J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
33 (1986), 429-439.

Propagation of microanalyticity at the boundary for solutions of linear differential equations

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(Communicated by H. Komatsu)

Abstract Let $M=R^n$, $iS^*M=R^n\times iS^{n-1}$. For coordinates $(x;i\eta)=(x_1, x';i\eta_1, i\eta')$ in iS^*M, we set $N=\{x_1=0\}$, $M^+=\{x_1\geqq 0\}$, $S^{n-2}=\{\eta_1=0\}$, $iS^*N=R^{n-1}\times iS^{n-2}$. Let $P=P(D)$ be a differential operator with constant coefficients and order m for which N is non-characteristic. Let $\mathcal{A}_M$ be the sheaf of real analytic functions on M, denote by $\mathcal{A}_M^P$ the kernel sheaf of P, and, for $u\in\Gamma(U\cap\mathring{M}^+, \mathcal{A}_M^P)$, $U\subset M$ open, let $\gamma(u)$ be the m traces of u on $U\cap N$. For $(x';i\eta')\in iS^*N$ with $(0,x')\in U$ we discuss the condition:

(0.1) $$(x', i\eta')\notin \mathrm{SS}\gamma(u) \qquad \text{for any} \quad u\in\Gamma(U\cap\mathring{M}^+, \mathcal{A}_M^P).$$

We prove that "$-\eta'$-semihyperbolicity" to N^+ of P implies (0.1). Under some additional hypotheses we also prove the converse.

The first part of the statement was conjectured by Kaneko in [2]; its proof is a consequence of the results of [11] on "N-regularity" of non-microcharacteristic operators. The second part is obtained by means of a microlocally-null solution.

I wish to thank Prof. P. Schapira for frequent and invaluable discussions on this subject.

§1. Review on microlocal boundary value problems (cf. [5], [6])

Let M be an n-dimensional real analytic manifold, $N\subset M$ an analytic hypersurface, $M^{\pm}$ the pair of closed half spaces of M with boundary N, X and Y complexifications of M and N respectively. Let T_M^*X, $T_{M^\pm}^*X$, T_N^*X be the conormal boundles of M, $M^{\pm}$, N in X and T_N^*Y that of N in Y. Denote by $\varpi: Y\times_X T^*X\to T^*X$, $\rho: Y\times_X T^*X\to T^*Y$ and $\pi: T^*X\to X$ the natural mappings. We recall the sheaves $\mathcal{B}_M$, $\mathcal{B}_N$ of hyperfunctions on M, N, and the sheaves $\mathcal{C}_{M|X}$, $\mathcal{C}_{N|X}$, $\mathcal{C}_{M^\pm|X}$, $\mathcal{C}_{N|Y}$ of microfunctions on T_M^*X, T_N^*X, $T_{M^\pm}^*X$, T_N^*Y defined in [8], [5]. We collect in a Proposition all properties of such sheaves we need later (see [5] for the proof).

PROPOSITION 1.1. a) *We have an isomorphism*

$$\Gamma_M(\mathcal{B}_M)|_N \cong \dot{\pi}_*(C_{M^\pm|X})|_N, \qquad (\textit{where } \dot{\pi}=\pi|_{T^*X\setminus T_Y^*X}).$$

b) *We have, in* $(N \times_M T_M^* X)\setminus T_Y^* X$, *injective morphisms*:

$$C_{N|X} \rightarrowtail C_{M^\pm|X} \rightarrowtail C_{M|X}, \tag{1.1}$$

and an exact sequence:

$$0 \longrightarrow C_{N|X} \longrightarrow C_{M^+|X} \oplus C_{M^-|X} \longrightarrow C_{M|X} \longrightarrow 0. \tag{1.2}$$

c) *The sections of* $C_{N|X}$, $C_{M^\pm|X}$ *have the unique continuation property along the fibers of* $\rho|_{T_N^*X\setminus T_Y^*X}$, $\rho|_{\varpi^{-1}(T_{M^\pm}^*X)\setminus T_Y^*X}$. *In particular if* $P(x, D)$ *is a differential operator with analytic coefficients for which* N *is non-characteristic (i.e. if* ρ *is proper on* $\varpi^{-1}(\text{char}\, P)$*), then*:

$$P: C_{N|X} \longrightarrow C_{N|X} \quad \textit{and} \quad P: C_{M^\pm|X} \longrightarrow C_{M^\pm|X} \tag{1.3}$$

are injective on $T_N^*X\setminus T_Y^*X$ *and* $\varpi^{-1}(T_{M^\pm}^*X)\setminus T_Y^*X$ *respectively.*

Let $x^* \in N \times_M T_M^* X$ and let $P=P(x, D)$, (where $D=-i\partial/\partial x$), be a differential operator with analytic coefficients in a neighborhood of $\pi(x^*)$.

DEFINITION 1.2 ([6]). P is said to be N^+-*regular* at x^* iff the following implication holds:

$$u \in (C_{M^+|X})_{x^*} \cap \Gamma_{N \times_M T_M^* X}(C_{M|X})_{x^*}, \quad Pu \in (C_{N|X})_{x^*} \Longrightarrow u \in (C_{N|X})_{x^*}. \tag{1.4}$$

Replacing $C_{M^+|X}$ by $C_{M^-|X}$ (resp. $C_{M|X}$) in (1.4), we obtain the definition of N^--regularity (resp. N-regularity). Note that P is N-regular iff it is N^+- and N^--regular due to the exactness of (1.2) at x^*.

