z_2, \cdots, z_n)$ of $X$ such that $f=z_1$ and that all $z_j$ are real-valued on $M$, and call such $z$ admissible. We use the notation $z'=(z_2, \cdots, z_n)$ and $z_j=x_j+\sqrt{-1}y_j$. There is a sheaf $\mathcal{B}\mathcal{O}$ on $\tilde{M}$ of hyperfunctions with holomorphic parameters $z'$. Let $\tilde{\iota}_{\pm}: \tilde{M}_{\pm}\to\tilde{M}$ be embeddings and set

$$\mathcal{B}\mathcal{O}_{Y|\tilde{M}_{\pm}}=(\tilde{\iota}_{\pm})_{*}(\tilde{\iota}_{\pm})^{-1}\mathcal{B}\mathcal{O}|_{Y}.$$

DEFINITION 1.1. $\tilde{\mathcal{B}}_{N|M_{\pm}}=\mathcal{H}_{N}^{n-1}(\mathcal{B}\mathcal{O}_{Y|\tilde{M}_{\pm}})$, $\tilde{\mathcal{B}}_{N|M}=\mathcal{H}_{N}^{n-1}(\mathcal{B}\mathcal{O}|_{Y})$. There exist injective homomorphisms $\alpha_{\pm}: \mathcal{B}_{N|M_{\pm}}\to\tilde{\mathcal{B}}_{N|M_{\pm}}$ (see Theorem 1.2 of [11], where we used the notation $\alpha$ instead of $\alpha_{+}$). By the same argument as the proof of Theorem 1.2 of [11], we can show that there exists a natural injective homomorphism $\alpha: \mathcal{B}_{M}|_{N}\to\tilde{\mathcal{B}}_{N|M}$. Hence, from now on, we regard $\mathcal{B}_{M}|_{N}$ and $\mathcal{B}_{N|M_{\pm}}$ as subsheaves of $\tilde{\mathcal{B}}_{N|M}$ and $\tilde{\mathcal{B}}_{N|M_{\pm}}$ respectively. Moreover there exist restriction maps

$$r_{\pm}: \mathcal{B}_{M}|_{N}\longrightarrow\mathcal{B}_{N|M_{\pm}}, \qquad \tilde{r}_{\pm}: \tilde{\mathcal{B}}_{N|M}\longrightarrow\tilde{\mathcal{B}}_{N|M_{\pm}}.$$

($r_{\pm}(u)=u|_{M_{\pm}}$ for $u\in\mathcal{B}_{M}|_{N}$, and $\tilde{r}_{\pm}$ are induced by the natural maps $\mathcal{B}\mathcal{O}|_{Y}\to\mathcal{B}\mathcal{O}_{Y|\tilde{M}_{\pm}}$.) Combining these maps, we get a commutative diagram

$$\begin{array}{ccccc}
\mathcal{B}_{N|M_{+}} & \xleftarrow{r_{+}} & \mathcal{B}_{M}|_{N} & \xrightarrow{r_{-}} & \mathcal{B}_{N|M_{-}} \\
\downarrow{\scriptstyle\alpha_{+}} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha_{-}} \\
\tilde{\mathcal{B}}_{N|M_{+}} & \xleftarrow{\tilde{r}_{+}} & \tilde{\mathcal{B}}_{N|M} & \xrightarrow{\tilde{r}_{-}} & \tilde{\mathcal{B}}_{N|M_{-}}.
\end{array}$$

LEMMA 1.1. *$(\tilde{r}_+, \tilde{r}_-)$ induces an isomorphism*

$$\tilde{\mathscr{B}}_{N|M}/\mathscr{B}_M|_N \xrightarrow{\sim} (\tilde{\mathscr{B}}_{N|M_+}/\mathscr{B}_{N|M_+}) \oplus (\tilde{\mathscr{B}}_{N|M_-}/\mathscr{B}_{N|M_-}) .$$

PROOF. Since $\mathscr{B}\mathscr{O}$ is flabby with respect to the real coordinate $x_1$ (cf. Lemma 1.2 of [11]), there is an exact sequence

$$0 \longrightarrow \Gamma_Y(\mathscr{B}\mathscr{O}) \longrightarrow \mathscr{B}\mathscr{O}|_Y \longrightarrow \mathscr{B}\mathscr{O}_{Y|\tilde{M}_+} \oplus \mathscr{B}\mathscr{O}_{Y|\tilde{M}_-} \longrightarrow 0 .$$

Applying the functor $\boldsymbol{R}\Gamma_N$ (the right derived functor of $\Gamma_N$), we get a commutative diagram

$$\text{(1.1)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_N(\mathscr{B}_M) & \longrightarrow & \mathscr{B}_M|_N & \longrightarrow & \mathscr{B}_{N|M_+} \oplus \mathscr{B}_{N|M_-} & \longrightarrow & 0 \\ & & \| & & \downarrow \alpha & & \downarrow (\alpha_+, \alpha_-) & & \\ 0 & \longrightarrow & \Gamma_N(\mathscr{B}_M) & \longrightarrow & \tilde{\mathscr{B}}_{N|M} & \longrightarrow & \tilde{\mathscr{B}}_{N|M_+} \oplus \tilde{\mathscr{B}}_{N|M_-} & \longrightarrow & 0 \end{array}$$

with exact rows since $N$ is purely $(n-1)$-codimensional with respect to $\mathscr{B}\mathscr{O}|_Y$ and $\mathscr{B}\mathscr{O}_{Y|\tilde{M}_\pm}$ (cf. Proposition 1.3 of [11]). Since $\alpha$ and $\alpha_\pm$ are injective, the assertion follows from this diagram.

Now let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module ($\mathscr{D}_X$ denotes the sheaf on $X$ of rings of linear partial differential operators of finite order with holomorphic coefficients) for which $N$ (i.e. $Y$) is non-characteristic. Let $\mathscr{M}_Y = \mathscr{M}/f\mathscr{M}$ be the tangential system of $\mathscr{M}$ to $Y$. Then $\mathscr{M}_Y$ becomes a coherent $\mathscr{D}_Y$-module. We have defined (Theorem 2.1 of [11]) injective homomorphisms $\gamma_\pm$ (boundary value maps) as composites of the homomorphisms

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{N|M_\pm}) \underset{\alpha_\pm}{\longrightarrow} \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \tilde{\mathscr{B}}_{N|M_\pm}) \underset{\tilde{\gamma}_\pm}{\xrightarrow{\sim}} \mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N) .$$

PROPOSITION 1.1. *Let $u_\pm$ be sections of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{N|M_\pm})$ respectively. Then there exists a section $u$ of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M|_N)$ such that $r_\pm(u) = u_\pm$ if and only if $\gamma_+(u_+) = \gamma_-(u_-)$. Moreover such $u$ is unique.*

PROOF. By virtue of Holmgren's theorem for hyperfunctions (Theorem 2.1.3 of Sato-Kawai-Kashiwara [12, Chapter III]) we have

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) = 0 .$$

Hence from (1.1) we get a commutative diagram

$$
\begin{array}{ll}
 & 0 \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M|_N) \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \oplus \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_-}) \\
(1.2) & \qquad\qquad \downarrow \alpha \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow (\alpha_+, \alpha_-) \\
 & 0 \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M}) \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \oplus \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_-}) \\
 & \longrightarrow \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{B}_M)) \\
 & \qquad\qquad \| \\
 & \longrightarrow \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{B}_M))
\end{array}
$$

with exact rows. On the other hand, by the proof of Proposition 2.1 of [11], we have a commutative diagram of isomorphisms

$$
(1.3) \qquad
\begin{tikzcd}
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \arrow[dr, "\tilde{\gamma}_+"'] & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M}) \arrow[l, "\sim", "\tilde{r}_+"'] \arrow[r, "\sim"', "\tilde{r}_-"] \arrow[d, "\wr"] & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_-}) \arrow[dl, "\tilde{\gamma}_-"] \\
 & \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N) . &
\end{tikzcd}
$$

By (1.2) and (1.3) we get injective homomorphisms

$$
\begin{aligned}
&(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \oplus \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_-}))/\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M|_N) \\
\xrightarrow[(\alpha_+, \alpha_-)]{} \; &(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \oplus \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_-}))/\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M}) \\
\xrightarrow[\tilde{\gamma}_+ - \tilde{\gamma}_-]{\sim} \; &\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N) .
\end{aligned}
$$

This completes the proof.

Let $\rho : T^*X|_Y \to T^*Y$ be the canonical projection. For the sake of simplicity, we sometimes identify the purely imaginary cosphere bundles $\sqrt{-1}S^*M = S^*_M X$ and $\sqrt{-1}S^*N$ with $\sqrt{-1}\mathring{T}^*M$ and $\sqrt{-1}\mathring{T}^*N$ respectively by identifying the points in a same orbit of the action of $\boldsymbol{R}_+$; here $\sqrt{-1}\mathring{T}^*M$ denotes the purely imaginary cotangent bundle with the zero section removed, and $\boldsymbol{R}_+ = \{t \in \boldsymbol{R} ;\ t > 0\}$.

DEFINITION 1.2 (cf. [11, § 3]). Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module with respect to which $N$ is non-characteristic. We define subsets $V^{\pm}_{N,A}(\mathcal{M})$ and $V_{N,A}(\mathcal{M})$ of $\sqrt{-1}S^*N$ as follows: A point $x^* = (\mathring{x}, \sqrt{-1}\mathring{\xi}'\infty)$ of $\sqrt{-1}S^*N$ is not contained in $V^{\pm}_{N,A}(\mathcal{M})$ if and only if

(MH1) $\mathcal{M}$ is micro-hyperbolic relative to $\tilde{M}_{\pm}$ in the direction $\pm dx_1$ at each point $y^*$ of $\rho^{-1}(x^*)\cap\sqrt{-1}S^*M\cap \mathrm{SS}(\mathcal{M})$ in the sense of Definition 3.1 of Ôaku [11]; i.e. there exist a neighborhood $U$ of $y^*$ in $T^*X$ and an open cone $\Gamma$ of $\boldsymbol{C}_z^n\times\boldsymbol{C}_\zeta^n$ such that

$$(0\,;\,\mp 1, 0, \cdots, 0)\in\Gamma\,,$$

$$((U\cap\sqrt{-1}T^*M)+\Gamma)\cap U\cap \mathrm{SS}(\mathcal{M})\cap T^*X|_{\tilde{M}_{\pm}}=\emptyset\,,$$

where $z$ is an admissible local coordinate system of $X$, and $\zeta=(\zeta_1,\cdots,\zeta_n)$ are its dual variables.

