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On removable singularities of stationary harmonic maps

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§ 0. Introduction

Let M and N be Riemannian manifolds of dimension m and n respectively. For each smooth map $u: M \to N$ the energy functional $E(u)$ of u is defined by

$$E(u)=\int_M |du|^2 dV. \tag{0.1}$$

A smooth map u of M into N is called a smooth *harmonic map* if u is a critical point of the energy functional E, that is, for every smooth one-parameter family u_t of $C^\infty(M, N)$ with $u_0=u$,

$$\frac{d}{dt}E(u_t)\Big|_{t=0}=0.$$

The Euler-Lagrange equation for the functional E is written as

$$\Delta u+\sum_{i,j} g^{ij}A_u(D_i u, D_j u)=0. \tag{0.2}$$

This is a nonlinear elliptic system for the smooth maps of M into N.

Among the problems on harmonic maps the existence problem is the most important one. It is the problem on the solvability of equation (0.2). The standard approach in analysis to solve nonlinear equations such as (0.2) is divided into two steps. The first step is the construction of 'weak solutions' for the equation. The second step is the regularity problem for the weak solutions, that is, to discuss whether the weak solutions are actually smooth solutions. Usually, weak solutions are defined in the space of distributions. For equation (0.2), a map $u: M \to N$ is called a weak solution (or *weakly harmonic map*) if u belongs to the space $H^{1,2}(M, N) \cap L^\infty(M, N)$ of Sobolev maps having bounded image and satisfies (0.2) in distribution sense. As for the regularity problem for the weakly harmonic maps, various results have been known (see [1], [5]).

For instance, Hildebrandt-Kaul-Widman [5] proved that a weakly harmonic map with sufficiently small oscillation is a smooth harmonic map. As a corollary, a continuous weakly harmonic map is smooth.

It seems impossible to define in general a *critical* point of E as a functional on $H^{1,2}(M, N) \cap L^{\infty}(M, N)$, while a critical point of E as a functional on $C^{\infty}(M, N)$ is well defined. Since smooth harmonic maps are defined to be critical points of E as a functional on $C^{\infty}(M, N)$, it is natural to consider weakly harmonic maps which are also, *in some sense*, critical points of E as a functional on $H^{1,2}(M, N) \cap L^{\infty}(M, N)$. One example is the case when u is an *energy minimizing map*, that is, u attains the minimum of the functional E on the class $H^{1,2}(M, N) \cap L^{\infty}(M, N)$. Another example is the case when u is a *stationary harmonic map* which has been defined rather recently (for the definition, see Section 1). In both examples, it is easily seen u is a weakly harmonic map. In 1948, Morrey [7] showed that if $m = \dim M = 2$, then every energy minimizing map is a smooth harmonic map. In the case m is greater than two, Schoen-Uhlenbeck [11] proved that an energy minimizing map $u: M \to N$ is smooth outside a closed subset $\mathcal{S}$ whose Hausdorff dimension is less than or equal to $n-3$. And they provided us a mechanism which can lower the Hausdorff dimension of $\mathcal{S}$. As for stationary harmonic maps, Schoen [10] showed that if $m = \dim M = 2$, then every stationary harmonic map is a smooth harmonic map. However, regularity problem on stationary harmonic maps are hardly understood in dimension greater than two. In general, Price stated in [8] without explicit proof that stationary harmonic maps satisfy the important inequality called the *monotonicity formula*.

In this paper we show that every stationary harmonic map of M into N with locally L^m integrable gradient can not have isolated singular points. When $m = \dim M = 2$, the theorem is contained in the result by Sacks-Uhlenbeck in [9]. Their proof relies upon techniques from complex function theory of one variable. Our proof makes use of the monotonicity formula. And also, our theorem may be considered as an extension of the regularity result [11] for energy minimizing maps to stationary harmonic maps. In [12] our theorem is stated for the special case when M is an open set of Euclidean space $\boldsymbol{R}^m$. Our proof is somewhat simplified than that of [12].

In Section 1 we recall the definitions of weakly harmonic and stationary harmonic maps and state our main theorem. In Section 2 we prove an a-priori estimate of C^2 harmonic maps under a certain assumption on local integrability of its gradient. Our method is similar to that of Uhlenbeck [13]. In Section 3 the monotonicity formula for stationary harmonic maps is proved following the idea of Price [8]. In Section 4 we prove our main theorem. In the final section we obtain several corollaries derived from our main theorem and from the result of Schoen-Uhlenbeck [11].

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§ 1. Statement of results

Let (M, g) and (N, h) be Riemannian manifolds of dimension m and n respectively. We may assume without loss of generality that (N, h) is isometrically imbedded in Euclidean space $\boldsymbol{R}^k$. For $p \geqq 1$, let $H^{1,p}(M, \boldsymbol{R}^k)$ be the Sobolev space of maps $u: M \to \boldsymbol{R}^k$ whose component functions belong to L^p together with their first derivatives.

For a C^1 map $u: M \to N$ the *energy density* $e(u)$ of u is defined by

$$\begin{aligned} e(u)(x) &= |du(x)|^2 = (du(x), du(x)) \\ &= \sum_{\alpha=1}^{k} \sum_{i,j=1}^{m} g^{ij}(x) D_i u^\alpha(x) D_j u^\alpha(x). \end{aligned}$$

The *energy functional* $E(u)$ is defined by

$$E(u) = \int_M e(u) dV,$$

where $dV = \det(g_{ij})^{1/2} dx$ is the volume element of (M, g). For the functional E the Euler-Lagrange equation is

$$\Delta u^\alpha(x) = \sum_{i,j=1}^{m} g^{ij}(x) A^\alpha_{u(x)}(D_i u(x), D_j u(x)), \quad \alpha = 1, \cdots, k, \tag{1.1}$$

where for $y \in N$, $A_y = (A^\alpha_y)$ is the second fundamental form of N in $\boldsymbol{R}^k$ at y. Equation (1.1) is a quasilinear elliptic system.