For understanding the meaning of N^+-regularity we recall the theory of boundary values of hyperfunction solutions of differential equations following [9] and [10]. For local coordinates $x=(x_1, x')$ in M we set $N=\{x_1=0\}$ and assume N non-characteristic for P. Let m be the order of P, P_m the principal part, and $\mathcal{B}_M^P$ the sheaf of $\mathcal{B}_M$-solutions of P. According to [9] we know that for $u \in \Gamma(\mathring{M}^+, \mathcal{B}_M^P)$ there exist a unique extension $[u]^+ \in \Gamma_{M^+}(\mathcal{B}_M)$ of u, and unique sections $\gamma(u)=(h_j) \in (\mathcal{B}_N)^m$ which give an equality of the form:

$$P[u]^+ = \sum_{j=0}^{m-1} h_j \otimes \delta_{x_1}^{(j)}. \tag{1.5}$$

We will call such $[u]^+$ the *canonical extension* of u and such $\gamma(u)$ the *traces* of u on N. Let $x^* \in N \times_M T_M^* X$; if $P_m(x^* + (0; \zeta_1, 0, \cdots))/\zeta_1^m$ is analytic and $\neq 0$ at $\zeta_1=0$, we can decompose $P=P'Q'$, Q' being invertible at x^* and P' being of Weierstrass type in D_{x_1} with degree μ (cf. [8]). Then N^+-regularity of P at x^* is equivalent to vanishing at $\rho(x^*)$ of the μ traces of $((\mathcal{C}_{M^+|X}\cap\Gamma_{N\underset{M}{\times}T^*_MX}(\mathcal{C}_{M|X}))/\mathcal{C}_{N|X})$-solutions of P' (which can be defined as in (1.5) by the aid of the division theorem for the sheaf $\mathcal{C}_{N|X}$ [4]). In particular for $y^*\in\dot{T}^*_NY$ we have (cf. [10]):

PROPOSITION 1.3. *Under the above hypotheses on P and N assume further*:

$$\rho^{-1}(y^*)\cap\operatorname{char}P\subset\bar{\omega}^{-1}(T^*_{M^+}X)\,, \tag{1.6}$$

(1.7) *P is N^+-regular at any point of $\rho^{-1}(y^*)\cap\bar{\omega}^{-1}(\operatorname{char}P\cap T^*_MX)$.*

It then follows

(1.8) *For any solution $u\in(\Gamma_{M^+}(\mathcal{B}_M)/\Gamma_N(\mathcal{B}_M))_{\pi(y^*)}$ of $Pu=0$, which satisfies $\overline{\mathrm{SS}\,u|_{\dot{M}^+}}\cap\rho^{-1}(y^*)=\emptyset$, we have $y^*\notin\mathrm{SS}\,\gamma(u)$.*

§ 2. N-regularity of constant coefficients operators

From now on we let $M=\boldsymbol{R}^n$, $X=\boldsymbol{C}^n$. We also assume that $P=P(D)$ has constant coefficients and that $N\subset M$ is a hyperplane. We denote by (z,ζ), $z=x+iy$, $\zeta=\xi+i\eta$ the coordinates in T^*X, put $S^{n-1}=\{\eta\in\boldsymbol{R}^n\,;\,|\eta|=1\}$ and write also $\boldsymbol{R}^n\times iS^{n-1}$ instead of $\dot{T}^*_MX=T^*_MX\setminus T^*_XX$ (by identifying the points of T^*_MX on the same orbit of the action of $\boldsymbol{R}^+$). Let $P(\zeta)$, $\zeta\in\boldsymbol{C}^n$ be the polynomial associated to $P(D)$, let $P_m(\zeta)$ be the principal part of $P(\zeta)$, and let $i\eta$ be a point in iS^{n-1}. We denote by $(P_m)_{i\eta}$ the first non-vanishing term of the expansion of P_m at $i\eta$ into a (Taylor) sum of homogeneous polynomials. If μ denotes the degree of $(P_m)_{i\eta}$ we then have:

$$P_m(\zeta)=(P_m)_{i\eta}(\zeta-i\eta)+o(|\zeta-i\eta|^{\mu})\,,\qquad \zeta\to i\eta\,.$$

DEFINITION 2.1. Let $i\eta$ and $i\theta$ belong to iS^{n-1}. We say that $i\theta$ is *non-micro-characteristic* for P at $i\eta$ iff

$$(P_m)_{i\eta}(i\theta)\neq 0\,. \tag{2.1}$$

REMARK 2.2. By the homogeneity of P_m it is obvious that the above property only depends on the image of $i\theta$ by the projection ρ of iS^{n-1} from the poles $\pm i\eta$ to the equator.

