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Boundary value problems for systems of linear partial differential equations and propagation of micro-analyticity

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Abstract. We give a general formulation of boundary value problems in the framework of hyperfunctions both for systems of linear partial differential equations with non-characteristic boundary and for Fuchsian systems of partial differential equations in a unified manner. We also give a microlocal formulation, which enables us to prove new results on propagation of micro-analyticity up to the boundary for solutions of systems micro-hyperbolic in a weak sense.

Contents

Introduction

Schapira [26] and Komatsu-Kawai [17] have formulated boundary value problems in the framework of hyperfunctions for single linear partial differential equations for which the boundary is non-characteristic and real analytic. They have shown the advantage of hyperfunctions in boundary value problem: any hyperfunction solution of such an equation has as many boundary values as the order of the equation, which are hyperfunctions on the boundary, and solutions are locally unique, i.e. Holmgren's type uniqueness theorem holds.

It seems, however, that general theory of boundary value problem for systems of linear partial differential equations has been still lacking except the work of Kashiwara-Kawai [6, 7] for elliptic systems.

In this paper we study boundary value problems both for general systems of linear partial differential equations with non-characteristic boundary and for Fuchsian systems in the sense of Tahara [29]. We define boundary values of solutions and prove Holmgren's type uniqueness theorem; we also give a microlocal formulation and study the propagation of micro-analyticity at the boundary for hyperfunction solutions. Our method is to introduce a new sheaf attached to the boundary for which the boundary value problem is always well-posed.

Let M be an n-dimensional paracompact real analytic manifold and let M_+ be its open subset with 1-codimensional real analytic boundary N. Let $\iota: M_+ \to M$ be the canonical embedding and put $\mathcal{B}_{M_+} = \iota_* \iota^{-1} \mathcal{B}_M$ and $\mathcal{B}_{N|M_+} = \mathcal{B}_{M_+}|_N$, where $\mathcal{B}_M$ is the sheaf of hyperfunctions on M. Then a section of $\mathcal{B}_{N|M_+}$ is a hyperfunction defined on the intersection of M_+ and a neighborhood of a point of N. Let X and Y be complexifications of M and N respectively and let

$$\mathcal{M}: \sum_{j=1}^{k} P_{ij} u_j = 0 \qquad (i=1, \cdots, m)$$

be a system of linear partial differential equations with analytic coefficients. Then $\mathcal{M}$ is regarded as a coherent $\mathcal{D}_X$ module $(\mathcal{D}_X)^k/(\mathcal{D}_X)^m(P_{ij})$, where $\mathcal{D}_X$ denotes the sheaf of linear partial differential operators with holomorphic coefficients on X. Boundary value problem (in the local sense) is to study the relation between $\mathcal{B}_{N|M_+}$-solutions $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+})$ of $\mathcal{M}$ and their boundary values.

In order to define a new sheaf, we assume that there exists a real valued real analytic function f on a neighborhood of N such that $f=0$ on N, $f>0$ on M_+, and $df \neq 0$. (We can slightly weaken this condition. See § 1.) We put $\tilde{M} = \{z \in X;\ \mathrm{Im} f(z) = 0\}$, $\tilde{M}_+ = \{z \in \tilde{M};\ f(z) > 0\}$. Then there is a sheaf $\mathcal{BO}$ on $\tilde{M}$ of hyperfunctions with holomorphic parameters since $\tilde{M}$ is locally isomorphic to $\boldsymbol{R} \times \boldsymbol{C}^{n-1}$. Using $\mathcal{BO}$, we define a sheaf $\tilde{\mathcal{B}}_{N|M_+}$ on N (Definition 1.2). We prove that there is an injective homomorphism α: $\mathcal{B}_{N|M_+} \to \tilde{\mathcal{B}}_{N|M_+}$ and that the boundary value problem is well-posed in $\tilde{\mathcal{B}}_{N|M_+}$.

First, let us assume that Y is non-characteristic for $\mathcal{M}$. Then there is an isomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N),$$

where $\mathcal{M}_Y$ is the tangential system of $\mathcal{M}$ to Y. Combined with the injectivity of α, this gives an injective homomorphism (boundary value map)

$$\gamma : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).$$

Hence for solutions of $\mathcal{M}$ we have defined their boundary values, which are hyperfunction solutions on N of the tangential system, and at the same time, proved Holmgren's type uniqueness theorem.

Next, let us consider Fuchsian systems: Let $z=(z_1, z')=(z_1, z_2, \cdots, z_n)$ be a local coordinate system of X with $f=z_1$. We use the notation $D=(D_1, D'), D'=(D_2, \cdots, D_n)$ with $D_j=\partial/\partial z_j$. We assume that $k=m$ and that $P=(P_{ij})$ has the form

$$P=z_1 D_1 I_m - A(z, D');$$

here I_m is the $m\times m$ unit matrix, $A=(A_{ij})$ is a matrix of sections of $\mathcal{D}_X$ free from D_1 such that each A_{ij} is of order $\leqq n_i-n_j+1$ with integers $n_1, \cdots, n_m$, and that $A_{ij}(0, z', D')$ is equal to a function $a_{ij}(z')$. We assume moreover that any pair of the eigenvalues of $A_0=(a_{ij})$ do not differ by integers. (These conditions are independent of the choice of local coordinate systems as above.) Then there exists an isomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \xrightarrow{\sim} (\mathcal{B}_N)^m,$$

and hence an injective homomorphism (boundary value map)

$$\gamma : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow (\mathcal{B}_N)^m.$$

This involves a new approach to boundary value problem for equations with regular singularities initiated by Kashiwara-Oshima [10] as long as the boundary is of codimension 1.

Note that previous works cited above ([6, 7, 10, 17, 26]) use so-called canonical extension of hyperfunction solutions in M_+ to M, which we can dispense with. Hence we hope our method will be useful for concrete expression of solutions.

In non-characteristic case we also prove that $\mathcal{B}_{N|M_+}$-solutions of $\mathcal{M}$ become F-mild in the sense of Ôaku [21, 22]; i.e. having boundary values in a natural way (F-mildness is a generalization of mildness due to Kataoka [13]). Hence in this case the boundary values are defined without the assumption of existence of f. We remark that in this case $\mathcal{B}_{N|M_+}$-solutions become mild in fact (see [13]).

In the same way as the theory of microfunctions (cf. Morimoto [19, 20] and Sato-Kawai-Kashiwara [25]), we define sheaves $\mathcal{C}_{M_+}$ on $S^*_M\tilde{M}$, $\mathcal{C}_{N|M_+}$ and $\tilde{\mathcal{C}}_{N|M_+}$ on $S^*_N Y$, microlocalizing $\mathcal{B}_{M_+}, \mathcal{B}_{N|M_+}, \tilde{\mathcal{B}}_{N|M_+}$ respectively (here $S^*_M\tilde{M}$ denotes the conormal sphere bundle of M in $\tilde{M}$). Then the argument above is also valid with $\mathcal{B}_{N|M_+}$, $\tilde{\mathcal{B}}_{N|M_+}$, $\mathcal{B}_N$ replaced by $\mathcal{C}_{N|M_+}$, $\tilde{\mathcal{C}}_{N|M_+}$, $\mathcal{C}_N$ respectively (here $\mathcal{C}_N$ denotes the sheaf on $S^*_N Y$ of microfunctions). In particular we get a microlocal version of Holmgren's type uniqueness theorem, which has been proved for single equations with non-characteristic boundary by Schapira [27] and Kataoka [33].

In § 3, using the sheaf $\mathcal{C}_{M_+}$, which is supported by $L_0 \cup L_+$ with $L_0 = S^*_M\tilde{M}|_N \cong S^*_N Y$, $L_+ = S^*_M\tilde{M}|_{M_+}$, we study propagation of micro-analyticity of solutions up to the boundary. First we prove

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{L_0}(\mathcal{C}_{M_+}))_{x^*} = 0$$

for a point x^* of L_0 under some conditions (a kind of micro-hyperbolicity) on $\mathcal{M}$ (Theorem 3.1). We remark that $\mathcal{M}$ may be degenerate on the boundary. As an application of this theorem we obtain the following: Suppose moreover that Y is non-characteristic for $\mathcal{M}$ or $\mathcal{M}$ is a Fuchsian system with the condition above concerning A_0, and that a $\mathcal{B}_{N|M_+}$-solution f of $\mathcal{M}$ is micro-analytic on $p^{-1}(U \cap L_+)$ with a neighborhood U of x^* in $S^*_M\tilde{M}$; here $p: S^*_M X \setminus S^*_{\tilde{M}} X \to S^*_M\tilde{M}$ is the canonical map. Then the boundary value $\gamma(f)$ of f is micro-analytic at x^* (Theorems 3.2 and 3.3). Results of this type have been proved by Kaneko [2], Schapira [27], Kataoka [14], Sjöstrand [28] for single equations (or for determined systems [28]) for which the boundary is non-characteristic. ([27] and [28] also treat the case where boundary conditions appear.) However there seems to have been no such result for degenerate equations including Fuchsian systems.

To prove Theorem 3.1, we modify the argument of prolongation of cohomology groups with holomorphic coefficients due to Kashiwara-Schapira [11] and apply it to cohomology groups with $\mathcal{B}\mathcal{O}$ coefficients (see Sect. 3.2).

Our method in § 1 and § 2 will work also in case of higher codimensional boundary to some extent. We shall study this case in a forthcoming paper.

§ 1. Several sheaves attached to the boundary

1.1. Definitions and vanishing theorems

In this section we define several sheaves attached to the boundary, which will be "function spaces" for boundary value problems. We use the notion and notation of derived categories and triangles in accordance with Hartshorne [1].

Let M be a paracompact n-dimensional real analytic manifold and M_+ be its open subset such that its boundary $N=\partial M_+$ is a one-codimensional real analytic closed submanifold of M. There exist complexifications X and Y of M and N respectively such that Y is a closed submanifold of X.

We assume that there are a real analytic closed submanifold $\tilde{M}$ of X and its open subset $\tilde{M}_+$ such that $M\subset\tilde{M}\subset X$, $\partial\tilde{M}_+=Y$, $\tilde{M}_+\cap M=M_+$, and that M, $\tilde{M}$, $\tilde{M}_+$, X are locally isomorphic to $\boldsymbol{R}^n$, $\boldsymbol{R}\times\boldsymbol{C}^{n-1}$, $\boldsymbol{R}_+\times\boldsymbol{C}^{n-1}$, $\boldsymbol{C}^n$ respectively with $\boldsymbol{R}_+=\{t\in\boldsymbol{R}\,;\, t>0\}$. More precisely, for any point $\mathring{z}$ of X we assume that there exists a local coordinate system $z=(z_1,\cdots,z_n)$ over a neighborhood Ω of $\mathring{z}$ such that

$$z(\Omega\cap M)=z(\Omega)\cap\boldsymbol{R}^n,\qquad z(\Omega\cap\tilde{M})=z(\Omega)\cap(\boldsymbol{R}\times\boldsymbol{C}^{n-1}),$$

$$z(\Omega\cap\tilde{M}_+)=z(\Omega)\cap(\boldsymbol{R}_+\times\boldsymbol{C}^{n-1}).$$

From now on we fix such $\tilde{M}$, $\tilde{M}_+$. We call such a local coordinate system as above *admissible* and use the notation $z=(z_1,z')$, $z_j=x_j+\sqrt{-1}\,y_j$.

REMARK. Suppose that there exists a real valued real analytic function f on a neighborhood of N in M such that $N=\{f=0\}$, $f>0$ in M_+, $df\neq0$ on N. Then replacing X by a neighborhood of N in X, we can take

$$\tilde{M}=\{z\in X\,;\, \mathrm{Im}\,f(z)=0\},\qquad \tilde{M}_+=\{z\in\tilde{M}\,;\, \mathrm{Re}\,f(z)>0\}.$$

Let $\iota: M_+\to M$, $\tilde{\iota}: \tilde{M}_+\to\tilde{M}$ be the natural embeddings. We denote by $\mathscr{B}_M$ the sheaf of hyperfunctions on M. Set $\mathscr{B}\mathscr{O}=\mathscr{B}\mathscr{O}_{\tilde{M}}=\mathscr{H}^1_{\tilde{M}}(\mathscr{O}_X)\otimes\omega_{\tilde{M}}$, where $\mathscr{O}_X$ is the sheaf of holomorphic functions on X, and $\omega_{\tilde{M}}$ is the orientation sheaf of $\tilde{M}$. Then $\mathscr{B}\mathscr{O}$ is the sheaf of hyperfunctions with holomorphic parameters.

DEFINITION 1.1. $\mathscr{B}_{M_+}=\iota_*\iota^{-1}\mathscr{B}_M$, $\qquad \mathscr{B}_{N|M_+}=\mathscr{B}_{M_+}|_N$,

$$\mathscr{B}\mathscr{O}_{\tilde{M}_+}=\tilde{\iota}_*\tilde{\iota}^{-1}\mathscr{B}\mathscr{O},\qquad \mathscr{B}\mathscr{O}_{Y|\tilde{M}_+}=\mathscr{B}\mathscr{O}_{\tilde{M}_+}|_Y.$$

Note that these sheaves are $\mathscr{D}_X$-modules, where $\mathscr{D}_X$ is the sheaf of rings of linear partial differential operators (of finite order) on X with holomorphic coefficients.

LEMMA 1.1. *Put* $\tilde{M}=\boldsymbol{R}\times\boldsymbol{C}^{n-1}$ *and* $M=\boldsymbol{R}^n$. *Then for any open set* U *of* $\boldsymbol{R}$ *and any Stein open set* Ω *of* $\boldsymbol{C}^{n-1}$ *we have*

$$H^\nu(U\times\Omega\,;\,\mathscr{B}\mathscr{O})=0\qquad(\nu\neq0).$$

PROOF. We use the notation $z=(z_1,z')$ with $z'=(z_2,\cdots,z_n)\in\boldsymbol{C}^{n-1}$, $z=x+\sqrt{-1}\,y$ with $x,y\in\boldsymbol{R}^n$ etc. Set

$$V=\{z\in \boldsymbol{C}^n\,;\ x_1=\operatorname{Re} z_1\in U,\, z'\in \Omega\}.$$

Then V is a Stein open set of $\boldsymbol{C}^n$, and we get

$$\boldsymbol{R}\Gamma(U\times\Omega\,;\ \mathcal{B}\mathcal{O})=\boldsymbol{R}\Gamma(V\,;\ \boldsymbol{R}\Gamma_{\tilde{M}}(\mathcal{O}_X))[1]=\boldsymbol{R}\Gamma_{\tilde{M}\cap V}(V\,;\ \mathcal{O}_X)[1].$$

For $\nu\geqq 2$, we have

$$H^{\nu}_{\tilde{M}\cap V}(V\,;\ \mathcal{O}_X)\cong H^{\nu-1}(V\backslash\tilde{M}\,;\ \mathcal{O}_X)=0$$

since $V\backslash\tilde{M}$ is also Stein. Hence we have

$$H^{\nu}(U\times\Omega\,;\ \mathcal{B}\mathcal{O})=H^{\nu+1}_{\tilde{M}\cap V}(V\,;\ \mathcal{O}_X)=0$$

for $\nu\geqq 1$. This completes the proof.

REMARK. The argument above implies that $H^{\nu}(\Omega\,;\ \mathcal{B}\mathcal{O})=0$ holds for $\nu\neq 0$ if there exists a Stein open set V of X such that $V\cap\tilde{M}=\Omega$.

LEMMA 1.2. $\qquad R^{\nu}\bar{\iota}_*\bar{\iota}^{-1}\mathcal{B}\mathcal{O}=0 \qquad$ *for* $\ \nu\neq 0$.

PROOF. We may assume $X=\boldsymbol{C}^n$, $\tilde{M}=\boldsymbol{R}\times\boldsymbol{C}^{n-1}$, $\tilde{M}_+=\boldsymbol{R}_+\times\boldsymbol{C}^{n-1}$. Let $\mathring{z}$ be a point of $\tilde{M}$. Then we have

$$(R^{\nu}\bar{\iota}_*\bar{\iota}^{-1}\mathcal{B}\mathcal{O})_{\mathring{z}}=\varinjlim_{U} H^{\nu}(U\cap\tilde{M}_+\,;\ \mathcal{B}\mathcal{O}),$$

where U runs on the system of neighborhoods of $\mathring{z}$. We may assume that U is the product of an open set of $\boldsymbol{R}$ and a Stein open set of $\boldsymbol{C}^{n-1}$. Thus this lemma follows from Lemma 1.1.

PROPOSITION 1.1. *There are natural isomorphisms*

$$\mathcal{B}_{M_+}\cong\mathcal{H}^{n-1}_{M}(\mathcal{B}\mathcal{O}_{\tilde{M}_+})\otimes\omega_{M/\tilde{M}},\qquad \mathcal{B}_{N|M_+}\cong\mathcal{H}^{n-1}_{M}(\mathcal{B}\mathcal{O}_{\tilde{M}_+})|_N\otimes\omega_N$$

where $\omega_{M/\tilde{M}}$ *denotes the sheaf of orientation of* M *relative to* $\tilde{M}$.

PROOF. First note that (cf. [25, Chapter I] and Komatsu [16])

$$\mathcal{B}_M\cong\mathcal{H}^{n-1}_M(\mathcal{B}\mathcal{O})\otimes\omega_{M/\tilde{M}}.$$

Hence we get

$$\begin{aligned}\mathcal{B}_{M_+}&\cong\boldsymbol{R}\bar{\iota}_*\bar{\iota}^{-1}\boldsymbol{R}\Gamma_M(\mathcal{B}\mathcal{O})[n-1]\otimes\omega_{M/\tilde{M}}\cong\boldsymbol{R}\bar{\iota}_*\boldsymbol{R}\Gamma_{M_+}(\bar{\iota}^{-1}\mathcal{B}\mathcal{O})[n-1]\otimes\omega_{M/\tilde{M}}\\ &\cong\boldsymbol{R}\Gamma_M(\boldsymbol{R}\bar{\iota}_*\bar{\iota}^{-1}\mathcal{B}\mathcal{O})[n-1]\otimes\omega_{M/\tilde{M}}=\boldsymbol{R}\Gamma_M(\mathcal{B}\mathcal{O}_{\tilde{M}+})[n-1]\otimes\omega_{M/\tilde{M}}.\end{aligned}$$

This proves the first isomorphism, from which the second follows.

Taking these isomorphisms into account, we introduce a new sheaf as follows:

DEFINITION 1.2. $\tilde{\mathscr{B}}_{N|M_+}=\mathscr{H}_N^{n-1}(\mathscr{B}\mathscr{O}_{Y|\tilde{M}_+})\otimes\omega_N$.

Note that $\tilde{\mathscr{B}}_{N|M_+}$ depends also on $\tilde{M}_+$ not only on M_+ and that $\tilde{\mathscr{B}}_{N|M_+}$ is a $\mathscr{D}_X$-module.

PROPOSITION 1.2. *There is a natural $\mathscr{D}_X$-linear sheaf homomorphism*

$$\alpha:\mathscr{B}_{N|M_+}\longrightarrow\tilde{\mathscr{B}}_{N|M_+}.$$

PROOF. There is a natural $\mathscr{D}_X$-linear homomorphism

$$\mathscr{H}_M^{\nu}(\mathscr{B}\mathscr{O}_{\tilde{M}_+})|_N\longrightarrow\mathscr{H}_N^{\nu}(\mathscr{B}\mathscr{O}_{\tilde{M}_+}|_Y)$$

induced by the embedding $Y\to\tilde{M}$. Since $\omega_{M/\tilde{M}}|_N\cong\omega_{N/Y}\cong\omega_N$, we get the homomorphism α.

To prove fundamental properties of $\tilde{\mathscr{B}}_{N|M_+}$, we need the vanishing of some cohomology groups whose coefficients are $\mathscr{B}\mathscr{O}_{Y|\tilde{M}_+}$. For this purpose we begin with the following:

LEMMA 1.3. *Let Ω be a Stein open set of $\boldsymbol{C}^m$. Then the set $\{0\}\times\Omega$ in $\boldsymbol{C}^d\times\boldsymbol{C}^m$ has a fundamental system of neighborhoods consisting of Stein open sets in $\boldsymbol{C}^d\times\boldsymbol{C}^m$.*

PROOF. This lemma is a special case of a theorem of Siu (Invent. Math. **38**, 89-100 (1976)). We give here an elementary proof. We use the notation $w=(w_1,\cdots,w_d)$ and $z=(z_1,\cdots,z_m)$. There exists a strictly plurisubharmonic C^∞ function φ on Ω such that the closure of $\{z\in\Omega\,;\,\varphi(z)<c\}$ is compact in Ω for any $c\in\boldsymbol{R}$. Let V be an open neighborhood of $\{0\}\times\Omega$. Our aim is to show that there is a Stein open set U of $\boldsymbol{C}^d\times\boldsymbol{C}^m$ such that $\{0\}\times\Omega\subset U\subset V$. We can take a C^∞ function g on $\boldsymbol{R}$ such that $g'>0$, $g''\geqq 0$, $U=\{(w,z)\in\boldsymbol{C}^d\times\Omega\,;\,|w|^2<\exp(-(g(\varphi(z))))\}\subset V$. Set $f=e^{-g}$. Then we have $f'<0$ and $f''\cdot f\leqq(f')^2$. We set

$$\psi(w,z)=|w|^2-f(\varphi(z)). \tag{1.1}$$

Let $(\mathring{w},\mathring{z})$ be a point of $\partial U\cap(\boldsymbol{C}^d\times\Omega)$ and let $(\theta,\zeta)\in\boldsymbol{C}^d\times\boldsymbol{C}^m$ satisfy

$$\sum_{j=1}^{d}\frac{\partial\psi}{\partial w_j}(\mathring{w},\mathring{z})\theta_j+\sum_{j=1}^{m}\frac{\partial\psi}{\partial z_j}(\mathring{w},\mathring{z})\zeta_j=0. \tag{1.2}$$

Let us calculate the Levi form $\mathscr{L}\psi(\mathring{w},\mathring{z})(\theta,\zeta)$ of ψ at $(\mathring{w},\mathring{z})$:

$$\mathscr{L}\psi(\mathring{w},\mathring{z})(\theta,\zeta)=\sum_{j,k}\frac{\partial^2\psi}{\partial w_j\partial\bar{w}_k}(\mathring{w},\mathring{z})\theta_j\bar{\theta}_k+\sum_{j,k}\frac{\partial^2\psi}{\partial z_j\partial\bar{z}_k}(\mathring{w},\mathring{z})\zeta_j\bar{\zeta}_k$$

$$=|\theta|^2-f'(\varphi(\mathring{z}))\sum_{j,k}\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k}(\mathring{z})\zeta_j\bar{\zeta}_k-f''(\varphi(\mathring{z}))\left|\sum_j\frac{\partial\varphi}{\partial z_j}(\mathring{z})\zeta_j\right|^2.$$

Since

$$\left| f'(\varphi(\mathring{z})) \sum_{j=1}^{m} \frac{\partial \varphi}{\partial z_j}(\mathring{z})\zeta_j \right| = \left| \sum_{j=1}^{d} \overline{\mathring{w}}_j \theta_j \right| \leqq |\mathring{w}| \cdot |\theta|$$

by (1.1) and (1.2), we get (note that we may assume $f''(\varphi(z)) \geqq 0$)

$$\mathcal{L}\psi(\mathring{w}, \mathring{z})(\theta, \zeta) \geqq |\theta|^2 - \frac{f''(\varphi(\mathring{z}))}{f'(\varphi(\mathring{z}))^2} |\mathring{w}|^2 |\theta|^2 \geqq |\theta|^2 - \frac{|\mathring{w}|^2 |\theta|^2}{f(\varphi(\mathring{z}))} = 0.$$

Thus the boundary of U is pseudo-convex in $C^d \times \Omega$, and U is Stein. This completes the proof.

PROPOSITION 1.3. *Put* $X = \boldsymbol{C}^n$, $\tilde{M} = \boldsymbol{R} \times \boldsymbol{C}^{n-1}$, $\tilde{M}_+ = \boldsymbol{R}_+ \times \boldsymbol{C}^{n-1}$, $M = \boldsymbol{R}^n$. *Then*

(i) $H^\nu(\Omega; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) = 0$ *holds for any Stein open set* Ω *of* $Y = \{0\} \times \boldsymbol{C}^{n-1} \cong \boldsymbol{C}^{n-1}$ *and any integer* $\nu \neq 0$.

(ii) *The flabby dimension of* $\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}$ *is* $n-1$.

(iii) $H^\nu_{\boldsymbol{R}^{n-1}+\sqrt{-1}G}(\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+})|_N = 0$ *for any proper convex closed cone* G *with vertex* 0 *in* $\boldsymbol{R}^{n-1}$ *and* $\nu \neq n-1$.

PROOF. (i) By the definition of $\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}$ and Lemma 1.2 we have

$$\begin{aligned} &\boldsymbol{R}\Gamma(\Omega; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) = \boldsymbol{R}\Gamma(\Omega; \boldsymbol{R}\iota_* \iota^{-1}\mathcal{B}\mathcal{O}|_Y) \\ &= \varinjlim_{U} \boldsymbol{R}\Gamma(U; \boldsymbol{R}\iota_* \iota^{-1}\mathcal{B}\mathcal{O}) = \varinjlim_{U} \boldsymbol{R}\Gamma(U \cap \tilde{M}_+; \mathcal{B}\mathcal{O}), \end{aligned}$$

where U runs on the system of neighborhoods of Ω in $\tilde{M}$. Let $\tilde{U}$ be an open set of $X = \boldsymbol{C}^n$ such that $\tilde{U} \cap \tilde{M} = U$ and set $X_+ = \{z \in X; x_1 > 0\}$. Then we get

$$\begin{aligned} &\boldsymbol{R}\Gamma(U \cap \tilde{M}_+; \mathcal{B}\mathcal{O}) = \boldsymbol{R}\Gamma(U \cap \tilde{M}_+; \boldsymbol{R}\Gamma_{\tilde{M}}(\mathcal{O}_X))[1] \\ &= \boldsymbol{R}\Gamma_{\tilde{M} \cap X_+ \cap \tilde{U}}(\tilde{U} \cap X_+; \mathcal{O}_X)[1]. \end{aligned}$$

By virtue of Lemma 1.3 we have for $\nu \geqq 1$,

$$\begin{aligned} &H^\nu(\Omega; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) = \varinjlim_{\tilde{U}} H^{\nu+1}_{\tilde{M} \cap X_+ \cap \tilde{U}}(\tilde{U} \cap X_+; \mathcal{O}_X) \\ &\cong \varinjlim_{\tilde{U}} H^\nu(\tilde{U} \cap X_+ \setminus \tilde{M}; \mathcal{O}_X) = 0, \end{aligned}$$

where $\tilde{U}$ runs on the system of neighborhoods of Ω in X. This completes the proof of (i).