DEFINITION 1.1. A smooth map u of M into N is called a *harmonic map* if u satisfies equation (1.1).

From the definition of the functional E, it is seen that the natural class of maps u for which $E(u)$ is defined consists of those which are bounded and have first derivatives in L^2. This is written as

$$\mathfrak{M}=\mathfrak{M}(M, N)=H^{1,2}(M, N)\cap L^{\infty}(M, N),$$

where

$$H^{1,2}(M, N)=\{u\in H^{1,2}(M, \boldsymbol{R}^k):\quad u(x)\in N \text{ a.e. } x\in M\},$$

and $L^{\infty}(M, N)$ is defined similarly.

DEFINITION 1.2. A map $u: M\to N$ is called a *weakly harmonic map* if $u\in\mathfrak{M}$ and u satisfies

$$(1.2)\qquad \int_M \sum_{\alpha=1}^{k}\sum_{i,j=1}^{m} g^{ij}(D_iu^{\alpha}D_j\eta^{\alpha}+A_u^{\alpha}(D_iu, D_ju)\eta^{\alpha})dV=0,$$

for all $\eta\in C_0^{\infty}(M, \boldsymbol{R}^k)$.

We consider the critical points of the functional E in $\mathfrak{M}$. We want to call $u\in\mathfrak{M}$ a critical point of E if u satisfies

$$\frac{d}{dt}E(u_t)\Big|_{t=0}=0$$

for every variation u_t of u. However, since $\mathfrak{M}$ is only defined as a subset of the Hilbert space, the above definition has no clear meaning. Thus, we have to restrict the class of variations u_t of u to be considered. There are two basic classes of variations which one would like to perform on a map $u\in\mathfrak{M}$. The first is defined by a map $\eta\in C_0^{\infty}(M, \boldsymbol{R}^k)$. Given such η we consider the map $u+t\eta$. However, the map $u+t\eta$ is not in $\mathfrak{M}$ in general. To correct this deficiency we project $u+t\eta$ onto N. More precisely, we choose a small tubular neighborhood $U\subset\boldsymbol{R}^k$ of N and let $\Pi: U\to N$ be the nearest point projection. Since η has bounded image, we have $u(x)+t\cdot\eta(x)\in U$ for sufficiently small $t\in\boldsymbol{R}$ and almost all $x\in M$. Thus, we get a variation $u_t=\Pi\circ(u+t\eta)$. There is the second type of variations which one can consider. This is a variation by reparametrization of M; that is, given one-parameter family $F_t: M\to M$ of diffeomorphisms which are equal to the identity outside a compact subset of M and satisfy $F_0=id.$, we set $u_t=u\circ F_t$.

DEFINITION 1.3. A map $u: M\to N$ is called a *stationary harmonic map* if $u\in\mathfrak{M}$ and u is critical with respect to both types of variations described above.

REMARK 1.4. (1) It can be seen without much difficulty that u is weakly harmonic map if and only if u is critical with respect to the first type of the variations; that is, for each $\eta \in C_0^\infty(M, \boldsymbol{R}^k)$, u satisfies

$$\frac{d}{dt}E(u_t)\Big|_{t=0}=0 \qquad \text{where } u_t=\Pi\circ(u+t\eta).$$

(2) If a weakly harmonic map is proved to be continuous, then it is a smooth harmonic map ([4], [5]).
(3) By (1) and Definition 1.3, a stationary harmonic map is weakly harmonic, but the converse is not known. It is known that a continuous weakly harmonic map is a stationary harmonic map (see [10]).

For a weakly harmonic map $u: M \to N$, a point $x \in M$ is called a *regular point* of u if u is continuous in a neighborhood of x. Otherwise a point x is called a *singular point* of u.

Our main result is as follows.

MAIN THEOREM. *Let M, N be Riemannian manifolds and let $m=\dim M \geqq 3$. Suppose that $u \in \mathfrak{M}(M, N)$ is a stationary harmonic map satisfying*

(1.3) $$\int_D |du|^m dV < \infty \quad \textit{for any compact subset } D \textit{ of } M.$$

Then, u can not have isolated singular points.

REMARK 1.5. (1) In case $m=2$, the above theorem is proved by Sacks-Uhlenbeck [9] for any weakly harmonic map. Furthermore, Schoen [10] showed that if $m=2$, a stationary harmonic map is always a smooth harmonic map.
(2) In the above theorem, condition (1.3) is crucial. Indeed, if $m \geqq 3$, the equator map

$$u_e : B=\{x \in \boldsymbol{R}^m : |x|<1\} \longrightarrow S^m$$

defined by $u_e(x)=(x/|x|, 0)$ is a stationary harmonic map with isolated singularity at the origin (see Appendix). However, u_e does not satisfy condition (1.3).

§ 2. The gradient estimate

In this section we prove the following

THEOREM 2.1. *Suppose that* $u \in C^2(B_{2R}, N)$ *is a harmonic map of* (B_{2R}, g) *into* N, *where* $B_{2R}=\{x \in \boldsymbol{R}^m : |x|<2R\}$ *and the Riemannian metric* g *satisfies*

$$\lambda^{-1}(\delta_{ij}) \leqq (g^{ij}) \leqq \lambda(\delta_{ij}) \qquad \textit{for some } \lambda \geqq 1. \tag{2.1}$$

There exists $\varepsilon_0>0$ *depending only on* n, λ, N *such that if*

$$\int_{B_{2R}} |du|^m dV \leqq \varepsilon_0, \tag{2.2}$$

then u *satisfies the inequality*

$$\sup_{B_R} |du|^2 \leqq C_1 R^{-m} \int_{B_{2R}} |du|^2 dV, \tag{2.3}$$

where C_1 *depends only on* m, λ, N, *the Ricci curvature of* (B_{2R}, g).