(MH2) $\rho^{-1}(x^*)\cap \mathrm{cl}(\mathrm{SS}(\mathcal{M})\cap T^*X|_{\tilde{M}_{\pm}})\subset\{(\zeta_1, x^*)\in\rho^{-1}(x^*)\,;\,\pm\mathrm{Re}\,\zeta_1\geqq 0\}$,

where cl denotes the closure in $T^*X$.

We set $V_{N,A}(\mathcal{M})=V_{N,A}^{+}(\mathcal{M})\cup V_{N,A}^{-}(\mathcal{M})$.

DEFINITION 1.3. We set

$$V_{N,B}^{\pm}(\mathcal{M})=V_{N,A}^{\pm}(\mathcal{M})$$
$$\cup\{x^*\in\sqrt{-1}S^*N\,;\,\rho^{-1}(x^*)\cap \mathrm{cl}(\mathrm{SS}(\mathcal{M})\cap\sqrt{-1}T^*M|_{M_{\pm}})\neq\emptyset\}$$

and $V_{N,B}(\mathcal{M})=V_{N,B}^{+}(\mathcal{M})\cup V_{N,B}^{-}(\mathcal{M})$.

REMARK 1.1. It is easy to see that $V_{N,A}^{\pm}(\mathcal{M})$ and $V_{N,B}^{\pm}(\mathcal{M})$ are closed sets. When $\mathcal{M}$ is a single equation, these sets coincide with the sets of $A$- and $B$-boundary characteristic points defined by Kaneko [2] by virtue of the local version of Bochner's tube theorem.

Theorem 3.2 of Ôaku [11] implies the following:

PROPOSITION 1.2. *If $u_{\pm}$ is a real analytic (resp. hyperfunction) solution of $\mathcal{M}$ on $M_{\pm}$, then the singular spectrum $\mathrm{SS}(\gamma_{\pm}(u_{\pm}))$ of the boundary value of $u_{\pm}$ is contained in $V_{N,A}^{\mp}(\mathcal{M})$ (resp. $V_{N,B}^{\mp}(\mathcal{M})$).*

Now we can generalize Theorem 3.1 of Kaneko [2] to systems.

THEOREM 1.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module with respect to which $N$ is non-characteristic. Let $\mathring{x}$ be a point of $N$ and $\varphi$ be a real valued $C^1$-function on $N$ such that $\varphi(\mathring{x})=0$ and $d\varphi(\mathring{x})\neq 0$. Assume that $K$ is a closed subset of $N$ such that $\varphi\leqq 0$ on $K$ and that $V_{N,A}(\mathcal{M})$ (resp. $V_{N,B}(\mathcal{M})$) does not contain both of the points $(\mathring{x}, \pm\sqrt{-1}d\varphi(\mathring{x})\infty)\in\sqrt{-1}S^*N$. Then any real analytic (resp. hyperfunction) solution $u$ of $\mathcal{M}$ on $U\backslash K$, $U$ being an open neighborhood of $\mathring{x}$ in $M$, is uniquely continued to a neighborhood* *of $\mathring{x}$ as a hyperfunction solution of $\mathcal{M}$.*

PROOF. By Proposition 1.2, either one of the points $(\mathring{x}, \pm\sqrt{-1}d\varphi(\mathring{x})\infty)$ is disjoint from both of $SS(\gamma_\pm(u_\pm))$ with $u_\pm = u|_{M_\pm}$. In view of Proposition 1.1, $\gamma_+(u_+) - \gamma_-(u_-) = 0$ on $U \cap (N \setminus K)$. Then by virtue of the following version of Holmgren's uniqueness theorem (Lemma 1.2), we get $\gamma_+(u_+) - \gamma_-(u_-) = 0$ on a neighborhood of $\mathring{x}$ in $N$. Hence the statement of the theorem follows from Proposition 1.1.

LEMMA 1.2. *Let $\varphi$ be a real valued $C^1$ function on $\boldsymbol{R}^n$ such that $\varphi(0) = 0$ and $d\varphi(0) \neq 0$. If $u$ is a hyperfunction defined on a neighborhood of 0 with support contained in $\{x \in \boldsymbol{R}^n ; \varphi(x) \leqq 0\}$, and if its singular spectrum does not contain both of $(0, \pm\sqrt{-1}d\varphi(0)\infty)$, then $u$ vanishes on a neighborhood of 0.*

PROOF. This lemma follows immediately from Holmgren's theorem for hyperfunctions (Proposition 3.5.2 of Kashiwara-Kawai-Kimura [4]) and the method of sweeping out (see e.g. the geometric arguments in Kashiwara-Schapira [7]).

Under some additional conditions, we get continuation as real analytic solutions. We denote by $\pi_M : \sqrt{-1}\mathring{T}^*M \to M$ and $\pi_N : \sqrt{-1}\mathring{T}^*N \to N$ the canonical projections.

THEOREM 1.2. *Let $\mathcal{M}, \varphi, K$ be as in Theorem 1.1 and assume that $V_{N,A}(\mathcal{M})$ does not contain both of the points $(\mathring{x}, \pm\sqrt{-1}d\varphi(\mathring{x}))$. Assume, moreover, that for each point $x^*$ of $\pi_M^{-1}(\mathring{x})$, there exist $a, b \in \boldsymbol{R}$ with $b \geqq 0$ such that $adx_1 + bd\varphi$ is micro-hyperbolic for $\mathcal{M}$ at $x^*$ in the sense of Kashiwara-Schapira [7]. Then any real analytic solution $u$ of $\mathcal{M}$ on $U \setminus K$, $U$ being an open neighborhood of $\mathring{x}$ in $M$, is uniquely continued to a neighborhood of $\mathring{x}$ as a real analytic solution of $\mathcal{M}$.*

PROOF. First we can continue $u$ to a neighborhood of $\mathring{x}$ as a hyperfunction solution $\tilde{u}$ in view of Theorem 1.1. Since the singular spectrum of $\tilde{u}$ is contained in $\{(x, \sqrt{-1}\xi) ; ax_1 + b\varphi(x') \leqq 0\}$, $\tilde{u}$ becomes real analytic by virtue of Theorem 2.2.1 of Kashiwara-Schapira [7].

Now let us consider isolated singularities of real analytic solutions. The following is a generalization of Theorem I of Kaneko [3]:

THEOREM 1.3. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module with respect to which*

*$N$ is non-characteristic and $\mathring{x}$ be a point of $N$. Assume that $\pi_N^{-1}(\mathring{x})$ is not contained in $V_{N,A}(\mathcal{M})$. Then any real analytic solution of $\mathcal{M}$ on $U\backslash\{\mathring{x}\}$, $U$ being an open neighborhood of $\mathring{x}$ in $M$, is uniquely continued to $U$ as a hyperfunction solution of $\mathcal{M}$. Moreover if, for any point $x^*$ of $\pi_N^{-1}(\mathring{x})$, there exists a vector $v\in \boldsymbol{R}^n\backslash\{0\}$ such that $\langle v, dx\rangle$ is micro-hyperbolic for $\mathcal{M}$ at $x^*$, then $u$ is continued to $U$ as a real analytic solution of $\mathcal{M}$.*

PROOF. There exists a real valued real analytic function $\varphi$ on $N$ such that $\varphi(\mathring{x})=0$ and $d\varphi(\mathring{x})\neq 0$ and that $(\mathring{x}, \sqrt{-1}d\varphi(\mathring{x}))$ is not contained in $V_{N,A}(\mathcal{M})$. Hence $u$ is continued to $U$ as a hyperfunction solution $\tilde{u}$ by virtue of Theorem 1.1. Under the additional condition, micro-analyticity of $\tilde{u}$ propagates up to $\mathring{x}$ from outside by virtue of Theorem 2.2.1 of [7].

*Example* 1.1. Let $M$ be an open set in $\boldsymbol{R}^4$ containing 0 and consider a system $\mathcal{M}$ defined by

$$\mathcal{M} : (D_1^3+D_3^3)u=D_2(D_3^2+D_4^2)u=0$$

with $D_j=\partial/\partial x_j$. Then any real analytic solution $u(x)$ of $\mathcal{M}$ defined on $M\backslash\{x\in M\,;\, x_1=0,\ x_2\leqq 0\}$ is continued to a neighborhood of 0 as a real analytic solution. In fact, it is easy to see that $(0, \sqrt{-1}dx_2\infty)$ is not contained in $V_{N,A}(\mathcal{M})$ (cf. Example 3.1 of [11]) and that $dx_1$ is micro-hyperbolic for $\mathcal{M}$ (in the sense of [7]) at $x^*=(0, \sqrt{-1}\langle\xi, dx\rangle\infty)\in\sqrt{-1}S^*M$ if $\xi_2\neq 0$, while $dx_2$ is micro-hyperbolic for $\mathcal{M}$ at $x^*$ if $\xi_2=0$.

*Example* 1.2. Let $M$ be as in Example 1.1 and consider

$$\mathcal{M} : ((D_1+\sqrt{-1}x_1^{2k-1}D_2)^m+D_3^m)u=(D_3+\sqrt{-1}D_4)u=0$$

with positive integers $k$ and $m$. Then any hyperfunction solution of $\mathcal{M}$ on $M\backslash\{x\in\boldsymbol{R}^4\,;\, x_1=0,\ x_2\leqq 0\}$ is uniquely continued to a neighborhood of 0 as a hyperfunction solution of $\mathcal{M}$. In fact, $(0, \sqrt{-1}dx_2)\notin V_{N,B}(\mathcal{M})$ holds in this case. Note that $\pm dx_1$ are not micro-hyperbolic for $\mathcal{M}$ at $(0, \sqrt{-1}dx_2)$ in the sense of [7].