(ii) follows from the fact the flabby dimension of $\mathcal{O}_X$ is equal to n.

(iii) By (ii) we have

$$\mathcal{H}^\nu_{\boldsymbol{R}^{n-1}+\sqrt{-1}G}(\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+})|_N = 0 \tag{1.3}$$

for $\nu \geqq n$. In order to prove (1.3) for $\nu \leqq n-2$, we use the following lemma, which is a part of the abstract edge of the wedge theorem due to Kashiwara-Laurent [9].

LEMMA 1.4 (Théorème 1.4.1 of [9]). *Let T be a topological space. Suppose that there is given a contravariant functor which associate with each complex manifold Y a sheaf $\mathcal{F}_Y$ on $Y \times T$ of $\mathcal{O}_Y$-modules satisfying the following* (H1)-(H3):

(H1) *If $U \supset V$ are open subsets of Y such that U is connected and V is not empty, and if W is an open subset of T, then we have*

$$\Gamma_{(U \setminus V) \times W}(U \times W;\ \mathcal{F}_Y) = 0 .$$

(H2) *Let $f: Y \to \boldsymbol{C}$ be a holomorphic function with $df \neq 0$ on Y and put $Z = f^{-1}(0)$. Let $i: Z \to Y$ be the canonical embedding. Then we have a short exact sequence*

$$0 \longrightarrow \mathcal{F}_Y \xrightarrow{f\cdot} \mathcal{F}_Y \xrightarrow{i^*} i_* \mathcal{F}_Z \longrightarrow 0 .$$

(H3) *Let Y and Z be complex manifolds with Z compact. Let f be the projection of $Y \times Z \times T$ to $Y \times T$. Then*

$$R^\nu f_* \mathcal{F}_{Y \times Z} \cong \mathcal{F}_Y \otimes_{\boldsymbol{C}} H^\nu(Z, \mathcal{O}_Z)$$

holds for any integer ν.

In addition to these conditions, suppose that G is a closed convex set of $\boldsymbol{C}^n$, $z \in G$, and that there is no $\boldsymbol{C}$-linear subvariety L of $\boldsymbol{C}^n$ of dimension $(n-q+1)$ containing z such that $L \cap G$ is a neighborhood of z in L. Then for any $t \in T$

$$\mathcal{H}^\nu_{G \times T}(\mathcal{F}_{\boldsymbol{C}^n})_{(z,t)} = 0$$

holds for any $\nu < q$.

PROOF OF PROPOSITION 1.3 (continued). Now let us prove (1.3) for $\nu \leqq n-2$ using Lemma 1.4. For each complex manifold Y we set

$$\mathcal{F}_Y = (\tilde{\iota}_Y)_* (\tilde{\iota}_Y)^{-1} \mathcal{B}\mathcal{O}_{\boldsymbol{R} \times Y} |_{\{0\} \times Y} ,$$

where $\tilde{\iota}_Y: \boldsymbol{R}_+ \times Y \to \boldsymbol{R} \times Y$ is the embedding. It is easy to see that $Y \mapsto \mathcal{F}_Y$ defines a contravariant functor. Hence it suffices to verify (H1)-(H3) of Lemma 1.4 for this $\mathcal{F}_Y$ with $T = \{0\}$.

(H1) follows from the unique continuation property of sections of $\mathcal{B}\mathcal{O}$ with respect to the holomorphic parameters. We can verify (H2) by applying the functor $\boldsymbol{R}(\tilde{\iota}_Y)_* \tilde{\iota}_Y^{-1} \boldsymbol{R}\Gamma_{\boldsymbol{R} \times Y}$ to the short exact sequence

$$0 \longrightarrow \mathcal{O}_{C\times Y} \xrightarrow{f\cdot} \mathcal{O}_{C\times Y} \xrightarrow{i^*} i_*\mathcal{O}_{C\times Z} \longrightarrow 0,$$

where i is the injection of $\boldsymbol{C}\times Z$ to $\boldsymbol{C}\times Y$.

Lastly let us verify (H3). Let $f: Y\times Z\to Y$ and $\tilde{f}: \boldsymbol{C}\times Y\times Z\to \boldsymbol{C}\times Y$ be the projections. We shall show the isomorphism

$$\boldsymbol{R}f_*\mathcal{O}_{Y\times Z}\cong\mathcal{O}_Y\otimes_{\boldsymbol{C}}\boldsymbol{R}\Gamma(Z;\mathcal{O}_Z). \tag{1.4}$$

Let us take the Dolbeault resolution

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{E}_Z^{(0,0)} \xrightarrow{\bar\partial} \mathcal{E}_Z^{(0,1)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}_Z^{(0,d)} \longrightarrow 0,$$

where d is the dimension of Z and $\mathcal{E}_Z^{(0,\nu)}$ denotes the sheaf of differential forms of type $(0,\nu)$ with C^∞ coefficients on Z. For $y\in Y$ we have a homomorphism

$$\mathcal{O}_{Y,y}\otimes_{\boldsymbol{C}}\boldsymbol{R}\Gamma(Z;\mathcal{O}_Z) \longrightarrow \boldsymbol{R}\Gamma(\{y\}\times Z;\mathcal{O}_{Y\times Z})\cong\boldsymbol{R}f_*(\mathcal{O}_{Y\times Z})_y.$$

Since $H^\nu(Z;\mathcal{O}_Z)$ is finite dimensional for any ν (theorem of Cartan-Serre), we get the isomorphisms

$$\mathcal{O}_{Y,y}\otimes_{\boldsymbol{C}}H^\nu(Z;\mathcal{O}_Z) \xrightarrow{\sim} \mathcal{O}_{Y,y}\hat{\otimes}_{\boldsymbol{C}}H^\nu(Z;\mathcal{O}_Z) \xrightarrow{\sim} H^\nu(\{y\}\times Z;\mathcal{O}_{Y\times Z})$$

applying the argument of Andreotti-Grauert. This proves (1.4).

Replacing Y with $\boldsymbol{C}\times Y$ and applying the functor $\boldsymbol{R}\Gamma_{\boldsymbol{R}\times Z}$, we get

$$\boldsymbol{R}\tilde{f}_*\mathcal{B}\mathcal{O}_{\boldsymbol{R}\times Y\times Z}\cong\mathcal{B}\mathcal{O}_{\boldsymbol{R}\times Y}\otimes_{\boldsymbol{C}}\boldsymbol{R}\Gamma(Z;\mathcal{O}_Z).$$

Applying the functor $\boldsymbol{R}(\iota_Y)_*\iota_Y^{-1}$, we get

$$\boldsymbol{R}\tilde{f}_*\mathcal{F}_{Y\times Z}\cong\mathcal{F}_Y\otimes_{\boldsymbol{C}}\boldsymbol{R}\Gamma(Z;\mathcal{O}_Z).$$

Thus (H3) is verified. Since $\boldsymbol{R}^{n-1}+\sqrt{-1}\,G$ does not contain $\boldsymbol{C}$-linear subvarieties of $\boldsymbol{C}^{n-1}$ of dimension $\geqq 1$, (iii) follows from Lemma 1.4. This completes the proof of Proposition 1.3.

By Proposition 1.3 we have the following:

PROPOSITION 1.4. (i) *$\tilde{\mathcal{B}}_{N|M_+}$ is a flabby sheaf on N.*
(ii) *Let U be an open set of N. Then*

$$\Gamma(U;\tilde{\mathcal{B}}_{N|M_+})\cong H_U^{n-1}(\tilde{U};\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+})$$

holds for any open set $\tilde{U}$ of Y such that $\tilde{U}\cap N=U$.

We can show similar results for $\mathcal{B}_{M_+}$:

PROPOSITION 1.5. (i) *$\mathscr{B}_{M_+}$ is a flabby sheaf on M supported by $\bar{M}_+$.*
(ii) *Let U be an open set of M. Then*

$$\Gamma(U;\ \mathscr{B}_{M_+})\cong H^{n-1}_{\tilde{U}\cap M_+}(\tilde{U}\cap\tilde{M}_+;\ \mathscr{B}\mathcal{O})$$

holds for any open set $\tilde{U}$ of $\tilde{M}$ such that $\tilde{U}\cap M=U$.

This is a consequence of the following:

PROPOSITION 1.6. *Put $X=\boldsymbol{C}^n$, $\tilde{M}=\boldsymbol{R}\times\boldsymbol{C}^{n-1}$, $\tilde{M}_+=\boldsymbol{R}_+\times\boldsymbol{C}^{n-1}$, $M=\boldsymbol{R}^n$. Then*

(i) *$H^\nu(U\times\Omega;\ \mathscr{B}\mathcal{O}_{\tilde{M}_+})=0$ holds for any open set U of $\boldsymbol{R}$ and any Stein open set Ω of $\boldsymbol{C}^{n-1}$, and for any $\nu\neq 0$.*

(ii) *The flabby dimension of $\mathscr{B}\mathcal{O}_{\tilde{M}_+}$ is $n-1$.*

(iii) *$\mathscr{H}^\nu_{\boldsymbol{R}\times(\boldsymbol{R}^{n-1}+\sqrt{-1}G)}(\mathscr{B}\mathcal{O}_{\tilde{M}_+})|_M=0$ for any proper convex closed cone G with vertex 0 in $\boldsymbol{R}^{n-1}$ and $\nu\neq n-1$.*

This can be proved by the same method as Proposition 1.3.

1.2. Microlocalization and concrete expression

Let

$$\pi_{M/\tilde{M}}:(\tilde{M}\backslash M)\cup S^*_M\tilde{M}\longrightarrow\tilde{M},\qquad \pi_{N/Y}:(Y\backslash N)\cup S^*_NY\longrightarrow Y$$

be comonoidal transforms of $\tilde{M}$ and Y with centers M and N respectively (cf. [25, Chapter I]). If $z=x+\sqrt{-1}\,y$ is an admissible local coordinate system of X, then $(x,\sqrt{-1}\eta'\infty)=(x,\sqrt{-1}\langle\eta',dz'\rangle\infty)$ (resp. $(0,x',\sqrt{-1}\eta'\infty)$) is the corresponding local coordinate system of $S^*_M\tilde{M}$ (resp. S^*_NY) with $\eta'=(\eta_2,\cdots,\eta_n)\in\boldsymbol{R}^{n-1}$ and $\langle\eta',dz'\rangle=\eta_2dz_2+\cdots+\eta_ndz_n$. We define sheaves on $S^*_M\tilde{M}$ and on S^*_NY as follows (we denote by a the antipodal map):

DEFINITION 1.3.

$$\mathcal{C}_{M_+}=\mathscr{H}^{n-1}_{S^*_M\tilde{M}}((\pi_{M/\tilde{M}})^{-1}\mathscr{B}\mathcal{O}_{\tilde{M}_+})^a\otimes\omega_{M/\tilde{M}},\qquad \mathcal{C}_{N|M_+}=\mathcal{C}_{M_+}|_{(\pi_{M/\tilde{M}})^{-1}(N)}$$

$$\tilde{\mathcal{C}}_{N|M_+}=\mathscr{H}^{n-1}_{S^*_NY}((\pi_{N/Y})^{-1}\mathscr{B}\mathcal{O}_{Y|\tilde{M}_+})^a\otimes\omega_N.$$

Note that these cohomology groups of order ν vanish for $\nu\neq n-1$ in view of Propositions 1.3 and 1.6. A natural map $T^*\tilde{M}\times_{\tilde{M}}Y\to T^*Y$ induces a (real analytic) diffeomorphism $S^*_M\tilde{M}\times_M N\simeq S^*_NY$. Hence we identify $(\pi_{M/\tilde{M}})^{-1}(N)$ with S^*_NY. By virtue of Propositions 1.3 and 1.6, the arguments in Morimoto [19, 20] and [25, Chapter I] work in these cases. Let

$$\tau_{M/\tilde{M}}: (\tilde{M}\backslash M)\cup S_M\tilde{M} \longrightarrow \tilde{M}, \qquad \tau_{N/Y}: (Y\backslash N)\cup S_N Y \longrightarrow Y$$

be the real monoidal transforms of $\tilde{M}$ and Y with centers M and N respectively (cf. [25]) and let

$$\varepsilon_{M/\tilde{M}}: \tilde{M}\backslash M \longrightarrow (\tilde{M}\backslash M)\cup S_M\tilde{M}, \qquad \varepsilon_{N/Y}: Y\backslash N \longrightarrow (Y\backslash N)\cup S_N Y$$

be the canonical embeddings. If (x_1, z') is a local coordinate system of $\tilde{M}$, then $x+\sqrt{-1}\,v'0=x+\sqrt{-1}\langle v', (\partial/\partial z')\rangle 0$ (resp. $(0, x')+\sqrt{-1}\,v'0$) is the corresponding local coordinate system of $S_M\tilde{M}$ (resp. $S_N Y$) with $v'\in \boldsymbol{R}^{n-1}$. Note that we can identify $(\tau_{M/\tilde{M}})^{-1}(N)$ with $S_N Y$. Set

$$\tilde{\mathcal{A}}_{M_+}=((\varepsilon_{M/\tilde{M}})_*(\mathcal{B}\mathcal{O}_{\tilde{M}_+}|_{\tilde{M}\backslash M}))|_{S_M\tilde{M}},$$

$$\tilde{\mathcal{A}}_{N|M_+}=((\varepsilon_{N/Y})_*(\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}|_{Y\backslash N}))|_{S_N Y}.$$

Then there exist injective homomorphisms (boundary value maps)

$$b_+: \tilde{\mathcal{A}}_{M_+} \longrightarrow (\tau_{M/\tilde{M}})^{-1}\mathcal{B}_{M_+}, \qquad \tilde{b}: \tilde{\mathcal{A}}_{N|M_+} \longrightarrow (\tau_{N/Y})^{-1}\tilde{\mathcal{B}}_{N|M_+}$$

and surjective homomorphisms (spectral maps)

$$\mathrm{sp}_+: (\pi_{M/\tilde{M}})^{-1}\mathcal{B}_{M_+} \longrightarrow \mathcal{C}_{M_+}, \qquad \widetilde{\mathrm{sp}}: (\pi_{N/Y})^{-1}\tilde{\mathcal{B}}_{N|M_+} \longrightarrow \tilde{\mathcal{C}}_{N|M_+}.$$

Here we define b_+ and $\tilde{b}$ by using Cech cohomology groups in the same way as [19, 20] (see also Martineau [18] and Komatsu [16]). A subset U of $S_M\tilde{M}$ (resp. $S_N Y$) is called convex if the intersection of U and each fiber of $\tau_{M/\tilde{M}}$ (resp. $\tau_{N/Y}$) is convex. We define the polar set $U^\circ\subset S_M^*\tilde{M}$ (resp. S_N^*Y) of U by

$$U^\circ=\{(x, \sqrt{-1}\,\eta'\infty);\ -\mathrm{Re}\langle\eta', v'\rangle<0 \quad \text{for any } x+\sqrt{-1}\,v'0\in U\}.$$

In the same way as in [19, 20] we obtain the following:

PROPOSITION 1.7. (i) *There exists an exact sequence*

$$0 \longrightarrow \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}|_N \xrightarrow{\tilde{b}} \tilde{\mathcal{B}}_{N|M_+} \xrightarrow{\widetilde{\mathrm{sp}}} (\pi_{N/Y})_*\tilde{\mathcal{C}}_{N|M_+} \longrightarrow 0.$$

(ii) *Let U be an open convex subset of $S_N Y$ and let f be a section of $\tilde{\mathcal{B}}_{N|M_+}$ over $\tau_{N/Y}(U)$. Then there exists a section F of $\tilde{\mathcal{A}}_{N|M_+}$ over U such that $\tilde{b}(F)=f$ if and only if the support* $\mathrm{supp}(\widetilde{\mathrm{sp}}(f))$ *is contained in U°.*

PROPOSITION 1.8. (i) *There exists an exact sequence*

$$0 \longrightarrow \mathcal{B}\mathcal{O}_{\tilde{M}_+}|_M \xrightarrow{b_+} \mathcal{B}_{M_+} \xrightarrow{\mathrm{sp}_+} (\pi_{M/\tilde{M}})_*\mathcal{C}_{M_+} \longrightarrow 0.$$

(ii) *Let U be an open convex subset of $S_M\tilde{M}$ and let f be a section of $\mathscr{B}_{M_+}$ over $\tau_{M/\tilde{M}}(U)$. Then there exists a section F of $\tilde{\mathscr{A}}_{M_+}$ over U such that $b_+(F)=f$ if and only if* $\operatorname{supp}(\mathrm{sp}_+(f))$ *is contained in U°.*

Now we give a concrete description of $\tilde{\mathscr{B}}_{N|M_+}$ following an idea of Kaneko [4] for usual hyperfunctions. Set $\tilde{M}_+=\boldsymbol{R}\times\boldsymbol{C}^{n-1}$, $M_+=\boldsymbol{R}_+\times\boldsymbol{R}^{n-1}$, $N=\{0\}\times\boldsymbol{R}^{n-1}$ and let U be an open subset of N. We define a $\boldsymbol{C}$-linear space $\mathscr{F}(U)$ as the direct sum

$$\mathscr{F}(U)=\bigoplus_{\Gamma}\tilde{\mathscr{A}}_{N|M_+}(U+\sqrt{-1}\,\Gamma 0),$$

where Γ runs over all the open convex cones of $\boldsymbol{R}^{n-1}$. Then an element f of $\mathscr{F}(U)$ is written as

$$f=\sum_{j=1}^{J}F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)$$

with $F_j\in\tilde{\mathscr{A}}_{N|M_+}(U+\sqrt{-1}\,\Gamma_j 0)$. Let $\mathscr{G}(U)$ be the minimal $\boldsymbol{C}$-linear subspace of $\mathscr{F}(U)$ containing all the elements of $\mathscr{F}(U)$ of the form

$$F_1(x_1, x'+\sqrt{-1}\,\Gamma_1 0)-F_2(x_1, x'+\sqrt{-1}\,\Gamma_2 0)$$

with $F_j\in\tilde{\mathscr{A}}_{N|M_+}(U+\sqrt{-1}\,\Gamma_j 0)$ $(j=1,2)$ such that $\Gamma_2\subset\Gamma_1$ and $F_1|_{U+\sqrt{-1}\Gamma_2}=F_2$. We denote $\mathscr{F}(U)/\mathscr{G}(U)$ by $F\tilde{\mathscr{B}}_{N|M_+}(U)$. Then it is easy to see that $\tilde{b}$ induces a $\boldsymbol{C}$-linear homomorphism Φ of $F\tilde{\mathscr{B}}_{N|M_+}(U)$ to $\tilde{\mathscr{B}}_{N|M_+}(U)$.

PROPOSITION 1.9. *$\Phi: F\tilde{\mathscr{B}}_{N|M_+}(U)\to\tilde{\mathscr{B}}_{N|M_+}(U)$ is an isomorphism.*

PROOF. Let $\eta^0,\cdots,\eta^{n-1}$ be vectors of $\boldsymbol{R}^{n-1}$ such that

$$\bigcup_{j=0}^{n-1}E_{\eta^j}=\boldsymbol{R}^{n-1}\setminus\{0\},\qquad \det(\eta^1,\cdots,\eta^{n-1})>0,$$

where $E_{\eta^j}=\{y'\in\boldsymbol{R}^{n-1};\ \langle y',\eta^j\rangle>0\}$. Let Ω be a Stein neighborhood of U in Y and set

$$U_j=\{(0,z')\in\Omega;\ \operatorname{Im}z'\in E_{\eta^j}\},$$

$$\mathcal{U}=\{\Omega, U_0,\cdots,U_{n-1}\},\qquad \mathcal{U}'=\{U_0,\cdots,U_{n-1}\}.$$

Since $(\mathcal{U},\mathcal{U}')$ is a relative Stein covering of $(\Omega,\Omega\setminus U)$, we get

$$\begin{aligned}\tilde{\mathscr{B}}_{N|M_+}(U)&\cong H^{n-1}(\mathcal{U}\bmod\mathcal{U}';\ \mathscr{B}\mathcal{O}_{Y|\tilde{M}_+})\\&=C^{n-1}(\mathcal{U}\bmod\mathcal{U}';\ \mathscr{B}\mathcal{O}_{Y|\tilde{M}_+})/\delta C^{n-2}(\mathcal{U}\bmod\mathcal{U}';\ \mathscr{B}\mathcal{O}_{Y|\tilde{M}_+}),\end{aligned}$$

here δ is the coboundary operator. Through this isomorphism an element of $\tilde{\mathscr{B}}_{N|M_+}(U)$ is expressed as the modulo class $[f]$ of an

$$f=\sum_{j=0}^{n-1}(-1)^j F_j(x_1, z')\Omega\wedge U_0\wedge\cdots\wedge U_{j-1}\wedge U_{j+1}\wedge\cdots\wedge U_{n-1}$$
$$\in C^{n-1}(\mathcal{U} \bmod \mathcal{U}' ; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}),$$

where F_j is a section of $\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}$ on $\bigcap_{\nu\neq j} U_\nu$. Set

$$\Psi([f])=\sum_{j=0}^{n-1}F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)\in F\tilde{\mathcal{B}}_{N|M_+}(U)$$

with $\Gamma_j=\bigcap_{\nu\neq j}E_{\eta^\nu}$. Then this defines a $\boldsymbol{C}$-linear map

$$\Psi : H^{n-1}(\mathcal{U} \bmod \mathcal{U}' ; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) \longrightarrow F\tilde{\mathcal{B}}_{N|M_+}(U).$$

Let us show that Ψ induces a homomorphism of $\tilde{\mathcal{B}}_{N|M_+}(U)$ to $F\tilde{\mathcal{B}}_{N|M_+}(U)$ independently of the choice of $\eta^0, \cdots, \eta^{n-1}$. By moving the frame $(\eta^0, \cdots, \eta^{n-1})$ step by step, we have only to show that Ψ defines the same map as before even if η^0 is replaced by η'^0 sufficiently close to η^0. Hence take η'^0 so that

$$E_{\eta'^0}\cup E_{\eta^1}\cup\cdots\cup E_{\eta^{n-1}}=\boldsymbol{R}^{n-1}\setminus\{0\}$$

and set

$$V_0=\{(0, z')\in\Omega ;\ \operatorname{Im} z'\in E_{\eta'^0}\},$$
$$W_0=\{(0, z')\in\Omega ;\ \operatorname{Im} z'\in E_{\eta^0}\cap E_{\eta'^0}\},$$
$$\mathcal{V}=\{\Omega, V_0, V_1, \cdots, V_{n-1}\}, \qquad \mathcal{V}'=\{V_0, V_1, \cdots, V_{n-1}\},$$
$$\mathcal{W}=\{\Omega, W_0, W_1, \cdots, W_{n-1}\}, \qquad \mathcal{W}'=\{W_0, W_1, \cdots, W_{n-1}\}$$

with $V_j=W_j=U_j$ for $j=1, \cdots, n-1$. Then both $(\mathcal{V}, \mathcal{V}')$ and $(\mathcal{W}, \mathcal{W}')$ are relative Stein coverings of $(\Omega, \Omega\setminus U)$. Suppose that

$$g=\sum_{j=0}^{n-1}(-1)^j G_j(x_1, z')\Omega\wedge V_0\wedge\cdots\wedge V_{j-1}\wedge V_{j+1}\wedge\cdots\wedge V_{n-1}$$
$$\in C^{n-1}(\mathcal{V} \bmod \mathcal{V}' ; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+})$$

defines the same element as $[f]$ in $\tilde{\mathcal{B}}_{N|M_+}(U)$. Then the image of $[f]$ by the natural map

$$H^{n-1}(\mathcal{U} \bmod \mathcal{U}' ; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) \longrightarrow H^{n-1}(\mathcal{W} \bmod \mathcal{W}' ; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+})$$

and that of $[g]$ by

$$H^{n-1}(\mathcal{V} \bmod \mathcal{V}' ; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) \longrightarrow H^{n-1}(\mathcal{W} \bmod \mathcal{W}' ; \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+})$$

coincide. Hence there exist sections H_{jk} of $\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}$ on $\bigcap_{\nu\neq j,k} W_\nu$ such that

$$F_j - G_j = \sum_{k=0}^{j-1} (-1)^{j+k+1} H_{jk} + \sum_{k=j+1}^{n-1} (-1)^{j+k} H_{jk}, \ H_{jk} + H_{kj} = 0 .$$

This implies

$$\sum_{j=0}^{n-1} (F_j(x_1, x' + \sqrt{-1}\,\Gamma_j 0) - G_j(x_1, x' + \sqrt{-1}\,\Gamma'_j 0)) = 0$$

in $F\tilde{\mathscr{B}}_{N|M_+}(U)$, where $\Gamma'_0 = \Gamma_0$ and

$$\Gamma'_j = E_{\eta'^0} \cap \bigcap_{1 \leq \nu \neq j} E_{\eta^\nu} \qquad (j = 1, \cdots, n-1) .$$

Thus Ψ is well-defined as a homomorphism of $\tilde{\mathscr{B}}_{N|M_+}(U)$ to $F\tilde{\mathscr{B}}_{N|M_+}(U)$. In view of this invariance of Ψ and the definition of $\tilde{b}$, it is easy to see that both $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are identities. This completes the proof.

From now on we identify $\tilde{\mathscr{B}}_{N|M_+}(U)$ with $F\tilde{\mathscr{B}}_{N|M_+}(U)$. The following lemma will be used together with Proposition 1.9 in the proof of the edge of the wedge theorem.