We make several preparations to prove this theorem.

PROPOSITION 2.2. *If* u *is a* C^2 *harmonic map of* M *into* N, *then we have*

$$\begin{aligned} |du|\Delta(|du|) \geqq Q(du) \equiv & \sum_{\mu} \mathrm{Ric}^M(u^*\theta^\mu, u^*\theta^\mu) \\ & - \sum_{i,j} \langle R^N(u_*e_i, u_*e_j)u_*e_i, u_*e_j \rangle, \end{aligned} \tag{2.4}$$

where $\{e_i\}$ *and* $\{\theta^\mu\}$ *denote orthonormal frames for* TM *and* T^*N *respectively*, u_* *and* u^* *are the differential and the pullback of* u *respectively*, $\mathrm{Ric}^M(\cdot, \cdot)$ *is the Ricci curvature tensor of* M *and* $R^N(\cdot, \cdot)$ *denotes the Riemannian curvature tensor of* N.

PROOF. The Bochner formula for harmonic maps (see [1] for its proof) is

$$\frac{1}{2}\Delta e(u) = |\nabla du|^2 + Q(du). \tag{2.5}$$

By simple computation we have

$$\frac{1}{2}\Delta e(u) = \frac{1}{2}\Delta |du|^2 = |du|\Delta(|du|) + |\nabla |du||^2. \tag{2.6}$$

We differentiate the both sides of the identity $|du|^2=(du, du)$ to obtain

$$|du|\nabla|du| = (du, \nabla du).$$

Using the Schwarz inequality, we have

$$|du||\nabla|du||=|(du, \nabla du)|\leqq|du||\nabla du|.$$

Dividing by $|du|$,

$$|\nabla|du||\leqq|\nabla du| \tag{2.7}$$

holds. By (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned}|du|\Delta(|du|)&=\frac{1}{2}\Delta e(u)-|\nabla|du||^2\geqq\frac{1}{2}\Delta e(u)-|\nabla du|^2\\&=Q(du).\end{aligned}$$

Q.E.D.

Suppose that u is a C^2 harmonic map of (B_{2R}, g) into N and the Riemannian metric g on B_{2R} satisfies condition (2.1). For simplicity we assume $R=1$. The general case can be proved by the coordinate transformation: $x\to x/R$. From Proposition 2.2

$$\begin{aligned}|du|\Delta|du|&\geqq\sum_{\mu}\mathrm{Ric}^g(u^*\theta^\mu, u^*\theta^\mu)-\sum_{i,j}\langle R^N(u_*e_i, u_*e_j)u_*e_i, u_*e_j\rangle\\&\geqq-a|du|^2-K|du|^4,\end{aligned}$$

where

$$\begin{aligned}a&=n\cdot\sup\{|\mathrm{Ric}^g(x)(\xi,\xi)|:(x,\xi)\in T^*B,\ \|\xi\|=1\},\\K&=m^2\cdot\sup\{\kappa(y):\ y\in\mathrm{Im}(u)\},\end{aligned}$$

and $\kappa(y)$ is the sectional curvature of N at y. Dividing by $|du|$, we have

$$\Delta|du|\geqq-a|du|-K|du|^3.$$

Setting $b=K|du|^2$ and regarding b as known function, we can write the above inequality as

$$\Delta f+(a+b)f\geqq 0 \qquad \text{in } B_2,$$

for $f=|du|$.

If b is in L^p for some $p>m/2$ in this inequality, Theorem 2.1 is derived from the following de Diorgi-Nash-Moser theorem ([4, §8.6]).

PROPOSITION 2.3. *Let Ω be a bounded domain in R^m and let g be a Riemannian metric on Ω satisfying* (2.1). *Let $a>0$ be a constant and*

$b \in L^p(\Omega)$ *for some* $p > m/2$. *Suppose that* $f \in H^{1,2}(\Omega)$ *is nonnegative and satisfies weakly*

$$\Delta f + (a+b)f \geqq 0 \quad in\ \Omega. \tag{2.8}$$

Then, for each $\Omega' \Subset \Omega$ *the following inequality holds.*

$$\sup_{\Omega'} f^2 \leqq C_2 d^{-m} \int_{\Omega} f^2 dV, \tag{2.9}$$

where $d = \operatorname{dist}(\Omega', \partial\Omega)$ *and* C_2 *depends only on* m, p, a, λ, $\operatorname{diam}(\Omega)$ *and* $\|b\|_{L^p}$.

However, in our case b is assumed to be in $L^{m/2}$, so we cannot apply the above theorem directly. We need the following lemma.

LEMMA 2.4. *Suppose that* g *is a Riemannian metric on* B_2 *satisfying condition* (2.1) *for some constant* $\lambda \geqq 1$. *Let* $a > 0$ *be a constant and* $b \in L^{m/2}(B_2)$. *And suppose that* $f \in H^{1,2}(B_2) \cap L^\infty_{\mathrm{loc}}(B_2)$ *is nonnegative and satisfies* (2.8) *weakly. Then, for each* $q > 2$, *there exists a constant* $\varepsilon_q = \varepsilon_q(m, \lambda) > 0$ *such that if* $\|b\|_{L^{m/2}(B_2)} \leqq \varepsilon_q$, *then,*

$$\|f\|_{L^q(B_{3/2})} \leqq C_3 \|f\|_{L^2(B_2)},$$

holds where C_3 *depends only on* m, q, λ *and* a.

PROOF. Since f satisfies inequality (2.8) weakly, we have

$$\int g^{ij} D_i f D_j \zeta dV \leqq a \int f \zeta dV + \int b f \zeta dV \tag{2.10}$$

for all $\zeta \in H_0^{1,2}(B_2)$, where the summation convention is understood.