The condition of micro-hyperbolicity in Theorems 1.2 and 1.3 is not always necessary :

*Example* 1.3. Let $M$ be as in Example 1.1 and put

$$P=D_1(D_1+\sqrt{-1}x_1D_2)+\sum_{j=1}^{4}a_j(x)D_j+b(x)\,;$$

here $a_j$ and $b$ are real analytic functions on $M$ such that $a_2(0)\notin\sqrt{-1}\boldsymbol{N}$

with $\boldsymbol{N}=\{0,1,2,\cdots\}$. Then any real analytic solution of the system

$$\mathcal{M}: Pu=(D_3+\sqrt{-1}D_4)u=0$$

on $M\backslash\{(0,x_2)\,;\,x_2\leqq 0\}$ is continued to a neighborhood of 0 as a real analytic solution. In fact, since $(0,\sqrt{-1}dx_2\infty)\notin V_{N,A}(\mathcal{M})$, the solution is continued to a neighborhood of 0 as a hyperfunction solution in view of Theorem 1.1. Then analyticity propagates up to 0 by Theorem 2.1 of Ôaku [9] (cf. Example 2.5 of [9]). Note that, in general, there are no $a, c\in\boldsymbol{R}$ such that $adx_1+cdx_2$ is micro-hyperbolic for $\mathcal{M}$ at $(0,\pm\sqrt{-1}dx_2)$ in the sense of [7].

REMARK 1.2. Theorems 1.1–1.3 still hold even if we replace $V_{N,*}(\mathcal{M})$ by

$$V_{N,*}(\mathcal{M})\cap\mathrm{Supp}(\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N))$$

for $*=A, B$; here $\mathcal{C}_N$ denotes the sheaf on $\sqrt{-1}S^*N$ of microfunctions, and Supp denotes the support of the sheaf. (This assertion follows immediately from the arguments above.) This makes no change for single equations, but for overdetermined systems, this provides, in some cases, another sufficient condition for continuation of solutions. For example, any hyperfunction (resp. real analytic) solution of the system

$$\mathcal{M}: (D_1+\sqrt{-1}D_2)u=(D_3+\sqrt{-1}x_3D_4)u=0$$

defined on $U\backslash\{x\in\boldsymbol{R}^4\,;\,x_1=0,\,x_4\leqq 0\}$ with an open neighborhood $U$ of 0 in $\boldsymbol{R}^4$ is continued to a neighborhood of 0 as a hyperfunction (resp. real analytic) solution. In fact, $\mathcal{M}_Y=\mathcal{M}/z_1\mathcal{M}$ is microlocally analytic hypo-elliptic (i.e. $\mathcal{H}om(\mathcal{M}_Y, \mathcal{C}_N)=0$) at $(0,\sqrt{-1}dx_4\infty)$, while both of $(0,\pm\sqrt{-1}dx_4\infty)$ belong to $V_{N,A}(\mathcal{M})$. Note also that the solution has $x_1+\sqrt{-1}x_2$ as a holomorphic parameter. See, e.g., Sato-Kawai-Kashiwara [12, Chapter III], Kashiwara-Kawai-Oshima [5], Ôaku [9] as to (sufficient) conditions for systems to be microlocally analytic hypo-elliptic.

## § 2. Boundary value problem for Fuchsian equations.

Let $M, M_\pm, N, X, \tilde{M}_\pm, Y$ be as in § 1. For an admissible local coordinate system $z=x+\sqrt{-1}y=(z_1, z')$, we use the notation $D=(D_1, D')$, $D'=(D_2,\cdots,D_n)$, where $D_j=\partial/\partial z_j$ in the complex domain, and $D_j=\partial/\partial x_j$ in the real domain (there will be no fear of confusion).

We assume that a linear partial differential operator $P$ with real analytic coefficients defined on $M$ is a *Fuchsian operator of weight $m-k$ with respect to $N$* in the sense of Baouendi-Goulaouic [1]: i.e. in a neighborhood of each point of $N$, $P$ is written in the form

$$P=a(x)(x_1^kD_1^m+A_1(x,D')x_1^{k-1}D_1^{m-1}+\cdots+A_k(x,D')D_1^{m-k}+\cdots+A_m(x,D'))$$

with an admissible local coordinate system (cf. § 1) $z=x+\sqrt{-1}y$; here $a(x)$ is a non-vanishing real analytic function, $k$ and $m$ are integers with $0\leqq k\leqq m$, $A_j(x,D')$ is an operator of order $\leqq j$ for $1\leqq j\leqq m$, and $A_j(0,x',D')$ is of order 0, i.e. equals a real analytic function $a_j(x')$, for $1\leqq j\leqq k$. The roots $\lambda=0,1,\cdots,m-k-1,\ \lambda_1(x'),\cdots,\lambda_k(x')$ of the *indicial equation*

$$\lambda(\lambda-1)\cdots(\lambda-m+1)+a_1(x')\lambda(\lambda-1)\cdots(\lambda-m+2)+\cdots$$
$$+a_k(x')\lambda(\lambda-1)\cdots(\lambda-m+k+1)=0$$

are called the *characteristic exponents* of $P$. For a point $\mathring{x}=(0,\mathring{x}')$ of $N$, we define a condition $\mathrm{C}(\mathring{x})$ by

$$\mathrm{C}(\mathring{x}):\lambda_i(\mathring{x}')\notin \boldsymbol{Z},\ \lambda_i(\mathring{x}')-\lambda_j(\mathring{x}')\notin \boldsymbol{Z}\backslash\{0\}\qquad \text{for any } 1\leqq i,\ j\leqq k\,.$$

The notion of Fuchsian operators and the characteristic exponents are independent of (not necessarily admissible) local coordinate systems fixing $N$. However, in this paper we fix an admissible local coordinate system since the problems of continuation of solutions treated in § 3 will be of local character.

We set $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_XP$; i.e. $\mathcal{M}$ is the equation $Pu=0$. Then, e.g., $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_{N|M_+})$ is the sheaf of $\mathcal{B}_{N|M_+}$-solutions of $\mathcal{M}$.

Let us recall the definition of the microlocalizations $\mathcal{C}_{N|M_\pm}$ and $\tilde{\mathcal{C}}_{N|M_\pm}$ of $\mathcal{B}_{N|M_\pm}$ and $\tilde{\mathcal{B}}_{N|M_\pm}$ respectively.

DEFINITION 2.1 (cf. [11]). Let $\pi_{M/\tilde{M}}:(\tilde{M}\backslash M)\cup S_M^*\tilde{M}\to\tilde{M}$ and $\pi_{N/Y}:(Y\backslash N)\cup S_N^*Y\to Y$ be comonoidal transformations (cf. [12]). We define sheaves $\mathcal{C}_{M_\pm}$ on $S_M^*\tilde{M}$, and $\mathcal{C}_{N|M_\pm}$ and $\tilde{\mathcal{C}}_{N|M_\pm}$ on $\sqrt{-1}S^*N=S_N^*Y$ by

$$\mathcal{C}_{M_\pm}=\mathcal{H}^{n-1}_{S_M^*\tilde{M}}((\pi_{M/\tilde{M}})^{-1}(\tilde{\iota}_\pm)_*(\tilde{\iota}_\pm)^{-1}\mathcal{BO})^a\,,\qquad \mathcal{C}_{N|M_\pm}=\mathcal{C}_{M_\pm}|_{S_N^*Y}\,,$$
$$\tilde{\mathcal{C}}_{N|M_\pm}=\mathcal{H}^{n-1}_{S_N^*Y}((\pi_{N/Y})^{-1}\mathcal{BO}_{Y|\tilde{M}_\pm})^a\,,$$

where $a$ denotes the antipodal map, and $S_N^*Y$ is regarded naturally as a subset of $S_M^*\tilde{M}$.

There exist short exact sequences (Propositions 1.7 and 1.8 of [11])

$$0\longrightarrow(\tilde{\iota}_\pm)_*(\tilde{\iota}_\pm)^{-1}\mathcal{BO}|_M\longrightarrow(\iota_\pm)_*(\iota_\pm)^{-1}\mathcal{B}_M\xrightarrow{\mathrm{sp}_\pm}(\pi_{M/\tilde{M}})_*\mathcal{C}_{M_\pm}\longrightarrow 0\,,$$
$$0\longrightarrow\mathcal{BO}_{Y|\tilde{M}_\pm}|_N\longrightarrow\tilde{\mathcal{B}}_{N|M_\pm}\xrightarrow{\widetilde{\mathrm{sp}}_\pm}(\pi_N)_*\tilde{\mathcal{C}}_{N|M_\pm}\longrightarrow 0$$

($\mathrm{sp}_\pm$ and $\widetilde{\mathrm{sp}}_\pm$ are called spectral maps), and injective homomorphisms

$\alpha_\pm : C_{N|M_\pm} \to \tilde{C}_{N|M_\pm}$ compatible with $\alpha_\pm : \mathcal{B}_{N|M_\pm} \to \tilde{\mathcal{B}}_{N|M_\pm}$.

DEFINITION 2.2 (Proposition 2.4 of [11]). The sections of the sheaf $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$ over an open subset $\Omega$ of $Y$ are the operators

$$P(z, D') = \sum_{j=0}^{\infty} P_j(z, D')$$

satisfying the following conditions:

(i) $P_j(z, \zeta')$ is holomorphic on a neighborhood of $\Omega \times \boldsymbol{C}^{n-1}$ in $X \times \boldsymbol{C}^{n-1}$, and homogeneous (hence a polynomial) of degree $j$ with respect to $\zeta'$.

(ii) For any compact set $K$ of $Y$, and for any $\varepsilon > 0$, there exist $C > 0$ and $\delta > 0$ such that

$$|P_j(z, \zeta')| \leqq \frac{C}{j!}(\varepsilon|\zeta'|)^j$$

for any $j$ and $\zeta'$ if $|z_1| < \delta$, $z' \in K$.

Note that $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$ has a natural ring structure and that it acts on $\mathcal{B}\mathcal{O}|_Y$ and $\mathcal{B}\mathcal{O}_{Y|\tilde{M}_\pm}$, and hence on $\tilde{\mathcal{B}}_{N|M}$, $\tilde{\mathcal{B}}_{N|M_\pm}$, and $\tilde{C}_{N|M_\pm}$ (cf. [11]).