LEMMA 1.5. *Let f be a section of $\tilde{\mathscr{B}}_{N|M_+}$ on a neighborhood of $\mathring{x} \in N$. Then $(\mathring{x}, \sqrt{-1}\,\mathring{\xi}'\infty) \in S_N^* Y$ is not contained in* $\mathrm{supp}(\tilde{\mathrm{sp}}(f))$ *if and only if there exist sections F_j of $\tilde{\mathscr{A}}_{N|M_+}$ on $U + \sqrt{-1}\,\Gamma_j 0$ such that U is a neighborhood of $\mathring{x}$, Γ_j is an open cone of $\boldsymbol{R}^{n-1}$ with $\mathring{\xi}' \notin \Gamma_j^\circ = \{\eta' \,;\, \langle y', \eta' \rangle > 0$ for any $y' \in \Gamma_j\}$ and that*

$$f(x) = \sum_{j=1}^{J} F_j(x_1, x' + \sqrt{-1}\,\Gamma_j 0) .$$

PROOF. We may assume $\mathring{x} = 0$. Suppose $x^* = (0, \sqrt{-1}\,\mathring{\xi}'\infty) \notin \mathrm{supp}(\tilde{\mathrm{sp}}(f))$. We take $\xi^1, \cdots, \xi^{n-1} \in \boldsymbol{R}^{n-1}$ so that

$$\boldsymbol{R}_+\xi^1 \cup \cdots \cup \boldsymbol{R}_+\xi^{n-1} = \{y' \in \boldsymbol{R}^{n-1} \backslash \{0\} \,;\, \langle y', \mathring{\xi}' \rangle = 0\} .$$

For $\varepsilon > 0$ putting $\eta^j = \xi^j + \mathring{\xi}'/\varepsilon$ for $j = 1, \cdots, n-1$ and $\eta^0 = -\mathring{\xi}'$, we set

$$E_\varepsilon^j = \{y' \in \boldsymbol{R}^{n-1} ;\ \langle y', \eta^j \rangle > 0\} ,$$

$$D_\varepsilon = \{(0, z') \in Y ;\ |z'| < \varepsilon\} ,$$

$$U_\varepsilon^j = \{(0, z') \in D_\varepsilon ;\ \mathrm{Im}\, z' \in E_\varepsilon^j\} ,$$

$$Z_\varepsilon = D_\varepsilon \backslash \bigcup_{j=1}^{n-1} U_\varepsilon^j .$$

Then we have

$$\tilde{C}_{N|M_+, x^*} \cong \varinjlim_{\varepsilon} H_{Z_\varepsilon}^{n-1}(D_\varepsilon ;\ \mathscr{B}\mathscr{O}_{Y|\tilde{M}_+}) .$$

Since $\widetilde{\mathrm{sp}}$ is induced by the natural map

$$H_N^{n-1}(D_\varepsilon\,;\ \mathscr{B}\mathcal{O}_{Y|\tilde{M}_+}) \longrightarrow H_{Z_\varepsilon}^{n-1}(D_\varepsilon\,;\ \mathscr{B}\mathcal{O}_{Y|\tilde{M}_+})\,, \tag{1.5}$$

the image of f by (1.5) vanishes for some $\varepsilon>0$. We fix such an ε. Then as a section of $\tilde{\mathscr{B}}_{N|M_+}(D_\varepsilon\cap N)$,

$$f(x)=\sum_{j=0}^{n-1} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)$$

holds with sections F_j of $\mathscr{B}\mathcal{O}_{Y|\tilde{M}_+}$ on $\bigcap_{\nu\neq j} U_\varepsilon^\nu$, where $\Gamma_j=\bigcap_{\nu\neq j} E_\varepsilon^\nu$. Set

$$\mathcal{U}=\{D_\varepsilon, U_\varepsilon^0, U_\varepsilon^1, \cdots, U_\varepsilon^{n-1}\}, \qquad \mathcal{U}'=\{U_\varepsilon^0, U_\varepsilon^1, \cdots, U_\varepsilon^{n-1}\},$$

$$\mathcal{V}=\{D_\varepsilon, U_\varepsilon^1, \cdots, U_\varepsilon^{n-1}\}, \qquad \mathcal{V}'=\{U_\varepsilon^1, \cdots, U_\varepsilon^{n-1}\}.$$

Then (1.5) is compatible with the natural map

$$H^{n-1}(\mathcal{U} \bmod \mathcal{U}'\,;\ \mathscr{B}\mathcal{O}_{Y|\tilde{M}_+}) \longrightarrow H^{n-1}(\mathcal{V} \bmod \mathcal{V}'\,;\ \mathscr{B}\mathcal{O}_{Y|\tilde{M}_+}).$$

Since the image of f by this homomorphism vanishes, there exist sections F_{0k} of $\mathscr{B}\mathcal{O}_{Y|\tilde{M}_+}$ on $\bigcap_{\nu\neq 0,k} U_\varepsilon^\nu$ for $k=1,\cdots,n-1$ such that $F_0=F_{01}+\cdots+F_{0,n-1}$. Hence we get

$$f(x)=\sum_{k=1}^{n-1} F_{0k}(x_1, x'+\sqrt{-1}\,\Gamma_{0k}0)+\sum_{j=1}^{n-1} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)$$

with $\Gamma_{0k}=\bigcap_{\nu\neq 0,k} E_\varepsilon^\nu$. Moreover neither Γ_j° $(j=1,\cdots,n-1)$ nor Γ_{0k}° contain $\mathring{\xi}'$. The converse implication is obvious by virtue of Proposition 1.7. This completes the proof.

1.3. Edge of the wedge theorem and injectivity of α

PROPOSITION 1.10 (Edge of the wedge theorem for $\tilde{\mathscr{B}}_{N|M_+}$). *Let $M, N, \tilde{M}$ be Euclidean spaces and let Γ_j $(j=1,\cdots,J)$ be open convex cones of $\boldsymbol{R}^{n-1}$ and V be a bounded open set of N.*

(i) *If f is a section of $\tilde{\mathscr{B}}_{N|M_+}$ on V such that* $\operatorname{supp}(\widetilde{\mathrm{sp}}(f))$ *is contained in the interior of $V\times\sqrt{-1}\left(\bigcup_{j=1}^{J}\Gamma_j^\circ\right)\infty$, then there exist sections F_j $(j=1,\cdots,J)$ of $\tilde{\mathscr{A}}_{N|M_+}$ on $V+\sqrt{-1}\,\Gamma_j 0$ such that*

$$f(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0).$$

(ii) *Let F_j be sections of $\tilde{\mathscr{A}}_{N|M_+}$ on $V+\sqrt{-1}\,\Gamma_j 0$ such that*

$$\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)=0$$

in $\tilde{\mathscr{B}}_{N|M_+}(V)$. *Then for any subcone* $\Gamma'_j \Subset \Gamma_j$ (*i.e. the closure of* $\Gamma'_j \cap S^{n-2}$ *is contained in* Γ_j), *there exist sections* F_{jk} *of* $\tilde{\mathscr{A}}_{N|M_+}$ *on* $V+\sqrt{-1}(\Gamma'_j+\Gamma'_k)0$ *such that*

$$F_j=\sum_{k=1}^{J} F_{jk}, \qquad F_{jk}+F_{kj}=0 \qquad (1 \leqq j, k \leqq J).$$

PROOF. We apply the theory of curvilinear wave expansion of holomorphic functions developed by Kataoka [12] and Kaneko [3] to hyperfunctions with holomorphic parameters. Putting $\xi'=(\xi_2, \cdots, \xi_n) \in S^{n-2}$, $z'=x'+\sqrt{-1}\,y'=(z_2, \cdots, z_n) \in \boldsymbol{C}^{n-1}$, $\langle z', \xi'\rangle = z_2\xi_2+\cdots+z_n\xi_n$, $z'^2=\langle z', z'\rangle$, we set

$$W(z'\,;\,\xi')=\frac{(n-2)!}{(-2\pi\sqrt{-1})^{n-1}} \times \frac{(1-\sqrt{-1}\langle z', \xi'\rangle)^{n-3}\{1-\sqrt{-1}\langle z', \xi'\rangle-(z'^2-\langle z', \xi'\rangle^2)\}}{\{\langle z', \xi'\rangle+\sqrt{-1}(z'^2-\langle z', \xi'\rangle^2)\}^{n-1}}.$$

First let us prove (i). Since $\tilde{\mathscr{B}}_{N|M_+}$ is flabby, there exists a section $\tilde{f}$ of $\tilde{\mathscr{B}}_{N|M_+}$ on N such that $\tilde{f}=f$ on V. Using the Cech cohomology we can take sections $G_\nu(x_1, z')$ of $\mathscr{B}\mathscr{O}_{Y|\tilde{M}_+}$ on $\{(0, z') \in Y\,;\, \operatorname{Im} z' \in \Delta_\nu\}$ with open convex cones Δ_ν of $\boldsymbol{R}^{n-1}$ such that

$$\tilde{f}(x)=\sum_{\nu=1}^{n} G_\nu(x_1, x'+\sqrt{-1}\,\Delta_\nu 0).$$

Let $r>0$ be large enough so that $D=\{x' \in \boldsymbol{R}^{n-1}\,;\, |x'|<r\}$ contains $\bar{V}$ (we sometimes regard V as a subset of $\boldsymbol{R}^{n-1}$). Set

$$\theta(t)=\begin{cases} t^2 & \text{if} \quad 0 \leqq t \leqq \dfrac{1}{2} \\[2ex] (16t^4+4t^2-1)^{1/2}/2(1+4t^2) & \text{if} \quad t>\dfrac{1}{2}. \end{cases}$$

Then $W(z'\,;\,\xi')$ is holomorphic on a neighborhood of

$$\{(z'\,;\,\xi') \in \boldsymbol{C}^{n-1} \times S^{n-2}\,;\, \langle y', \xi'\rangle > y'^2-\langle y', \xi'\rangle^2\}$$
$$\cup\{(z', \xi') \in \boldsymbol{C}^{n-1} \times S^{n-2}\,;\, |y'|<\theta(|x'|)\}$$

in $\boldsymbol{C}^{n-1} \times \tilde{S}$, where $\tilde{S}$ is a complex neighborhood of S^{n-2} (cf. §1 of [12]). Choosing sufficiently small $a_\nu \in \Delta_\nu$ so that $2|a_\nu|<\theta(\operatorname{dis}(V, \partial D))$ (here dis denotes the distance in Euclidean norm), we set

$$G(x_1, z'\,;\,\xi')=\sum_{\nu=1}^{n}\int_{D+\sqrt{-1}a_\nu} G_\nu(x_1, w')W(z'-w'\,;\,\xi')dw'.$$

Then by deforming paths of integration, G becomes a section of $\mathscr{B}\mathcal{O}_{Y\times\tilde{S}|\tilde{M}_+\times\tilde{S}}$ on a neighborhood of

$$\{(0, z'\,;\,\xi')\in Y\times S^{n-2}\,;\; x'\in V, \langle y', \xi'\rangle > y'^2-\langle y', \xi'\rangle^2\}.$$

Let Γ'_j be an open cone containing Γ_j such that

$$\bigcup_{j=1}^{J}\Gamma'^{\circ}_j=\bigcup_{j=1}^{J}\Gamma^{\circ}_j$$

and that the volume of $\Gamma'^{\circ}_j\cap\Gamma'^{\circ}_k$ is zero for $j\neq k$. Set

$$F_j(x_1, z'\,;\,\xi')=\int_{\Gamma'^{\circ}_j\cap S^{n-2}} G(x_1, z'\,;\,\xi')d\sigma(\xi'),$$

where $d\sigma(\xi')$ denotes the volume element of S^{n-2}. Then F_j defines a section of $\tilde{\mathscr{A}}_{N|M_+}$ on $V+\sqrt{-1}\,\Gamma'_j 0$. Put

$$\Delta_0=\boldsymbol{R}^{n-1}\setminus\bigcup_{j=1}^{J}\Gamma^{\circ}_j.$$

We decompose $\bar{\Delta}_0$ into the union of closed cones $\Delta_{01},\cdots,\Delta_{0n}$ so that the convex hull of $\Delta_{0\nu}$ is proper and that the volume of $\Delta_{0\nu}\cap\Delta_{0\mu}$ is zero for $\nu\neq\mu$, and set

$$F_{0\nu}(x_1, z')=\int_{\Delta_{0\nu}\cap S^{n-2}} G(x_1, z'\,;\,\xi')d\sigma(\xi').$$

Then $F_{0\nu}$ defines a section of $\tilde{\mathscr{A}}_{N|M_+}$ on $V+\sqrt{-1}\,\Gamma_{0\nu}$ with $\Gamma_{0\nu}=\mathrm{int}(\Delta_{0\nu})^{\circ}$. We can prove that the inverse formula for curvilinear wave expansion (Theorem 1.1.8 of [12]) also applies to $\mathscr{B}\mathcal{O}_{Y|\tilde{M}_+}$ by expressing sections of $\mathscr{B}\mathcal{O}_{Y|\tilde{M}_+}$ as a sum of boundary values of holomorphic functions. Hence as a section of $\tilde{\mathscr{B}}_{N|M_+}$ on V we have

$$f(x)=\sum_{j=1}^{J}F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)+\sum_{\nu=1}^{n}F_{0\nu}(x_1, x'+\sqrt{-1}\,\Gamma_{0\nu}0).$$

Now let us show that G is extended to a section of $\mathscr{B}\mathcal{O}_{Y\times\tilde{S}|\tilde{M}_+\times\tilde{S}}$ on a neighborhood of

$$\{(0, z', \xi')\in Y\times S^{n-2}\,;\; x'\in V, y'=0, \xi'\in S^{n-2}\cap\bar{\Delta}_0\}.$$

Choose arbitrary $\mathring{x}\in V$ and $\mathring{\xi}'\in\bar{\Delta}_0\cap S^{n-2}$. Then by Lemma 1.5 there exist $\varepsilon>0$ and sections H_j $(j=1,\cdots,j_0)$ of $\mathscr{B}\mathcal{O}_{Y\cdot\tilde{M}_+}$ on $\{(0, z')\in Y\,;\,|z'-\mathring{x}|<\varepsilon,$ $\mathrm{Im}\,z'\in V_j\}$ such that

$$f(x)=\sum_{j=1}^{j_0} H_j(x_1, x'+\sqrt{-1}\,V_j 0)$$

on $\{(0, x')\in N;\ |x'-\mathring{x}'|<\varepsilon\}$, where V_j are open cones with $\mathring{\xi}'\notin V_j^\circ$. Then using Proposition 1.9 and deforming paths of integration, we can verify that $G(x_1, z', \xi')$ is extended to a section of $\mathscr{B}\mathcal{O}_{Y\times \tilde{S}|\tilde{M}_+\times \tilde{S}}$ on a neighborhood of $(\mathring{x}, \mathring{\xi}')$ (cf. [3]). Thus each $F_{0\nu}$ is extended to a section of $\mathscr{B}\mathcal{O}_{Y|\tilde{M}_+}$ on a neighborhood of V, i.e. a section of $\tilde{\mathscr{A}}_{N|M_+}$ on $V+\sqrt{-1}\,S^{n-2}\infty$. This completes the proof of (i).

Now let us prove (ii) by induction on J. The statement for $J=2$ follows from Proposition 1.7. Assume $J\geqq 3$. Since

$$-F_J(x_1, x'+\sqrt{-1}\,\Gamma_J 0)=\sum_{j=1}^{J-1} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0),$$

the support of $\tilde{\mathrm{sp}}(\tilde{b}(F_J))$ is contained in $V\times\sqrt{-1}\bigcup_{j=1}^{J-1}(\Gamma_j^\circ\cap\Gamma_J^\circ)\infty$. Let Γ_j'' be an open convex cone such that $\Gamma_j'\Subset\Gamma_j''\Subset\Gamma_j$. Then by virtue of (i) there exist sections G_j of $\tilde{\mathscr{A}}_{N|M_+}$ on $V+\sqrt{-1}\,(\Gamma_j''+\Gamma_J'')0$ such that

$$F_J=\sum_{j=1}^{J-1} G_j.$$

Then since

$$\sum_{j=1}^{J-1}(F_j+G_j)(x_1, x'+\sqrt{-1}\,\Gamma_j''0)=0,$$

there exist sections F_{jk} of $\tilde{\mathscr{A}}_{N|M_+}$ on $V+\sqrt{-1}\,(\Gamma_j'+\Gamma_k')0$ such that

$$F_j+G_j=\sum_{k=1}^{J-1} F_{jk}, \qquad F_{jk}+F_{kj}=0 \qquad (1\leqq j, k\leqq J-1)$$

by virtue of the induction hypothesis. Set $F_{j,J}=-F_{J,j}=-G_j$ for $j=1,\cdots,J-1$. Then we get

$$F_j=\sum_{k=1}^{J} F_{jk} \qquad (j=1,\cdots,J).$$

This completes the proof.

In the same way we get the following:

PROPOSITION 1.11. *Suppose $M, N, \tilde{M}$ are Euclidean spaces. Let Γ_j be open convex cones of $\boldsymbol{R}^{n-1}$ and I, V be bounded open sets of $\boldsymbol{R}$ and $\boldsymbol{R}^{n-1}$ respectively.*

(i) *If f is a section of $\mathscr{B}_{M_+}$ on $I\times V$ such that* $\mathrm{supp}(\mathrm{sp}_+(f))$ *is contained in the interior of $(I\times V)\times\sqrt{-1}\left(\bigcup_{j=1}^{J}\Gamma_j^\circ\right)\infty$, then there exist sections*

F_j $(j=1,\cdots,J)$ of $\widetilde{\mathcal{A}}_{M_+}$ on $(I\times V)+\sqrt{-1}\,\Gamma_j 0$ such that

$$f(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0).$$

(ii) *Let F_j be sections of $\widetilde{\mathcal{A}}_{M_+}$ on $(I\times V)+\sqrt{-1}\,\Gamma_j 0$ such that*

$$\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)=0$$

in $\mathcal{B}_{M_+}(I\times V)$. Then for any subcone $\Gamma'_j \Subset \Gamma_j$, there exist sections F_{jk} of $\widetilde{\mathcal{A}}_{M_+}$ on $(I\times V)+\sqrt{-1}\,(\Gamma'_j+\Gamma'_k)0$ such that

$$F_j=\sum_{k=1}^{J} F_{jk}, \qquad F_{jk}+F_{kj}=0 \qquad (1\leqq j, k\leqq J).$$

Now we are ready to prove the injectivity of α. First let us micro-localize α. The following proposition follows from the definitions of $\mathcal{C}_{N|M_+}$ and $\mathcal{C}_{N|M_+}$.

PROPOSITION 1.12. *There exists a $\mathcal{D}_X$-linear homomorphism*

$$\alpha: \mathcal{C}_{N|M_+} \longrightarrow \widetilde{\mathcal{C}}_{N|M_+}$$

compatible with $\alpha: \mathcal{B}_{N|M_+}\to\widetilde{\mathcal{B}}_{N|M_+}$.

THEOREM 1.1. *$\alpha: \mathcal{C}_{N|M_+}\to\widetilde{\mathcal{C}}_{N|M_+}$ is injective.*

PROOF. We may assume that $M, N, \tilde{M}$ are Euclidean spaces. We shall prove that α is injective at $x^*=(0, \sqrt{-1}\,\mathring{\xi}'\infty)\in S_N^*Y$ with $\mathring{\xi}'\in S^{n-2}$. Let f be a germ of $\mathcal{C}_{N|M_+}$ at x^* such that $\alpha(f)=0$. Expressing the stalk $\mathcal{C}_{N|M_+,x^*}$ as a Cech cohomology group as in the proof of Lemma 1.5, we can take a section F of $\mathcal{B}\mathcal{O}_{\tilde{M}_+}$ on $\{(x_1, z')\in\tilde{M}\,;\ |x_1|<\varepsilon, |z'|<\varepsilon, \operatorname{Im} z'\in\Gamma\}$ with an open convex cone Γ containing $\mathring{\xi}'$ such that $\mathrm{sp}_+(b_+(F))=f$ on a neighborhood of x^* in $S_{\tilde{M}}^*\tilde{M}$. Since $\widetilde{\mathrm{sp}}(\tilde{b}(F|_Y))=0$ on a neighborhood of x^* in S_N^*Y, by virtue of Proposition 1.10 there exist δ with $0<\delta\leqq\varepsilon$, open convex cones $\Gamma_1,\cdots,\Gamma_J$ containing $\mathring{\xi}'$, and sections F_j of $\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}$ on $\{(0, z')\in Y\,;\ |z'|<\delta,\ \operatorname{Im} z'\in\Gamma_j\}$, such that

$$\Gamma_j\cap\{y'\in\boldsymbol{R}^{n-1}\,;\ \langle y', \mathring{\xi}'\rangle<0\}\neq\emptyset \qquad (j=1,\cdots,J)$$

and that

$$\tilde{b}(F|_Y)=\tilde{b}(F_1)+\cdots+\tilde{b}(F_J)$$

on $\{(0, x')\in N\,;\ |x'|<\delta\}$. Since $\tilde{b}$ is injective and the sections of $\mathcal{B}\mathcal{O}_{\tilde{M}_+}$ have unique continuation property with respect to z',

$$F(x_1, z') = F_1(x_1, z') + \cdots + F_J(x_1, z') \tag{1.6}$$

holds as sections of $\mathscr{B}\mathcal{O}_{\tilde{M}_+}$ on a neighborhood of $\{(0, z') \in Y;\ |z'| < \delta,\ \operatorname{Im} z' \in \Gamma_0\}$ in $\tilde{M}$ with $\Gamma_0 = \Gamma \cap \Gamma_1 \cap \cdots \cap \Gamma_J$. Note that Γ_0 contains $\mathring{\xi}'$. Choosing $a > 0$ so that $2a < \theta(\delta/12)$ and $a < \delta/4$, put $C = \{z' \in \boldsymbol{C}^{n-1};\ |x'| \leqq \delta/2,\ y' = a\mathring{\xi}'\}$. Set

$$G(x_1, z'\ ;\ \xi') = \int_C F(x_1, w') W(z' - w'\ ;\ \xi') dw'.$$

Then G becomes a section of $\mathscr{B}\mathcal{O}_{\tilde{M}_+ \times \tilde{S}}$ on a neighborhood in $\tilde{M} \times \tilde{S}$ of $\{(x_1, z'\ ;\ \xi') \in \tilde{M} \times S^{n-2};\ |x_1| < \varepsilon,\ |x'| < \delta/4,\ \langle y', \xi' \rangle > y'^2 - \langle y', \xi' \rangle^2\}$. By (1.6) we get

$$G(x_1, z'\ ;\ \xi') = \sum_{j=1}^{J} \int_C F_j(x_1, w') W(z' - w'\ ;\ \xi') dw'$$

on a neighborhood in $\tilde{M} \times \tilde{S}$ of $\{(0, z'\ ;\ \xi') \in Y \times S^{n-2};\ |x'| < \delta/4,\ \langle y', \xi' \rangle > y'^2 - \langle y', \xi' \rangle^2\}$.

Choose $\mathring{\xi}'_j \in \Gamma_j \cap S^{n-2}$ with $\langle \mathring{\xi}'_j, \mathring{\xi}' \rangle < 0$. Take c_j such that $0 < 2c_j < -\langle \mathring{\xi}'_j, \mathring{\xi}' \rangle$ and $2c_j < \theta(\delta/12)$. By deforming C for each j so that $\operatorname{Im} C = c_j \mathring{\xi}'_j$ if $|\operatorname{Re} C| \leqq \delta/3$, we see that G is extended to a section of $\mathscr{B}\mathcal{O}_{\tilde{M}_+ \times \tilde{S}}$ on a neighborhood of $\{(x_1, z'\ ;\ \xi') \in \tilde{M} \times S^{n-2};\ |x_1| < \delta',\ |z'| < \delta',\ |\xi' - \mathring{\xi}'| < \delta'\}$ in $\tilde{M} \times \tilde{S}$ with some $\delta' > 0$ since $y' = 0 - c_j \mathring{\xi}'_j$ satisfies $\langle y', \mathring{\xi}' \rangle > y'^2 - \langle y', \mathring{\xi}' \rangle^2$. Set

$$F_0(x_1, z') = \int_\Delta G(x_1, z'\ ;\ \xi') d\sigma(\xi')$$

with $\Delta = \{\xi' \in S^{n-2};\ |\xi' - \mathring{\xi}'| \leqq \delta'/2\}$. Then F_0 is a section of $\mathscr{B}\mathcal{O}_{\tilde{M}_+}$ on $\{(x_1, z') \in \tilde{M}\ ;\ |x_1| < \delta',\ |z'| < \delta'\}$. On the other hand by virtue of the inverse formula for curvilinear wave expansion, $\mathrm{sp}_+(b_+(F_0)) = \mathrm{sp}_+(b_+(F))$ holds on a neighborhood of x^* in $S^*_M\tilde{M}$. Thus we get $f = \mathrm{sp}_+(b_+(F_0)) = 0$ at x^*. This completes the proof.

THEOREM 1.2. *$\alpha : \mathscr{B}_{N|M_+} \to \tilde{\mathscr{B}}_{N|M_+}$ is injective.*

PROOF. Let f be a germ of $\mathscr{B}_{N|M_+}$ at $\mathring{x} \in N$ such that $\alpha(f) = 0$ at $\mathring{x}$. Then since $\alpha(\mathrm{sp}_+(f)) = 0$ we get $\mathrm{sp}_+(f) = 0$ on $(\pi_{M/\tilde{M}})^{-1}(\mathring{x})$ in view of Theorem 1.1, which implies that f is a germ of $\mathscr{B}\mathcal{O}_{\tilde{M}_+}$ at $\mathring{x}$ in view of Proposition 1.8. The restriction of α to $\mathscr{B}\mathcal{O}_{\tilde{M}_+}|_N$ is obviously injective. Hence we get $f = 0$. This completes the proof.