We take $\zeta = \eta^2 f^{2\gamma-1}$ where $\eta \in C_0^1(B_2)$, $0 \leqq \eta \leqq 1$ and $\gamma \geqq 1$. By the chain rule ([4, §7]), ζ is a valid test function and

$$D\zeta = (2\gamma-1)\eta^2 f^{2\gamma-2} Df + 2\eta f^{2\gamma-1} D\eta. \tag{2.11}$$

By (2.10) and (2.11), we have

$$\begin{aligned}(2\gamma-1)\int \eta^2 f^{2\gamma-2} g^{ij} D_i f D_j f dV + 2\int \eta f^{2\gamma-1} g^{ij} D_i f D_j \eta dV \\ \leqq a \int (\eta f^\gamma)^2 dV + \int b(\eta f^\gamma)^2 dV.\end{aligned}$$

Using (2.1) and the chain rule, we obtain

$$(2.12)\quad \frac{2\gamma-1}{\gamma^2\lambda}\int|\eta Df^\gamma|^2 dV+2\int\eta f^{2\gamma-1}g^{ij}D_i f D_j\eta dV\leqq a\int(\eta f^\gamma)^2 dV+\int b(\eta f^\gamma)^2 dV.$$

By the Schwarz inequality and (2.1),

$$|g^{ij}D_i f D_j\eta|\leqq(g^{ij}D_i f D_j f)^{1/2}(g^{hk}D_h\eta D_k\eta)^{1/2}\leqq\lambda|Df||D\eta|$$

holds. Thus, we have

$$\begin{aligned}\frac{2\gamma-1}{\gamma^2\lambda}\int|\eta Df^\gamma|^2 dV&\leqq 2\lambda\int\eta f^{2\gamma-1}|Df||D\eta|dV+a\int(\eta f^\gamma)^2 dV+\int b(\eta f^\gamma)^2 dV\\&=\frac{2\lambda}{\gamma}\int f^\gamma|\eta Df^\gamma||D\eta|dV+a\int(\eta f^\gamma)^2 dV+\int b(\eta f^\gamma)^2 dV\\&\leqq\frac{2\lambda}{\gamma}\int(f^\gamma|D(\eta f^\gamma)||D\eta|+f^{2\gamma}|D\eta|^2)dV+a\int(\eta f^\gamma)^2 dV+\int b(\eta f^\gamma)^2 dV.\end{aligned}$$

Using the relation,

$$|D(\eta f^\gamma)|^2=|\eta Df^\gamma|^2+2f^\gamma D(\eta f^\gamma)\cdot D\eta-f^{2\gamma}|D\eta|^2,$$

we obtain

$$\begin{aligned}\frac{2\gamma-1}{\gamma^2\lambda}\int|D(\eta f^\gamma)|^2 dV\leqq 2\Big(1+\frac{\lambda}{\gamma}\Big)\int f^\gamma|D(\eta f^\gamma)||D\eta|dV+\Big(1+\frac{2\lambda}{\gamma}\Big)\int f^{2\gamma}|D\eta|^2 dV\\+a\int(\eta f^\gamma)^2 dV+\int b(\eta f^\gamma)^2 dV.\end{aligned}$$

Applying the arithmetic-geometric mean inequality, we have

$$\begin{aligned}\frac{2\gamma-1}{\gamma^2\lambda}\int|D(\eta f^\gamma)|^2 dV\leqq\Big(1+\frac{\lambda}{\gamma}\Big)\delta\int|D(\eta f^\gamma)|^2 dV+\Big[\Big(1+\frac{\lambda}{\gamma}\Big)\frac{1}{\delta}+\Big(1+\frac{2\lambda}{\gamma}\Big)\Big]\\\times\int f^{2\gamma}|D\eta|^2 dV+a\int(\eta f^\gamma)^2 dV+\int b(\eta f^\gamma)^2 dV,\end{aligned}$$

for any $\delta>0$. Taking $\delta=(2\gamma-1)/2\gamma\lambda(\lambda+\gamma)$, we obtain

$$\frac{2\gamma-1}{2\gamma^2\lambda}\int|D(\eta f^\gamma)|^2 dV\leqq 2(\lambda+\gamma)^3\int f^{2\gamma}|D\eta|^2 dV+a\int(\eta f^\gamma)^2 dV+\int b(\eta f^\gamma)^2 dV.$$

Applying the Hölder inequality to the third term of the right hand side, we have

$$\frac{2\gamma-1}{2\gamma^2\lambda}\int|D(\eta f^\gamma)|^2 dV\leqq 2(\lambda+\gamma)^3\int f^{2\gamma}|D\eta|^2 dV$$

$$+a\int(\eta f^{\gamma})^2dV+\|b\|_{L^{m/2}(B_2)}\left(\int(\eta f^{\gamma})^{2\nu}dV\right)^{1/\nu},$$

where $\nu=m/(m-2)$. From the Sobolev inequality corresponding to the imbedding: $H_0^{1,2}\to L^{2\nu}$, we obtain

$$\left(\frac{2\gamma-1}{2\gamma^2\lambda S}-\|b\|_{L^{m/2}(B_2)}\right)\|\eta f^{\gamma}\|^2_{L^{2\nu}(B_2)}\leqq a\int(\eta f^{\gamma})^2dV+2(\lambda+\gamma)^3\int f^{2\nu}|D\eta|^2dV, \tag{2.13}$$

where $S=S(m)$ is the constant of the Sobolev inequality.