THEOREM 2.1. *Let $\mathcal{M}$ be as above and assume the condition* C($\mathring{x}$) *for a point $\mathring{x}$ of $N$. Then, on a neighborhood of $\mathring{x}$, there exist injective homomorphisms (boundary value maps)*

$$\gamma_\pm : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_\pm}) \longrightarrow (\mathcal{B}_N)^m,$$

$$\gamma_\pm : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_{N|M_\pm}) \longrightarrow (C_N)^m,$$

*and these maps commute with the spectral maps.* (*We decompose $\gamma_\pm$ into the form $\gamma_\pm = (\gamma_{\pm\mathrm{reg}}, \gamma_{\pm\mathrm{sing}})$ with*

$$\gamma_{\pm\mathrm{reg}} : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_\pm}) \longrightarrow (\mathcal{B}_N)^{m-k} \ \ (\textit{or}\ \ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_{N|M_\pm}) \longrightarrow (C_N)^{m-k})$$

$$\gamma_{\pm\mathrm{sing}} : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_\pm}) \longrightarrow (\mathcal{B}_N)^{k} \ \ (\textit{or}\ \ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_{N|M_\pm}) \longrightarrow (C_N)^{k}),$$

*and the meaning of this decomposition will be clarified in Propositions* 2.1 *and* 3.1.)

PROOF. Let us assume $k < m$ since the case $k = m$ is easier. We use the technique due to Tahara (see the proof of Theorem 1.2.12 of [13]). As a $\mathcal{D}_X$-module, the equation $z_1 Pu = 0$ is equivalent to a system

$$\left(\begin{pmatrix} D_1 I_{m-k-1} & 0 \\ 0 & z_1 D_1 I_{k+1} \end{pmatrix} - \begin{pmatrix} A_{11} & A_{12} \\ z_1 A_{21} & A_{22} \end{pmatrix}\right)\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} = 0,$$

where $I_\nu$ is the $\nu\times\nu$ unit matrix, $A_{ij}$ are $m_i\times m_j$ matrices with $m_1=m-k$, $m_2=k$ defined by

$$A_{11}=\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix},\quad A_{12}=\begin{pmatrix} 0 & & \\ \vdots & & 0 \\ 0 & & \\ 1 & 0\cdots\cdots & 0 \end{pmatrix},$$

$$A_{21}=\begin{pmatrix} 0 & \cdots\cdots\cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ -A_m & \cdots & -A_{k+1} \end{pmatrix},\quad A_{22}=\begin{pmatrix} 1 & 1 & & & \\ & 2 & \ddots & & \\ & & \ddots & 1 & \\ & & & k-1 & 1 \\ -A_k & \cdots & & -A_2, & -A_1+k \end{pmatrix},$$

by the relations

$$u^{(1)}=\begin{pmatrix} u \\ D_1u \\ \vdots \\ D_1^{m-k-1}u \end{pmatrix},\qquad u^{(2)}=\begin{pmatrix} z_1D_1^{m-k}u \\ z_1^2D_1^{m-k+1}u \\ \vdots \\ z_1^kD_1^{m-1}u \end{pmatrix}.$$

We define a system $\mathcal{L}$ by

$$\mathcal{L}:(z_1D_1I_m-B)\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}=0$$

with

$$B=\begin{pmatrix} z_1I_{m-k-1} & 0 \\ 0 & I_{k+1} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ z_1A_{21} & A_{22} \end{pmatrix}=\begin{pmatrix} z_1A_{11} & A_{12} \\ z_1A_{21} & A_{22} \end{pmatrix}.$$

Then $\mathcal{L}$ is a Fuchsian system in the sense of Tahara [13]. In fact

$$B_0(z')=B(0,z',D')=\begin{pmatrix} 0 & A_{12} \\ 0 & E(z') \end{pmatrix}$$

is a matrix of holomorphic functions in $z'$ by the definition of Fuchsian operators, where $E(z')=A_{22}(0,z',D')$. It is easy to see

$$\det(\lambda I_m-B_0(z'))=\lambda^{m-k}\{(\lambda-1)(\lambda-2)\cdots(\lambda-k)$$
$$+(\lambda-1)(\lambda-2)\cdots(\lambda-k+1)a_1(z')+\cdots+(\lambda-1)a_{k-1}(z')+a_k(z')\}.$$

Hence the eigenvalues of $B_0(z')$ are 0 (with multiplicity $m-k$) and $\lambda_j(z')-m+k+1$ ($j=1,\cdots,k$). Under the condition C($\mathring{x}$), there exists an invertible $m\times m$ matrix $Q=Q(z,D')$ of germs of $\mathcal{O}\widetilde{\mathcal{D}}_{Y|\tilde{X}}$ at $\mathring{x}$ such that

$$Q^{-1}(z_1D_1I_m-B)Q=z_1D_1I_m-B_0,\qquad Q(0,z',D')=I_m$$

by virtue of Theorem 1.3.6 of Tahara [13]. Put

$$R=Q\begin{pmatrix} I_{m-k} & A_{12}E(z')^{-1} \\ 0 & I_k \end{pmatrix}.$$

Then we have

$$R^{-1}(z_1D_1I_m-B)R=z_1D_1I_m-\begin{pmatrix} 0 & 0 \\ 0 & E(z') \end{pmatrix}. \tag{2.1}$$

Hence any $\tilde{\mathscr{B}}_{N|M_+}$-solution of $\mathscr{L}$ is uniquely written in the form

$$\begin{pmatrix} u_+^{(1)} \\ u_+^{(2)} \end{pmatrix}=R\begin{pmatrix} a_+(x') \\ x_1^{E(x')}b_+(x') \end{pmatrix}$$

with $a_+(x')\in(\mathscr{B}_N)^{m-k}$, $b_+(x')\in(\mathscr{B}_N)^k$ on a neighborhood of $\mathring{x}$. We get an isomorphism

$$\tilde{\gamma}_+:\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\tilde{\mathscr{B}}_{N|M_+})\xrightarrow{\sim}(\mathscr{B}_N)^m$$

on a neighborhood of $\mathring{x}$ defined by

$$\tilde{\gamma}_+(u_+)=\begin{pmatrix} \tilde{\gamma}_{+\mathrm{reg}}(v_+) \\ \tilde{\gamma}_{+\mathrm{sing}}(v_+) \end{pmatrix}=\begin{pmatrix} a_+(x') \\ b_+(x') \end{pmatrix}$$

since $x_1:\tilde{\mathscr{B}}_{N|M_+}\to\tilde{\mathscr{B}}_{N|M_+}$ is an isomorphism. Combining $\tilde{\gamma}_+$ with the injective homomorphism $\alpha_+$, we get an injective homomorphism

$$\gamma_+:\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_{N|M_+})\longrightarrow(\mathscr{B}_N)^m$$

with $\gamma_+(u_+)=(\gamma_{+\mathrm{reg}}(u_+),\ \gamma_{+\mathrm{sing}}(u_+))\in(\mathscr{B}_N)^{m-k}\oplus(\mathscr{B}_N)^k$. $\gamma_-$ is defined in the same way. Since $\tilde{C}_{N|M_\pm}$ are $\mathscr{O}\tilde{\mathscr{D}}_{Y|\tilde{M}}$-modules, the above argument also holds even if we replace $\mathscr{B}_{N|M_\pm}$, $\tilde{\mathscr{B}}_{N|M_\pm}$, $\mathscr{B}_N$ by $C_{N|M_\pm}$, $\tilde{C}_{N|M_\pm}$, $C_N$ respectively. This completes the proof.

Now let us recall the definition of F-mild hyperfunctions introduced in Ôaku [10]:

DEFINITION 2.3. Let $z$ be an admissible local coordinate system around $\mathring{x}\in N$. Then a section $f_\pm$ of $\mathscr{B}_{N|M_\pm}$ is F-*mild at* $\mathring{x}$ if and only if there exist an integer $J$, open convex cones $\Gamma_j$ of $\boldsymbol{R}^{n-1}$, $\varepsilon>0$, and holomorphic functions $F_j(z)$ $(j=1,\cdots,J)$ on a neighborhood of

$$\{z\in\boldsymbol{C}^n\,;\,|z-\mathring{x}|<\varepsilon,\ \pm z_1=\pm x_1\geqq 0,\ \mathrm{Im}\,z'\in\Gamma_j\}$$

such that on $\{x\in M_\pm\,;\,|x-\mathring{x}|<\varepsilon\}$,

$$f_\pm(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\Gamma_j 0)\,.$$

We denote by $\mathscr{B}^F_{N|M_\pm}$ the subsheaves of $\mathscr{B}_{N|M_\pm}$ consisting of sections which are F-mild at each point of $N$.

The notion of F-mildness is invariant under local coordinate transformations fixing $N$; for a section $f_\pm(x)$ of $\mathscr{B}^F_{N|M_\pm}$, the boundary value $f_\pm(\pm 0, x')$ is naturally defined as a hyperfunction on $N$ (cf. [10]).

PROPOSITION 2.1. *Let $\mathscr{M}$ be the equation $Pu=0$ as above and assume the condition* C$(\mathring{x})$ *for a point $\mathring{x}$ of $N$. Let $u_\pm(x)$ be a $\mathscr{B}_{N|M_\pm}$-solution of $\mathscr{M}$. Then $u_\pm(x)$ is* F-*mild at $\mathring{x}$ if and only if $\gamma_{\pm\mathrm{sing}}(u_\pm)=0$ on a neighborhood of $\mathring{x}$. Moreover, if $u_\pm(x)$ is* F-*mild, then $\gamma_{\pm\mathrm{reg}}(u_\pm)$ is equal to the set of boundary values*

$$(u_\pm(\pm 0, x'), \cdots, D_1^{m-k-1}u_\pm(\pm 0, x'))$$

*in the sense of* F-*mild hyperfunctions.*

PROOF. We define a sheaf $\tilde{\mathscr{B}}^A$ on $N$ by

$$\tilde{\mathscr{B}}^A=\mathscr{H}^{n-1}_N(\mathscr{O}_X|_Y)\,,$$

where $\mathscr{O}_X$ denotes the sheaf on $X$ of holomorphic functions. Then there exist natural injective homomorphisms

$$\beta_\pm : \tilde{\mathscr{B}}^A \longrightarrow \tilde{\mathscr{B}}_{N|M_\pm}$$

(cf. Lemma 2.1 of [11]). Moreover $\alpha_\pm$ induce injective homomorphisms

$$\mathscr{B}_{N|M_\pm}/\mathscr{B}^F_{N|M_\pm} \longrightarrow \tilde{\mathscr{B}}_{N|M_\pm}/\beta_\pm(\tilde{\mathscr{B}}^A)$$

(cf. Proposition 2.3 of [11]). Thus a section $f_\pm$ of $\mathscr{B}_{N|M_\pm}$ is F-mild if and only if $\alpha_\pm(f)$ is contained in the image of $\beta_\pm$.