§ 2. Formulation of boundary value problems

2.1. Non-characteristic boundary value problem for systems

It is well-known (cf. Kashiwara [5]) that systems of linear partial differential equations with holomorphic coefficients on a complex manifold X is nothing but sheaves of coherent modules over $\mathcal{D}_X$. (Hereafter we simply call them systems.) For a system $\mathcal{M}$, we denote its characteristic variety (in T^*X) by $\mathrm{SS}(\mathcal{M})$. A complex submanifold Y of X is called non-characteristic for $\mathcal{M}$ if $\mathring{T}^*_Y X \cap \mathrm{SS}(\mathcal{M}) = \emptyset$, where $\mathring{T}^*_Y X$ denotes the conormal bundle of Y with its zero-section removed. When Y is non-characteristic for $\mathcal{M}$, the tangential system $\mathcal{M}_Y$ of $\mathcal{M}$ to Y is defined by

$$\mathcal{M}_Y = \mathcal{D}_{Y \to X} \otimes_{\mathcal{D}_X} \mathcal{M} = \mathcal{M}/(f_1\mathcal{M} + \cdots + f_d\mathcal{M}),$$

where $\mathcal{D}_{Y\to X} = \mathcal{D}_X/(f_1\mathcal{D}_X + \cdots + f_d\mathcal{D}_X)$ as a $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodule if $Y = \{z \in X;\ f_1(z) = \cdots = f_d(z) = 0\}$ with holomorphic functions $f_1, \cdots, f_d$. Then $\mathcal{M}_Y$ becomes a coherent $\mathcal{D}_Y$-module. In general, for a sheaf $\mathcal{F}$ of $\mathcal{D}_X$-modules, $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$ is a complex of sheaves whose j-th cohomology group is $\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$ ($\boldsymbol{R}\mathcal{H}om$ denotes the right derived functor of $\mathcal{H}om$). In particular $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$ is the sheaf of $\mathcal{F}$-solutions of $\mathcal{M}$.

Now let us return to the original situation of § 1.

PROPOSITION 2.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined on a neighborhood Ω in X of an open subset U of N. Assume that $Y \cap \Omega$ is non-characteristic for $\mathcal{M}$. Then for any $j \in \boldsymbol{Z}$ there exists on U an isomorphism*

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \xrightarrow{\sim} \mathcal{E}xt^j_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).$$

PROOF. Put $\tilde{M}_- = \tilde{M} \setminus \tilde{M}_+$. Then from the exact sequence

$$0 \longrightarrow \Gamma_{\tilde{M}_-}(\mathcal{B}\mathcal{O})|_Y \longrightarrow \mathcal{B}\mathcal{O}|_Y \longrightarrow \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+} \longrightarrow 0$$

we get a triangle (cf. [1])

$$\begin{tikzcd}[column sep=small] & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\tilde{M}_-}(\mathcal{B}\mathcal{O})|_Y) \arrow[dl] & \\ \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}|_Y) \arrow[rr] & & \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) . \arrow[ul, "+1"'] \end{tikzcd} \tag{2.1}$$

First let us show

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\tilde{M}_-}(\mathcal{B}\mathcal{O})|_Y) = 0. \tag{2.2}$$

For this purpose let us take $X \times X$ as a complexification of X. If $z = x + \sqrt{-1}\,y$ is a local coordinate system of X, then (z, w) is the corresponding one of $X\times X$, where w is the copy of z, and X is embedded to $X\times X$ by the map $z\to(z, \bar{z})$. Let π be the projection of $X\times X$ to X defined by $\pi(z, w)=z$. Let $\tilde{M}^C$ be a complexification of $\tilde{M}$ in $X\times X$ and f be the restriction of π to $\tilde{M}^C$. Let $\tilde{x}$, $\tilde{y}$ be complexifications of x, y. Then $(\tilde{x}, \tilde{y})$ is also a local coordinate system of $X\times X$ with $z=\tilde{x}+\sqrt{-1}\,\tilde{y}$, $w=\tilde{x}-\sqrt{-1}\,\tilde{y}$, and (x_1, z', w') is a local coordinate system of $\tilde{M}^C$. Set

$$f^*\mathcal{M}=\mathcal{D}_{\tilde{M}^C\to X}\otimes_{f^{-1}\mathcal{D}_X}f^{-1}\mathcal{M},$$

$$\mathcal{D}_{\tilde{M}^C\to X}=\mathcal{D}_{\tilde{M}^C}\Big/\Big(\mathcal{D}_{\tilde{M}^C}\Big(\frac{\partial}{\partial w_2}\Big)+\cdots+\mathcal{D}_{\tilde{M}^C}\Big(\frac{\partial}{\partial w_n}\Big)\Big).$$

Then Théorème 3.1.4 of Tajima [30] implies

$$\begin{aligned}(2.3)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\tilde{M}^C}}(f^*\mathcal{M}, \mathcal{B}_{\tilde{M}}) &\cong \boldsymbol{R}\Gamma_{\tilde{M}}(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))\otimes\omega_{\tilde{M}}[1]\\ &=\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}).\end{aligned}$$

Let us show that $\pm dx_1$ is hyperbolic for $f^*\mathcal{M}$ in the sense of Kashiwara-Schapira [11]. A point of $T^*_{\tilde{M}}\tilde{M}^C$ is written as

$$\begin{aligned}x^*&=(x_1, z'\,;\ \sqrt{-1}\,\eta_1 d\tilde{x}_1+\sqrt{-1}\,\langle\eta', d\tilde{x}'\rangle+\sqrt{-1}\,\langle\xi', d\tilde{y}'\rangle)\\ &=(x_1, z'\,;\ \sqrt{-1}\,\eta_1 d\tilde{x}_1+\frac{1}{2}\langle\xi'+\sqrt{-1}\,\eta', dz'\rangle+\frac{1}{2}\langle-\xi'+\sqrt{-1}\,\eta', dw'\rangle)\end{aligned}$$

with $(x_1, z')\in\boldsymbol{R}\times\boldsymbol{C}^{n-1}$, $\eta=(\eta_1, \eta')\in\boldsymbol{R}^n$, $\xi'\in\boldsymbol{R}^{n-1}$. On the other hand we have

$$\mathrm{SS}(f^*\mathcal{M})\subset\{(\tilde{x}_1, z', w'\,;\ \zeta_1 d\tilde{x}_1+\langle\zeta', dz'\rangle)\in T^*\tilde{M}^C\,;\ (\tilde{x}_1, z', \langle\zeta, dz\rangle)\in\mathrm{SS}(\mathcal{M})\}.$$

Take a vector $\theta=(\delta z, \delta\zeta, \delta\zeta')\in\boldsymbol{C}^n\times\boldsymbol{C}^n\times\boldsymbol{C}^{n-1}$ such that $|\delta z|<c$, $|\delta\zeta'|<c$, $|\delta\zeta'|<c$, $\mathrm{Re}\,(\delta\zeta_1)=\pm1$ with $c>0$. If c is small enough, there exists $\varepsilon>0$ (independent of x^* when x^* moves over a compact subset of $T^*_{\tilde{M}}\tilde{M}^C$) such that $x^*+t\theta\notin\mathrm{SS}(f^*\mathcal{M})$ if $0<t<\varepsilon$. In fact if $x^*+t\theta\in\mathrm{SS}(f^*\mathcal{M})$, then $\xi'-\sqrt{-1}\,\eta'=2t\delta\zeta'$ holds and

$$z^*=\Big((x_1, z')+t\delta z, (\sqrt{-1}\,\eta_1+\delta\zeta_1)d\tilde{x}_1+\Big\langle\frac{1}{2}(\xi'+\sqrt{-1}\,\eta')+t\delta\zeta', dz'\Big\rangle\Big)$$

belongs to $\mathrm{SS}(\mathcal{M})$. Since

$$\Big|\frac{1}{2}(\xi'+\sqrt{-1}\,\eta')+t\delta\zeta'\Big|<2ct\leq 2c|\sqrt{-1}\,\eta_1+t\delta\zeta_1|$$

and Y is non-characteristic for $\mathcal{M}$, z^* does not belong to $\mathrm{SS}(\mathcal{M})$ if c and ε are small enough. Thus $\pm dx_1$ is hyperbolic for $f^*\mathcal{M}$. We get (2.2) from (2.3) and Corollary 2.2.2 of [11]. From (2.1) and (2.2) we get an isomorphism

$$(2.4)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}|_Y) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}).$$

Next let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X|_{\tilde{M}} \longrightarrow \mathcal{B}\mathcal{O} \longrightarrow (\pi_{\tilde{M}/X})_*\mathcal{C}\mathcal{O} \longrightarrow 0\,;$$

here $\mathcal{C}\mathcal{O}$ is the sheaf on $S^*_{\tilde{M}}X$ of microfunctions with holomorphic parameters z', and $\pi_{\tilde{M}/X}$ is the projection of $S^*_{\tilde{M}}X$ to $\tilde{M}$. From this exact sequence we get

$$(2.5)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O})|_Y$$

since $S^*_{\tilde{M}}X \cap \mathrm{SS}(\mathcal{M}) = \emptyset$ on a neighborhood of $\Omega \cap Y$. By the Cauchy-Kovalevskaja theorem due to Kashiwara (Theorem 2.5.16 of [5]), we have

$$(2.6)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

Combining (2.4)–(2.6) we get an isomorphism

$$(2.7)\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

Applying to this the functor $\boldsymbol{R}\Gamma_N \otimes \omega_N[n-1]$ we get finally

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).$$

This completes the proof.

REMARK. (2.2) can also be proved by using the sheaf $C_{M_+|X}$ of Kataoka [12].

THEOREM 2.1. *Under the same assumption as in Proposition* 2.1, *for any* $j \in \boldsymbol{Z}$ *there exists on* U *a homomorphism*

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow \mathcal{E}xt^j_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).$$

In particular there exists on U *an injective homomorphism*

$$\gamma : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).$$

PROOF. This is an immediate consequence of Theorem 1.2 and Proposition 2.1.

Let us microlocalize this formulation.

PROPOSITION 2.2. *Under the same assumption as in Proposition* 2.1 *there exists on* $(\pi_{N/Y})^{-1}(U)$ *an isomorphism*

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{C}_{N|M_+}) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, C_N),$$

where C_N denotes the sheaf on S_N^*Y of microfunctions.

PROOF. From (2.7) we get

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{C}_{N|M_+}) &\cong \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \boldsymbol{R}\Gamma_{S_N^*Y}((\pi_{N/Y})^{-1}\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+})^a)\otimes\omega_N[n-1] \\ &\cong \boldsymbol{R}\Gamma_{S_N^*Y}((\pi_{N/Y})^{-1}(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+})))^a\otimes\omega_N[n-1] \\ &\cong \boldsymbol{R}\Gamma_{S_N^*Y}((\pi_{N/Y})^{-1}(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)))^a\otimes\omega_N[n-1] \\ &\cong \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, C_N).\end{aligned}$$

This completes the proof.

From this proposition and Theorem 1.1 we get the following:

THEOREM 2.2. *Under the same assumption as in Proposition* 2.1, *for any $j\in\boldsymbol{Z}$ there exists on $(\pi_{N/Y})^{-1}(U)$ a homomorphism*

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, C_{N|M_+}) \longrightarrow \mathcal{E}xt^j_{\mathcal{D}_Y}(\mathcal{M}_Y, C_N)$$

compatible with the one in Theorem 2.1. *In particular there exists on $(\pi_{N/Y})^{-1}(U)$ an injective homomorphism*

$$\gamma:\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_{N|M_+}) \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, C_N)$$

compatible with the γ in Theorem 2.1.

A natural map $T^*X\underset{X}{\times}\tilde{M}\to T^*\tilde{M}$ associated with the embedding $\tilde{M}\to X$ induces a map

$$p:\ S_M^*X\backslash S_{\tilde{M}}^*X \longrightarrow S_M^*\tilde{M}.$$

Put $L_+=(\pi_{M/\tilde{M}})^{-1}(M_+)\subset S_M^*\tilde{M}$. Then there exists a natural homomorphism

$$\phi:\ p^{-1}(C_{M_+}|_{L_+}) \longrightarrow C_M|_{p^{-1}(L_+)}$$

such that $\phi(\mathrm{sp}_+(f))=\mathrm{sp}(f)$ for a section f of $\mathcal{B}_{M_+}|_{M_+}=\mathcal{B}_M|_{M_+}$, where C_M denotes the sheaf on S_M^*X of microfunctions and $\mathrm{sp}:\pi^{-1}\mathcal{B}_M\to C_M$ is the spectral map. More concretely ϕ is defined as follows (cf. Komatsu [16]): Suppose a germ $u(x)$ of C_{M_+} at $x^*=(\mathring{x}, \sqrt{-1}\,\mathring{\xi}'\infty)\in L_+$ is defined by $u=\mathrm{sp}_+(b_+(F))$, where F is a section of $\mathcal{B}\mathcal{O}$ on $\{(x_1, z')\in\tilde{M}\,;\ \mathrm{Im}\,z'\in\Gamma, |x-\mathring{x}|<\varepsilon\}$ with an open convex cone Γ of $\boldsymbol{R}^{n-1}$ and $\varepsilon>0$. Then there exist holomorphic functions $F_\pm(z)$ on $\{z\in X;\ \mathrm{Im}\,z'\in\Gamma, |x-\mathring{x}|<\varepsilon, \pm\mathrm{Im}\,z_1>0\}$ such that

$$F(x_1, z')=F_+(x_1+\sqrt{-1}0, z')-F_-(x_1-\sqrt{-1}0, z').$$

Set $\Gamma_\pm=\{y\in \boldsymbol{R}^n;\ \pm y_1>0, y'\in\Gamma\}$. Then a germ $\psi(u)$ of $p_*(C_M|_{p^{-1}(L_+)})$ at x^* is defined by

$$\psi(u)=\mathrm{sp}(F_+(x+\sqrt{-1}\,\Gamma_+0)-F_-(x+\sqrt{-1}\,\Gamma_-0)).$$

COROLLARY 2.1. *Let f be a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_{N|M_+})$ at $x^*=(\mathring{x}, \sqrt{-1}\,\mathring{\xi}'\infty)$ with $\mathring{x}=(0,\mathring{x}')\in N$ and $\mathring{\xi}'\in S^{n-2}$. Assume that $\gamma(f)$ vanishes on a neighborhood of x^*. Then there exists $\varepsilon>0$ such that $\psi(f)$ vanishes on $\{(x,\sqrt{-1}\,\xi\infty)\in S_M^*X;\ 0<x_1<\varepsilon, |x'-\mathring{x}'|<\varepsilon, \xi_1\in \boldsymbol{R}, |\xi'-\mathring{\xi}'|<\varepsilon\}$ as a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_M)$.*

COROLLARY 2.2. *If Y is non-characteristic for $\mathcal{M}$ and if $\mathcal{M}_Y$ is microlocally hypoelliptic at $x^*\in S_N^*Y$, i.e. $\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, C_N)_{x^*}=0$, then the conclusion of Corollary 2.1 holds. Moreover if $\mathcal{M}_Y$ is hypoelliptic, i.e. $\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N/\mathcal{A}_N)=0$, then there is an isomorphism*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_N \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}).$$

2.2. F-mild hyperfunctions

We clarify the meaning of the boundary value homomorphism γ of Theorem 2.1 using F-mild hyperfunctions introduced by Ôaku [21, 22].

DEFINITION 2.1 ([21, 22]). Let z be an admissible local coordinate system of X. Then a germ f of $\mathcal{B}_{N|M_+}$ at $\mathring{x}\in N$ is called F-*mild* at $\mathring{x}$ if and only if f is written in the form

$$f=b_+(F_1)+\cdots+b_+(F_J)$$

on $\{x\in M_+;\ |x-\mathring{x}|<\varepsilon\}$, where F_j is holomorphic on a neighborhood in X of

$$D_+(\mathring{x}, \Gamma_j, \varepsilon)=\{z\in X;\ |z-\mathring{x}|<\varepsilon,\ \mathrm{Re}\,z_1\geqq 0,\ \mathrm{Im}\,z_1=0,\ \mathrm{Im}\,z'\in\Gamma_j\}$$

with $\varepsilon>0$ and an open cone Γ_j of $\boldsymbol{R}^{n-1}$ for $j=1,\cdots,J$. Here F_j is regarded as a section of $\mathcal{BO}_{\tilde{M}_+}$ by the composition of natural homomorphisms $\mathcal{O}_X|_{\tilde{M}}\to\mathcal{BO}_{\tilde{M}}$ and $\mathcal{BO}_{\tilde{M}}\to\mathcal{BO}_{\tilde{M}_+}$. F-mildness does not depend neither on local coordinate system nor on the choice of $\tilde{M}$. The subsheaf of $\mathcal{B}_{N|M_+}$ consisting of its sections which are F-mild at each point of N is denoted by $\mathcal{B}^F_{N|M_+}$.

DEFINITION 2.2. $\tilde{\mathcal{B}}^A=\mathcal{H}^{n-1}_N(\mathcal{O}_X|_Y)\otimes\omega_N$,

$$\tilde{\mathcal{A}}^A=((\varepsilon_{N/Y})_*(\mathcal{O}_X|_{Y\setminus N}))|_{S_NY}.$$

There are homomorphisms $\beta:\tilde{\mathcal{B}}^A\to\tilde{\mathcal{B}}_{N|M_+}$ and $\beta:\tilde{\mathcal{A}}^A\to\tilde{\mathcal{A}}_{N|M_+}$ induced by $\mathcal{O}_X|_Y \to \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}$. By virtue of Lemmas 1.3 and 1.4, there exists an injective homomorphism $b_A : \tilde{\mathcal{A}}^A \to \tilde{\mathcal{B}}^A$ such that $\beta \circ b_A = \tilde{b} \circ \beta$ by the same argument as in [19, 20].

LEMMA 2.1. *$\beta : \tilde{\mathcal{A}}^A \to \tilde{\mathcal{A}}_{N|M_+}$ and $\beta : \tilde{\mathcal{B}}^A \to \tilde{\mathcal{B}}_{N|M_+}$ are injective.*

PROOF. The injectivity of the first homomorphism is obvious. Hence let us prove the injectivity of the second one. Let f be a germ of $\tilde{\mathcal{B}}^A$ at $0 \in N$. Then by expressing $\tilde{\mathcal{B}}^A$ by a Cech cohomology group we can take holomorphic functions F_j defined on a neighborhood in X of $\{(0, z') \in Y;\ |z'| < \varepsilon, \operatorname{Im} z' \in \Gamma_j\}$ with open convex cones Γ_j of $\boldsymbol{R}^{n-1}$ such that $f = b_A(F_1) + \cdots + b_A(F_J)$. Suppose $\beta(f) = 0$ and let Γ'_j be an open subcone of Γ_j. Then by virtue of Proposition 1.10 there exist sections F_{jk} of $\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}$ on $\{(0, z') \in Y;\ |z| < \varepsilon, \operatorname{Im} z' \in \Gamma'_j + \Gamma'_k\}$ such that

$$F_j = \sum_{k=1}^{J} F_{jk}, \qquad F_{jk} + F_{kj} = 0 \qquad (1 \leqq j, k \leqq J). \tag{2.8}$$

Take $a_j \in \Gamma'_j$ so that $2|a_j| < \theta(\varepsilon/4)$ and set

$$F(x_1, z'\,;\ \xi') = \sum_{j=1}^{J} \int_{C_j} F_j(x_1, w') W(z' - w'\,;\ \xi') dw'$$

with chains $C_j = \{z' \in \boldsymbol{C}^{n-1};\ |x'| \leqq \varepsilon/2, y' = a_j\}$. Then F defines a holomorphic function on a neighborhood in $X \times \tilde{S}$ of

$$\{(0, z', \xi') \in Y \times S^{n-2};\ |x'| < \varepsilon/4, \langle y', \xi' \rangle > y'^2 - \langle y', \xi' \rangle^2\}.$$

On the other hand it follows from (2.8) that

$$F(x_1, z'\,;\ \xi') = \sum_{j<k} \int_{C_j - C_k} F_{jk}(x_1, w') W(z' - w'\,;\ \xi') dw'$$

defines a section of $\mathcal{B}\mathcal{O}_{Y \times \tilde{S} | \tilde{M}_+ \times \tilde{S}}$ on a neighborhood of $\{(0, z', \xi') \in Y \times S^{n-2};\ |z'| < \varepsilon/4\}$. By the unique continuation property of $\mathcal{C}\mathcal{O}$, F becomes holomorphic on a neighborhood in $X \times \tilde{S}$ of $\{(x_1, z', \xi') \in \tilde{M} \times S^{n-1};\ 0 < x_1 < \delta, |z'| < \delta\}$ with some $\delta > 0$.

Let $\Delta_1, \cdots, \Delta_n$ be open convex cones of $\boldsymbol{R}^{n-1}$ such that $\bigcup_{\nu=1}^{n} \bar{\Delta}_\nu = \boldsymbol{R}^{n-1}$ and $\Delta_\nu \cap \Delta_\mu = \emptyset$ for $\nu \neq \mu$ and set

$$G_\nu(x_1, z') = \int_{\Delta_\nu \cap S^{n-2}} F(x_1, z'\,;\ \xi') d\sigma(\xi').$$

Then G_ν defines a section of $\tilde{\mathcal{A}}^A$ on $\{(0, x') \in N;\ |x'| < \varepsilon/4\} + \sqrt{-1} \operatorname{int} \Delta_\nu^\circ$ and $f = \sum_{\nu=1}^{n} b_A(G_\nu)$ holds. Moreover, each G_ν is also holomorphic on a neighborhood of $\{(x_1, z')\,;\, 0<x_1<\delta, |z'|<\delta\}$, and we have

$$\sum_{\nu=1}^{n} \tilde{b}(G_\nu)=\beta(f)=0.$$

This implies $G_1+\cdots+G_n=0$ on a neighborhood of $\{(x_1, z')\,;\, 0<x_1<\delta, |z'|<\delta\}$ since $\tilde{b}$ is injective. Let V_ν be an open cone such that $V_\nu \Subset \Delta_\nu^\circ$. Then by virtue of the edge of the wedge theorem for F-mild hyperfunctions (Theorem 1 of [21]), there exist holomorphic functions $G_{\nu\mu}$ on a neighborhood of $D_+(0, V_\nu+V_\mu, \varepsilon')$ with $\varepsilon'>0$ such that

$$G_\nu=\sum_{\mu=1}^{n} G_{\nu\mu}, \qquad G_{\nu\mu}+G_{\mu\nu}=0 \qquad (1\leqq\nu, \mu\leqq n).$$

This implies

$$f=\sum_{\nu=1}^{n} b_A(G_\nu)=\sum_{\nu,\mu} b_A(G_{\nu\mu})=0.$$

This completes the proof.

In view of this lemma we can regard $\tilde{\mathscr{B}}^A$ as a subsheaf of $\tilde{\mathscr{B}}_{N|M_+}$. Then it is easy to see that $\alpha(\mathscr{B}^F_{N|M_+})$ is contained in $\tilde{\mathscr{B}}^A$.

PROPOSITION 2.3. *α induces an injective homomorphism*

$$\mathscr{B}_{N|M_+}/\mathscr{B}^F_{N|M_+} \longrightarrow \tilde{\mathscr{B}}_{N|M_+}/\tilde{\mathscr{B}}^A.$$

PROOF. Let f be a germ of $\mathscr{B}_{N|M_+}$ at $\mathring{x}\in N$ such that $\alpha(f)\in(\tilde{\mathscr{B}}^A)_{\mathring{x}}$. Using an admissible local coordinate system z we may assume $\mathring{x}=0$. There exist sections F_j of $\mathscr{B}\mathscr{O}_{\tilde{M}_+}$ on $\{(x_1, z')\,;\, |x_1|<\varepsilon, |z'|<\varepsilon, \mathrm{Im}\, z'\in\Gamma_j\}$ with $\varepsilon>0$ and open cones Γ_j such that $f=\sum_{j=1}^{J} b_+(F_j)$. Since $\alpha(f)\in\tilde{\mathscr{B}}^A$, we can assume that there exist holomorphic functions G_k on a neighborhood in X of $\{(0, z')\in Y\,;\, |z'|<\varepsilon, \mathrm{Im}\, z'\in V_k\}$ with open cones V_k such that

$$\alpha(f)=\sum_{k=1}^{k_0} \tilde{b}(G_k).$$

Put

$$F(x_1, z'\,;\, \xi')=\sum_{j=1}^{J}\int_{C_j} F_j(x_1, w')W(z'-w'\,;\, \xi')dw',$$

$$G(x_1, z'\,;\, \xi')=\sum_{k=1}^{k_0}\int_{C'_k} G_k(x_1, w')W(z'-w'\,;\, \xi')dw',$$

where C_j and C'_k are chains as in the proof of Lemma 2.1. Since

$$\sum_{j=1}^{J} \tilde{b}(F_j)=\sum_{k=1}^{k_0} \tilde{b}(G_k),$$

Proposition 1.10 implies that $H=F-G$ is a section of $\mathcal{BO}_{Y\times\tilde{S}|\tilde{M}_+\times\tilde{S}}$ on a neighborhood of $\{(0,z',\xi')\in Y\times S^{n-2}\,;\ |z'|<\varepsilon/4\}$. Let $\Delta_1,\cdots,\Delta_n$ be as in the proof of Lemma 2.1 and set

$$g_\nu(x_1,z')=\int_{\Delta_\nu\cap S^{n-2}}G(x_1,z'\,;\ \xi')d\sigma(\xi')\,,$$

$$h_\nu(x_1,z')=\int_{\Delta_\nu\cap S^{n-2}}H(x_1,z'\,;\ \xi')d\sigma(\xi')\,,$$

and $f_\nu=g_\nu+h_\nu$. Then f_ν and g_ν are sections of $\tilde{\mathcal{A}}_{M_+}$ and of $\tilde{\mathcal{A}}^A$ respectively on $\{(0,x')\in N\,;\ |x'|<\varepsilon/4\}+\sqrt{-1}\operatorname{int}\Delta_\nu^\circ 0$ and h_ν is a section of $\mathcal{BO}_{\tilde{M}_+}$ on $\{(0,x')\in N\,;\ |x'|<\varepsilon/4\}$. Hence g_ν becomes holomorphic on a neighborhood of $D_+(0,\Delta'_\nu,\delta)$ with some $\delta>0$ for any open cone $\Delta'_\nu\Subset\Delta_\nu^\circ$ by virtue of the unique continuation property of $\mathcal{CO}$. In view of the inverse formula of curvilinear wave expansion we get

$$f=\sum_{\nu=1}^n b_+(f_\nu)=\sum_{\nu=1}^n b_+(g_\nu)+\sum_{\nu=1}^n b_+(h_\nu)\,.$$

Since

$$\alpha\Big(\sum_{\nu=1}^n b_+(h_\nu)\Big)=\sum_{\nu=1}^n \tilde{b}(h_\nu)=\sum_{\nu=1}^n(\tilde{b}(f_\nu)-\tilde{b}(g_\nu))=\alpha(f)-\alpha(f)=0\,,$$

and α is injective, we get

$$f=\sum_{\nu=1}^n b_+(g_\nu)\,.$$

Hence f is F-mild at $\mathring{x}=0$. This completes the proof.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined on a complex neighborhood $\Omega\subset X$ of $U\subset N$ such that $Y\cap\Omega$ is non-characteristic for $\mathcal{M}$. Since $\mathcal{B}^F_{N|M_+}/x_1\mathcal{B}^F_{N|M_+}\cong\mathcal{B}_N$ (this isomorphism is induced by the boundary value map for F-mild hyperfunctions (cf. [21])), there is a natural homomorphism

$$\gamma_0:\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}^F_{N|M_+})\longrightarrow\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{B}_N)\,.$$

THEOREM 2.3. *Let* $\mathcal{M}$ *be as above. Then we have*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_{N|M_+}/\mathcal{B}^F_{N|M_+})=0\,,$$

and hence

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}^F_{N|M_+})=\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_{N|M_+})\,.$$

Moreover the homomorphism γ *of Theorem* 2.1 *coincides with* γ_0.