For each $q>2$, we choose i_0 with $\nu^{i_0}<q/2\leqq\nu^{i_0+1}$. We take $\gamma=\gamma_i=\nu^i$ and set the cut off function $\eta=\eta_i$ such that

$$\begin{aligned}&\eta_i(x)=1 \quad \text{if} \quad |x|\leqq R_i,\\ &\eta_i(x)=0 \quad \text{if} \quad |x|\geqq R_{i-1},\\ &\qquad \text{where } R_i=2-(i/2i_0),\ (i=0,1,\cdots,i_0).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\left(\frac{2\gamma_i-1}{2\gamma_i^2\lambda S}-\|b\|_{L^{m/2}(B_2)}\right)\|f\|_{L^{2\gamma_{i+1}}(B_{R_i})}\\ &\leqq(a\cdot\sup|\eta_i|^2+2(\gamma_i+\lambda)^3\sup|D\eta_i|^2)\|f\|_{L^{2\gamma_i}(B_{R_{i-1}})}\end{aligned}$$

for $i=1,2,\cdots$cdot, i_0. Taking

$$\varepsilon_q=\min\left\{\frac{2\gamma_i-1}{4\gamma_i^2\lambda S}:\ i=1,2,\cdots,i_0\right\}$$

we obtain the desired result. Q.E.D.

PROOF OF THEOREM 2.1. We first consider the case $R=1$. We recall that the function $f=|du|$ satisfies the inequality

$$\Delta f+(a+b)f\geqq 0 \qquad \text{in } B_2.$$

We fix a number $q>m$. From Lemma 2.4 we have an estimate of $\|f\|_{L^q(B_{3/2})}$ under the condition $\|b\|_{L^{m/2}(B_2)}\leqq\varepsilon_q$. Noting the relation $b=Kf^2$, we obtain a bound of $\|b\|_{L^{q/2}(B_{3/2})}$. We apply Proposition 2.3 with $\Omega=B_{3/2}$ and $\Omega'=B_1$ to get the desired result.

For any $R>0$, we use the scaling argument. We set $v(x)=u(Rx)$. Applying the coordinate transformation: $x\to x/R$ to (2.4), we have

$$\Delta_R|dv|\geqq-a|dv|-K|dv|^3 \qquad \text{in } B_2, \tag{2.14}$$

where Δ_R is the Laplacian with respect to the metric g_R on B_2 given by $g_R(x)=g(Rx)$. We remark that the constants a, K, λ are not changed under the coordinate transformation. Hence, using the above argument to (2.14), we obtain the desired result. Q.E.D.

§ 3. Monotonicity formula

By the definition of stationary harmonic maps the following formula is proved in [8].

PROPOSITION 3.1 (The first variation formula). *Let $u\in\mathfrak{M}(M,N)$ be a stationary harmonic map. Suppose that $F_t: M\to M$ is a one-parameter family of diffeomorphisms which are equal to the identity outside some compact set and with $F_0=\mathrm{id}$. If X is the variation vector field of $\{F_t\}$, then we have*

$$(3.1)\qquad \frac{d}{dt}E(u\circ F_t)\Big|_{t=0}=-\int_M(|du|^2\operatorname{div}X-2\sum_{i=1}^m\langle du(\nabla_{e_i}X), du(e_i)\rangle)dV=0,$$

where $\{e_i\}$ is an orthonormal frame of TM.

For $x\in M$, $\rho>0$ we denote by $B(x,\rho)$ the geodesic ball of M with the center x and the radius ρ. We set

$$E_\rho^x(u)=\int_{B(x,\rho)}|du|^2dV.$$

THEOREM 3.2 (The monotonicity formula). *Let $x_0\in M$ and $R>0$. Suppose that the distance from x_0 to the cut locus or the boundary ∂M is at least R. Let $u\in\mathfrak{M}(B(x_0,R),N)$ be a stationary harmonic map. Then, there exists a constant $\Lambda=\Lambda(m,g)\geqq 0$ such that for each $x\in B(x_0,R/2)$ and $0<\sigma<\rho<R/2$*

$$(3.2)\qquad e^{\Lambda\rho}\rho^{2-m}E_\rho^x(u)-e^{\Lambda\sigma}\sigma^{2-m}E_\rho^x(u)\geqq 2\int_{B(x,\rho)-B(x,\sigma)}e^{\Lambda r}r^{2-m}|D_ru|^2dV,$$

holds where r is the geodesic radial coordinate on $B(x,R/2)$.

PROOF. We choose a one-parameter family F_t satisfying

$$X=\frac{d}{dt}F_t\Big|_{t=0}=\xi(r)r\frac{\partial}{\partial r}$$

where $\xi \in C_0^\infty(\boldsymbol{R})$. Let $\{\partial/\partial r, e_1, \cdots, e_{m-1}\}$ be an orthonormal frame of TM. From (3.1), we have

$$\int_M |du|^2 \operatorname{div} X dV = 2\int_M \langle du(\nabla_{\partial/\partial r}X),\ du(\partial/\partial r)\rangle dV + 2\sum_{i=1}^{m-1}\int_M \langle du(\nabla_{e_i}X),\ du(e_i)\rangle dV. \tag{3.3}$$

We remark

$$\nabla_{\partial/\partial r}X = (\xi r)'\frac{\partial}{\partial r},$$

and

$$\langle \nabla_{re_i}\frac{\partial}{\partial r},\ e_j\rangle(y) = \delta_{ij} + \int_0^r \varepsilon_{ij}(s, \theta)ds,$$

$$\text{where } \varepsilon_{ij} = D_r\langle \nabla_{re_i}\frac{\partial}{\partial r},\ e_j\rangle,\ y = (r, \theta).$$

Since $|\varepsilon_{ij}|$ depends only on C^2 norm of the metric g, we can take $\tilde{\Lambda}$ such that

$$\tilde{\Lambda} = \max_{i,j}\ \sup\{|\varepsilon_{ij}(x)| :\ x \in B(x_0, R)\} < \infty.$$