REMARK 2.3. For a point x^* and subsets S, V of T^*X, with V smooth, one defines a closed cone $C_V(S)_{x^*}$ in the normal boundle $(T_V T^*X)_{x^*}= T_{x^*}T^*X/T_{x^*}V$, (with the real underlying structure), in the following way. A vector $\delta\in(T_V T^*X)_{x^*}$ does not belong to $C_V(S)_{x^*}$ if and only if there exist an open cone $\Gamma\subset T_{x^*}T^*X$, invariant under $T_{x^*}V$ and verifying $\Gamma/T_{x^*}V\supset\{\delta\}$, and a neighborhood U of x^*, such that:

$$((U\cap V)+\Gamma)\cap U\cap S=\emptyset \qquad \text{(cf. [4])}.$$

Let η, θ be as in Definition 2.1, choose coordinates such that $\eta=(0,\cdots,0,1)$, $\rho(\theta)=(1,0,\cdots)$, and take $x^*\in\boldsymbol{R}^n\times\{i\eta\}$. In view of the homogeneity of P_m it is immediately seen that (2.1) is equivalent to:

$$\lambda\partial/\partial\zeta_1+\bar{\lambda}\partial/\partial\bar{\zeta}_1\notin C_V(\mathrm{char}\,P)_{x^*}$$
$$\text{for } V=\{\zeta_1=\cdots=\zeta_{n-1}=0\} \text{ and for any } \lambda\in\dot{\boldsymbol{C}},$$

or else to:

$$\theta\cdot\partial/\partial\xi\notin C_{V'}(\mathrm{char}\,P)_{x^*} \qquad \text{for } V'=\{\xi_1=\zeta_2=\cdots=\zeta_{n-1}=\xi_n=0\}$$
$$\text{and for any } \theta\in S^{n-1} \text{ with } \rho(\theta)=(\pm1,0,\cdots).$$

(For the second statement cf. the proof of Lemma 3.3.)

Let η, θ belong to S^{n-1} and set $N=\{x\cdot\theta=0\}$. In view of Remark 2.3, the following is a particular case of Theorem 4.3 of [11].

THEOREM 2.4. *Let $(P_m)_{i\eta}(i\theta)\neq0$. Then P is N-regular at $i\eta$ (i.e. at any $x^*\in N\times\{i\eta\}$).*

To obtain a partial converse we construct in next theorem "micro-locally-null" solutions. Let $\eta,\theta\in S^{n-1}$, set $N=\{x\cdot\theta=0\}$ and denote by $M^{\pm}$ the pair of closed half spaces of M with boundary N.

THEOREM 2.5. *Let P_m have real coefficients and assume:*

$$(2.2) \qquad (P_m)_{i\eta}(i\theta)=0\,,\quad \partial((P_m)_{i\eta})(i\theta)\neq0\,,\quad (\partial=(\partial/\partial\zeta_i)_i)\,.$$

Then there exist hyperfunctions $u^{\pm}$, in a neighborhood of 0, which satisfy:

$$(2.3) \qquad Pu^{\pm}=0\,,\quad \mathrm{SS}\,u^{\pm}\subset M^{\mp}\times\{i\eta\}\,,\quad (0\,;i\eta)\in\mathrm{SS}\,u^{\pm}\,.$$

PROOF. We will prove the statement for $u=u^+$. First we construct $u\in\Gamma(M,\mathcal{B}_M)$ verifying:

$$(2.4) \qquad Pu=0\,,\quad (0\,;i\eta)\in\mathrm{SS}\,u\,,\quad \mathrm{SS}\,u\cap(\mathring{M}^+\times\{i\eta\})=\emptyset\,.$$

In the proof we will replace $(P_m)_{i\eta}$ by $(P_m)_\eta=i^{-m+\mu}(P_m)_{i\eta}$ (μ being the degree of $(P_m)_{i\eta}$), and $i\theta$ by θ for simplicity. Let us choose $\eta^1\in S^{n-1}$ with

$$(P_m)_\eta(\eta^1)\neq 0 \quad \text{and} \quad (\eta^1\cdot\partial)((P_m)_\eta)(\theta)\neq 0\,.$$

We can then write

(2.5) $P_m(\eta+\sigma\theta+\tau\eta^1)=\tau Q(\sigma,\tau)+R(\sigma,\tau)$, $\sigma,\tau\in\boldsymbol{C}$, where degree $Q=\mu-1$, $|Q(\sigma,\tau)|\geqq c|(\sigma,\tau)|^{\mu-1}$ for $|\tau/\sigma|\ll 1$, $R(\sigma,\tau)=o(|(\sigma,\tau)|^{\mu})$ and finally Q and R are real for real arguments.

Thus when $|\tau/\sigma|\ll 1$, the equation (2.5) for τ is equivalent to:

(2.6) $\tau-r(\sigma,\tau)=0$ with $r(\sigma,\tau)=o(|\sigma|)$ and with r analytic and real for real argument.

We denote by $\tau_1^0=\tau_1^0(\sigma)$ the small solution of (2.6) for τ, i.e. the only solution $\tau(\sigma)$ of (2.5) with $\tau(\sigma)=o(|\sigma|)$; $\tau_1^0(\sigma)$ is clearly real for real σ.