PROOF. By (2.4) and (2.5) we get

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}^A) \cong \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}),$$

and hence

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}/\tilde{\mathcal{B}}^A)=0.$$

Combined with Proposition 2.3 this implies

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}/\mathcal{B}^F_{N|M_+})=0.$$

Next let us show that γ coincides with γ_0. Note that $\tilde{\mathcal{B}}^A/x_1\tilde{\mathcal{B}}^A \cong \mathcal{B}_N$. Hence there is a homomorphism

$$\gamma' : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}^A) \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).$$

This γ' coincides with the one induced by the isomorphism

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X|_Y) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^F_{N|M_+}) \arrow[r, "\alpha_0"'] \arrow[d, "\beta_0"', "\wr"] & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}^A) \arrow[r, "\gamma'"'] \arrow[d, "\beta"', "\wr"] & \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N) \\
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \arrow[r, "\alpha"] & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}). &
\end{tikzcd}$$

Since $\gamma_0=\gamma'\circ\alpha_0$ and $\gamma=\gamma'\circ\beta^{-1}\circ\alpha$, we get $\gamma_0=\gamma\circ\beta_0$. This completes the proof.

REMARK. Kataoka has proved that $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathring{\mathcal{B}}_{N|M_+})=\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+})$ holds under the above assumption, where $\mathring{\mathcal{B}}_{N|M_+}$ is the sheaf of mild hyperfunctions (cf. [13]). Since $\mathring{\mathcal{B}}_{N|M_+}$ is a proper subsheaf of $\mathcal{B}^F_{N|M_+}$ (cf. [21]), the second equality of Theorem 2.3 is contained in his result.

COROLLARY 2.3. *The boundary value homomorphism γ of Theorem 2.1 is independent of the choice of $\tilde{M}$, and hence γ is well-defined for any real analytic manifold $\bar{M}_+$ with 1-codimensional real analytic boundary N.*

2.3. Fuchsian systems of partial differential equations

Let $z=(z_1, z')$ be an admissible local coordinate system of X. We use the notation $D_z=(D_{z_1}, D_{z'}), D_{z'}=(D_{z_2}, \cdots, D_{z_n})$ with $D_{z_j}=\partial/\partial z_j$. Let $\mathcal{O}\mathcal{D}_{\tilde{M}}$ be the sheaf of rings consisting of the sections of $\mathcal{D}_X|_{\tilde{M}}$ commuting with z_1 (i.e. not containing D_{z_1}). We denote by $\mathcal{O}\mathcal{D}_{\tilde{M}}[D_1]$ the sheaf of sections of

$\mathcal{D}_X|_{\tilde{M}}$ which are polynomials of D_{z_1} with coefficients in $\mathcal{O}\mathcal{D}_{\tilde{M}}$. It is easy to see that $\mathcal{O}\mathcal{D}_{\tilde{M}}$ and $\mathcal{O}\mathcal{D}_{\tilde{M}}[D_1]$ are well-defined as sheaves on $\tilde{M}$ not depending on z. In fact, if $\tilde{z}$ is another admissible local coordinate system, we have relations of the form

$$\tilde{z}_1=\varphi_1(z_1), \qquad \tilde{z}_j=\varphi_j(z) \qquad (j=2, \cdots, n), \tag{2.9}$$

and hence

$$D_{z_1}=\sum_{k=1}^{n}\frac{\partial\varphi_k}{\partial z_1}D_{\tilde{z}_k}, \qquad D_{z_j}=\sum_{k=2}^{n}\frac{\partial\varphi_k}{\partial z_j}D_{\tilde{z}_k} \qquad (j=2, \cdots, n).$$

Let P be an $m\times m$ matrix whose components are sections of $\mathcal{O}\mathcal{D}_{\tilde{M}}[D_1]$ on a neighborhood $\Omega\subset\tilde{M}$ of $U\subset N$. After Tahara [29] P is called a *Fuchsian system* (with respect to $\tilde{M}_+$) if P is written in the form

$$P=a(z)(z_1D_{z_1}I_m-A(z, D_{z'}))$$

by an admissible local coordinate system z, where $a(z)$ is a non-vanishing holomorphic function, I_m is the $m\times m$ unit matrix, $A=(A_{ij})$ is a matrix of sections of $\mathcal{O}\mathcal{D}_{\tilde{M}}$ such that

(A.1) $(A_{ij}(0, z', D_{z'}))$ is of order $\leqq 0$, i.e. equals a matrix $A_0(z')$ of holomorphic functions.

(A.2) There exist integers $n_1, \cdots, n_m$ such that $A_{ij}(z, D_{z'})$ is of order $\leqq n_i-n_j+1$.

The eigenvalues of $A_0(z')$, which we denote by $\lambda_1(z'), \cdots, \lambda_m(z')$ counting their multiplicities, are called the *characteristic eigenvalues* of P. The notion of Fuchsian systems and the matrix A_0 are independent of admissible local coordinate systems. In fact, if $\tilde{z}$ is another admissible local coordinate system, then we get

$$z_1D_{z_1}=c_1(\tilde{z}_1)\tilde{z}_1D_{\tilde{z}_1}+\tilde{z}_1\sum_{j=2}^{n}c_j(\tilde{z})D_{\tilde{z}_j}$$

with $c_1(0)=1$, and hence

$$z_1D_{z_1}I_m-A=c_1(\tilde{z}_1)(\tilde{z}_1D_{\tilde{z}_1}I_m-c_1(\tilde{z}_1)^{-1}(A-\tilde{z}_1\sum_{j=2}^{n}c_j(\tilde{z})D_{\tilde{z}_j}I_m)).$$

In the sequel we also assume

$$\lambda_i(x')-\lambda_j(x')\notin \boldsymbol{Z} \quad \text{if} \quad i\neq j \quad \text{and} \quad (0, x')\in U. \tag{A.3}$$

We define an m-dimensional complex vector bundle $L(A_0)$ on $U\subset N$ as follows: Let $z=x+\sqrt{-1}\,y$ and $\tilde{z}=\tilde{x}+\sqrt{-1}\,\tilde{y}$ be admissible local coordinate systems. Then each point of $L(A_0)$ is written as $(dx_1)^{A_0}c$ with a column vector $c\in \boldsymbol{C}^m$; two points $(dx_1)^{A_0}c$ and $(d\tilde{x}_1)^{A_0}\tilde{c}$ in the same fiber define a same point if and only if

$$c=\left(\frac{d\tilde{x}_1}{dx_1}(0)\right)^{A_0(x')}\tilde{c}\,.$$

(Note that $\tilde{x}_1$ depends only on x_1.) Set $\mathcal{M}=\mathcal{D}_X^m/\mathcal{D}_X^m P$, where $\mathcal{D}_X^m$ is the sheaf consisting of the row vectors with components in $\mathcal{D}_X$.

THEOREM 2.4. *Under the above assumptions, there exist injective sheaf homomorphisms (boundary value maps)*

$$\gamma:\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_{N|M_+})\longrightarrow \mathcal{B}_N\otimes_C L(A_0)$$

$$\gamma:\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},C_{N|M_+})\longrightarrow C_N\otimes_C(\pi_{N/Y})^{-1}L(A_0)$$

on U and on $(\pi_{N/Y})^{-1}(U)$ respectively, and these γ's commute with spectral maps.

In order to prove this theorem we introduce an extension ring of $\mathcal{O}\mathcal{D}_{\tilde{M}}|_Y$. For a moment let us assume that there exists a holomorphic function f on X such that $Y=\{f=0\}$, $\tilde{M}=\{\operatorname{Im} f=0\}$, and that $df\neq 0$ on X. Put

$$X\times_f X=\{(z,w)\in X\times X;\ f(z)=f(w)\}$$

and let $\pi_1: X\times_f X\to X$ be the projection to the first component. Let $\mathcal{O}_{X\times_f X}^{(0,n-1)}$ be the sheaf of $(n-1)$-forms with respect to the fiber coordinates of π_1 with coefficients in $\mathcal{O}_{X\times_f X}$. We denote by Δ_Y the diagonal set of $Y\times Y$.

DEFINITION 2.3. $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}=\mathcal{H}_{\Delta_Y}^{n-1}(\mathcal{O}_{X\times_f X}^{(0,n-1)}|_{Y\times Y})$.

Identifying Δ_Y with Y we regard $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$ as a sheaf on Y. It is easy to see that $\mathcal{O}_{X\times_f X}^{(0,n-1)}|_{Y\times Y}$ does not depend on f. Hence $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$ is well-defined without the assumption of existence of f. By the same argument as in §1, we have

$$\mathcal{H}_{\Delta_Y}^{\nu}(\mathcal{O}_{X\times_f X}^{(0,n-1)}|_{Y\times Y})=0 \qquad (\nu\neq n-1)\,.$$

By the same argument as in [11] we can verify that $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$ has a ring structure by virtue of residue maps. Note that

$$\tilde{\mathcal{D}}_{Y|\tilde{M}}=\mathcal{H}_{\Delta_Y}^{n-1}(\mathcal{O}_{X\times_f X}^{(0,n-1)}[D_1]|_{Y\times Y})=\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}[D_1]$$

is the sheaf of polynomials of D_{z_1} with coefficients in $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$, independent of f and z, and contains $\mathcal{D}_X|_Y$ as a subring.

Let $\mathcal{D}_X^\infty$ be the sheaf on X of linear partial differential operators of possibly infinite order with holomorphic coefficients. Then the sheaf $\mathcal{O}\mathcal{D}_{\tilde{M}}^\infty$ of sections of $\mathcal{D}_X^\infty|_{\tilde{M}}$ commuting with f is cohomologically defined by

$$\mathcal{O}\mathcal{D}_{\tilde{M}}^\infty = \mathcal{H}_{\Delta_X}^{n-1}(\mathcal{O}_{X\times_f X}^{(0,n-1)})|_{\tilde{M}}.$$

Hence $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$ is an extension ring of $\mathcal{O}\mathcal{D}_{\tilde{M}}^\infty|_Y$ since the ring structure of $\mathcal{O}\mathcal{D}_{\tilde{M}}^\infty$ is also defined through residue maps (cf. [11] and [8]).

LEMMA 2.2. *$\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}$ has a structure of $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$-module compatible with that of $\mathcal{O}\mathcal{D}_{\tilde{M}}^\infty$-module.*

PROOF. To distinguish the first and second components put $X=X_1=X_2$, etc. Using the residue map

$$\boldsymbol{R}(\pi_1)_!\mathcal{O}_{X\times_f X}^{(0,n-1)}[n-1] \longrightarrow \mathcal{O}_X$$

we get homomorphisms

$$\begin{aligned}\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}\otimes_C\mathcal{B}\mathcal{O}_{\tilde{M}}|_Y &\cong \boldsymbol{R}\Gamma_{\Delta_Y}(\mathcal{O}_{X_1\times_f X_2}^{(0,n-1)}|_{Y_1\times Y_2})[n-1]\otimes_C\mathcal{B}\mathcal{O}_{\tilde{M}_2}|_{Y_2}\\ &\longrightarrow \boldsymbol{R}\Gamma_{\Delta_Y}(\mathcal{B}\mathcal{O}_{\tilde{M}_1\times_f\tilde{M}_2}^{(0,n-1)}|_{Y_1\times Y_2})[n-1]\\ &\cong \boldsymbol{R}\Gamma_{\Delta_Y}(\boldsymbol{R}\Gamma_{\tilde{M}\times_f\tilde{M}}(\mathcal{O}_{X\times_f X}^{(0,n-1)})|_{Y\times Y})[n]\otimes\omega_{\tilde{M}}\\ &\longrightarrow \boldsymbol{R}\Gamma_{\tilde{M}}(\boldsymbol{R}(\pi_1)_!(\mathcal{O}_{X\times_f X}^{(0,n-1)}))|_Y[n]\otimes\omega_{\tilde{M}}\\ &\longrightarrow \boldsymbol{R}\Gamma_{\tilde{M}}(\mathcal{O}_X)[1]\otimes\omega_{\tilde{M}}\cong\mathcal{B}\mathcal{O}_{\tilde{M}},\end{aligned}$$

and in the same way,

$$\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}\otimes_C\Gamma_{\tilde{M}_-}(\mathcal{B}\mathcal{O}_{\tilde{M}})|_Y \longrightarrow \Gamma_{\tilde{M}_-}(\mathcal{B}\mathcal{O}_{\tilde{M}}),$$

where $\tilde{M}_-=\tilde{M}\backslash\tilde{M}_+$. Since

$$\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}\cong(\mathcal{B}\mathcal{O}_{\tilde{M}}/\Gamma_{\tilde{M}_-}(\mathcal{B}\mathcal{O}_{\tilde{M}}))|_Y,$$

these homomorphisms induce

$$\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}\otimes_C\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+} \longrightarrow \mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}.$$

We can verify that $\mathcal{B}\mathcal{O}_{Y|\tilde{M}_+}$ becomes an $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$-module by this homomorphism. Since the action of $\mathcal{O}\mathcal{D}_{\tilde{M}}^\infty$ is defined in the same way (cf. [8, Chap. III]) the actions of $\mathcal{O}\mathcal{D}_{\tilde{M}}^\infty$ and of $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$ are compatible with each other. This completes the proof.

Now let us give a concrete description of $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$. For this purpose fixing an admissible local coordinate system z, we may regard X as a

Stein open set of $\boldsymbol{C}^n$. Let $w=(w_1, w')$ be a copy of z. Then (z, w') serves as a local coordinate system of $X\times_f X$. Let U_1 be a Stein neighborhood of Δ_Y in $Y\times Y$ and set

$$U_j=\{(0, z', w')\in U_1\,;\ z_j\neq w_j\}\qquad (j=2, \cdots, n),$$
$$\mathcal{U}=\{U_1, U_2, \cdots, U_n\},\qquad \mathcal{U}'=\{U_2, \cdots, U_n\}.$$

Then by virtue of Lemma 1.3 we have

$$\begin{aligned}\mathcal{O}\widetilde{\mathcal{D}}_{Y|\tilde{M}}(Y) &\cong \varinjlim_{U_1} H^{n-1}(\mathcal{U} \bmod \mathcal{U}'\,;\ \mathcal{O}^{(0,n-1)}_{X\times_f X}|_{Y\times Y})\\ &\cong \varinjlim_{U_1}\Big(\Gamma(U, \mathcal{O}^{(0,n-1)}_{X\times_f X}|_{Y\times Y})/\bigoplus_{j=2}^{n}\Gamma(U_{\hat{j}}, \mathcal{O}^{(0,n-1)}_{X\times_f X}|_{Y\times Y})\Big),\end{aligned}$$

where $U=\bigcap_{j=1}^{n} U_j$ and $U_{\hat{j}}=\bigcap_{k\neq j} U_k$, and U_1 runs on Stein neighborhoods of Δ_Y in $Y\times Y$. For $\alpha=(\alpha_2, \cdots, \alpha_n)\in \boldsymbol{N}^{n-1}$ with $\boldsymbol{N}=\{0, 1, 2, \cdots\}$ set

$$\Phi_\alpha(w')=\prod_{j=2}^{n}\Big(\frac{1}{2\pi\sqrt{-1}}\frac{\alpha_j!}{(-w_j)^{\alpha_j+1}}\Big).$$

Using Laurent series we can identify $\mathcal{O}\widetilde{\mathcal{D}}_{Y|\tilde{M}}(Y)$ with the set of the functions of the form

$$\sum_{\alpha\in \boldsymbol{N}^{n-1}} a_\alpha(z)\Phi_\alpha(z'-w'), \tag{2.10}$$

where $a_\alpha(z)$ are holomorphic on a neighborhood of Y in X, and this series converges on a neighborhood of U in $X\times_f X$. We write an element A defined by (2.10) as

$$A=\sum_{\alpha\in \boldsymbol{N}^{n-1}} a_\alpha(z)D^\alpha_{z'}$$

with $D^\alpha_{z'}=D^{\alpha_2}_{z_2}\cdots D^{\alpha_n}_{z_n}$. Note that for any $\varepsilon>0$ and any $K\Subset Y$ there exist $C>0$ and $\delta>0$ such that

$$|a_\alpha(z)|\leqq C\frac{\varepsilon^{|\alpha|}}{\alpha!}$$

for any z with $|z_1|<\delta$ and $z'\in K$, and $\alpha\in \boldsymbol{N}^{n-1}$. Hence we get

PROPOSITION 2.4. *Each section of $\mathcal{O}\widetilde{\mathcal{D}}_{Y|\tilde{M}}$ over a Stein open set Ω of Y is uniquely expressed in the form*

$$A(z, D_{z'})=\sum_{j=0}^{\infty} A_j(z, D_{z'}),$$

where A_j satisfy the following conditions:

(i) *$A_j(z, \zeta')$ is holomorphic on a neighborhood of $\Omega \times \boldsymbol{C}^{n-1}$ in $X \times \boldsymbol{C}^{n-1}$, and homogeneous of order j with respect to ζ'.*

(ii) *For any $K \Subset Y$ and $\varepsilon > 0$ there exist $C > 0$ and $\delta > 0$ such that for any j*

$$|A_j(z, \zeta')| \leqq \frac{C}{j!}(\varepsilon|\zeta'|)^j$$

holds if $|z_1| < \delta$, $z' \in K$, and $\zeta' \in \boldsymbol{C}^{n-1}$.

Thus $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$ contains the formal differential operators introduced by Tahara (Proposition 1.1.9 of [29]).

PROOF OF THEOREM 2.4. First let us fix a local coordinate system z around $x \in U$ as above. Then by Theorem 1.3.6 of [29], there exists an invertible matrix $Q = Q(z, D_{z'})$ of germs of $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$ at $\mathring{x}$ such that

$$\left\{\begin{array}{l} Q^{-1}(z_1 D_{z_1} I_m - A)Q = z_1 D_{z_1} I_m - A_0, \\ Q(0, z', D_{z'}) = I_m \end{array}\right. \tag{2.11}$$

as matrices of germs of $\tilde{\mathcal{D}}_{Y|\tilde{M}}$. In fact Tahara proves that such Q satisfying (2.11) exists uniquely as a matrix of formal differential operators. Since his proof of the uniqueness of Q also applies to $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$, such Q is unique as a matrix of $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$.

Let us show that there exists an isomorphism

$$\tilde{\gamma} : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \xrightarrow{\sim} \mathcal{B}_N \otimes_{C} L(A_0).$$

Let u be a column vector of m germs of $\tilde{\mathcal{B}}_{N|M_+}$ at $\mathring{x}$ such that $Pu = 0$. Since $\tilde{\mathcal{B}}_{N|M_+}$ is an $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$-module, it follows from (2.11)

$$D_{x_1}(x_1^{-A_0} Q^{-1} u(x)) = 0.$$

Hence by Proposition 2.1 u is written in the form

$$u = Q(x_1^{A_0} v(x')) \tag{2.12}$$

uniquely with a column vector v of germs $v_1, \cdots, v_m$ of $\mathcal{B}_N$ at $\mathring{x}$. We put $\tilde{\gamma}(u) = (dx_1)^{A_0} v \in \mathcal{B}_N \otimes_C L(A_0)$. Thus we have defined an isomorphism $\tilde{\gamma}$ with respect to z.

Let us prove that this $\tilde{\gamma}$ is independent of z. For this purpose, we may assume that A_0 is a diagonal matrix in view of (A.3). Let $\tilde{z}$ be another admissible local coordinate system around $\mathring{x}$ with the relations (2.9). Then (2.11) is written in the form

$$Q^{-1}\Big(c_1(\tilde z_1)\tilde z_1 D_{\tilde z_1}+\tilde z_1\sum_{j=2}^{n}c_j(\tilde z)D_{\tilde z_j}-A\Big)Q$$

$$=c_1(\tilde z_1)\tilde z_1 D_{\tilde z_1}+\tilde z_1\sum_{j=2}^{n}c_j(\tilde z)D_{\tilde z_j}-A_0(\psi'(\tilde z))\,;$$

here $\psi=(\psi_1,\psi')$ is the inverse transformation of $\varphi=(\varphi_1,\cdots,\varphi_n)$. First let us assume $\varphi_1(z_1)=z_1$, i. e. $c_1=1$. Then there exists a diagonal matrix R of invertible sections of $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde M}$ such that

$$R^{-1}\Big(\tilde z_1 D_{\tilde z_1}+\tilde z_1\sum_{j=2}^{n}c_j(\tilde z)D_{\tilde z_j}-A_0(\psi'(\tilde z))\Big)R=\tilde z_1 D_{\tilde z_1}-A_0(\psi'(0,\tilde z'))\,,$$

and that $R(0,\tilde z',D_{\tilde z'})=I_m$. In particular we have

$$(z_1D_{z_1}-A_0(z'))R=R(\tilde z_1D_{\tilde z_1}-A_0(\psi'(0,\tilde z'))). \tag{2.13}$$

Since

$$(QR)^{-1}(\tilde z_1 D_{\tilde z_1}+\tilde z_1\sum_{j=2}^{n}c_j(\tilde z)D_{\tilde z_j}-A)QR=\tilde z_1 D_{\tilde z_1}-A_0(\psi'(0,\tilde z'))\,,$$

the above u is written as

$$u=QR(\tilde x_1^{A_0(\psi'(0,\tilde x'))}\tilde v(\tilde x'))\,, \tag{2.14}$$

and $\tilde v$ is the boundary value of u with respect to $\tilde z$. By (2.12) and (2.14) we have

$$v(x')=x_1^{-A_0(x')}R\tilde x_1^{A_0(\psi'(0,\tilde x'))}\tilde v(\tilde x')\,. \tag{2.15}$$

Replacing the condition (i) of Proposition 2.4 by a weaker one that $A_j(z,\zeta')$ is holomorphic (and homogeneous of order j in ζ') on the intersection of $\tilde M_+\times \boldsymbol{C}^{n-1}$ and a neighborhood of $\Omega\times\boldsymbol{C}^{n-1}$, we get an extension ring $\mathcal{A}\tilde{\mathcal{D}}_{Y|\tilde M_+}$ of $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde M}$. The operator

$$S=\sum_{j=0}^{\infty}S_j(\tilde z,D_{\tilde z'})=\tilde z_1^{-A_0(\psi'(\tilde z))}R\tilde z_1^{A_0(\psi'(0,\tilde z'))}$$

is a diagonal matrix of sections of $\mathcal{A}\tilde{\mathcal{D}}_{Y|\tilde M_+}$. From (2.13) we get

$$\Big(\frac{\partial}{\partial\tilde z_1}+\sum_{j=2}^{n}c_j(\tilde z)D_{\tilde z_j}\Big)S=0\,,$$

where $(\partial/\partial\tilde z_1)S=[\tilde z_1,S]$. Hence each component of S becomes a section of $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde M}$. On the other hand, using Leibniz's formula we can verify that the j-th homogeneous part $S_j(\tilde x_1,\tilde z,D_{\tilde z'})$ tends to 0 if $j\geqq 1$, and to I_m if $j=0$ as $\tilde x_1>0$ tends to 0. Hence we have $v=S\tilde v$ with $S(0,\tilde z',D_{\tilde z'})=I_m$. Restricting this relation to $x_1=0$ (note that both v and $S\tilde{v}$ can be regarded as sections of $\tilde{\mathcal{B}}^A$), we get $v(x')=\tilde{v}(\psi'(0,\tilde{x}'))$. Thus $\tilde{\gamma}$ is invariant under a local coordinate transformation (2.9) if $\varphi_1(z_1)=z_1$.

Next let us assume $\varphi_j(z)=z_j$ for $j=2,\cdots,n$. There exists a diagonal matrix R of non-vanishing holomorphic functions such that $R(0,\tilde{z}')=I_m$ and that

$$R^{-1}(\tilde{z}_1 D_{\tilde{z}_1}-c_1(\tilde{z}_1)^{-1}A_0(\tilde{z}'))R=\tilde{z}_1 D_{\tilde{z}_1}-A_0(\tilde{z}')\,.$$

Let $u, v, \tilde{v}$ be as above. Then we have

$$v(x')=x_1^{-A_0(x')}R\tilde{x}_1^{A_0(\tilde{x}')}\tilde{v}(\tilde{x}')=\left(\frac{\tilde{x}_1}{x_1}\right)^{A_0(x')}R\tilde{v}(\tilde{x}')\,.$$

Restricting this relation to $x_1=0$ we get

$$v(x')=\left(\frac{d\tilde{x}_1}{dx_1}(0)\right)^{A_0(x')}\tilde{v}(\tilde{x}')\,.$$

This proves that $\tilde{\gamma}$ is independent of z. By Theorem 1.1 $\gamma=\tilde{\gamma}\circ\alpha$ satisfies the statements of the theorem. Since $\tilde{C}_{N|M_+}$ is also an $\mathcal{O}\tilde{\mathcal{D}}_{Y|\tilde{M}}$-module, the statement for $C_{N|M_+}$ follows from the above argument and Theorem 1.2. This completes the proof of Theorem 2.4.

REMARK. If we can take and fix an admissible local coordinate system over U, then Theorem 2.4 holds even if we replace (A.3) with a weaker condition

(A.3)′ $\qquad \lambda_i(x')-\lambda_j(x')\notin \boldsymbol{Z}\setminus\{0\}$ for any i, j and $(0,x')\in U$.