Thus, we have

$$\begin{aligned}\operatorname{div}X &= \langle \nabla_{\partial/\partial r}X,\ \frac{\partial}{\partial r}\rangle + \sum_{i=1}^{m-1}\langle \nabla_{e_i}X,\ e_i\rangle \\ &= (\xi r)' + \sum_{i=1}^{m-1}\xi\langle \nabla_{re_i}\frac{\partial}{\partial r},\ e_i\rangle \\ &\geqq \xi' r + \xi + \xi\sum_{i=1}^{m-1}(1 - \tilde{\Lambda} r)\end{aligned}$$

$$\operatorname{div}X \geqq \xi' r + m\xi - (m-1)\tilde{\Lambda}\xi r \tag{3.4}$$

and

$$\begin{aligned}\sum_{i=1}^{m-1}\langle du(\nabla_{e_i}X,\ du(e_i)\rangle &= \sum_{i,j}^{m-1}\xi\langle \nabla_{re_i}\frac{\partial}{\partial r},\ e_j\rangle\langle du(e_i),\ du(e_j)\rangle \\ &\leqq \sum_{i,j}^{m-1}\xi(\delta_{ij} + r\tilde{\Lambda})|du(e_i)||du(e_j)| \\ &\leqq \xi\sum_{i=1}^{m-1}|du(e_i)|^2 + \frac{1}{2}r\xi\tilde{\Lambda}\sum_{i,j}^{m-1}(|du(e_i)|^2 + |du(e_j)|^2),\end{aligned}$$

$$\sum_{i=1}^{m-1} \langle du(\nabla_{e_i} X),\ du(e_i)\rangle \leqq \xi(|du|^2 - |D_r u|^2) + (m-1)\tilde{\Lambda}\xi r |du|^2. \tag{3.5}$$

From (3.4), (3.5) and (3.3), we obtain

$$\begin{aligned} &\int |du|^2(\xi' r + m\xi - (m-1)\tilde{\Lambda}\xi r)dV \\ &\leqq 2\int (|D_r u|^2 \xi' r + |du|^2 \xi + (m-1)|du|^2 \tilde{\Lambda}\xi r) dV. \end{aligned} \tag{3.6}$$

We choose, for $\tau \in [\sigma, \rho]$

$$\xi(r) = \xi_\tau(r) = \varphi(r/\tau)$$

where $\varphi \in C_0^\infty[0, \infty)$ satisfying

$$\varphi(r) = 1 \ \ r \in [0, 1],\ \varphi(r) = 0 \ \ r \in [1+\delta, \infty) \ \ (\delta > 0),\ \varphi'(r) \leqq 0.$$

Then,

$$\tau \frac{\partial}{\partial \tau}(\xi_\tau(r)) = -r\xi'_\tau(r),$$

holds. Thus,

$$\begin{aligned} -2\int |D_r u|^2 \tau\Big(\frac{\partial}{\partial \tau}\xi_\tau\Big)dV \geqq & -\int |du|^2 \tau \Big(\frac{\partial}{\partial \tau}\xi_\tau\Big) dV \\ & + (m-2)\int |du|^2 \xi_\tau dV - 3(m-1)(1+\delta)\tilde{\Lambda}\tau \int |du|^2 \xi_\tau dV. \end{aligned}$$

$$\begin{aligned} 2\tau \frac{\partial}{\partial \tau}\int |D_r u|^2 \xi_\tau dV \leqq & \ \tau \frac{\partial}{\partial \tau}\int |du|^2 \xi_\tau dV - (m-2)\int |du|^2 \xi_\tau dV \\ & + 3(m-1)(1+\delta)\tilde{\Lambda}\tau \int |du|^2 \xi_\tau dV. \end{aligned}$$

$$\begin{aligned} 2\tau \frac{\partial}{\partial \tau}\int |D_r u|^2 \xi_\tau dV \leqq & \ \tau \frac{\partial}{\partial \tau}\int |du|^2 \xi_\tau dV + (2-m)\int |du|^2 \xi_\tau dV \\ & + 3(m-1)\tilde{\Lambda}(1+\delta)\tau \int |du|^2 \xi_\tau dV. \end{aligned} \tag{3.7}$$

We set $\Lambda = 6(m-1)\tilde{\Lambda}$ as Λ. By multiplying $e^{\Lambda\tau}\tau^{1-m}$ in (3.7), we have, for any $\delta \leqq 1$

$$\frac{\partial}{\partial \tau}\Big(e^{\Lambda\tau}\tau^{2-m}\int |du|^2 \xi_\tau dV\Big) \geqq 2e^{\Lambda\tau}\tau^{2-m}\frac{\partial}{\partial \tau}\int |D_r u|^2 \xi_r dV.$$

By integrating from $\tau=\sigma$ to $\tau=\rho$ and taking the limit $\delta\to 0$, the desired result (3.2) follows. Q.E.D.

When the metric g is flat, $\tilde{\Lambda}=0$ and so is Λ. Hence, we have the following.

COROLLARY 3.3. *Let Ω be an open set of $\boldsymbol{R}^m$ and $u\in\mathfrak{M}(\Omega, N)$ be a stationary harmonic map. For each $x_0\in\Omega$ and $0<\sigma<\rho<\mathrm{dist}(x_0, \partial\Omega)$, we have the following equality.*

$$\rho^{2-m}\int_{B(x_0,\rho)}|du|^2dx-\sigma^{2-m}\int_{B(x_0,\sigma)}|du|^2dx=2\int_{B(x_0,\rho)-B(x_0,\sigma)}r^{2-m}|D_r u|^2dx, \tag{3.8}$$

where $r=|x-x_0|$.

§ 4. Proof of Main Theorem

Let $u\in\mathfrak{M}(M, N)$ be a stationary harmonic map satisfying condition (1.3). For a point $x_0\in M$, we assume there exists a neighborhood V of x_0 such that u is continuous in $V-\{x_0\}$. By Remark 1.4 (2), u is smooth in $V-\{x_0\}$. We take a normal coordinate with the origin x_0. We set $B(\rho)=B(0, \rho)$ and $S(\rho)=\partial B(\rho)$.