When $k=m$, $u_\pm(x)$ is F-mild if and only if $u_\pm(x)=0$ (Theorem 2 of Ôaku [10]). Hence let us assume $k<m$ and use the same notation as in the proof of Theorem 2.1. Now let $u_\pm$ be a $\mathscr{B}_{N|M_\pm}$-solution of $\mathscr{M}$ and set

$$\gamma_{\pm\mathrm{reg}}(u_\pm)=a_\pm(x')\,, \qquad \gamma_{\pm\mathrm{sing}}(u_\pm)=b_\pm(x')\,.$$

Then $u_\pm(x)$ is the first component of the vector

$$R(x, D')\begin{pmatrix} a_\pm(x') \\ x_1^{E(x')}b_\pm(x')\end{pmatrix}.$$

It is easy to see that $\tilde{\mathscr{B}}^A$ is an $\mathcal{O}\tilde{\mathscr{D}}_{Y|\tilde{M}}$-module. Hence $u_\pm$ is F-mild if and only if $a_\pm(x')$ and $v_\pm^{(2)}(x)=x_1^{E(x')}b_\pm(x')$ are contained in the image of $\beta_\pm$.

Let us consider a system

$$\mathcal{N}^{(2)}:(z_1D_1I_k-E(z'))v^{(2)}=0\,.$$

Since $E(\mathring{x})$ has no integer as an eigenvalue, we get

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{N}^{(2)};\tilde{\mathscr{B}}^A)=\boldsymbol{R}\Gamma_N(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{N}^{(2)};\mathcal{O}_X|_Y))[n-1]=0\,,$$

where $\boldsymbol{R}\mathscr{H}om$ denotes the right derived functor of $\mathscr{H}om$. Thus $v_\pm^{(2)}(x)$ is contained in the image of $\beta_\pm$ if and only if $b_\pm(x')=0$ since $v_\pm^{(2)}(x)$ is a solution of $\mathcal{N}^{(2)}$. On the other hand, $a_\pm(x')$ is always contained in the image of $\beta_\pm$. Thus $u_\pm(x)$ is F-mild if and only if $b_\pm(x')=0$.

Now assume $b_\pm(x')=0$. Let us write $R$ in the form

$$R=\begin{pmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{pmatrix}$$

with $m_i\times m_j$ matrices $R_{ij}$. Then we have

$$\alpha_\pm\left(\begin{pmatrix}u_\pm(x)\\ D_1u_\pm(x)\\ \vdots\\ D_1^{m-k-1}u_\pm(x)\end{pmatrix}\right)=R_{11}(x,D')a_\pm(x') \tag{2.2}$$

as sections of $\tilde{\mathscr{B}}_{N:M_\pm}$. Since $u_\pm(x)$ is F-mild, the components of both sides of (2.2) are contained in the image of $\beta_\pm$. Hence we can regard (2.2) as an equality in $\tilde{\mathscr{B}}^A$. Applying the natural homomorphism

$$\tilde{\mathscr{B}}^A\longrightarrow\mathscr{B}_N\cong\tilde{\mathscr{B}}^A/x_1\tilde{\mathscr{B}}^A\,,$$

which is compatible with the boundary value homomorphism

$$\mathscr{B}^F_{N:M_\pm}\longrightarrow\mathscr{B}_N\cong\mathscr{B}^F_{N|M_\pm}/x_1\mathscr{B}^F_{N|M_\pm}$$

defined by $u_\pm(x)\mapsto u_\pm(\pm0,x')$, we get

$$a_\pm(x')={}^t(u_\pm(\pm0,x'),\cdots,D_1^{m-k-1}u(\pm0,x'))$$

since $R_{11}(0,x',D')=I_{m-k}$. This completes the proof.

DEFINITION 2.4. Let $P$ be a Fuchsian operator of weight $m-k$ with respect to $N$. We define sets of *A-boundary characteristic points* $V^*_{N,A}(P)\subset\sqrt{-1}S^*N$ with $*=+,-,\emptyset$ as follows: A point $x^*=(\mathring{x},\sqrt{-1}\mathring{\xi}'\infty)$ of $\sqrt{-1}S^*N$ is not contained in $V^\pm_{N,A}(P)$ if and only if there exists $\varepsilon>0$ such that $\sigma(P)(x_1, \zeta_1, \sqrt{-1}\xi')\neq 0$ for any $x\in M_{\pm}$ with $|x-\mathring{x}|<\varepsilon$, $\xi'\in \boldsymbol{R}^{n-1}$ with $|\xi'-\mathring{\xi}'|<\varepsilon$, and for any $\zeta_1\in \boldsymbol{C}$ with $\pm\mathrm{Re}\,\zeta_1<0$; here $\sigma(P)$ denotes the principal symbol of $P$. We put

$$V_{N,A}(P)=V_{N,A}^{+}(P)\cup V_{N,A}^{-}(P)\,.$$

DEFINITION 2.5. We define sets of *B-boundary characteristic points* $V_{N,B}^{*}(P)\subset\sqrt{-1}S^{*}N$ with $*=+,\,-,\,\emptyset$ by replacing the condition $\pm\mathrm{Re}\,\zeta_1<0$ in Definition 2.4 by $\pm\mathrm{Re}\,\zeta_1\leqq 0$.

REMARK. When $k=0$, the sets defined above coincide with the sets of $A$- and $B$-boundary characteristic points defined by Kaneko [2].

We denote by $p:\sqrt{-1}S^{*}M\backslash S_{\tilde{M}}^{*}X\to S_{M}^{*}\tilde{M}$ the canonical map. Then there exist natural sheaf homomorphisms

$$\phi_{\pm}: p^{-1}(\mathcal{C}_{M_{\pm}}|_{L_{\pm}})\longrightarrow \mathcal{C}_{M}|_{p^{-1}(L_{\pm})}\,,$$

such that $\phi_{\pm}(\mathrm{sp}_{\pm}(f_{\pm}))=\mathrm{sp}(f_{\pm})$ for sections $f_{\pm}$ of $\mathcal{B}_{M_{\pm}}=\mathcal{B}_{M}|_{M_{\pm}}$; here $\mathcal{C}_{M}$ denotes the sheaf on $\sqrt{-1}S^{*}M$ of microfunctions, $\mathrm{sp}:\pi_{M}^{-1}\mathcal{B}_{M}\to\mathcal{C}_{M}$ is the spectral map, and $L_{\pm}=S_{M}^{*}\tilde{M}|_{M_{\pm}}$.

THEOREM 2.2. *Assume the condition* $\mathrm{C}(\mathring{x})$ *with a point* $\mathring{x}=(0,\mathring{x}')$ *of* $N$ *and suppose that a point* $x^{*}=(\mathring{x},\sqrt{-1}\mathring{\xi}')$ *of* $\sqrt{-1}S^{*}N$ *is not contained in* $V_{N,A}^{\pm}(P)$. *Let* $u_{\pm}(x)$ *be a section of* $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{M_{\pm}})$ *on a neighborhood of* $x^{*}$ *in* $S_{M}^{*}\tilde{M}$ *such that* $\phi_{\pm}(u_{\pm})$ *vanishes on* $U\cap L_{\pm}$. *Then* $\gamma_{\pm}(u_{\pm})$ *vanishes on a neighborhood of* $x^{*}$, *where*

$$\gamma_{\pm}:\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|M_{\pm}})\longrightarrow(\mathcal{C}_{N})^{m}$$

*are the boundary value homomorphisms defined in Theorem* 2.1.

PROOF. Let $\mathcal{L}$ be the system defined in the proof of Theorem 2.1. Then $\mathcal{L}$ is a Fuchsian system in the sense of Tahara [13], and no pair of the eigenvalues of $B_0(\mathring{x})$ differ by a non-zero integer. Hence this theorem follows immediately from Theorem 3.3 (cf. also the remark after it) of Ôaku [11].

COROLLARY 2.1. *Under the condition* $\mathrm{C}(\mathring{x})$ *with an* $\mathring{x}\in N$, *let* $u_{\pm}(x)$ *be a real analytic* (*resp. hyperfunction*) *solution of* $\mathcal{M}$ *on* $M_{\pm}$. *Then the singular spectrum of* $\gamma_{\pm}(u_{\pm})$ *is contained in* $V_{N,A}^{\pm}(P)$ (*resp. in* $V_{N,B}^{\pm}(P)$).

## § 3. Continuation of solutions of Fuchsian equations.

In this section we study the continuation of solutions of Fuchsian equations by using the theory of boundary value problem for Fuchsian equations developed in § 2, and the theory of micro-hyperbolic boundary value problem developed in [11]. We use the same notation as in § 2.

PROPOSITION 3.1. *Let $\mathcal{M}$ be the equation $Pu=0$ with a Fuchsian partial differential operator $P$ of weight $m-k$ with respect to $N$. Assume the condition* C$(\mathring{x})$ *for a point $\mathring{x}\in N$. Let $u_\pm(x)$ be $\mathcal{B}_{N|M_\pm}$-solutions of $\mathcal{M}$. Then there exists a hyperfunction solution $u(x)$ of $\mathcal{M}$ defined on a neighborhood of $\mathring{x}$ in $M$ such that $u(x)=u_\pm(x)$ on $M_\pm$ if and only if $\gamma_{+\mathrm{reg}}(u_+)=\gamma_{-\mathrm{reg}}(u_-)$ holds on a neighborhood of $\mathring{x}$ in $N$. Moreover, such $u(x)$ is unique.*

PROOF. We use the same notation as in the proof of Theorem 2.1. Then, as a $\mathcal{D}_X$-module, $\mathcal{M}$ is equivalent to a system

$$\mathcal{M}' : \left(\begin{pmatrix} D_1 I_{m-k} & 0 \\ 0 & z_1 D_1 I_k \end{pmatrix} - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}-I_k \end{pmatrix}\right) v=0$$

by the relations

$$v=\begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix}, \quad v^{(1)}=\begin{pmatrix} u \\ \vdots \\ D_1^{m-k-1}u \end{pmatrix}, \quad v^{(2)}=\begin{pmatrix} D_1^{m-k} \\ \vdots \\ z_1^{k-1}D_1^{m-1}u \end{pmatrix}.$$