COROLLARY 2.3. *Under the same assumptions as Theorem* 2.4 *the same conclusion as Corollary* 2.1 *holds.*

For the later use let us consider a singular coordinate transformation: Let z be an admissible local coordinate system around $\mathring{x}\in U$ and set

$$\tilde{z}_1=z_1^{1/k},\qquad \tilde{z}_j=z_j\qquad (j=2,\cdots,n)$$

with a positive integer k. Then $P=z_1 D_{z_1}I_m-A$ is transformed into

$$\tilde{P}=\frac{1}{k}\tilde{z}_1 D_{\tilde{z}_1}I_m-A(\tilde{z}_1^k,\tilde{z}',D_{\tilde{z}'})\,.$$

Hence $\tilde{P}$ is also a Fuchsian system with characteristic eigenvalues $k\lambda_1,\cdots,k\lambda_m$. Set $\tilde{\mathcal{M}}=\mathcal{D}_X^m/\mathcal{D}_X^m\tilde{P}$. Then we easily get

PROPOSITION 2.5. *In the above situation let u be a section of*

$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+})$ (*resp.* $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_{N|M_+})$). *Assume* $k(\lambda_i-\lambda_j)\notin \boldsymbol{Z}$ *for* $i\neq j$ *and that* $\gamma(u)=(dx_1)^{A_0}v$ *with* $v\in(\mathcal{B}_N)^m$ (*resp.* $(C_N)^m$). *Then* $\tilde{u}=u(\tilde{x}_1^k, \tilde{x}')$ *is a section of* $\mathcal{H}om_{\mathcal{D}_X}(\tilde{\mathcal{M}}, \mathcal{B}_{N|M_+})$ (*resp.* $\mathcal{H}om_{\mathcal{D}_X}(\tilde{\mathcal{M}}, C_{N|M_+})$) *and we have* $\gamma(\tilde{u}) =(d\tilde{x}_1)^{kA_0}v$.

§ 3. Propagation of micro-analyticity up to the boundary

3.1. Micro-hyperbolic systems and main results

First let us define the notion of (relative) micro-hyperbolicity. Let

$$H:\ T^*(T^*X) \xrightarrow{\sim} T(T^*X)$$

be the Hamilton map defined by

$$\langle\theta, v\rangle=\langle d\omega, v\wedge H(\theta)\rangle$$

for $v\in T(T^*X), \theta\in T^*(T^*X)$, where $\omega=\sum_{j=1}^{n}\zeta_j dz_j$ is the fundamental 1-form on T^*X (z is a local coordinate system of X and ζ is its dual variable). Let $z=x+\sqrt{-1}\,y$ be an admissible local coordinate system of X. Then $\theta_0=dz_1\in T^*(T^*X)$ is invariant up to positive constant under the change of admissible local coordinate systems. It is easy to see that $H(\theta_0)\in T(T^*X)$ belongs to $T(T^*X|_{\tilde{M}})$ with $T^*X|_{\tilde{M}}=T^*X\underset{X}{\times}\tilde{M}$. For a point x^* and subsets S, V of T^*X we denote by $C_{x^*}(S\,;\,V)$ the normal cone of S along V at x^* after Kashiwara-Schapira (Definition 1.1.1 of [11]). Note that $C_{x^*}(S\,;\,V)$ is a closed cone of the tangent space $T_{x^*}(T^*X)$ of T^*X at x^*. We denote by $\mathcal{E}_X$ the sheaf on T^*X of microdifferential operators of finite order.

DEFINITION 3.1. A coherent $\mathcal{E}_X$-module $\mathcal{M}$ defined on a neighborhood of $x^*\in T_M^*X|_N$ in T^*X is called *micro-hyperbolic relative to* $\tilde{M}_+$ *in the direction* $\theta\in T_{x^*}^*(T^*X)$ *at* x^* if and only if

$$H(\theta)\notin C_{x^*}(\mathrm{Supp}(\mathcal{M})\cap T^*X|_{\tilde{M}_+}\,;\ T_M^*X).$$

REMARK. (i) Let z be an admissible local coordinate system. Then $\mathcal{M}$ is micro-hyperbolic relative to $\tilde{M}_+$ in the direction $\theta_0=dz_1$ at x^* if and only if there exist an open neighborhood U of x^* in $T^*X|_{\tilde{M}}$ and an open cone Γ in $T_{x^*}(T^*X|_{\tilde{M}})\cong\{(x_1, z'\,;\,w)\in \boldsymbol{R}\times \boldsymbol{C}^{n-1}\times \boldsymbol{C}^n\}$ containing $(0, 0\,;\,-1, 0, \cdots, 0)$ such that

$$((U\cap T_M^*X)+\Gamma)\cap U\cap \mathrm{Supp}(\mathcal{M})\cap T^*X|_{\tilde{M}_+}=\emptyset.$$

(ii) If $\theta\in T_{x^*}^*(T^*X)$ is micro-hyperbolic for $\mathcal{M}$ in the sense of [11], then $\mathcal{M}$ is micro-hyperbolic relative to $\tilde{M}_+$ in the direction θ at x^*.

(iii) Assume that $\mathcal{M}$ is a single equation $Pu=0$ with a microdifferential operator P defined on a neighborhood of $x^*=(\mathring{x}, \sqrt{-1}\,\mathring{\xi})\in T_M^*X|_N$. Assume moreover that the principal symbol of P is written in the form $\sigma(P)(z,\zeta)=z_1^k p(z,\zeta)$, where k is a nonnegative integer, and $p(z,\zeta)$ is a holomorphic function such that $p(\mathring{x},\zeta_1,\sqrt{-1}\,\mathring{\xi}')$ is not identically zero as a function of ζ_1. Then $\mathcal{M}$ is micro-hyperbolic relative to $\tilde{M}_+$ in the direction θ_0 at x^* if and only if there exists $\varepsilon>0$ such that

$$p(x, \sqrt{-1}\,\xi-(t,0,\cdots,0))\neq 0$$

whenever $x\in \boldsymbol{R}^n$, $|x-\mathring{x}|<\varepsilon$, $x_1>0$, $\xi\in\boldsymbol{R}^n$, $|\xi-\mathring{\xi}|<\varepsilon$, $0<t<\varepsilon$. In fact, this assertion follows from the local version of Bochner's tube theorem due to Kashiwara and Komatsu (cf. [15]) applied to $1/p(z_1^2, z', \zeta)$ (cf. [31], [2]).

In this section we sometimes identify S_M^*X with $\mathring{T}_M^*X=T_M^*X\backslash 0$. We denote by $\pi: T^*X\to X$ and $\rho: T^*X|_Y\to T^*Y$ the canonical projections. Note that $\mathcal{C}_{M_+}$ is supported by $L_0\cup L_+$ with $L_0=S_M^*\tilde{M}|_N$ and $L_+=S_M^*\tilde{M}|_{M_+}$.

THEOREM 3.1. *Let x^* be a point of $\mathring{T}_N^*Y$ and $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined on a neighborhood of $\pi_{N/Y}(x^*)$. Assume the following conditions:*

(C.1) $$\mathring{T}_Y^*X\cap \mathrm{cl}\,(\mathrm{SS}(\mathcal{M})\cap T^*X|_{\tilde{M}_+})=\emptyset,$$

where cl *denotes the closure in T^*X.*

(C.2) *$\mathcal{E}_X\otimes_{\pi^{-1}\mathcal{D}_X}\pi^{-1}\mathcal{M}$ is micro-hyperbolic relative to $\tilde{M}_+$ in the direction $\theta_0=dz_1$ at each point of $\rho^{-1}(x^*)\cap T_M^*X$.*

(C.3) *If z is an admissible local coordinate system of X around $\pi_{N/Y}(x^*)$ and $\zeta=(\zeta_1,\zeta')$ is the dual variable of z with ζ_1 being the fiber coordinate of ρ, then*

$$\rho^{-1}(x^*)\cap\mathrm{cl}(\mathrm{SS}(\mathcal{M})\cap T^*X|_{\tilde{M}_+})\subset\{(\zeta_1, x^*)\in\rho^{-1}(x^*)\,;\ \mathrm{Re}\,\zeta_1\geqq 0\}$$

holds. (This condition is independent of z.)
Under these conditions we have

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{L_0}(\mathcal{C}_{M_+}))_{x^*}=0.$$

We shall prove this theorem in Sect. 3.3. Using this theorem we shall prove some results on the propagation of micro-analyticity of solutions up to the boundary. For this purpose we first prove the following:

LEMMA 3.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined on a neighborhood* *of* $\mathring{x}\in N$ *satisfying* (C.1). *Then there exists a neighborhood* V *of* $\mathring{x}$ *in* M *such that the homomorphism*

$$\phi:\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_{M_+})|_{L_+} \longrightarrow p_*(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_M)|_{p^{-1}(L_+)})$$

is injective on $L_+\cap(\pi_{M/\tilde{M}})^{-1}(V)$.

PROOF. Let z be an admissible local coordinate system around $\mathring{x}=0$ and set $Y_t=\{z\in X;\ z_1=t\}$, $\tilde{M}_t=\{(x_1, z')\in\tilde{M}\ ;\ x_1>t\}$, $N_t=\{x\in M;\ x_1=t\}$ for $t>0$. By (C.1) there exists $c>0$ such that Y_t is non-characteristic for $\mathcal{M}$ if $0<t<c$. Fix such t and put $L_t=S_M^*\tilde{M}|_{N_t}, K_t=S_M\tilde{M}|_{N_t}$. Let

$$\pi_t:\ (Y_t\backslash N_t)\cup L_t \longrightarrow Y_t,$$

$$\tau_t:\ (Y_t\backslash N_t)\cup K_t \longrightarrow Y_t,$$

$$\varepsilon_t:\ Y_t\backslash N_t \longrightarrow (Y_t\backslash N_t)\cup K_t$$

be canonical maps. Set

$$\tilde{\mathcal{B}}_t=\mathcal{H}^{n-1}_{N_t}(\mathcal{B}\mathcal{O}_{\tilde{M}}|_{Y_t}), \qquad \tilde{\mathcal{B}}^A_t=\mathcal{H}^{n-1}_{N_t}(\mathcal{O}_X|_{Y_t}),$$

$$\tilde{\mathcal{A}}_t=(\varepsilon_{t*}(\mathcal{B}\mathcal{O}_{\tilde{M}}|_{Y_t\backslash N_t}))|_{K_t}, \qquad \tilde{\mathcal{A}}^A_t=(\varepsilon_{t*}(\mathcal{O}_X|_{Y_t\backslash N_t}))|_{K_t},$$

$$\tilde{C}_t=\mathcal{H}^{n-1}_{L_t}(\pi_t^{-1}(\mathcal{B}\mathcal{O}_{\tilde{M}}|_{Y_t}))^a, \qquad \tilde{C}^A_t=\mathcal{H}^{n-1}_{L_t}(\pi_t^{-1}(\mathcal{O}_X|_{Y_t}))^a.$$

Since the arguments in § 1 apply to these sheaves, there exist injective homomorphisms

$$b_t:\ \tilde{\mathcal{A}}_t \longrightarrow \tau_t^{-1}\tilde{\mathcal{B}}_t, \qquad b^A_t:\ \tilde{\mathcal{A}}^A \longrightarrow \tau_t^{-1}\tilde{\mathcal{B}}^A_t$$

and surjective homomorphisms

$$\mathrm{sp}_t:\ \pi_t^{-1}\tilde{\mathcal{B}}_t \longrightarrow \tilde{C}_t, \qquad \mathrm{sp}^A_t:\ \pi_t^{-1}\tilde{\mathcal{B}}^A_t \longrightarrow \tilde{C}^A_t,$$

and Propositions 1.7 and 1.10 are also valid for $\tilde{\mathcal{B}}_t$. By the same argument as the proof of Theorem 1.1 we can prove that there exists an injective $\mathcal{D}_X$-linear homomorphism

$$\alpha_t:\ C_{M_+}|_{L_t} \longrightarrow \tilde{C}_t.$$

The homomorphism $\mathcal{O}_X|_{Y_t}\to\mathcal{B}\mathcal{O}_{\tilde{M}}|_{Y_t}$ induces $\beta_t:\ \tilde{C}^A_t\to\tilde{C}_t$.

Let f be a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_{M_+})$ at $y^*=(\mathring{y}, \sqrt{-1}\,\mathring{\xi}'\infty)\in L_t$. Let us take a system of generators $u_1, \cdots, u_m$ of $\mathcal{M}$ over $\mathcal{D}_X$ and put $f_j=f(u_j)$. Then f_j is a germ of C_{M_+} at y^*. From (2.5) we get an isomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{C}^A_t) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{C}_t).$$

Hence there exists a germ g_j of $\tilde{C}_t^A$ at y^* such that $\alpha_t(f_j)=\beta_t(g_j)$. There exists a section F of $\mathscr{B}\mathscr{O}$ on

$$\{(x_1, z')\in\tilde{M}\ ;\ |x_1-t|<\varepsilon, |z'-\mathring{y}'|<\varepsilon, \operatorname{Im} z'\in\Gamma\}$$

with an open cone $\Gamma\subset\boldsymbol{R}^{n-1}$ containing $\mathring{\xi}'$ and an $\varepsilon>0$ such that $f_j=\mathrm{sp}_+(b_+(F))$ at y^*. On the other hand we may assume that there exists a section G of $\mathscr{O}_X|_{Y_t}$ on $\{(t, z')\in Y_t\ ;\ |z'-\mathring{y}'|<\varepsilon, \operatorname{Im} z'\in\Gamma\}$ such that

$$g_j=\mathrm{sp}_t^A(b_t^A(G)).$$

From $\alpha_t(f_j)=\beta_t(g_j)$ we get

$$\mathrm{sp}_t(b_t(F|_{Y_t}-G))=0$$

at y^*. Hence by the $\tilde{\mathscr{B}}_t$ version of Proposition 1.10 there exist sections F_ν $(\nu=1,\cdots,\nu_0)$ of $\mathscr{B}\mathscr{O}_{\tilde{M}}|_{Y_t}$ on $\{(t, z')\in Y_t\ ;\ |z'-\mathring{y}'|<\delta, \operatorname{Im} z'\in\Gamma_\nu\}$ with a $\delta>0$ and open cones Γ_ν containing $\mathring{\xi}'$ such that

$$\Gamma_\nu\cap\{y'\in\boldsymbol{R}^{n-1}\ ;\ \langle y', \mathring{\xi}'\rangle<0\}\neq\emptyset,$$

$$F-G=\sum_{\nu=1}^{\nu_0}F_j.$$

Choosing $a>0$ so that $2a<\theta(\delta/12)$ and $a<\delta/4$, set

$$\tilde{F}(x_1, z'\ ;\ \xi')=\int_C F(x_1, w')W(z'-w'\ ;\ \xi')dw',$$

$$\tilde{G}(x_1, z'\ ;\ \xi')=\int_C G(x_1, w')W(z'-w'\ ;\ \xi')dw'$$

with $C=\{z'\in\boldsymbol{C}^{n-1}\ ;\ |x'|\leqq\delta/2, y'=a\mathring{\xi}'\}$ and set

$$F_0=\int_\Delta\tilde{F}(x_1, z'\ ;\ \xi')d\sigma(\xi'),\qquad G_0=\int_\Delta\tilde{G}(x_1, z'\ ;\ \xi')d\sigma(\xi'),$$

where $\Delta=\{\xi'\in S^{n-2}\ ;\ |\xi'-\mathring{\xi}'|<c\}$ with sufficiently small $c>0$. Then the same argument as the proof of Theorem 1.1 implies that F_0-G_0 is a section of $\mathscr{B}\mathscr{O}$ on $\{(x_1, z')\in\tilde{M}\ ;\ |x_1-t|<\delta', |z'|<\delta'\}$ for some $\delta'>0$. By the unique continuation property of $\mathscr{C}\mathscr{O}$, G_0 becomes holomorphic on a neighborhood of

$$\{(x_1, z')\in\tilde{M}\ ;\ |x_1-t|<\delta', |z'-\mathring{y}'|<\delta', \operatorname{Im} z'\in\Gamma'\}$$

with some open cone Γ' containing $\mathring{\xi}'$. Hence we get

$$f_j=\mathrm{sp}_+(b_+(F))=\mathrm{sp}_+(b_+(F_0))=\mathrm{sp}_+(b_+(G_0))$$

at y^*. By virtue of the local version of Bochner's tube theorem (cf. Komatsu [15]) there exists an open cone $\tilde{\Gamma}$ in $\boldsymbol{R}^n$ containing $\Gamma' \subset \boldsymbol{R}^{n-1} \cong \{0\} \times \boldsymbol{R}^{n-1}$ such that G_0 is holomorphic on

$$D(\mathring{y}, \varepsilon', \tilde{\Gamma}) = \{z \in \boldsymbol{C}^n ;\ |z_1 - t| < \varepsilon',\ |z' - \mathring{y}'| < \varepsilon',\ \operatorname{Im} z \in \tilde{\Gamma}\}$$

with some $\varepsilon' > 0$.

Assume $\phi(f) = 0$ on $p^{-1}(y^*)$. Then we have

$$\mathrm{sp}(G_0(x + \sqrt{-1}\,\tilde{\Gamma}0)) = \phi(f_j) = 0$$

on a neighborhood of $p^{-1}(y^*)$. By the flabbiness of $\mathcal{C}_M$ there exist holomorphic functions G_ν $(\nu = 1, \cdots, \nu_0)$ on $D(\mathring{y}, \varepsilon', \tilde{\Gamma}_\nu)$ with open convex cones $\tilde{\Gamma}_\nu \subset \boldsymbol{R}^n$ containing $\mathring{\xi}'$ such that

$$\tilde{\Gamma}_\nu \cap \{(0, y') ;\ \langle y', \mathring{\xi}' \rangle < 0\} \neq \emptyset,$$

$$G_0 = \sum_{\nu=1}^{\nu_0} G_\nu .$$

Hence we get

$$f_j = \mathrm{sp}_+(b_+(G_0)) = \sum_{\nu=1}^{\nu_0} \mathrm{sp}_+(b_+(G_\nu|_{\tilde{M}})) = 0,$$

and consequently $f = 0$, at y^*. This completes the proof.

THEOREM 3.2. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined on a neighborhood of $\pi_{M/\tilde{M}}(U)$ with an open set U of $S^*_M\tilde{M}$ such that Y is non-characteristic for $\mathcal{M}$. Suppose moreover that $\mathcal{M}$ satisfies* (C.2) *and* (C.3) *of Theorem* 3.1 *for any $x^* \in U \cap S^*_N Y$. Under these assumptions if f is a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{M_+})$ on U such that $\phi(f)$ vanishes on $p^{-1}(U \cap L_+)$, then $\gamma(f)$ vanishes as a section of $\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$ on $U \cap S^*_N Y$.*

PROOF. This follows immediately from Theorem 3.1 and Lemma 3.1.

COROLLARY 3.1. *Suppose that $\mathcal{M}$ satisfies the conditions of Theorem* 3.2 *with $U = (\pi_{M/\tilde{M}})^{-1}(V)$, where V is an open set of M. Let f be a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)$ on $V \cap M_+$, where $\mathcal{A}_M$ denotes the sheaf on M of real analytic functions. Then f is uniquely continued to a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ on a neighborhood in X of $V \cap N$.*

The following example was suggested to us by a problem posed by G. Zampieri (communicated by K. Kataoka).

Example 3.1. We use the notation $(z, w) \in \boldsymbol{C}^n \times \boldsymbol{C}^d$ with $z = x + \sqrt{-1}\, y$

and $z=(z_1, z')\in \boldsymbol{C}\times\boldsymbol{C}^{n-1}$. Let ζ and θ be the variables dual to z and to w respectively. Let $P=P(z, w, D_z, D_w)$ be a linear partial differential operator with holomorphic coefficients defined on a neighborhood of $(0, 0)$, and assume that $\{(z, w)\,;\, z_1=0\}$ is non-characteristic for P. Assume, moreover, that there exist $C, \varepsilon>0$ such that the principal symbol $\sigma(P)(z, w, \zeta, \theta)$ never vanishes if $z=(x_1, z')\in\boldsymbol{R}\times\boldsymbol{C}^{n-1}$ with $|z|<\varepsilon$ and $x_1>0$, $|w|<\varepsilon$, $\zeta\in\boldsymbol{C}^n$ and $\theta\in\boldsymbol{C}^d$ with $\zeta_n\neq 0$, $|\zeta_j/\zeta_n|<\varepsilon$ $(j=2, \cdots, n-1)$, $|\theta_k/\zeta_n|<\varepsilon$ $(k=1, \cdots, d)$, and if

$$\mathrm{Im}\Big(\frac{\zeta_1}{\zeta_n}\Big)>C\Big(\sum_{j=2}^{n}|\mathrm{Im}\, z_j|+\sum_{j=2}^{n-1}\Big|\mathrm{Im}\Big(\frac{\zeta_j}{\zeta_n}\Big)\Big|+\sum_{k=1}^{d}\Big|\frac{\theta_k}{\zeta_n}\Big|\Big).$$

Under these assumptions, suppose that a hyperfunction $u(x, w)$ with holomorphic parameters w satisfies

$$P(x, w, D_x, D_w)u(x, w)=0$$

on $\{(x, w)\in\boldsymbol{R}^n\times\boldsymbol{C}^d\,;\, |x|<\varepsilon, |w|<\varepsilon, x_1>0\}$. Then $u(x, w)$ is micro-analytic on

$$\{(x, w, \sqrt{-1}\,\langle\eta, dx\rangle\infty\in\sqrt{-1}\,S^*(\boldsymbol{R}^n\times\boldsymbol{C}^d)\,;\, |x|<\delta, |w|<\delta,$$
$$x_1>0, \eta=(\eta_1, \eta')\in\boldsymbol{R}\times\boldsymbol{R}^{n-1}, |\eta'-(0, \cdots, 0, 1)|<\delta\}$$

for some $\delta>0$, if and only if the boundary values $D_1^\nu u(+0, x', w)$ $(0\leqq\nu\leqq \mathrm{ord}\, P-1)$, which are hyperfunctions with holomorphic parameters w, are micro-analytic at $(0, 0, \sqrt{-1}dx_n\infty)\in\sqrt{-1}S^*(\boldsymbol{R}^{n-1}\times\boldsymbol{C}^d)$. In fact, it is easy to verify that the system

$$P(x, w, D_x, D_w)u=\frac{\partial}{\partial\overline{w}_1}u=\cdots=\frac{\partial}{\partial\overline{w}_d}u=0$$

satisfies the conditions of Theorem 3.2. A typical example is

$$P=D_{x_1}^m+D_{w_1}^m+\cdots+D_{w_d}^m+(\text{lower order terms})$$

with $(x, w)\in\boldsymbol{R}^n\times\boldsymbol{C}^d$ and a positive integer m.

Now let P be a Fuchsian system defined on a neighborhood of $\pi_{M/\tilde{M}}(U)$ with an open set U of $S_M^*\tilde{M}$ satisfying (A.1)-(A.3) and set $\mathcal{M}=\mathcal{D}_X^m/\mathcal{D}_X^m P$. The determinant $\det P$ is defined by

$$(\det P)(z, \zeta)=\det(z_1\zeta_1 I_m-(\sigma_{n_i-n_j+1}(A_{ij})(z, \zeta'))),$$

where σ_j denotes the principal symbol of order j (cf. Sato-Kashiwara [24]).

THEOREM 3.3. *In addition to the above assumptions, assume that*

$$(\det P)(x, \zeta_1, \sqrt{-1}\,\xi')\neq 0$$

whenever $(x, \sqrt{-1}\,\xi') \in U \cap L_+$ *and* $\mathrm{Re}\,\zeta_1 < 0$. *Let* f *be a section of* $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, C_{M_+})$ *on* U *such that* $\psi(f)$ *vanishes on* $p^{-1}(U \cap L_+)$. *Then* $\gamma(f) \in C_N \otimes_C (\pi_{N/Y})^{-1} L(A_0)$ *vanishes on* $U \cap S_N^* Y$.

PROOF. We fix $x^* \in U \cap S_N^* Y$ and take an admissible local coordinate system z around $\pi_{M/\tilde{M}}(x^*)$. We transform P into $\tilde{P}$ by a singular coordinate transformation

$$z_1 = \tilde{z}_1^k \quad \text{and} \quad z_j = \tilde{z}_j \quad (j = 2, \cdots, n)$$

with an integer $k \geqq m$ such that $k(\lambda_i - \lambda_j) \in \boldsymbol{Z}$ for $i \neq j$. Then $\det \tilde{P}$ is written in the form

$$(\det \tilde{P})(\tilde{z}, \tilde{\zeta}) = \det\left(\frac{1}{k}\tilde{z}_1\tilde{\zeta}_1\delta_{ij} - \sigma_{n_i - n_j + 1}(A_{ij})(\tilde{z}_1^k, \tilde{z}', \tilde{\zeta}')\right)$$

$$= k^{-m}\tilde{z}_1^m \tilde{p}(\tilde{z}, \tilde{\zeta})$$

with a monic polynomial $\tilde{p}$ in $\tilde{\zeta}_1$ with coefficients holomorphic in $(\tilde{z}, \tilde{\zeta}')$. Thus $\tilde{\mathcal{M}} = \mathcal{D}_X^m / \mathcal{D}_X^m \tilde{P}$ satisfies (C.1). Applying the local Bochner theorem to $\tilde{p}(\tilde{z}_1^{2k}, \tilde{z}', \tilde{\zeta})$ we can verify (C.2) (cf. Kaneko [2]). Now let us verify (C.3). By the continuity of the roots of the equation $\tilde{p}(\tilde{z}, \tilde{\zeta}) = 0$ in $\tilde{\zeta}_1$, we have

$$\{(\tilde{\zeta}_1, x^*) \in \rho^{-1}(x^*)\} \cap \mathrm{cl}(\mathrm{SS}(\tilde{\mathcal{M}}) \cap T^*X|_{\tilde{M}_+})$$

$$\subset \{(\tilde{\zeta}_1, x^*)\,;\ \tilde{p}(\tilde{\zeta}_1, x^*) = 0\} \subset \{(\tilde{\zeta}_1, x^*)\,;\ \mathrm{Re}\,\tilde{\zeta}_1 \geqq 0\}.$$

Thus we have verified (C.3). From Theorem 3.1, Lemma 3.1 and Proposition 2.5 it follows that $\gamma(f)$ vanishes on a neighborhood of x^*. This completes the proof.