Let $\lambda\in\boldsymbol{R}^+$, $\lambda>C$, and $\sigma\in\boldsymbol{C}$, $|\sigma|<c$. Denote by $\tau_j^0(\lambda(\eta+\sigma\theta))$, $j=1,\cdots,\mu$, and $\tau_j(\lambda(\eta+\sigma\theta))$ the μ zeros for τ of $P_m(\lambda(\eta+\sigma\theta)+\tau\eta^1)$ and $P(\lambda(\eta+\sigma\theta)+\tau\eta^1)$ respectively, with order $\lambda 0(|\sigma|)$. For suitable labelling we have $|\tau_j^0-\tau_j|<c_1\lambda^{1-1/\mu}$ for $\lambda>C$. On the other hand for some small positive δ there exists $c=c_\delta$, $(c<1/3)$, such that for $2c_1\lambda^{-1/\mu}/\delta<|\sigma|<c$ and for $j\neq 1$, we have: $|\tau_j^0-\tau_1^0|\geqq\delta\lambda|\sigma|\geqq 2c_1\lambda^{1-1/\mu}$.

Thus for $\lambda>C\gg 0$ and $2c_1\lambda^{-1/\mu}/\delta<|\sigma|<c\ll 1$, $\tau_1(\lambda(\eta+\sigma\theta))$ is an analytic function of λ and σ which verifies:

$$|\tau_1(\lambda(\eta+\sigma\theta))|=\lambda 0(|\sigma|^2)\,, \qquad (\text{if } |\sigma^2|>\lambda^{-1/\mu})\,, \tag{2.7}$$

$$|\mathrm{Im}\,\tau_1(\lambda(\eta+\sigma\theta))|=\lambda|\mathrm{Im}\,\sigma|0(|\sigma|)\,, \qquad (\text{if } |\mathrm{Im}\,\sigma||\sigma|>\lambda^{-1/\mu})\,. \tag{2.8}$$

(2.7) is obvious. To prove (2.8) we note that Cauchy inequalities give: $\left|\frac{\partial}{\partial\sigma}\tau_1^0(\eta+\sigma\theta)\right|=0(|\sigma|)$ due to $|\tau_1^0(\eta+\sigma\theta)|=0(|\sigma|^2)$. Since we also have $\mathrm{Im}\,\tau_1^0(\eta+\sigma\theta)=0$ for $\sigma\in\boldsymbol{R}$, we then obtain: $|\mathrm{Im}\,\tau_1^0(\eta+\sigma\theta)|=|\mathrm{Im}\,\sigma|0(|\sigma|)$ which obviously implies (2.8).

We put in the following $\sigma=s+i\lambda^{-(1-\nu)}$, $|s|<c$, $\lambda>C$; then for $1>\nu>1-1/2\mu$ and for suitable $C\gg 0$, $c\ll 1$, all above requirements are satisfied. We set $J=\{(\lambda,s)\,;\,\lambda>C,\,|s|<c\}$ and $I=\{\lambda(\eta+s\theta+i\lambda^{-(1-\nu)}\theta)\,;\,(\lambda,s)\in J\}$; we also denote by $\zeta=\zeta(\lambda,s)$ the points of I. We put, for $z,\zeta\in\boldsymbol{C}^n$ with ζ close to I:

$$F(z,\zeta)=\exp[i\langle z,\zeta+\tau_1(\zeta)\eta^1\rangle]\,. \tag{2.9}$$

Then because of (2.7), (2.8) we have, with a new constant c_1:

$$|F(z,\zeta)|\leqq\exp[-\lambda\langle y,\eta\rangle+2c\lambda|y|+c_1\lambda^{\nu}|z|]\,, \qquad \zeta\in I\,. \tag{2.10}$$

Thus for $y\cdot\eta>3c|y|$ the integral

$$(2.11)\qquad G(z)=\int_I F(z,\zeta)d\zeta\,,\quad (\text{where } \int_I F(z,\zeta)d\zeta \text{ stands for } \int_J F(z,\zeta(\lambda,s))d\lambda ds),$$

converges absolutely to define an analytic function of z. We put $\Gamma=\{y: y\cdot\eta>3c|y|\}$ and set:

$$(2.12)\qquad u(x)=G(x+i\Gamma 0)\,,$$

in the sense of hyperfunctions. Clearly $Pu=0$ and $\mathrm{SS}\,u\subset M\times i\Gamma^0$.

Let us remark now that it is not restrictive to assume η^1 orthogonal to η. Let ρ^1 be the projection of S^{n-1} from the poles $\pm\eta^1$ and let $I^1=\{\lambda(\eta+s\rho^1(\theta)),\ |s|<c,\ \lambda>0\}$ and $N^1=\{x\cdot\eta^1=0\}$. Then

$$u\Big|_{N^1}=\Big(\int_I e^{ix\cdot\zeta}d\zeta\Big)\Big|_{N^1}=\int_{I^1}e^{ix\cdot\zeta}d\zeta\,,$$

modulo microfunctions vanishing on $N^1\times\{i\eta\}$. Thus $(0;i\eta)\in\mathrm{SS}\,u|_{N^1}$ and therefore $\mathrm{SS}\,u\cap(\{0\}\times\{i(\rho^1)^{-1}(\eta)\})\neq\emptyset$. Recall the hypothesis $(P_m)_\eta(\eta^1)\neq 0$; then $(\rho^1)^{-1}(\eta)\cap P_m^{-1}(0)\cap B(\eta)\subset\{\eta\}$ for a suitably small neighborhood $B(\eta)$ of η in S^{n-1}. We can also assume $B(\eta)\supset\Gamma^0$; then $\mathrm{SS}\,u\subset M\times i(P_m^{-1}(0)\cap B(\eta))$ by Sato's theorem and by construction. Collecting the above remarks we then conclude: $(0;i\eta)\in\mathrm{SS}\,u$.