We define a system $\mathcal{N}$ by

$$\mathcal{N} : (D_1 z_1 I_m - B)w=0 .$$

Note that

$$D_1 z_1 I_m - B = \left(\begin{pmatrix} D_1 I_{m-k} & 0 \\ 0 & z_1 D_1 I_k \end{pmatrix} - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}-I_k \end{pmatrix}\right) T(z_1)$$

with

$$T(z_1)=\begin{pmatrix} z_1 I_{m-k} & 0 \\ 0 & I_k \end{pmatrix}.$$

Now let $v_\pm(x)$ be $\mathcal{B}_{N|M_\pm}$-solutions of $\mathcal{M}'$ and put

$$w_\pm(x)=T(x_1)^{-1}v_\pm(x) .$$

Let us prove that there exists a $\mathcal{B}_M|_N$-solution $v(x)$ of $\mathcal{M}'$ such that $v(x)=v_\pm(x)$ on $M_\pm$ if and only if there exists a $\mathcal{B}_M|_N$-solution $w(x)$ of $\mathcal{N}$ such that $w(x)=w_\pm(x)$ on $M_\pm$. First let us assume that there exists such $v(x)$. Then there exists a column vector $w(x)$ of $\mathscr{B}_M|_N$ such that

$$v(x)=T(x_1)w(x)$$

since $x_1:\mathscr{B}_M\to\mathscr{B}_M$ is surjective (cf. [4]). Then $w(x)$ is a $\mathscr{B}_M|_N$-solution of $\mathscr{N}$ such that $w(x)=w_\pm(x)$ on $M_\pm$. Next let us assume there exists a $\mathscr{B}_M|_N$-solution $w(x)$ of $\mathscr{N}$ such that $w(x)=w_\pm(x)$ on $M_\pm$. Then $v(x)=T(x)w(x)$ is a $\mathscr{B}_M|_N$-solution of $\mathscr{M}'$ such that $v(x)=v_\pm(x)$ on $M_\pm$. In view of (2.1) we have

$$R^{-1}(D_1z_1I_m-B)R=z_1D_1I_m-\begin{pmatrix}-I_{m-k} & 0\\ 0 & E(z')-I_k\end{pmatrix}. \tag{3.1}$$

Let us define a system $\tilde{\mathscr{N}}$ by

$$\tilde{\mathscr{N}}:\left(z_1D_1I_m-\begin{pmatrix}-I_{m-k} & 0\\ 0 & E(z')-I_k\end{pmatrix}\right)\tilde{w}=0.$$

Since $\mathscr{O}\tilde{\mathscr{D}}_{Y:\tilde{X}}$ acts on $\tilde{\mathscr{B}}_{N|M_\pm}$ and $\tilde{\mathscr{B}}_{N|M}$, there exists such $w(x)$ as above if and only if there exists a $\tilde{\mathscr{B}}_{N|M}$-solution $\tilde{w}(x)$ of $\tilde{\mathscr{N}}$ such that $\tilde{r}_\pm(\tilde{w})=R^{-1}\alpha_\pm(w_\pm)$ (see Lemma 1.1).

Let $\tilde{w}_\pm(x)$ be $\tilde{\mathscr{B}}_{N|M_\pm}$-solutions of $\tilde{\mathscr{N}}$. Then $\tilde{w}_\pm(x)$ are uniquely written in the form

$$\tilde{w}_\pm(x)=\begin{pmatrix}x_1^{-1}a_\pm(x')\\ x_1^{E(x')-1}b_\pm(x')\end{pmatrix}$$

with $a_\pm(x')\in(\mathscr{B}_N)^{m-k}$, $b_\pm(x')\in(\mathscr{B}_N)^k$. Let us prove that there exists a $\tilde{\mathscr{B}}_{N|M}$-solution $\tilde{w}(x)$ of $\tilde{\mathscr{N}}$ such that $\tilde{r}_\pm(\tilde{w})=\tilde{w}_\pm(x)$ if and only if $a_+(x')=a_-(x')$. Since

$$(x_1)_\pm^{E(x')-1}b_\pm(x')$$

make sense as vectors of hyperfunctions (or of sections of $\tilde{\mathscr{B}}_{N|M}$), there always exists $\tilde{w}^{(2)}(x)\in(\mathscr{B}_M|_N)^k$ such that

$$(x_1D_1-E(x')+I_k)\tilde{w}^{(2)}(x)=0$$

and that $\tilde{w}^{(2)}(x)=x_1^{E(x')-1}b_\pm(x')$ on $M_\pm$. Thus we have only to prove that there exists $\tilde{w}^{(1)}(x)\in(\mathscr{B}_M|_N)^{m-k}$ such that

$$D_1x_1\tilde{w}^{(1)}(x)=0,$$

and that $\tilde{w}^{(1)}(x)=x_1^{-1}a_\pm(x')$ on $M_\pm$ if and only if $a_+(x')=a_-(x')$. Assume that there exists such $\tilde{w}^{(1)}(x)$. Then $\tilde{v}^{(1)}(x')=x_1\tilde{w}^{(1)}(x)$ is a section of $(\mathscr{B}_N)^{m-k}$

and hence we get

$$\tilde{v}^{(1)}(x')=a_+(x')=a_-(x')\,.$$

Conversely, if $a_+(x')=a_-(x')$, then

$$\tilde{w}^{(1)}(x)=(x_1+\sqrt{-1}0)^{-1}a_+(x')$$

satisfies $D_1x_1\tilde{w}^{(1)}(x)=0$ and $\tilde{w}^{(1)}(x)=x_1^{-1}a_\pm(x')$ on $M_\pm$.

In view of (3.1) we get

$$v_\pm(x)=T(x_1)R\begin{pmatrix} x_1^{-1}a_\pm(x') \\ x_1^{E(x')-1}b_\pm(x')\end{pmatrix}=x_1^{-1}T(x_1)R\begin{pmatrix} a_\pm(x') \\ x_1^{E(x')}b_\pm(x')\end{pmatrix}. \tag{3.2}$$

On the other hand, by the argument of the proof of Theorem 2.1, we have

$$\begin{pmatrix} I_{m-k} & 0 \\ 0 & x_1I_k\end{pmatrix}v_\pm(x)=R\begin{pmatrix} \gamma_{\pm\mathrm{reg}}(u_\pm) \\ x_1^{E(x')}\gamma_{\pm\mathrm{sing}}(u_\pm)\end{pmatrix}. \tag{3.3}$$

Comparing (3.2) with (3.3) as sections of $\tilde{\mathscr{B}}_{N|M_\pm}$, we get

$$a_\pm(x')=\gamma_{\pm\mathrm{reg}}(u_\pm)\,,\qquad b_\pm(x')=\gamma_{\pm\mathrm{sing}}(u_\pm)\,.$$

Hence we have proved the first statement of the proposition. To prove the uniqueness of $\tilde{u}(x)$, it suffices to show

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_N(\mathscr{B}_M))=0\,. \tag{3.4}$$

Since $x_1^{m-k}P$ is a Fuchsian operator of weight 0, and none of its characteristic exponents is a negative integer, there exists no $\Gamma_N(\mathscr{B}_M)$-solution of $x_1^{m-k}Pu=0$ by virtue of Corollary 4.6 of Kashiwara-Oshima [6]. Hence we have proved (3.4). This completes the proof.

REMARK. Using the duality argument, Tahara defined boundary values equivalent to $\gamma_{\pm\mathrm{reg}}$ and proved a result similar to Proposition 3.1 under a condition weaker than $\mathrm{C}(\mathring{x})$ (Proposition 2.3.11 of [13]). Here we have chosen to prove it from our viewpoint employing the sheaves $\tilde{\mathscr{B}}_{N|M_\pm}$ instead of showing the equivalence of his definition and $\gamma_{\pm\mathrm{reg}}$.

As a special case, let us study the continuation of F-mild solutions.

PROPOSITION 3.2. *Under the condition* $\mathrm{C}(\mathring{x})$, *let* $u_\pm(x)$ *be* $\mathscr{B}^F_{N|M_\pm}$-*solutions of* $\mathscr{M}$ *such that*

$$\gamma_{+\mathrm{reg}}(u_+)=\gamma_{-\mathrm{reg}}(u_-)\,.$$

*Let* $u(x)$ *be the unique* $\mathcal{B}_M|_N$*-solution of* $\mathcal{M}$ *such that* $u(x)=u_\pm(x)$ *on* $M_\pm$. *Then* $u(x)$ *has* $x_1$ *as a real analytic parameter; i.e., the singular spectrum of* $u(x)$ *is disjoint from* $S_Y^*M$.

PROOF. By the Cauchy-Kowalevsky theorem for Fuchsian operators due to Baouendi-Goulaouic [1], we have a sheaf isomorphism

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X|_Y) \xrightarrow{\sim} (\mathcal{O}_Y)^{m-k}$$

in the derived category. Applying the right derived functor $\boldsymbol{R}\Gamma_N$, we get an isomorphism

$$\tilde{\gamma}^A : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}^A) \xrightarrow{\sim} (\mathcal{B}_N)^{m-k}.$$

(See the proof of Proposition 2.1 for the definition of $\tilde{\mathcal{B}}^A$.) There exists a commutative diagram

$$\begin{tikzcd} \tilde{\mathcal{B}}_{N|M_+} & \tilde{\mathcal{B}}^A \arrow[l, "\beta_+"'] \arrow[r, "\beta_-"] \arrow[d, "\beta"] & \tilde{\mathcal{B}}_{N|M_-} \\ & \tilde{\mathcal{B}}_{N|M} \arrow[ul, "\tilde{\gamma}_+"] \arrow[ur, "\tilde{\gamma}_-"'] & \end{tikzcd} \qquad (3.5)$$

On the other hand, we see from the proof of Proposition 2.1 that

$$\begin{tikzcd} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}^A) \arrow[r, "\sim", "\tilde{\gamma}^A"'] \arrow[d, "\beta_\pm"] & (\mathcal{B}_N)^{m-k} \arrow[d, "\begin{pmatrix}\mathrm{id}\\0\end{pmatrix}"] \\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_\pm}) \arrow[r, "\sim", "\tilde{\gamma}_\pm"'] & (\mathcal{B}_N)^m \end{tikzcd} \qquad (3.6)$$

is commutative. Put $a(x')=\gamma_{+\mathrm{reg}}(u_+)=\gamma_{-\mathrm{reg}}(u_-)$. Then there exists a $\tilde{\mathcal{B}}^A$-solution $\tilde{u}(x)$ of $\mathcal{M}$ such that $\tilde{\gamma}^A(\tilde{u})=a(x')$. It follows from (3.5) and (3.6) that

$$\tilde{\gamma}_\pm(\tilde{r}_\pm(\beta(\tilde{u})))=\tilde{\gamma}_\pm(\beta_\pm(\tilde{u}))=\begin{pmatrix} a(x') \\ 0 \end{pmatrix}.$$