REMARK. (i) If we fix a local coordinate system, Theorem 3.3 is also valid if we replace the condition (A.3) with a weaker one (A.3)′ (cf. § 2).

(ii) A single equation $Pu = 0$ with regular singularities in a weak sense (cf. Kashiwara-Oshima [10]) is equivalent to a Fuchsian system (cf. Tahara [29]). Hence Theorem 3.3 also holds for such equations.

3.2. Prolongation theorem

In order to prove Theorem 3.1 we modify the prolongation theorem for cohomology groups in the complex domain due to Kashiwara-Schapira [11] so as to apply to cohomology groups with $\mathcal{BO}$ coefficients. In this section we restrict our attention to Euclidean spaces and put $X = \boldsymbol{C}^n \ni z = (z_1, z') = x + \sqrt{-1}\,y$, $\tilde{M} = \boldsymbol{R} \times \boldsymbol{C}^{n-1}$, $Y = \{0\} \times \boldsymbol{C}^{n-1}$. We use the notion and notation in [11]. Let G be a proper convex closed cone in $\tilde{M}$ with vertex at the origin. Then X and $\tilde{M}$ are equipped with G-topology as follows: An open subset U of X (or of $\tilde{M}$) is called *G-open* if and only if $z+G\subset U$ holds for any $z\in U$. A subset D of X (or of $\tilde{M}$) is called *G-round* if and only if

$$(z+G)\cap(w-G)\subset D$$

for any $z, w\in D$. For a G-round open set $D\subset X$ we put

$$\mathcal{E}(G\,;\,D)=H_Z^n(D\times D\,;\,\mathcal{O}_{X\times X}^{(0,n)})$$

with $Z=\{(z,w)\in X\times X\,;\,w-z\in G\}$, where $\mathcal{O}_{X\times X}^{(0,n)}$ is the sheaf of holomorphic n-forms with respect to w. $\mathcal{E}(G\,;\,D)$ becomes a ring. In order to specify the topology we denote by Ω_G the subset Ω of X equipped with G-topology, and by $\phi_G: X\to X_G$, $\phi_G: \tilde{M}\to\tilde{M}_G$ the identity maps. First let us show an analogue of Theorem 3.2.4 of [11].

PROPOSITION 3.1. *Let G be a proper convex closed cone in $\tilde{M}$ and let $\Omega_0\subset\Omega$ be two G-open sets in $\tilde{M}$. Let D be a bounded G-round open set in X such that $\Omega\backslash\Omega_0\subset D$. Suppose that there exists an open convex set ω of $\tilde{M}$ such that*

$$\omega\cap(\Omega\backslash\Omega_0)=\emptyset,\qquad(\omega+G)\cap\Omega\cap D\subset\omega,\qquad\omega\supset\Omega_0\cap\partial D.$$

Then $\boldsymbol{R}\Gamma_{\Omega\backslash\Omega_0}((\phi_G)_*\mathcal{BO})$ is well-defined in the derived category of the abelian category of the sheaves of $\mathcal{E}(G\,;\,D)$-modules on Ω.

PROOF. First let us show that $H_{\Omega\backslash\Omega_0}^k(\Omega\,;\,\mathcal{BO})$ is an $\mathcal{E}(G\,;\,D)$-module for any $k\in\boldsymbol{Z}$. Set $\tilde{\Omega}=\{z\in X\,;\,(x_1,z')\in\Omega\}$ and $\tilde{\Omega}_0=\{z\in\tilde{\Omega}\,;\,(x_1,z')\in\Omega_0$ or $y_1\neq 0\}$. Then $\tilde{\Omega}$ and $\tilde{\Omega}_0$ are G-open sets in X and $\tilde{\Omega}\backslash\tilde{\Omega}_0=\Omega\backslash\Omega_0$ holds. Hence we obtain

$$H_{\Omega\backslash\Omega_0}^k(\Omega\,;\,\mathcal{BO})=H_{\tilde{\Omega}\backslash\tilde{\Omega}_0}^{k+1}(\tilde{\Omega}\,;\,\mathcal{O}_X).$$

Theorem 3.2.1 of [11] implies that $H_{\Omega\backslash\Omega_0}^k(\Omega\,;\,\mathcal{BO})$ is an $\mathcal{E}(G\,;\,D)$-module.

Secondly note that for any open convex set ω of $\tilde{M}$,

$$H^\nu(\omega\,;\,\mathcal{BO})=0\qquad(\nu>0)$$

holds (see the remark after Lemma 1.1).

By virtue of these facts the proof of Theorem 3.2.4 of [11] still works if we replace $\mathcal{O}_X$ with $\mathcal{BO}$, and pseudo-convex open sets in X by open convex sets in $\tilde{M}$. This completes the proof.

Let Q be an open convex cone of $T\tilde{M}$ (i.e. its fiber $Q(z)$ at each point z of $\tilde{M}$ is a convex cone). Then an open set Ω of $\tilde{M}$ is called *locally Q-flat* at $z\in\tilde{M}$ if

$$C_z(\tilde{M}\backslash\Omega\,;\ \Omega)\cap Q(z)=\emptyset\,;$$

and a locally closed set $Z\subset\tilde{M}$ is called *Q-flat* on an open set $V\subset\tilde{M}$ if there are two open sets Ω_1, Ω_0 which are locally Q-flat at any point in V such that

$$V\cap(\Omega\backslash\Omega_0)=V\cap Z.$$

Note that an extension ring $\mathcal{E}^R$ of $\mathcal{E}_X$ is defined by

$$\mathcal{E}^R=\mathcal{C}^R_{X|X\times X}\otimes_{\mathcal{O}_{X\times X}}\mathcal{O}^{(0,n)}_{X\times X},$$

and that there is a natural ring homomorphism

$$\mathcal{E}(G\,;\ D)\longrightarrow\Gamma(D\times\operatorname{int}G^\circ\,;\ \mathcal{E}^R),$$

where $\operatorname{int}G^\circ=\{\zeta\,;\ \operatorname{Re}\langle\zeta, z\rangle<0$ for any $z\in G\backslash\{0\}\}$ (cf. [11]). Let

$$\mathcal{M}^\cdot:\ 0\longleftarrow\mathcal{E}(G\,;\ D)^{N_0}\longleftarrow\mathcal{E}(G\,;\ D)^{N_1}\longleftarrow\cdots\longleftarrow\mathcal{E}(G\,;\ D)^{N_r}\longleftarrow 0$$

be a bounded complex of free $\mathcal{E}(G\,;\ D)$-modules of finite rank. Then we denote by $\mathrm{SS}(\mathcal{M}^\cdot)$ the closure in $D\times\operatorname{int}G^\circ\subset T^*X$ of

$$\{x^*=(z,\zeta)\in D\times\operatorname{int}G^\circ\,;\ \mathcal{E}^R_{x^*}\otimes_{\mathcal{E}(G;\,D)}\mathcal{M}^\cdot\ \text{is exact}\}.$$

THEOREM 3.4. *Let G be a proper convex closed cone in $\tilde{M}$, and D a bounded G-round open set in X. Let $\mathcal{M}^\cdot$ be a bounded complex of free $\mathcal{E}(G\,;\ D)$-modules of finite rank. Put $D'=D\cap\tilde{M}$ and let $\{\Omega_t\}_{0\leq t\leq 1}$ be a family of open sets of $\tilde{M}$. We assume the following:*

(a) *There is an open convex cone R of TD' such that $R(z)\neq\emptyset$ and $R(z)\supset G\backslash\{0\}$ for any $z\in D'$ and that either Ω_0 or Ω_1 is R-flat on D'.*

(b) $\mathrm{cl}(\Omega_1\backslash\Omega_0)\subset D'$, *where* cl *denotes the closure in* $\tilde{M}$.

(c) *There is an open convex set $\omega\subset\tilde{M}$ such that*

$$\omega\cap(\Omega_1\backslash\Omega_0)=\emptyset,\qquad(\omega+G)\cap\Omega_1\subset\omega,\qquad\mathrm{cl}(\Omega_0)\cap\partial D'\subset\omega.$$

(d) $\Omega_{t_0}=\bigcup_{t<t_0}\Omega_t$ *holds for any t_0 with $0<t_0\leqq 1$, and* $\bigcap_{t>t_0}\Omega_t\subset\mathrm{cl}(\Omega_{t_0})$ *holds for any t_0 with $0\leqq t_0<1$.*

(e) *There is an open convex cone Q of TD' containing R such that $\Omega_{t_1}\backslash\Omega_{t_0}$ is Q-flat on a neighborhood of any point of* $\mathrm{cl}(\Omega_1\backslash\Omega_0)$ *for any $0\leqq t_0\leqq t_1<1$, and that*

$$\{(z,\zeta)\in T^*X\,;\ z\in\mathrm{cl}(\Omega_1\backslash\Omega_0),\ \operatorname{Re}\langle\zeta, Q(z)\rangle<0\}\cap\mathrm{SS}(\mathcal{M}^\cdot)=\emptyset.$$

(f) $T^*_{\tilde{M}}X\cap\mathrm{cl}(\mathrm{SS}(\mathcal{M}^\cdot))=\emptyset$, *where* cl *denotes the closure in* T^*X.

Under these conditions $\Omega_t\cap D$ is open in D_G and

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G;\,D)}(\mathcal{M}^{\cdot},\boldsymbol{R}\Gamma_{\Omega_t\setminus\Omega_0}(\Omega_t\,;\,(\phi_G)_*\mathcal{B}\mathcal{O}))=0$$

holds for any $0\leqq t\leqq 1$.

PROOF. Condition (c) assures that $\boldsymbol{R}\Gamma_{\Omega_t\setminus\Omega_0}((\phi_G)_*\mathcal{B}\mathcal{O})$ is well-defined in the derived category of sheaves of $\mathcal{E}(G\,;\,D)$-modules. In view of (e) and (f) we can verify that for any point $\mathring{z}$ of $\mathrm{cl}(\Omega_1\setminus\Omega_0)$ there exist a proper convex closed cone G' of $\tilde{M}$ such that $G'\Supset G$, and a neighborhood V of $\mathring{z}$ in $\tilde{M}$, such that $G'\setminus\{0\}\subset Q(z)$ and that

$$\{(z,\zeta)\in T_z^*X\,;\ \mathrm{Re}\langle\zeta,G'\rangle\leqq 0,(\mathrm{Re}\,\zeta_1,\zeta')\neq 0\}\cap\mathrm{SS}(\mathcal{M}^{\cdot})=\emptyset$$

for any $z\in V$. Hence by virtue of the geometric arguments in [11, § 4] we have only to prove the following:

PROPOSITION 3.2. *Let G be a proper convex closed cone in $\tilde{M}$, and D a bounded G-round open set in X. Let G' be a proper convex closed cone of $\tilde{M}$ such that $G\Subset G'$ and* int $G'\neq\emptyset$ *(here* int *denotes the interior in $\tilde{M}$). Put $\Omega=\mathrm{int}\,G'$ and let f be a real valued linear function such that $f>0$ on $G'\setminus\{0\}$. Set $\omega=\{z\in\Omega\,;\,f(z)>1\}$ and assume that* $\mathrm{cl}(\Omega\setminus\omega)\subset D$. *Let $\mathcal{M}^{\cdot}$ be a bounded complex of free $\mathcal{E}(G\,;\,D)$-modules of finite rank satisfying* (f) *and*

$$\text{(e)}'\quad \{(z,\zeta)\,;\ z\in\mathrm{cl}(\Omega\setminus\omega),\mathrm{Re}\langle\zeta,G'\rangle\leqq 0,(\mathrm{Re}\,\zeta_1,\zeta')\neq 0\}\cap\mathrm{SS}(\mathcal{M}^{\cdot})=\emptyset\,.$$

Then we have

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G;\,D)}(\mathcal{M}^{\cdot},\boldsymbol{R}\Gamma_{\Omega\setminus\omega}((\phi_{G'})_*\mathcal{B}\mathcal{O}))=0\,.$$

PROOF. We extend f to a linear function $\tilde{f}$ on X by $\tilde{f}(z)=f(x_1,z')$ with $x_1=\mathrm{Re}\,z_1$. In view of (f) and (e)$'$ we can take a proper convex cone $\tilde{G}$ in X so that $\tilde{G}\cap\tilde{M}=G'$, $\mathrm{int}\,\tilde{G}\neq\emptyset$ (here int denotes the interior in X), and that

$$\{(z,\zeta)\,;\ z\in\tilde{G},\tilde{f}(z)\leqq 1,\mathrm{Re}\langle\zeta,G'\rangle\leqq 0,(\mathrm{Re}\,\zeta_1,\zeta')\neq 0\}\cap\mathrm{SS}(\mathcal{M}^{\cdot})=\emptyset\,.$$

Set $\tilde{\Omega}=\mathrm{int}\,\tilde{G}$, $\tilde{\omega}=\{z\in\tilde{\Omega}\,;\,\tilde{f}(z)>1\}$. Let us show

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G;\,D)}(\mathcal{M}^{\cdot},\boldsymbol{R}\Gamma_{\tilde{\Omega}\setminus\tilde{\omega}}((\phi_{G'})_*\mathcal{O}_X))=0\,. \tag{3.1}$$

Fix a point $\mathring{z}$ of $\tilde{\Omega}\setminus\tilde{\omega}$ and let V be a G'-open neighborhood of $\mathring{z}$ in X. Then there exist a point w of $\tilde{\Omega}\setminus\tilde{\omega}$ and a proper convex cone $\tilde{G}'$ of X such that

$$G'\subset\tilde{G}'\subset\tilde{G},\qquad \mathring{z}\in w+\mathrm{int}\,\tilde{G}'\subset V.$$

Note that

$$\{(z, \zeta)\,;\ z \in \tilde{\Omega} \backslash \tilde{\omega}, \operatorname{Re}\langle \zeta, \tilde{G}' \rangle \leqq 0, \zeta \neq 0\} \cap \mathrm{SS}(\mathcal{M}^{\cdot}) = \emptyset$$

since $\operatorname{Re}\langle \zeta, \tilde{G}' \rangle \leqq 0$ and $\zeta \neq 0$ imply $(\operatorname{Re}\zeta_1, \zeta') \neq 0$. Hence from Proposition 4.3.1 of [11]

$$\boldsymbol{R}\operatorname{Hom}_{\mathcal{E}(G;\,D)}(\mathcal{M}^{\cdot}, \boldsymbol{R}\Gamma_{\tilde{\Omega}'\backslash\tilde{\omega}}(\tilde{\Omega}'\,;\ (\phi_{G'})_* \mathcal{O}_X)) = 0$$

follows with $\tilde{\Omega}' = w + \operatorname{int} \tilde{G}'$. Consequently we get

$$\begin{aligned} &\boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G;\,D)}(\mathcal{M}^{\cdot}, \boldsymbol{R}\Gamma_{\tilde{\Omega}\backslash\tilde{\omega}}((\phi_{G'})_* \mathcal{O}_X))_{\hat{z}} \\ &= \varinjlim_{V} \boldsymbol{R}\operatorname{Hom}_{\mathcal{E}(G;\,D)}(\mathcal{M}^{\cdot}, \boldsymbol{R}\Gamma_{V\backslash\tilde{\omega}}(V\,;\ (\phi_{G'})_* \mathcal{O}_X)) = 0, \end{aligned}$$

where V runs on the system of neighborhoods of $\hat{z}$ in $X_{G'}$. Thus we have proved (3.1). Applying the functor $\boldsymbol{R}\Gamma_{\tilde{M}}$ to (3.1) (note that $\tilde{M}$ is closed in $X_{G'}$) we get

$$\begin{aligned} &\boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G;\,D)}(\mathcal{M}^{\cdot}, \boldsymbol{R}\Gamma_{\Omega\backslash\omega}((\phi_{G'})_* \mathcal{B}\mathcal{O})) \\ &\cong \boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G;\,D)}(\mathcal{M}^{\cdot}, \boldsymbol{R}\Gamma_{(\tilde{\Omega}\backslash\tilde{\omega})\cap\tilde{M}}((\phi_{G'})_* \mathcal{O}_X))[1] \\ &\cong \boldsymbol{R}\Gamma_{\tilde{M}}(\boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G;\,D)}(\mathcal{M}^{\cdot}, \boldsymbol{R}\Gamma_{\tilde{\Omega}\backslash\tilde{\omega}}((\phi_{G'})_* \mathcal{O}_X)))[1] = 0. \end{aligned}$$

This completes the proof of Proposition 3.2, and at the same time, the proof of Theorem 3.4.

Note that the arguments in this subsection also apply to the case where $\tilde{M} = \boldsymbol{R}^d \times \boldsymbol{C}^{n-d}$ with $d \geqq 2$.

3.3. Proof of Theorem 3.1

Let us begin with an expression of global sections of $\mathcal{C}_{M_+}$.

LEMMA 3.2. *Let U be an open set of $S_M^* \tilde{M}$ with proper convex fibers. Then we have*

$$\varinjlim_{\Omega, Z} H_Z^{\nu}(\Omega \cap \tilde{M}_+\,;\ \mathcal{B}\mathcal{O}) \cong \begin{cases} \Gamma(U\,;\ \mathcal{C}_{M_+}) & (\nu = n-1), \\ 0 & (\nu \neq n-1), \end{cases}$$

and $H^{\nu}(U\,;\ \mathcal{C}_{M_+}) = 0$ for $\nu \geqq 1$; here Ω runs on the system of open neighborhoods of U in $\tilde{M}$, and Z runs on the closed sets in $\tilde{M}$ such that $Z \supset \pi_{M/\tilde{M}}(U)$ and that $(\Omega \backslash Z) \cup U^a$ is a neighborhood of U^a in the comonoidal transform of $\tilde{M}$ with center M.

PROOF. We apply Proposition 1.2.4 of [25] (it is stated without proof there, while it follows immediately from Lemma 4.1.10 of Kataoka [12]).

Namely we get

$$\varinjlim_{\Omega, Z} H_Z^\nu(\Omega \cap \tilde{M}_+ ; \mathscr{B}\mathcal{O}) = \varinjlim_{\Omega, Z} H_Z^\nu(\Omega ; \mathscr{B}\mathcal{O}_{\tilde{M}_+}) \cong H^{\nu-n+1}(U ; \mathcal{C}_{M_+}).$$

Hence the fact that the flabby dimension of $\mathscr{B}\mathcal{O}_{M_+}$ is $n-1$ implies the lemma.

LEMMA 3.3. *Let x^* be a point of L_0. Then we have $\mathscr{H}_{L_0}^\nu(\mathcal{C}_{M_+})=0$ for $\nu \geqq 1$ and*

$$\varinjlim H_{Z\cap(\Omega\setminus\Omega_+)}^\nu(\Omega \cap \tilde{M}_+ ; \mathscr{B}\mathcal{O}) \cong \begin{cases} \Gamma_{L_0}(\mathcal{C}_{M_+})_{x^*} & (\nu = n-1), \\ 0 & (\nu \neq n-1), \end{cases}$$

where the inductive limit is taken with respect to the family of (Ω, Ω_+, Z) satisfying the following: Ω is an open neighborhood of $\pi_{M/\tilde{M}}(x^*)$ in $\tilde{M}$, Ω_+ is an open neighborhood of $\Omega \cap M_+$ in $\tilde{M}$, and Z is a closed set in $\tilde{M}$ containing $\Omega \cap M$ such that $S_M^*\tilde{M} \cup (\Omega \setminus Z)$ is a neighborhood of $(x^*)^a$ in the comonoidal transform of $\tilde{M}$ with center M.

PROOF. Let $U \subset S_M^*\tilde{M}$ be an open neighborhood of x^* with proper convex fibers. Let Ω and Ω_+ be open neighborhoods in $\tilde{M}$ of U and of $U \cap M_+$ respectively, and let Z be a closed subset of $\tilde{M}$ containing U such that $(\Omega \setminus Z) \cup U^a$ is a neighborhood of U^a. Taking inductive limits with respect to the family of (U, Ω, Ω_+, Z) satisfying the above conditions, we get a commutative diagram

$$\begin{array}{ccccccc} \cdots \to \varinjlim H_Z^{\nu-1}(\Omega_+, \mathscr{B}\mathcal{O}) & \to & \varinjlim H_{Z\cap(\Omega\setminus\Omega_+)}^\nu(\Omega ; \mathscr{B}\mathcal{O}_{\tilde{M}_+}) & \to & \varinjlim H_Z^\nu(\Omega ; \mathscr{B}\mathcal{O}_{\tilde{M}_+}) \to \cdots \\ \downarrow\wr & & \downarrow & & \downarrow\wr \\ \cdots \to \varinjlim H^{\nu-n}(U_+ ; \mathcal{C}_{M_+}) & \to & \varinjlim H_{L_0}^{\nu-n+1}(U ; \mathcal{C}_{M_+}) & \to & \varinjlim H^{\nu-n+1}(U ; \mathcal{C}_{M_+}) \to \cdots \end{array}$$

with exact rows. In view of Lemma 3.2 the right and left vertical arrows are isomorphisms. Hence we get

$$\varinjlim H_{Z\cap(\Omega\setminus\Omega_+)}^\nu(\Omega \cap \tilde{M}_+ ; \mathscr{B}\mathcal{O}) \cong \varinjlim H_{L_0}^{\nu-n+1}(U ; \mathcal{C}_{M_+}) \cong \mathscr{H}_{L_0}^{\nu-n+1}(\mathcal{C}_{M_+})_{x^*}.$$

When U runs on the system of open neighborhoods of x^*, the family of (Ω, Ω_+, Z) such that (U, Ω, Ω_+, Z) satisfies the conditions above is equivalent to the family described in the statement of the lemma. Hence the above isomorphism proves Lemma 3.3.

In order to prove Theorem 3.1 we may assume $X=\boldsymbol{C}^n$, $Y=\{0\}\times\boldsymbol{C}^{n-1}$, $\tilde{M}=\boldsymbol{R}\times\boldsymbol{C}^{n-1}$, $\tilde{M}_+=\boldsymbol{R}_+\times\boldsymbol{C}^{n-1}$, etc. Let G_0 be a proper convex closed cone in Y and let D be a G_0-round open set in X. Then by virtue of Proposition 3.1 and Lemma 3.2, the sheaf $\mathcal{C}_{M_+}$ restricted to

$$U(G_0, D)=\{(x, \sqrt{-1}\,\xi'\infty)\in S^*_M\tilde{M}\ ;\ x\in D, \mathrm{Re}\langle\sqrt{-1}\,\xi', w'\rangle<0$$
$$\text{for any } (0, w')\in G_0\backslash\{0\}\}$$

becomes an $\mathcal{E}(G_0\,;\,D)$-module. Note that there is a natural homomorphism

$$\mathcal{D}^\infty(D)=\mathcal{E}(\{0\}\,;\,D)\longrightarrow\mathcal{E}(G_0\,;\,D),$$

and the action of $\mathcal{E}(G_0, D)$ on $\mathcal{C}_{M_+}$ is compatible with that of $\mathcal{D}^\infty(D)$.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module satisfying the assumptions of Theorem 3.1. Then there is a free resolution

$$0\longleftarrow\mathcal{M}\longleftarrow\mathcal{D}_X^{N_0}\xleftarrow{P_1}\mathcal{D}_X^{N_1}\longleftarrow\cdots\xleftarrow{P_r}\mathcal{D}_X^{N_r}\longleftarrow 0$$

on a neighborhood D of $\pi(x^*)$ in X with matrices P_j of sections of $\mathcal{D}_X$ on D. Let G_0 be a proper convex closed cone such that $x^*\in U(G_0\,;\,D)$ (under the assumptions of Theorem 3.1 we can of course take $G_0=\{0\}$). We may assume that D is G_0-round. Let $\mathcal{M}^\cdot$ be the complex

$$0\longleftarrow\mathcal{E}(G_0\,;\,D)^{N_0}\xleftarrow{P_1}\mathcal{E}(G_0\,;\,D)^{N_1}\longleftarrow\cdots\xleftarrow{P_r}\mathcal{E}(G_0\,;\,D)^{N_r}\longleftarrow 0,$$

where P_j are regarded as matrices of elements of $\mathcal{E}(G_0\,;D)$ by the natural homomorphism $\mathcal{D}_X(D)\to\mathcal{E}(G_0\,;\,D)$. Note that $\mathrm{SS}(\mathcal{M}^\cdot)\subset\mathrm{SS}(\mathcal{M})$ holds since $\mathcal{E}^R$ is flat over $\mathcal{E}_X$. Hence Theorem 3.1 follows from the following more general result.