Now we prove the last part of (2.4). We set $\Omega_\varepsilon=\{\lambda(\eta+s\theta)+it\theta\,;\,\lambda\geqq C_\varepsilon,\ |s|\leqq\varepsilon,\ \lambda^\nu\leqq t\leqq\varepsilon\lambda\}$. As already seen, for any $\varepsilon\ll1$ we can find $C_\varepsilon\gg0$, with $C_\varepsilon^{1-\nu}>\varepsilon^{-1}$, in such a way that $\tau_1(\zeta)$ is an analytic function of $\zeta\in\Omega_\varepsilon$ which satisfies:

$$(2.13)\qquad |\tau_1(\zeta)|=\lambda 0(\varepsilon^2)\,,\quad |\mathrm{Im}\,\tau_1(\zeta)|=t0(\varepsilon)\,,\qquad \zeta\in\Omega_\varepsilon\,.$$

Then for F defined by (2.9) we have the estimate:

$$(2.14)\qquad |F(z,\zeta)|\leqq\exp[-\lambda y\cdot\eta+\lambda|y|0(\varepsilon)-tx\cdot\theta+t|x|0(\varepsilon)]\,,\qquad \zeta\in\Omega_\varepsilon\,.$$

Let $0<\alpha<1$; for $x\cdot\theta>\varepsilon^\alpha|x|$, $y\cdot\eta>0(\varepsilon)|y|$, $\varepsilon\ll1$, we then conclude that $Fd\zeta$ is integrable in Ω_ε. Under the same conditions we also have: $\lim_{j\to\infty}\int_{\Omega_\varepsilon\cap\{\lambda=j\}}Fd\zeta=0$. Thus we obtain:

$$(2.15)\qquad \int_{\Omega_\varepsilon\cap\{t=\lambda^\nu\}}Fd\zeta=\int_{\Omega_\varepsilon\cap\{t=\varepsilon\lambda\}}Fd\zeta+\int_{\Omega_\varepsilon\cap\{\lambda=C_\varepsilon\}}Fd\zeta+\int_{\Omega_\varepsilon\cap\{|s|=\varepsilon\}}Fd\zeta\,.$$

The second term in the right hand side of (2.15) is entire and the third is null on $\mathring{M}^+\times\{i\eta\}$ as a section of $C_{M|X}$.

For treating the first we set $t=\varepsilon\lambda$ in (2.14). Assuming $x\cdot\theta>\varepsilon^\alpha|x|$ we then have:

$$|F(z,\zeta)|\leq\exp[-\lambda((\varepsilon^{1+\alpha}-0(\varepsilon^2))|x|-2|y|)].$$

Thus the first integral on the right side of (2.15) defines a real analytic function on $x\cdot\theta>\varepsilon^{\alpha}|x|$ since, for such x and for any y with $|y|<(\varepsilon^{1+\alpha}-0(\varepsilon^2))/2$, it converges absolutely.

To complete the proof of (2.4) we only need to notice that, $\forall\varepsilon$, the hyperfunction u of (2.12) differs from the term on the left side of (2.15) by a term which is null on $M\times\{i\eta\}$ as a section of $\mathcal{C}_{M|X}$.

Last the statement of the theorem can be deduced from (2.4) by the following:

LEMMA 2.6. *Assume that there exists a hyperfunction u in a neighborhood of* 0 *which verifies*

(2.16) $$Pu=0\,,\quad (0\,;i\eta)\in \mathrm{SS}\,u\,,\quad \mathrm{SS}\,u\cap(\mathring{M}^+\times\{i\eta\})=\emptyset\,.$$

Then we can find v, in a neighborhood of 0, *which verifies*

(2.17) $$Pv=0\,,\quad (0\,;i\eta)\in \mathrm{SS}\,v\,,\quad \mathrm{SS}\,v\subset M^-\times\{i\eta\}\,.$$

PROOF. Let $W(x,\omega)$, $(x,\omega)\in M\times S^{n-1}$, be the component of a curve wave decomposition of $\delta(x)$ and let $J(D_\omega)$ be a local operator on S^{n-1} with constant coefficients (cf. [2]). For u as in (2.16) we take $\tilde{u}\in\mathcal{B}_M$ with $\tilde{u}-u=0$ on $B_\varepsilon=\{|x|<\varepsilon\}$ and $\tilde{u}=0$ on $M\backslash\bar{B}_\varepsilon$ due to the flabbiness of $\mathcal{B}_M$. For a suitable $J(D_\omega)$ and for $v'(x)=u(x)\underset{M}{*}J(D_\omega)W(x,\omega)|_{\omega=\eta}$, we then have $(0\,;i\eta)\in\mathrm{SS}\,v'$ due to Lemma 1.1 of [2]. We also have: $\mathrm{SS}\,v'|_{B_\varepsilon}\subset(M^-\cap B_\varepsilon)\times\{i\eta\}$, $\mathrm{SS}\,Pv'|_{B_\varepsilon}=0$. Thus if we replace v' by $v=v'+h$ where h is an analytic solution of $Ph=-Pv'$ on $B_{\varepsilon'}$, $\varepsilon'<\varepsilon$, then (2.17) is satisfied by such v.