On the other hand, we have

$$\tilde{\gamma}_\pm(\alpha_\pm(u_\pm))=\gamma_\pm(u_\pm)=\begin{pmatrix} a(x') \\ 0 \end{pmatrix}.$$

Hence $\alpha_\pm(u_\pm)=\tilde{r}_\pm(\beta(\tilde{u}))$ holds. In view of Lemma 1.1, this implies that there exists a section $u(x)$ of $\mathcal{B}_M|_N$ such that $\beta(\tilde{u})=\alpha(u)$. Since $\alpha(u)$ is contained in the image of $\beta$, we can prove that $u(x)$ has $x_1$ as a real analytic parameter by the same argument as the proof of Proposition 2.3 of [11] (see also the proof of Lemma 3.1 of [11]). Moreover, $u(x)=u_{\pm}(x)$ holds on $M_{\pm}$ since

$$\gamma_{\pm}(u|_{M_{\pm}})=\tilde{\gamma}_{\pm}(\alpha_{\pm}(u|_{M_{\pm}}))=\tilde{\gamma}_{\pm}(\tilde{r}_{\pm}(\alpha(u)))=\tilde{\gamma}_{\pm}(\alpha_{\pm}(u_{\pm}))=\gamma_{\pm}(u_{\pm}).$$

This completes the proof.

Now we can generalize Theorem 3.1 of Kaneko [2] to Fuchsian equations:

THEOREM 3.1. *Let $\mathcal{M}: Pu=0$ be the equation with a Fuchsian operator $P$ of weight $m-k$ with respect to $N$ and assume the condition* C($\mathring{x}$) *for a $\mathring{x}\in N$. Let $\varphi$ be a real valued $C^1$-function on $N$ such that $\varphi(\mathring{x})=0$ and that $d\varphi(\mathring{x})\neq 0$. Assume that $K$ is a closed subset of $N$ such that $\varphi\leqq 0$ on $K$ and that $V_{N,A}(P)$ (resp. $V_{N,B}(P)$) does not contain both of the points $(\mathring{x}, \pm\sqrt{-1}d\varphi(\mathring{x}))\in\sqrt{-1}S^*N$. Then any real analytic (resp. hyperfunction) solution $u(x)$ of $\mathcal{M}$ on $U\backslash K$, $U$ being an arbitrary open neighborhood of $\mathring{x}$ in $M$, is uniquely continued to a neighborhood of $\mathring{x}$ as a hyperfunction solution $\tilde{u}(x)$ of $\mathcal{M}$. Moreover, if $u(x)$ is real analytic on $U\backslash K$, $\tilde{u}(x)$ has $x_1$ as a real analytic parameter on a neighborhood of $\mathring{x}$.*

PROOF. By Theorem 2.2, not both of the two points $(\mathring{x}, \pm\sqrt{-1}d\varphi(\mathring{x}))$ are contained in

$$\mathrm{SS}(\gamma_{+}(u|_{M_{+}}))\cup\mathrm{SS}(\gamma_{-}(u|_{M_{-}})),$$

where SS denotes the singular spectrum of a vector of hyperfunctions. On the other hand, by Proposition 3.1,

$$\gamma_{+\mathrm{reg}}(u|_{M_{+}})=\gamma_{-\mathrm{reg}}(u|_{M_{-}}) \tag{3.7}$$

holds on $U\cap(N\backslash K)$. Hence by Lemma 1.2, (3.7) holds also on a neighborhood of $\mathring{x}$. Using Proposition 3.1 once again, we get an extension $\tilde{u}(x)$ of $u(x)$ to a neighborhood of $\mathring{x}$ as a hyperfunction solution of $\mathcal{M}$. Proposition 3.1 also implies the uniqueness of $\tilde{u}(x)$. Lastly, let us assume that $u(x)$ is real analytic on $U\backslash K$. Then, in particular, $u|_{M_{\pm}}$ are F-mild on $N\cap(U\backslash K)$, and hence

$$\gamma_{+\mathrm{sing}}(u|_{M_{+}})=\gamma_{-\mathrm{sing}}(u|_{M_{-}})=0 \tag{3.8}$$

holds on $N\cap(U\backslash K)$ by virtue of Proposition 2.1. By Lemma 1.2, (3.8) holds also on a neighborhood of $\mathring{x}$ in $N$. Thus Proposition 3.1 implies that $\tilde{u}(x)$ has $x_1$ as a real analytic parameter on a neighborhood in $M$ of $\mathring{x}$.

This completes the proof.

COROLLARY 3.1. *Let $\mathcal{M}: Pu=0$ be as in Theorem 3.1 with the condition* C$(\mathring{x})$ *for a point $\mathring{x}$ of $N$. Assume that there exists $\mathring{\xi}'\in S^{n-2}$ such that $(\mathring{x}, \sqrt{-1}\mathring{\xi}'\infty)$ is not contained in $V_{N,A}(P)$ (resp. $V_{N,B}(P)$). Then any real analytic (resp. hyperfunction) solution $u(x)$ of $\mathcal{M}$ defined on $U\backslash\{\mathring{x}\}$, $U$ being a neighborhood of $\mathring{x}$ in $M$, is uniquely continued to $U$ as a hyperfunction solution of $\mathcal{M}$. Moreover, the continued solution has $x_1$ as a real analytic parameter if $u(x)$ is real analytic on $U\backslash\{\mathring{x}\}$.*

PROOF. Put $\varphi(x')=\langle x', \mathring{\xi}'\rangle$ and $K=\{\mathring{x}\}$. Then the assumptions of Theorem 3.1 are satisfied. This completes the proof.

*Example* 3.1. Let $M$ be an open set of $\boldsymbol{R}^3$ containing 0 and put

$$P=x_1(D_1^2-x_1^{2k}(D_2^2-D_3^2))+\sum_{j=1}^{3}a_j(x)D_j+b(x),$$

where $a_j$ and $b$ are real analytic on $M$ with $a_1(0)\notin\boldsymbol{Z}$ and an integer $k\geqq 0$. Then any real analytic solution of $Pu=0$ on $M\backslash\{x\in M;\ x_1=0,\ x_2\leqq 0\}$ is uniquely continued to a neighborhood of 0 as a hyperfunction solution.

Let us give some sufficient conditions which guarantee that the continued solutions are real analytic. For this purpose let us begin with the following lemma.

LEMMA 3.1. *Let $A(x, D)$ be a microdifferential operator of finite order defined on a neighborhood of $x^*=(\mathring{x}, \sqrt{-1}\mathring{\xi}\infty)\in\sqrt{-1}S^*M$ with $\mathring{\xi}=(\mathring{\xi}_1, \mathring{\xi}')$ such that $\mathring{\xi}'\neq 0$. Assume that the principal symbol of $A$ is written in the form $\sigma(A)(x, \xi)=x_1^k a(x, \xi)$ with a nonnegative integer $k$ and an analytic function $a(x, \xi)$. We also assume the following two conditions:*

(3.9) *There exists $v\in\boldsymbol{R}^n\backslash\{0\}$ such that $a(x, \xi)$ is micro-hyperbolic in the direction $\langle v, dx\rangle$ at $x^*$; i.e. there exists $\varepsilon>0$ such that $a(x, \xi+\sqrt{-1}tv)\neq 0$ if $|x-\mathring{x}|<\varepsilon$, $|\xi-\mathring{\xi}|<\varepsilon$, and $0<t<\varepsilon$.*

(3.10) *$a(\mathring{x}, \xi_1, \mathring{\xi}')$ is not identically zero as a function of $\xi_1$.*

*Under these conditions, let $u(x)$ be a microfunction defined on a neighborhood of $x^*$ which satisfies $A(x, D)u(x)=0$ and vanishes on $\{(\mathring{x}, \sqrt{-1}\xi_1, \sqrt{-1}\mathring{\xi}')\in\sqrt{-1}S^*M;\ \xi_1<\mathring{\xi}_1\}\cup\{(x, \sqrt{-1}\xi);\ x\neq\mathring{x}\}$. Then $u(x)$ vanishes on a neighborhood of $\mathring{x}$.*

PROOF. We may assume $\mathring{x}=0$ and $\mathring{\xi}_1=0$. By Weierstrass' preparation theorem, we may also assume that $a(z, \zeta)$ is written as

$$a(z, \zeta)=\zeta_1^\mu+a_1(z, \zeta')\zeta_1^{\mu-1}+\cdots+a_\mu(z, \zeta'),$$

where $\mu\geqq 0$ is an integer, and $a_j(z, \zeta')$ are holomorphic on

$$\{(z, \zeta')\in \boldsymbol{C}^n\times \boldsymbol{C}^{n-1};\ |z|<\varepsilon,\ |\zeta'-\mathring{\xi}'|<\varepsilon\}$$

with $a_j(0, \mathring{\xi}')=0$ $(j=1, \cdots, \mu)$. We prove the lemma by induction on $\mu$. First let us assume $\mu=0$. Then the equation $A(x, D)u=0$ is equivalent to an equation $x_1^k u=0$ on a neighborhood of $x^*$ (cf. [12]). Since the bicharacteristics of $x_1$ are the integral curves of $\partial/\partial\xi_1$, $u(x)$ vanishes at $x^*$.