THEOREM 3.5. *Let $X, \tilde{M}, M, Y$ be Euclidean spaces as above and let x^* be a point of L_0. Let G_0 be a proper convex closed cone in Y and let D be a G_0-round open set in X. Suppose $x^*\in U(G_0\,;\,D)$ and that a bounded complex $\mathcal{M}^\cdot$ of free $\mathcal{E}(G_0\,;\,D)$-modules of finite rank satisfies the following conditions* (*cf. Theorem* 3.1):

(C.1) $\quad \mathring{T}^*_Y X\cap\mathrm{cl}(\mathrm{SS}(\mathcal{M}^\cdot)\cap T^*X|_{\tilde{M}_+})=\emptyset,$

(C.2) $\quad H(\theta_0)\notin C_{y^*}(\mathrm{SS}(\mathcal{M}^\cdot)\cap T^*X|_{\tilde{M}_+}\,;\,T^*_MX)$ *for any* $y^*\in\rho^{-1}(x^*)\cap T^*_MX.$

(C.3) $\quad \rho^{-1}(x^*)\cap\mathrm{cl}(\mathrm{SS}(\mathcal{M}^\cdot)\cap T^*X|_{\tilde{M}_+})\subset\{(\zeta_1, x^*)\,;\,\mathrm{Re}\,\zeta_1\geqq 0\}.$

Uuder these assumptions we have

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{E}(G_0;\,D)}(\mathcal{M}^\cdot\,;\,\Gamma_{L_0}(\mathcal{C}_{M_+}))_{x^*}=0.$$

PROOF. We may assume $x^*=(0, -\sqrt{-1}\,dx_n)$. By (C.1)-(C.3) there are C_0 and c_0 with $C_0>4$ and $0<c_0<1/4$ such that

$$\{z\in X\,;\,|z|\leqq 4nc_0\}\subset D, \tag{3.2}$$

$$G_0\setminus\{0\}\subset\{(0,z')\in Y\,;\ \operatorname{Im} z_n<-c_0(|\operatorname{Re} z'|+|\operatorname{Im} z''|)\},\tag{3.3}$$

$$\{(x_1,z',\zeta)\in T^*X|_{\tilde{M}_+}\,;\ 0<x_1\leqq c_0,\ |z'|\leqq c_0,\ \zeta\in \boldsymbol{C}^n\setminus\{0\},\ |\zeta_1|\geqq C_0|\zeta'|\}\cap \mathrm{SS}(\mathcal{M}^\cdot)=\emptyset,\tag{3.4}$$

$$\begin{aligned}\{(x_1,z',\zeta)\,;\ &0<x_1\leqq c_0,\ |z'|\leqq c_0,\ |\operatorname{Re}\zeta'|\leqq c_0|\operatorname{Im}\zeta_n|,\\ &|\operatorname{Im}\zeta''|\leqq c_0|\operatorname{Im}\zeta_n|,\ \operatorname{Im}\zeta_n<0,\\ &\operatorname{Re}\zeta_1<-C_0(|\operatorname{Re}\zeta'|+|\operatorname{Im} z'||\operatorname{Im}\zeta_n|)\}\cap\mathrm{SS}(\mathcal{M}^\cdot)=\emptyset,\end{aligned}\tag{3.5}$$

where we use the notation $z=(z_1,z')$, $\zeta=(\zeta_1,\zeta')$, $z'=(z'',z_n)$, etc. Let a,b,C be parameters such that

$$0<b<\frac{c_0}{8},\qquad C\geqq 2C_0,\qquad 0<a\leqq a_0=a_0(b)=\frac{bc_0}{8C_0},\tag{3.6}$$

and put

$$Z=Z(b)=\left\{(x_1,z')\in\tilde{M}\,;\ y_n\geqq\frac{b}{a_0}(x_1+a_0)|y''|\right\},$$

$$\begin{aligned}\Omega=\Omega(a,C)=\Big\{(x_1,z')\,;\ &x_1>0,\ x_1+a>\frac{1}{4C_0}\Big(|x'|+\frac{1}{c_0}y_n\Big),\\ &x_1+a>C\Big(|y''|+\frac{1}{c_0}y_n\Big)\Big\}\end{aligned}$$

with the notation $z=x+\sqrt{-1}\,y$, $y''=(y_2,\cdots,y_{n-1})$. Note that

$$\left\{(x_1,z')\,;\ x_1>0,\ |x'|<2C_0a,\ |y'|<\frac{c_0}{2C}a\right\}\subset\Omega,\tag{3.7}$$

$$\begin{aligned}&\left\{(x_1,z')\in\Omega\,;\ x_1\leqq a,\ \ y_n>-\frac{c_0a}{C}\right\}\\ &\quad\subset\left\{(x_1,z')\,;\ 0<x_1\leqq a,\ |x'|<8C_0a,\ |y''|<\frac{3a}{C},\ |y_n|<\frac{2c_0a}{C}\right\}.\end{aligned}\tag{3.8}$$

We denote by $H(a,C)$ the set of C^1-functions h on $[0,a]$ such that

$$\begin{cases} h(0)=0,\quad 0<h(x_1)\leqq C_0 & \text{for}\quad 0<x_1\leqq a,\\ 0\leqq h'(x_1)\leqq C_0 & \text{for}\quad 0\leqq x_1\leqq a,\quad 2\dfrac{c_0a}{C}<h(a).\end{cases}\tag{3.9}$$

For the sake of the simplicity of the notation, we use the convention that $h(x_1)=\infty$ for $x_1>a$. For $h\in H(a,C)$ put

$$\omega=\omega(a,b,C,h)=\left\{(x_1,z')\in\Omega(a,C)\,;\ y_n<h(x_1)\exp\left(\frac{8C_0}{b}x_1\right)\right.$$
$$\left.\text{or}\quad y_n<\frac{b}{a_0}(x_1+a_0)|y''|\right\}.$$

(Note that ω is open.) Let $\mathcal{J}$ be the set of (a,b,C,h) with a,b,C satisfying (3.6) and $h\in H(a,C)$. We define an order $\geqq$ in $\mathcal{J}$ by

$$(a_1,b_1,C_1,h_1)\geqq(a_2,b_2,C_2,h_2)$$

if and only if $a_1\leqq a_2, b_1\leqq b_2, C_1\geqq C_2, h_j\in H(a_j,C_j)$, and

$$\frac{1}{b_1}h_1(x_1)\exp\left(\frac{8C_0}{b_1}x_1\right)\leqq\frac{1}{b_2}h_2(x_1)\exp\left(\frac{8C_0}{b_2}x_1\right)\quad\text{for}\quad 0\leqq x_1\leqq a_1,$$

$$\frac{c_0}{C_1}(x_1+a_1)\leqq h_2(x_1)\exp\left(\frac{8C_0}{b_2}x_1\right)\quad\text{for}\quad a_1<x_1\leqq a_2.$$

Note that in this case $\Omega(a_1,C_1)\subset\Omega(a_2,C_2)$, $\omega(a_1,b_1,C_1,h_1)\subset\omega(a_2,b_2,C_2,h_2)$. We have $h(x_1)\exp(8C_0x_1/b)\leqq 2h(x_1)$ for $x_1\leqq a$ and

$$\Omega\backslash\omega=Z\cap\left(\left\{(x_1,z')\in\Omega\,;\ x_1<a,\ \ y_n>-\frac{c_0a}{C}\right\}\Big\backslash\left\{(x_1,z')\in\Omega\,;\right.\right.$$
$$\left.\left.x_1<a,\ |y_n|<h(x_1)\exp\left(\frac{8C_0}{b}x_1\right),\ |y''|<\frac{1}{b}h(x_1)\exp\left(\frac{8C_0}{b}x_1\right)\right\}\right).$$

Hence by virtue of (3.7), (3.8), the family $\{\Omega(a,C)\backslash\omega(a,b,C,h)\}$ with $(a,b,C,h)\in\mathcal{J}$ (with respect to the order defined above) is equivalent to the family $\{Z\cap(\Omega\backslash\Omega_+)\cap\tilde{M}_+\}$ with (Ω,Ω_+,Z) described in Lemma 3.3. In view of (3.3), (3.6), (3.8), we can verify that $\Omega=\Omega(a,C)$ and $\omega=\omega(a,b,C,h)$ are G_0-open and $\Omega\backslash\omega\subset D$. Hence $\boldsymbol{R}\Gamma_{\Omega\backslash\omega}(\Omega\,;\,(\phi_{G_0})_*\mathcal{BO})$ is well-defined as an $\mathcal{E}(G_0\,;\,D)$-module. By virtue of Lemma 3.3 we have

$$\text{(3.10)}\qquad \boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G_0;\,D)}(\mathcal{M}^{\cdot},\Gamma_{L_0}(\mathcal{C}_{M_+}))_{x^*}$$
$$\cong\varinjlim_{\mathcal{J}}\boldsymbol{R}\mathrm{Hom}_{\mathcal{E}(G_0;\,D)}(\mathcal{M}^{\cdot},\boldsymbol{R}\Gamma_{\Omega\backslash\omega}(\Omega\,;\,(\phi_{G_0})_*\mathcal{BO}))[n-1].$$

In the sequel we shall prove

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G_0;\,D)}(\mathcal{M}^{\cdot},\boldsymbol{R}\Gamma_{\Omega\backslash\omega}(\Omega\,;\,(\phi_{G_0})_*\mathcal{BO}))=0$$

for any $(a,b,C,h)\in\mathcal{J}$.

Let us fix $(a,b,C,h)\in\mathcal{J}$ and take $t_0>0$ such that $0<t_0\leqq a$. Put

$$\Omega_{t_0}=\left\{(x_1,z')\in\Omega\,;\ x_1>\frac{t_0}{4C_0b}y_n+t_0,\ \ y_n+c_0(|x'|+|y''|)<\frac{1}{2}c_0^2\right\}$$

and $\omega_{t_0}=\Omega_{t_0}\cap\omega$, and for $t_0\leqq t\leqq a$,

$$\Omega_{t_0,t}=\Omega_t\cup\omega_{t_0}.$$

Now our aim is to show

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{E}(G_0,D)}(\mathcal{M}^{\cdot}, \boldsymbol{R}\Gamma_{\Omega_{t_0}\setminus\omega_{t_0}}(\Omega_{t_0}\,;\,(\phi_{G_0})_*\mathcal{B}\mathcal{O}))=0 \tag{3.11}$$

applying Theorem 3.4. (And then we shall take the inductive limit as $t_0\to+0$.) It is easy to see

$$\Omega_{t_0,t_0}=\Omega_{t_0}, \qquad \Omega_{t_0,a}=\omega_{t_0},$$

$$\Omega_{t_0,t_1}=\bigcup_{t>t_1}\Omega_{t_0,t} \qquad \text{for} \quad t_1\geqq t_0,$$

$$\mathrm{cl}(\Omega_{t_0,t_1})\supset\bigcap_{t_0<t<t_1}\Omega_{t_0,t} \qquad \text{for} \quad t_0<t_1.$$

Put

$$D(a)=\{z\in X\,;\ 0<x_1<4a,\ |y_1|<c_0,\ \ y_n>-c_0^2,\ y_n+c_0(|x'|+|y''|)<c_0^2\},$$

$$V=\left\{(x_1,z')\in\tilde{M}\,;\ y_n<\frac{4c_0}{C_0}(x_1-2a)\right\},$$

and $D'(a)=D(a)\cap\tilde{M}$. Then $D(a)$ is G_0-round and contained in $\{|z|<4nc_0\}\subset D$, and V is convex and G_0-open. Note that

$$\Omega_{t_0}\setminus\omega_{t_0}\subset D'(a), \qquad V\cap(\Omega_{t_0}\setminus\omega_{t_0})=\emptyset$$

in view of (3.8). We can easily verify

$$\partial D'(a)\cap\mathrm{cl}(\Omega_{t_0})\subset V.$$

We define open sets Q, R of $TD'(a)$ by

$$Q(z)=\left\{w=(u_1,w')\in\boldsymbol{R}\times\boldsymbol{C}^{n-1}\,;\ |u'|+\frac{v_n}{c_0}<\min\left(4C_0u_1,\ \frac{16C_0}{t_0}u_1\right),\right.$$

$$\left.|v''|+\frac{v_n}{c_0}<\min\left(\frac{4C_0|y'|}{c_0}u_1,\ \frac{1}{C}u_1\right)\right\},$$

$$R(z)=Q(z)\cap\{(u_1,w')\,;\ v_n+c_0(|u'|+|v''|)<0\}$$

for $z=(x_1,z')\in D'(a)$ with the notation $w=(u_1,w')$, $w_j=u_j+\sqrt{-1}\,v_j$, $u'=(u_2,\cdots,u_n)$, $v''=(v_2,\cdots,v_{n-1})$. It is obvious that Q and R have convex fibers and that $R(z)\supset G_0\setminus\{0\}$ for any $z\in D'(a)$.

Let us verify that $\Omega_{t_0,t}$ is R-flat on $D'(a)$ and that $\Omega_{t_0,t_1}\setminus\Omega_{t_0,t_2}$ is Q-flat on $D'(a)$. First note that

$$\Omega_{t_0, t}=\Omega^0\cap\Omega^1\cap(\Omega^2\cup\Omega^3\cup\Omega^4)$$

with

$$\Omega^0=\left\{(x_1, z')\in\Omega\,;\ x_1>\frac{t_0}{4C_0b}y_n+t_0\right\},$$

$$\Omega^1=\left\{(x_1, z')\,;\ y_n+c_0(|x'|+|y''|)<\frac{1}{2}c_0^2\right\},$$

$$\Omega^2=\left\{(x_1, z')\,;\ x_1>\frac{t}{4C_0b}y_n+t\right\},$$

$$\Omega^3=\left\{(x_1, z')\,;\ y_n<\frac{b}{a_0}(x_1+a_0)|y''|\right\},$$

$$\Omega^4=\left\{(x_1, z')\,;\ y_n<h(x_1)\exp\left(\frac{8C_0}{b}x_1\right)\right\}.$$

It is easy to see that Ω^0 is Q-flat on $D'(a)$, Ω^1 is R-flat on $D'(a)$, and that Ω^2 is Q-flat on $D'(a)$ (note that $x_1>0$ on $D'(a)$).

Let us verify that Ω^3 is Q-flat on $\bar{\Omega}_{t_0}\cap D'(a)$. Assume $z=(x_1, z')\in\partial\Omega^3\cap\bar{\Omega}_{t_0}\cap D'(a)$ and $w=(u_1, w')\in Q(z)$. If $0\leqq u_1<a_0$, we have

$$y_n+v_n<\frac{b}{a_0}(x_1+a_0)|y''|+4C_0|y'|u_1-c_0|v''|$$

$$\leqq\frac{b}{a_0}(x_1+a_0+u_1)(|y''|-|v''|)\leqq\frac{b}{a_0}(x_1+a_0+u_1)|y''+v''|$$

since

$$4C_0|y'|\leqq 4C_0(5b+1)|y''|<\frac{b}{a_0}|y''|,\quad \frac{b}{a_0}(x_1+a_0+u_1)<c_0$$

in view of (3.6). If $u_1<0$, we have

$$y_n+v_n<\frac{b}{a_0}(x_1+a_0)|y''|+\frac{c_0}{C}u_1-c_0|v''|\leqq\frac{b}{a_0}(x_1+u_1+a_0)|y''+v''|$$

since $(x_1, z')\in\bar{\Omega}$ and $y_n=b(x_1+a_0)|y''|/a_0$ imply

$$\frac{b}{a_0}|y''|\leqq\frac{b}{a_0}\frac{4a+a_0}{C}<\frac{c_0}{C}.$$

Next let us verify that $\Omega^3\cup\Omega^4$ is Q-flat on $D'(a)\cap\Omega_{t_0}$. Since Ω^3 is Q-flat, it suffices to show that Ω^4 is Q-flat on

$$\Omega^5=\left\{(x_1, z')\in D'(a)\,;\ y_n>\frac{2b}{3}|y''|\right\}.$$

Put $f=-y_n+h(x_1)\exp(8C_0x_1/b)$ and

$$\partial f=\frac{1}{2}\frac{\partial f}{\partial x_1}dz_1+\sum_{j=2}^{n}\frac{\partial f}{\partial z_j}dz_j.$$

Then we have only to show $\mathrm{Re}\langle\partial f(z), w\rangle>0$ for any $z\in\partial\Omega^4\cap\Omega^5\cap\Omega$ and $w\in Q(z)$ (cf. [11, §3]). Note that in this case $x_1<a$ in view of (3.8) and (3.9). If $u_1\geqq 0$, we have

$$\begin{aligned}2\,\mathrm{Re}\langle\partial f(z), w\rangle&=\Big(h'(x_1)+\frac{8C_0}{b}h(x_1)\Big)\exp\Big(\frac{8C_0}{b}x_1\Big)u_1-v_n\\&>\frac{8C_0}{b}h(x_1)u_1-(4C_0|y'|u_1-c_0|v''|)\\&\geqq\Big(\frac{8C_0}{b}y_n-4C_0|y'|\Big)u_1\geqq 0\end{aligned}$$

since $z\in\Omega^5$. If $u_1<0$, we have

$$2\mathrm{Re}\langle\partial f(z), w\rangle>2\Big(h'(x_1)+\frac{8C_0}{b}h(x_1)\Big)u_1-\frac{16C_0c_0}{t_0}u_1\geqq 0$$

in view of (3.6) and (3.9). Thus we have proved that $\Omega_{t_0,t}$ is R-flat on $D'(a)$. Since

$$\Omega_{t_0,t_1}\backslash\Omega_{t_0,t_2}\subset\Omega_{t_0}\backslash\omega_{t_0}\Subset\Omega^1,$$

it follows that $\Omega_{t_0,t_1}\backslash\Omega_{t_0,t_2}$ is Q-flat on $D'(a)$.

Let us show

$$\begin{aligned}Q(z)^\circ&=\{\zeta=(\xi_1,\zeta')\in\boldsymbol{R}\times\boldsymbol{C}^{n-1};\ \mathrm{Re}\langle\zeta, w\rangle<0 \text{ for any } w\in Q(z)\}\\&\subset\{\zeta=(\xi_1,\zeta');\ |\xi'|\leqq-c_0\eta_n,\ |\eta''|\leqq-c_0\eta_n,\ \xi_1\leqq-2C_0(|\xi'|-|y'|\eta_n)\}=\varDelta\end{aligned}$$

for any $z\in\mathrm{cl}(\Omega_{t_0}\backslash\omega_{t_0})$. It suffices to show $Q(z)\supset\varDelta^\circ$. Suppose $w\notin Q(z)$. Then at least one of the inequalities appearing in the definition of Q fails. First assume

$$|u'|+\frac{v_n}{c_0}\geqq 4C_0u_1.$$

If $u_1\geqq 0$, take $\zeta\in\varDelta$ so that

$$\begin{aligned}&\eta_n<0,\ \ \langle u',\xi'\rangle=|u'||\xi'|,\ \ |\xi'|=c_0|\eta_n|,\ \ \eta''=0,\\&\xi_1=-2C_0(|\xi'|+|y'||\eta_n|).\end{aligned}$$

Then we get

$$\begin{aligned}\operatorname{Re}\langle w,\zeta\rangle &= u_1\xi_1+\langle u',\xi'\rangle-\langle v'',\eta''\rangle-v_n\eta_n\\ &\geqq -2C_0(|\xi'|+|y'||\eta_n|)u_1+(4C_0c_0u_1-c_0|u'|)|\eta_n|+|u'||\xi'|\\ &=2C_0(c_0-|y'|)|\eta_n|u_1\geqq 0.\end{aligned}$$

If $u_1<0$, we can choose $\zeta\in\Delta$ so that $\operatorname{Re}\langle w,\zeta\rangle>0$ since $-\xi_1$ can be arbitrarily large for $\zeta\in\Delta$.

In the same way we can choose $\zeta\in\Delta$ so that $\operatorname{Re}\langle w,\zeta\rangle\geqq 0$ if one of the other inequalities fails since

$$\frac{16C_0c_0}{t_0}-2C_0(c_0+|y'|)\geqq 0,\qquad \frac{c_0}{C}-2C_0|y'|\geqq 0$$

for $z\in\mathrm{cl}(\Omega_{t_0}\backslash\omega_{t_0})$. Thus we have $Q(z)\supset\Delta^\circ$, and consequently $Q(z)^\circ\subset\Delta$ for $z\in\mathrm{cl}(\Omega_{t_0}\backslash\omega_{t_0})$.

Hence in view of (3.4) and (3.5) we obtain (3.11) applying Theorem 3.4 to the family $\{\Omega_{t_0,t}\}_t$. Now set

$$\mathcal{F}^\cdot=\boldsymbol{R}\,\mathcal{H}om_{\mathcal{E}(G_0;\,D)}(\mathcal{M}^\cdot,\boldsymbol{R}\Gamma_{\Omega\backslash\omega}((\phi_{G_0})_*\mathcal{B}\mathcal{O})).$$

Then we have

$$\boldsymbol{R}\Gamma(\Omega_t;\,\mathcal{F}^\cdot)=\boldsymbol{R}\,\mathcal{H}om_{\mathcal{E}(G_0;\,D)}(\mathcal{M}^\cdot,\boldsymbol{R}\Gamma_{\Omega_t\backslash\omega_t}(\Omega_t;\,(\phi_{G_0})_*\mathcal{B}\mathcal{O}))=0$$

for any $t>0$. In particular $\{H^{\nu-1}(\Omega_t;\,\mathcal{F}^\cdot)\}_t$ satisfies the Mittag-Leffler condition. Hence by Lemma 4.2.5 of [11] we get

$$\begin{aligned}&H^\nu(\boldsymbol{R}\operatorname{Hom}_{\mathcal{E}(G_0;\,D)}(\mathcal{M}^\cdot,\boldsymbol{R}\Gamma_{\Omega\backslash\omega}(\Omega;\,(\phi_{G_0})_*\mathcal{B}\mathcal{O})))\\ &=H^\nu(\Omega;\,\mathcal{F}^\cdot)=\varprojlim_{t\to+0}H^\nu(\Omega_t;\,\mathcal{F}^\cdot)=0\end{aligned}$$

for any ν. In view of (3.10), this completes the proof of Theorem 3.5, and at the same time, the proof of Theorem 3.1.

Notes. (i) Theorems 3.2 and 3.3 are applied to the continuation problem of real analytic solutions of systems and single Fuchsian partial differential equations in our forthcoming paper [34], where results of Kaneko [2] are extended. For this purpose, boundary value problems for Fuchsian equations are also formulated there.

(ii) Professor Schapira has kindly suggested to us that Theorem 3.1 might be proved more neatly by using the machinery of the recent theory of Kashiwara-Schapira [32].

References

[1] Hartshorne, R., Residues and duality, Lecture Notes in Math. Vol. 20, Springer Verlag, Berlin-Heidelberg-New York, 1966.

[2] Kaneko, A., Singular spectrum of boundary values of solutions of partial differential equations with real analytic coefficients, Sci. Papers College Gen. Ed. Univ. Tokyo **25** (1975), 59-68.

[3] Kaneko, A., Introduction to Hyperfunctions. I, University of Tokyo Press, Tokyo, 1980 (in Japanese).

[4] Kaneko, A., Introduction to Hyperfunctions. II, University of Tokyo Press, Tokyo, 1982 (in Japanese).

[5] Kashiwara, M., Systems of Microdifferential Equations, Birkhäuser, Boston-Basel-Stuttgart, 1983.

[6] Kashiwara, M. and T. Kawai, On the boundary value problem for elliptic system of linear differential equations. I, Proc. Japan Acad. **48** (1972), 712-715.

[7] Kashiwara, M. and T. Kawai, On the boundary value problem for elliptic system of linear differential equations. II, Proc. Japan Acad. **49** (1973), 164-168.

[8] Kashiwara, M. and T. Kawai, On holonomic systems of microdifferential equations. III, Publ. Res. Inst. Math. Sci. **17** (1981), 813-979.

[9] Kashiwara, M. and Y. Laurent, Théorèmes d'annulation et deuxième microlocalisation, Prépublications, Univ. Paris-Sud, Orsay, 1983.

[10] Kashiwara, M. and T. Oshima, Systems of differential equations with regular singularities and their boundary value problems, Ann. of Math. **106** (1977), 145-200.

[11] Kashiwara, M. and P. Schapira, Micro-hyperbolic systems, Acta Math. **142** (1979), 1-55.

[12] Kataoka, K., On the theory of Radon transformations of hyperfunctions, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), 331-413.

[13] Kataoka, K., Micro-local theory of boundary value problems I, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 355-399.

[14] Kataoka, K., Micro-local theory of boundary value problems II, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), 31-56.

[15] Komatsu, H., A local version of Bochner's tube theorem, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **19** (1972), 201-214.

[16] Komatsu, H., An introduction to the theory of hyperfunctions, Lecture Notes in Math. Vol. 287, Springer Verlag, Berlin-Heidelberg-New York, 1973, 3-40.

[17] Komatsu, H. and T. Kawai, Boundary values of hyperfunction solutions of linear partial differential equations, Publ. Res. Inst. Math. Sci. **7** (1971), 95-104.

[18] Martineau, A., Le "edge of the wedge theorem" en théorie des hyperfonctions de Sato, Proceedings of the International Conference on Functional Analysis and Related Topics, University of Tokyo Press, Tokyo, 1970, 95-106.

[19] Morimoto, M., Sur la décomposition du faisceau des germes de singularités d'hyperfonctions, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **17** (1970), 215-239.

[20] Morimoto, M., Edge of the wedge theorem and hyperfunction, Lecture Notes in Math. Vol. 287, Springer Verlag, Berlin-Heidelberg-New York, 1973, 41-81.

[21] Ôaku, T., F-mild hyperfunctions and Fuchsian partial differential equations, Advanced Studies in Pure Math. **4** (1984), 223-242.

[22] Ôaku, T., Microlocal boundary value problem for Fuchsian operators I, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **32** (1985), 287-317.

[23] Ôaku, T., A new formulation of local boundary value problem in the framework of hyperfunctions. I, Proc. Japan Acad. **60** (1984), 283-286.

[24] Sato, M. and M. Kashiwara, The determinant of matrices of pseudodifferential

operators, Proc. Japan Acad. **51** (1975), 17-19.

[25] Sato, M., Kawai, T. and T. Kashiwara, Microfunctions and pseudodifferential equations, Lecture Notes in Math. Vol. 287, Springer Verlag, Berlin-Heidelberg-New York, 1973, 265-529.

[26] Schapira, P., Problème de Dirichlet et solutions hyperfonctions des équations elliptiques, Boll. Un. Mat. Ital. (Serie 4) **2** (1969), 367-372.

[27] Schapira, P., Propagation at the boundary and reflection of analytic singularities of solutions of linear partial differential equations I, Publ. Res. Inst. Math. Sci. **12** Suppl. (1977), 441-453.

[28] Sjöstrand, J., Analytic singularities and microhyperbolic boundary value problems, Math. Ann. **254** (1980), 211-256.

[29] Tahara, H., Fuchsian type equations and Fuchsian hyperbolic equations, Japan. J. Math. **5** (1979), 245-347.

[30] Tajima, S., Analyse microlocale sur les variétés de Cauchy-Riemann et problème du prolongement des solutions holomorphes des équations aux dérivées partielles, Publ. Res. Inst. Math. Sci. **18** (1982), 911-945.

[31] Bony, J.M. and P. Schapira, Solutions analytiques et hyperfonctions du problème de Cauchy pour les opérateurs hyperboliques non stricts, Astérisque Vol. 2 et 3 Soc. Math. France, Paris, 1973, 108-116.

[32] Kashiwara, M. and P. Schapira, Microlocal study of sheaves, Astérisque Vol. 128, Soc. Math. France, Paris, 1985, 1-235.

[33] Kataoka, K., A microlocal approach to general boundary value problems, Publ. Res. Inst. Math. Sci. **12** Suppl (1977), 147-153.

[34] Ôaku, T., Removable singularities of solutions of linear partial differential equations —Systems and Fuchsian equations—, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986) (to appear).

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