REMARK 2.7. In the proof of Theorem 2.5 we only need to assume that the restriction of $\mathrm{char}\,P(\zeta-i\eta)$ to some imaginary homogeneous 2-dimensional plane through $i\theta$ has an analytic branch tangent to the $i\theta$-axis. This condition covers a wider class of polynomials than those considered in Theorem 2.5. For instance all polynomials which are locally hyperbolic at $i\eta$ and such that $\pm i\theta\in\pm i\partial\Gamma$ satisfy the above condition. (If $\pm iv$ are directions of local hyperbolicity, we denote here by $\pm i\Gamma$ the components of $\pm iv$ in the complement of $i\boldsymbol{R}^n\cap(P_m)_{i\eta}^{-1}(0)$ in $i\boldsymbol{R}^n$.

REMARK 2.8. Let N be non-characteristic for P. The boundary values $\gamma(u)=(h_j)_j$ of $u\in\Gamma(\mathring{M}^+,\mathcal{B}_M^P)$ (cf. (1.5)) are calculated as $h_j=B_ju|_N$ for a normal system of boundary operators B_j. For $u(x)=G(x+i\Gamma0)$ with $\Gamma'=\Gamma\cap\{x\cdot\theta=0\}\neq\emptyset$, one easily obtains: $B_j(x,D)u(x)|_N=(B_j(z,D)G(z)|_Y)(x'+i\Gamma'0)$

(where x' is the variable in N). Thus it is easily seen that for the hyperfunction $u=u^+$ of Theorem 2.5 one has (in a neighborhood of 0):

$$u|_{\dot{M}^+}\in \mathcal{A}_M^P|_{\dot{M}^+}\,; \quad (0\,; i\eta)\in \mathrm{SS}\,\gamma(u)\,, \tag{2.18}$$

(where $\mathcal{A}_M$ is the sheaf of analytic functions on M and $\mathcal{A}_M^P$ the kernel sheaf of P).

REMARK 2.9. Let N be non-characteristic for P. Instead of the hypotheses of Theorem 2.5 assume that a root τ of $P_m(i\eta+\tau\theta)=0$ verifies $\mathrm{Re}\,\tau<0$. Then one can give a much simpler construction of an analytic solution $G(z)$ of $PG(z)=0$ on a set of the form $\boldsymbol{R}^n+i\Gamma^+$ where $\Gamma^+=\{y\cdot\eta>\varepsilon(|y|+Y(-x_1)|x_1|)$ (Y being the Heaviside function). Moreover one can prove that $G(x+i\Gamma^+0)|_{\dot{M}^+}$ is analytic near 0 and that $(0\,;i\eta')\in \mathrm{SS}((G(z)|_Y)(x'+i\Gamma'0))$ but one cannot expect any more that $G(x+i\Gamma^+0)|_{\dot{M}^+}$ extends as a hyperfunction solution of P to a neighborhood of 0. However since $G(x+i\Gamma^+0)|_{\dot{M}^+}$ is mild from N^+ (cf. [6]) then the calculus of its boundary values can be performed as in Remark 2.8 according to Proposition 2.6 of [2]. In particular (2.18) is satisfied by $u(x)=G(x+i\Gamma^+0)|_{\dot{M}^+}$.

COROLLARY 2.10. *In the hypotheses of Theorem 2.5, P is neither N^+- nor N^--regular at $i\eta$.*

PROOF. We fix $y^*=(y\,;i\eta)$, $y\in N$, take $u(x)=u^+(x-y)$ with u^+ as in (2.3) and prove that P is not N^+-regular at y^*. (The proof of non-N^--regularity is analogous.)

By flabbiness of $\mathcal{B}_M$ we write $u=u_1+u_2$ with $u_1\in\Gamma_{M^+}(\mathcal{B}_M)$, $u_2\in\Gamma_{M^-}(\mathcal{B}_M)$. We consider u as a section of $\mathcal{C}_{M|X}$ at y^* and u_1 (u_2 resp.), as a section of $\mathcal{C}_{M^+|X}$ ($\mathcal{C}_{M^-|X}$ resp.) (cf. § 1). The injectivity of $P:\mathcal{C}_{M^\pm|X}\to\mathcal{C}_{M^\pm|X}$ at y^* (cf. (1.3)) implies $u\notin\mathcal{C}_{M^+|X}\cup\mathcal{C}_{M^-|X}$ for $Pu=0$ and $u\neq 0$ as a section of $\mathcal{C}_{M|X}$ at y^*. It follows $u_1\notin\mathcal{C}_{N|X}$, $u_2\notin\mathcal{C}_{N|X}$ for $\mathcal{C}_{N|X}=\mathcal{C}_{M^+|X}\cap\mathcal{C}_{M^-|X}$ at y^* (by the exactness of (1.2)).

Note that, because of (2.3): $u_1\in\Gamma_{N\underset{M}{\times}T^*_MX}(\mathcal{C}_{M|X})\cap\mathcal{C}_{M^+|X}$ at y^* and $Pu_1\in\mathcal{C}_{N|X}$ at y^* (for $\Gamma_N(\mathcal{B}_M)\subset\mathcal{C}_{M^+|X}\cap\mathcal{C}_{M^-|X}=\mathcal{C}_{N|X}$ by Proposition 1.1). This contradicts the N^+-regularity at y^*.