Next let us assume the lemma with $\mu$ replaced by $0, 1, \cdots, \mu-1$. We may assume that $u(x)$ satisfies $A(x, D)u(x)=0$ on $\{(x, \sqrt{-1}\xi\infty);\ |x|<\varepsilon, |\xi-\mathring{\xi}|<\varepsilon\}$. There exists $0<\delta<\varepsilon$ such that any root $\zeta_1=\rho(\xi')$ of the equation $a(0, \zeta_1, \xi')=0$ satisfies $|\rho(\xi')|<\varepsilon/2$ if $|\xi'-\mathring{\xi}'|<\delta$, and that $u(x)$ vanishes on $\{(0, \sqrt{-1}\xi\infty)\in\sqrt{-1}S^*M;\ |\xi'-\mathring{\xi}'|<\delta, \xi_1<-\varepsilon/2\}$. Now let us fix $\theta'\in S^{n-2}$ with $|\theta'-\mathring{\xi}'|<\delta$ and denote by $\rho_1, \cdots, \rho_r$ the real distinct roots (if any) of the equation $a(0, \zeta_1, \theta')=0$ in $\zeta_1$ with $\rho_1<\rho_2<\cdots<\rho_r$. We denote by $\mu_j$ the multiplicity of the root $\rho_j$. Then we have $\mu_1+\cdots+\mu_r\leqq\mu$. Note that, by the first part of the proof, $u(x)$ vanishes on

$$\{(0, \sqrt{-1}\xi\infty)\in\sqrt{-1}S^*M;\ \xi'=\theta', \xi_1<\rho_1\}.$$

Let us assume $\mu_j<\mu$ for any $1\leqq j\leqq r$. Then $A(x, \xi)$ and $u(x)$ satisfy the assumptions of the lemma at $(0, \sqrt{-1}(\rho_1, \theta')\infty)$ with $\mu$ replaced by $\mu_1$. Hence by the induction hypothesis, $u(x)$ vanishes at $(0, \sqrt{-1}(\rho_1, \theta')\infty)$, and hence on $\{(0, \sqrt{-1}(\xi_1, \theta')\infty;\ \xi_1<\rho_2\}$. Using this argument step by step, we conclude that $u(x)$ vanishes on $\{(0, \sqrt{-1}(\xi_1, \theta')\infty;\ |\xi_1|<\varepsilon\}$.

Consequently, the support of $u(x)$ is contained in

$$\{(0, \sqrt{-1}\xi\infty);\ \xi'\in S_\delta,\ \xi_1\geqq\rho(\xi')\},$$

where $S_\delta$ is the subset of $\{\xi'\in S^{n-2};\ |\xi'-\mathring{\xi}'|<\delta\}$ consisting of such $\xi'$ that the equation $a(0, \zeta_1, \xi')=0$ in $\zeta_1$ has only one real root $\zeta_1=\rho(\xi')$ with multiplicity $\mu$. If $\xi'\in S_\delta$, we have

$$a(0, \xi)=(\xi_1-\rho(\xi'))^\mu.$$

Hence $\rho(\xi')=-a_1(0, \xi')/\mu$ holds and $a_1(0, \xi')$ is real for $\xi'\in S_\delta$. Put

$$h(x, \xi)=\langle v, x\rangle-c\Big(\xi_1+\frac{1}{\mu}\operatorname{Re} a_1(0, \xi')\Big)$$

with sufficiently small $c>0$ so that $dh$ is micro-hyperbolic for $A(x, D)$ at

$x^*$. Since the support of $u(x)$ is contained in $\{(x, \sqrt{-1}\xi\infty)\,;\, h(x, \xi) \leqq 0\}$, $u(x)$ vanishes at $x^*$ by virtue of Theorem 2.2.1 of [7]. This completes the proof.

THEOREM 3.2. *Let $P$ be a Fuchsian operator of weight $m-k$ with respect to $N$ satisfying the condition* $\mathrm{C}(\mathring{x})$ *with $\mathring{x} \in N$. Assume that the principal symbol of $P$ is written in the form*

$$\sigma(P)(x, \xi) = x_1^k p(x, \xi)$$

*with an analytic function $p(x, \xi)$ on a neighborhood of $\pi_M^{-1}(\mathring{x})$. Assume the following two conditions:*

(3.11) *There exists $\xi' \in S^{n-2}$ such that $(\mathring{x}, \sqrt{-1}\xi'\infty) \notin V_{N,A}(P)$.*

(3.12) *For any $\xi \in S^{n-1}$ with $p(\mathring{x}, \xi) = 0$, there exists $v \in \boldsymbol{R}^n \setminus \{0\}$ such that $p(x, \xi)$ is micro-hyperbolic in the direction $\langle v, dx \rangle$ at $(\mathring{x}, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*M$.*

*Under these conditions, any real analytic solution of $Pu=0$ on $U \setminus \{\mathring{x}\}$, where $U$ is an arbitrary open neighborhood of $\mathring{x}$ in $M$, is continued to $U$ as a real analytic function.*

PROOF. Let $u(x)$ be a real analytic solution of $Pu=0$ on $U \setminus \{\mathring{x}\}$. Then by (3.11) and Theorem 3.1, $u(x)$ is continued to $U$ as a hyperfunction solution $\tilde{u}(x)$ of $Pu=0$. Moreover $\tilde{u}(x)$ has $x_1$ as a real analytic parameter. Let us fix an arbitrary $\mathring{\xi}' \in S^{n-2}$ and let $\rho_1 < \cdots < \rho_r$ be the real distinct roots of the equation $p(\mathring{x}, \zeta_1, \mathring{\xi}') = 0$ in $\zeta_1$. Then $\tilde{u}(x)$ is micro-analytic on $\{(\mathring{x}, \sqrt{-1}(\xi_1, \mathring{\xi}')\infty) \in \sqrt{-1}S^*M\,;\, \xi_1 < \rho_1\}$ since $\tilde{u}(x)$ has $x_1$ as a real analytic parameter. Then by (3.12) and Lemma 3.1, $\tilde{u}(x)$ becomes micro-analytic at $(\mathring{x}, \sqrt{-1}(\rho_1, \mathring{\xi}')\infty)$, and hence on $\{(\mathring{x}, \sqrt{-1}(\xi_1, \mathring{\xi}')\infty)\,;\, \xi_1 < \rho_2\}$ (note that $p(z, \zeta)$ is a polynomial in $\zeta_1$ and satisfies (3.10)). Proceeding successively, we conclude that $\tilde{u}(x)$ is micro-analytic on $\{(\mathring{x}, \sqrt{-1}(\xi_1, \mathring{\xi}')\infty)\,;\, \xi_1 \in \boldsymbol{R}\}$. Since $\mathring{\xi}'$ is arbitrary, $\tilde{u}(x)$ becomes real analytic on $U$. This completes the proof.

The following corollary extends Theorem I of Kaneko [3] to Fuchsian equations:

COROLLARY 3.2. *Let $P$ be a Fuchsian operator of weight $m-k$ with respect to $N$ satisfying* $\mathrm{C}(\mathring{x})$ *with $\mathring{x} \in N$. Assume that the principal symbol of $P$ is written in the form*

$$\sigma(P)(x, \xi) = x_1^k p(x, \xi)$$

*with a real valued real analytic function $p(x, \xi)$ such that*

(3.13) $\mathrm{grad}_\xi p(\mathring{x}, \xi) \neq 0$ *if* $\xi \in S^{n-1}$ *and* $p(\mathring{x}, \xi) = 0$.

(3.14) *There exists* $\mathring{\xi}' \in S^{n-2}$ *such that the equation* $p(\mathring{x}, \zeta_1, \mathring{\xi}') = 0$ *in* $\zeta_1$ *has* $m$ *real distinct roots.*

*Under these assumptions, any real analytic solution of* $Pu=0$ *defined on* $U\backslash\{\mathring{x}\}$ *is continued to* $U$ *as a real analytic function, where* $U$ *is an arbitrary open neighborhood of* $\mathring{x}$ *in* $M$.

*Example* 3.2. Let $M$ be an open subset of $\boldsymbol{R}^n$ containing 0 and put

$$P = x_1(D_1^2 + \cdots + D_k^2 - D_{k+1}^2 - \cdots - D_n^2) + \sum_{j=1}^{n} a_j(x) D_j + b(x),$$

where $a_j(x)$ and $b(x)$ are real analytic on $M$ with $a_1(0) \in \boldsymbol{Z}$, and $1 \leqq k < n$. Then any real analytic function $u(x)$ on $M\backslash\{0\}$ satisfying $Pu(x)=0$ is continued to $U$ as a real analytic function.

*Example* 3.3. Let $M$ be an open subset of $\boldsymbol{R}^3$ containing 0 and put

$$P = x_1(D_1^3 - 3D_1D_2^2 + D_2^3 + D_3^3) + \sum_{1 \leqq i, j \leqq 3} a_{ij}(x) D_i D_j + \sum_{j=1}^{3} a_j(x) D_j + b(x),$$

where $a_{ij}$, $a_j$, $b$ are real analytic on $M$ with $a_{11}(0) \in \boldsymbol{Z}$. Then any real analytic function $u(x)$ on $M\backslash\{0\}$ satisfying $Pu(x)=0$ is continued to $M$ as a real analytic function.

Finally let us give an example where the assumption (3.12) of Theorem 3.2 does not hold, but real analytic solutions are continued real analytically:

*Example* 3.4. Put $x=(x_1, x_2) \in \boldsymbol{R}^2$ and consider an operator

$$P = x_1(D_1 + c(x)D_2)(D_1 + \sqrt{-1}x_1 D_2) + a_1(x)D_1 + a_2(x)D_2 + b(x),$$

where $a_1$, $a_2$, $b$, $c$ are real analytic on $\boldsymbol{R}^2$. Moreover, we assume that $c(x)$ is real valued and

$$c(0, x_2) \neq 0, \quad a_1(0, x_2) \in \boldsymbol{Z}, \quad -\frac{a_2(0, x_2)}{c(0, x_2)} \in \boldsymbol{N}$$

for any $x_2 \leqq 0$. Then any real analytic solution $u(x)$ of $Pu=0$ on $\boldsymbol{R}^2\backslash\{(0, x_2); x_2 \leqq 0\}$ extends to $\boldsymbol{R}^2$ as a real analytic function. In fact, since $(0, x_2, \sqrt{-1}dx_2) \in V_{N,A}(P)$, $u(x)$ extends to $\boldsymbol{R}^2$ as a hyperfunction solution $\tilde{u}(x)$ of $Pu=0$. Since $\tilde{u}(x)$ has $x_1$ as a real analytic parameter, the singular spectrum of $u(x)$ is contained in $\{(0, x_2, \sqrt{-1}(\xi_1, \xi_2)\infty); x_2 \leqq 0, \xi_1=0, \xi_2 \neq 0\}$ by virtue of Lemma 3.1. In view of Theorem 2.1 of Ôaku [9], $\tilde{u}(x)$ becomes microanalytic also at $\xi_1=0$, and hence real analytic on $\boldsymbol{R}^2$.

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