From condition (1.3), we get the following estimate.

PROPOSITION 4.1. *There exists $R_0>0$ such that for any $x\in B(R_0)-\{0\}$, u satisfies the inequality*

$$|du(x)|^2\leqq C_4|x|^{-m}\int_{B(2|x|)}|du|^2dV\leqq C_5|x|^{-2}\left(\int_{B(2|x|)}|du|^m dV\right)^{2/m}. \tag{4.1}$$

Here R_0, C_4 and C_5 depend only on m, g, N.

PROOF. By (1.3) the integral $\int_{B(r)}|du|^m dV$ is finite for any $r>0$. Then, the dominated convergence theorem implies

$$\lim_{r\to 0}\int_{B(r)}|du|^m dV=0.$$

We may choose $R_0>0$ satisfying

$$\int_{B(2R_0)}|du|^m dV\leqq\varepsilon_0,$$

where ε_0 is the constant in Theorem 2.1. For each $x\in B(R_0)-\{0\}$,

$$\int_{B(x,|x|)} |du|^m dV \leqq \int_{B(2R_0)} |du|^m dV \leqq \varepsilon_0.$$

Thus, we apply Theorem 2.1 for $B(x, |x|)$ to obtain

$$|du(x)|^2 \leqq C_4 |x|^{-m} \int_{B(x,|x|)} |du|^2 dV$$
$$\leqq C_4 |x|^{-m} \int_{B(2|x|)} |du|^2 dV. \qquad \text{Q.E.D.}$$

For $\rho \leqq R_0$ we set

$$F(\rho) = e^{A\rho} \rho^{2-m} \int_{B(\rho)} |du|^2 dV, \tag{4.2}$$

where A is the constant in Theorem 3.2. By Theorem 3.2 we observe that F is a non-decreasing function of ρ. Furthermore, we have

PROPOSITION 4.2. *For any* $\rho \in (0, R_0]$, F *satisfies*

$$F(\rho/2) \leqq \mu F(\rho) \tag{4.3}$$

where $\mu \in (0, 1)$ *is independent of* ρ.

PROOF. For each $\rho > 0$, we denote by ω_ρ the average value of u in $B(\rho)$. We fix $\rho \in (0, R_0]$. Using the Stokes theorem in $B(\rho) - B(\varepsilon)$ $(0 < \varepsilon < \rho)$, we obtain

$$\int_{B(\rho)-B(\varepsilon)} |d(u-\omega_\rho)|^2 dV = \int_{S(\sigma)} (u-\omega_\rho) \cdot D_r u dS \Big|_{\sigma=\varepsilon}^{\sigma=\rho} - \int_{B(\rho)-B(\varepsilon)} (u-\omega_\rho) \cdot \Delta u dV. \tag{4.4}$$

By Lemma 4.1, we have

$$\left| \int_{S(\varepsilon)} (u-\omega_\rho) \cdot D_r u dS \right| \leqq C_6 \|u\|_{L^\infty} \varepsilon^{m-3} \left(\int_{B(2\varepsilon)} |du|^m dV \right)^{2/m}.$$

Letting ε tend to 0 in (4.4) we have

$$\int_{B(\rho)} |du|^2 dV \leqq I_1 + I_2,$$
$$\text{where } I_1 = \int_{S(\rho)} |u-\omega_\rho| |D_r u| dS,$$
$$I_2 = \left| \int_{B(\rho)} (u-\omega_\rho) \cdot \Delta u dV \right|.$$

We note that integral I_2 exists since u satisfies (1.1) in $B(\rho)-\{0\}$. From the arithmetic-geometric mean inequality

$$I_1 \leqq \frac{\beta}{2}\int_{S(\rho)} |D_r u|^2 dS + \frac{1}{2\beta}\int_{S(\rho)} |u-\omega_\rho|^2 dS, \tag{4.5}$$

for any $\beta>0$. Since u is a smooth harmonic map in $B(\rho)-\{0\}$, (1.1) implies

$$I_2 = \left|\int_{B(\rho)} (u-\omega_\rho)\cdot A_u(du, du)dV\right| \leqq \|A\|_\infty \int_{B(\rho)} |u-\omega_\rho||du|^2 dV,$$

where $\|A\|_\infty$ is the bound of the second fundamental form A of N on the image of u. Using the arithmetic-geometric mean inequality, we have

$$I_2 \leqq \frac{1}{2}\int_{B(\rho)} |du|^2 dV + \frac{1}{2}\|A\|_\infty^2 \int_{B(\rho)} |u-\omega_\rho|^2 |du|^2 dV.$$

Applying the Hölder and the Sobolev inequality in the second term of the right hand side, we have

$$\begin{aligned} I_2 &\leqq \frac{1}{2}\int_{B(\rho)} |du|^2 dV + \frac{1}{2}\|A\|_\infty^2 \left(\int_{B(\rho)} |du|^m dV\right)^{2/m} \left(\int_{B(\rho)} |u-\omega_\rho|^{2m/(m-2)} dV\right)^{(m-2)/m} \\ &\leqq \frac{1}{2}\left(1+S\|A\|_\infty^2 \left(\int_{B(\rho)} |du|^m dV\right)^{2/m}\right)\int_{B(\rho)} |du|^2 dV, \end{aligned}$$

$$I_2 \leqq \frac{1}{2}(1+S\|A\|_\infty^2 \varepsilon_0^{2/m})\int_{B(\rho)} |du|^2 dV, \tag{4.6}$$

where $S=S(m)$ is the constant of the Sobolev inequality corresponding to the imbedding: $H^{1,2}\to L^{2m/(m-2)}$.