§ 3. Regularity of the traces of solutions of constant coefficients equations

We put here $\theta=(1,0,\cdots)$ and write $z=(z_1,z')$, $\zeta=(\zeta_1,\zeta')$, $z=x+iy$, $\zeta=\xi+i\eta$; sometimes we also write $\zeta=\mathrm{Re}\,\zeta+i\,\mathrm{Im}\,\zeta$. We set $N=\{x_1=0\}$,

$M^+=\{x_1\geqq 0\}$, $S^{n-1}=\{|\eta|=1\}$, $S^{n-2}=\{\eta\in S^{n-1}\,;\eta_1=0\}$ and also identify S^{n-2} with $\{\eta'\in R^{n-1}\,;|\eta'|=1\}$. We denote by σ the projection $(\zeta_1,\zeta')\to\zeta'$ from $\boldsymbol{C}^n$ to $\boldsymbol{C}^{n-1}$. Let $P=P(D)$ be an operator with constant coefficients, denote by P_m the principal part of P, and assume N non-characteristic for P. Let $i\eta'=i(0,\eta')\in iS^{n-2}$.

DEFINITION 3.1 (cf. [2]). P is said to be *$-\eta'$-semihyperbolic to N^+* iff, for a suitable constant c:

$$-\operatorname{Re}\zeta_1\leqq c[|\operatorname{Re}\zeta'|+((\operatorname{Im}\zeta')^2-(\operatorname{Im}\zeta'\cdot\eta')^2)^{1/2}] \quad \text{when } \operatorname{Im}\zeta'\cdot\eta'\geqq 0 \text{ and } P_m(\zeta_1,\zeta')=0\,. \tag{3.1}$$

REMARK 3.2. The former is weaker than the notion of "semihyperbolicity to N^+ at $-i\eta'$" which is defined as follows ([2]):

$$\operatorname{Re}\zeta_1\geqq 0 \qquad \text{when } \zeta\in\sigma^{-1}(i\tilde\eta')\cap P_m^{-1}(0) \text{ and } |\tilde\eta'-\eta'|\ll 1\,. \tag{3.2}$$

In fact by use of the local Bochner's tube theorem (and also by remembering that $P_m(i\theta)\neq 0$, $\theta=(1,0,\cdots)$) it is easily seen that (3.2)$\Rightarrow$(3.1). Let $\theta=(1,0,\cdots)\in S^{n-1}$, $\eta'\in S^{n-2}$ and assume $P_m(i\theta)\neq 0$.

LEMMA 3.3. (3.1) *is equivalent to*:

$$\begin{cases}\operatorname{Re}\zeta_1\geqq 0 & \text{for } \zeta\in\sigma^{-1}(i\eta')\cap P_m^{-1}(0) \qquad (3.3)\\ (P_m)_{i\eta}(i\theta)\neq 0 & \text{for } \eta\in\sigma^{-1}(i\eta')\cap P_m^{-1}(0)\cap i\boldsymbol{R}^n\,. \qquad (3.4)\end{cases}$$

PROOF. We let, in a suitable coordinate system, $\theta=(1,0,\cdots)$, $\eta'=(0,\cdots,0,1)$, and write $\zeta=(\zeta_1,\zeta'',\zeta_n)$, $\zeta'=(\zeta'',\zeta_n)$.

First we prove that (3.3) and (3.4) imply (3.1). Let $i\eta\in\sigma^{-1}(i\eta')\cap P_m^{-1}(0)\cap i\boldsymbol{R}^n$. The condition $(P_m)_{i\eta}(i\theta)\neq 0$ is equivalent to: $|\zeta_1|<c|\zeta'|$ for $(P_m)_{i\eta}(\zeta)=0$, $\zeta\neq 0$. Since $P_m(\zeta)=(P_m)_{i\eta}(\zeta-i\eta)+o(|\zeta-i\eta|^\mu)$, $|\zeta-i\eta|\to 0$ (μ being the degree of $(P_m)_{i\eta}$) then the former is also equivalent to: $|\zeta_1-i\eta_1|\leqq c|\zeta'-i\eta'|$ for $P_m(\zeta)=0$, $|\zeta-i\eta|\ll 1$. Taking into account the homogeneity of P_m we then obtain:

$$|\operatorname{Re}\zeta_1|\leqq c(|\zeta''|+|\operatorname{Re}\zeta_n|) \qquad \text{if } P_m(\zeta)=0\,,\ |\zeta-i\eta|\ll 1\,. \tag{3.5}$$

On the other hand for any $\zeta^0\in\sigma^{-1}(i\eta')\cap P_m^{-1}(0)\cap(\boldsymbol{C}^n\backslash i\boldsymbol{R}^n)$, we have $\operatorname{Re}\zeta_1^0>0$ by (3.3). It follows:

$$-\operatorname{Re}\zeta_1\leqq c(|\zeta''|+|\operatorname{Re}\zeta_n|) \qquad \text{if } P_m(\zeta)=0\,,\ |\zeta-\zeta^0|\ll 1\,. \tag{3.6}$$

By (3.5) and (3.6) we then conclude that (3.1) is satisfied by any $\zeta\in P_m^{-1}(0)$ with $|\zeta'/|\zeta'|-i\eta'|\ll 1$. In consequence it is also satisfied by any $\zeta\in P_m^{-1}(0)$

with $\operatorname{Im}\zeta_n \geqq 0$ since, when $|\zeta'/|\zeta'|-i\eta'|>\varepsilon$, $\operatorname{Im}\zeta_n\geqq 0$, we have $|\zeta'|< c(|\zeta''|+|\operatorname{Re}\zeta_n|)$ for a suitable $c=c_\varepsilon$.

(3.1)⇒(3.3) is obvious. Finally let us prove (3.1)⇒(3.4).

Let $i\eta\in\sigma^{-1}(i\eta')\cap P_m^{-1}(0)\cap i\boldsymbol{R}^n$ and consider $P_m(i\eta+\zeta+\tau\theta)$ for $\zeta\in\boldsymbol{C}^n$, $\tau\in\boldsymbol{C}$, $|\zeta|+|\tau|\ll 1$. Note that $P_m(i\eta+\tau\theta)$ cannot vanish identically in τ due to (3.1). Then for some integer $\nu\geqq 0$, $P_m(i\eta+\tau\theta)/\tau^\nu$ is analytic and $\neq 0$ at $\tau=0$. We write, in view of Weierstrass's theorem:

$$P_m(i\eta+\zeta+\tau\theta)=F(\zeta,\tau)(\tau^\nu+G(\zeta,\tau)),$$

where F is analytic and $\neq 0$ at $(\zeta,\tau)=(0,0)$ and G is a polynomial in τ of degree $\leqq\nu-1$ whose coefficients all vanish at $\zeta=0$.

To prove our statement we need to show that $\nu=\mu$ (μ being the degree of $(P_m)_{i\eta}$). Clearly $\nu\geqq\mu$. To show the opposite we take $\eta^1\in S^{n-1}$ such that $(P_m)_{i\eta}(i\eta^1)\neq 0$ and write:

$$P_m(i\eta+is\eta^1+\tau\theta)=F(is\eta^1,\tau)\prod_{j=1,\cdots,\nu}(\tau-\tau_j(s))$$

with $\tau_j(s)=a_j s^{b_j}(1+o(1))$, $s\in\boldsymbol{C}$, $s\to 0$, for some constants $a_j\in\boldsymbol{C}$, $b_j\in\boldsymbol{Q}$, $b_j>0$. We also note that $P_m(i\eta+is\eta^1)=(P_m)_{i\eta}(i\eta^1)s^\mu+o(|s^\mu|)$. Therefore if we suppose $\nu>\mu$, we then have $b_j<1$ and $a_j\neq 0$ for some j. Take $d\in\boldsymbol{C}$ such that $k=-\operatorname{Re}a_j d^{b_j}>0$ and let $s=td$, $t\in\boldsymbol{R}^+$; it is immediately seen that $-\operatorname{Re}\tau_j(s)=kt^{b_j}>c|s|$ for any c when $t\to 0$. This contradicts (3.1). The proof is complete.

Let $(x';i\eta')\in N\times iS^{n-2}$ and let $\rho: iS^{n-1}\to iS^{n-2}$ be the projection. We are now ready to compare (3.1) with the condition:

$$\text{(3.7)}\qquad \begin{aligned}&u\in(\Gamma_{M^+}(\mathcal{B}_M)/\Gamma_N(\mathcal{B}_M))_{(0,x')},\quad Pu=0,\\ &\overline{\operatorname{SS}u|_{\dot{M}^+}}\cap(\{(0,x')\}\times\{\rho^{-1}(i\eta')\})=\emptyset\Rightarrow(x';i\eta')\notin\operatorname{SS}\gamma(u).\end{aligned}$$

In fact by Lemma 3.3, Theorem 2.4, Proposition 1.3 and Remarks 2.8, 2.9, one immediately obtains:

THEOREM 3.4. *Let* $P_m(i\theta)\neq 0$, $(\theta=(1,0,\cdots))$, $N=\{x_1=0\}$, *and take* $(x';i\eta')\in N\times iS^{n-2}$. *Then* (3.1)⇒(3.7). *Conversely* (3.7)⇒(3.1) *if we assume in addition*: $P_m\in\boldsymbol{R}[\zeta]$ *and* $\partial((P_m)_{i\eta})(i\theta)\neq 0$ *for any* $i\eta\in\rho^{-1}(i\eta')\cap P_m^{-1}(0)$.

Example. The following is not included in the counterexamples to (3.7) considered by Kaneko in [2]. Let us consider in $\boldsymbol{R}^4$:

$$N=\{x_1=0\},\quad i\eta=i(0,0,0,1),\quad P(D)=D_1^3-D_4(D_1D_3-D_2^2).$$

We have:

$$\rho^{-1}(i\eta')\cap P^{-1}(0)=\{i\eta\},\ P_{i\eta}(i\theta)=[-i(\zeta_1\zeta_3-\zeta_2^2)]_{\zeta=(i,0,0)}=0,$$

$$\partial P_{i\eta}(i\theta)=[-i(\zeta_3,\ -2\zeta_2,\ \zeta_1)]_{\zeta=(i,0,0)}\neq 0.$$

Thus (3.7) is not satisfied and P is not N^+-regular at $i\eta$.

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(Received September 11, 1985)

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