From (4.5), (4.6) and (4.4), we obtain

$$(1-S\|A\|_\infty^2 \varepsilon_0^{2/m})\int_{B(\rho)} |du|^2 dV \leqq \beta\int_{S(\rho)} |D_r u|^2 dS + \frac{1}{\beta}\int_{S(\rho)} |u-\omega_\rho|^2 dS.$$

Replacing ε_0 by smaller one if necessary, we may assume that $S\|A\|_\infty^2 \varepsilon_0^{2/m} \leqq 1/2$. Thus, we have

$$\int_{B(\rho)} |du|^2 dV \leqq 2\beta\int_{S(\rho)} |D_r u|^2 dS + \frac{2}{\beta}\int_{S(\rho)} |u-\omega_\rho|^2 dS, \tag{4.7}$$

for any $\beta>0$. By the definition of ω_ρ we observe that

$$\frac{d}{d\rho}\int_{B(\rho)}|u-\omega_\rho|^2dV=\int_{S(\rho)}|u-\omega_\rho|^2dS-2\int_{B(\rho)}(u-\omega_\rho)dV\Big(\frac{d\omega_\rho}{d\rho}\Big)$$
$$=\int_{S(\rho)}|u-\omega_\rho|^2dS.$$

Thus, (4.7) is rewritten as follows.

$$\int_{B(\rho)}|du|^2dV\leqq 2\beta\int_{S(\rho)}|D_r u|^2dS+\Big(\frac{2}{\beta}\Big)\frac{d}{d\rho}\int_{B(\rho)}|u-\omega_\rho|^2dV. \tag{4.8}$$

We multiply $e^{A\rho}\rho^{2-m}$ in (4.8) and integrate from $\rho/2$ to ρ. By Theorem 3.2, we have

$$\int_{\rho/2}^{\rho}F(t)dt\leqq 2\beta\int_{B(\rho)-B(\rho/2)}e^{Ar}r^{2-m}|D_r u|^2dV$$
$$+\Big(\frac{2}{\beta}\Big)\int_{\rho/2}^{\rho}e^{At}t^{2-n}\Big(\frac{d}{dt}\int_{B(t)}|u-\omega_t|^2dV\Big)dt,$$

$$F(\rho/2)(\rho/2)\leqq\beta(F(\rho)-F(\rho/2))+\Big(\frac{2^{m-1}}{\beta}\Big)e^{A\rho}\rho^{2-m}\int_{B(\rho)}|u-\omega_\rho|^2dV.$$

Using the Poincaré inequality, we obtain

$$F(\rho/2)(\rho/2)\leqq\beta(F(\rho)-F(\rho/2))+\Big(\frac{C_7}{\beta}\Big)\rho^2F(\rho).$$

Setting $\beta=4C_7\rho$, we have the desired result

$$F\Big(\frac{\rho}{2}\Big)\leqq\Big(\frac{16C_7+1}{16C_7+2}\Big)F(\rho). \qquad \text{Q.E.D.}$$

We next need the following simple lemma.

LEMMA 4.3 (See [4], for the proof). *Let F be a non-decreasing function on an interval $(0, R_0]$ satisfying, for all $R\leqq R_0$, the inequality*

$$F(\tau R)\leqq\mu F(R)+G(R),$$

where G is also non-decreasing and $0<\tau$, $\mu<1$. Then, for any $\gamma\in(0,1)$ and $R\leqq R_0$ we have

$$F(R)\leqq C_8[(R/R_0)^\alpha F(R_0)+G(R^\gamma R_0^{1-\gamma})],$$

where $C_8=C_8(\mu,\tau)$ and $\alpha=\alpha(\mu,\tau,\gamma)$ are positive constants.

PROOF OF MAIN THEOREM. Applying Lemma 4.3 to (4.3), we have

$$F(\rho) \leqq C_9 \rho^{\alpha} \qquad \text{for some } \alpha > 0.$$

To combine this inequality with (4.1) we obtain

$$|du(x)|^2 \leqq C_4 |x|^{-m} F(2|x|) \leqq C_{10} |x|^{-2+\alpha},$$

for $0<|x|\leqq R_0$. This implies that u belongs to the Sobolev space $H^{1,p}(B(R_0), \boldsymbol{R}^k)$ for some $p>m$. From the Sobolev imbedding theorem: $H^{1,p} \to C^0$ $(p>m)$, u is continuous in $B(R_0)$. Q.E.D.

§ 5. Some remarks

We first state the following result obtained as a corollary of Main Theorem.

PROPOSITION 5.1. *Let $u \in \mathfrak{M}(M, N)$ be a weakly harmonic map and let $x_0 \in M$. Suppose that there exists ρ_0 such that u satisfies the followings:*

(1) *u is smooth in $B(x_0, \rho_0) - \{0\}$.*

(2) *The integral $\int_{B(x_0, \rho_0)} |du|^m dV$ is finite.*

(3) *For any $\rho \in (0, \rho_0]$, we have*

$$\int_{B(x_0,\rho)} r(x)^{2-m} |du(x)|^2 dV(x) \leqq C_{11} \rho^{2-m} \int_{B(x_0,\rho)} |du|^2 dV, \tag{5.1}$$

where $r(x)$ denotes the geodesic distance of x from x_0 and the constant C_{11} is independent of ρ.

Then, u is smooth in $B(x_0, \rho_0)$, that is, x_0 is a regular point of u.

PROOF. We take a normal coordinate with the origin x_0. For $\rho \in (0, \rho_0]$, we consider the function

$$H(\rho) = \int_{B(x_0,\rho)} r(x)^{2-m} |du(x)|^2 dV(x),$$

instead of F in the previous section. By Theorem 2.1, we have

$$|du(x)|^2 \leqq C_{12} |x|^{-2} H(2|x|)$$

for sufficiently small x. As in the proof of Proposition 4.2, we are able to derive the following inequality:

$$H(\rho/2) \leqq \mu H(\rho) \qquad 0<\mu<1,$$

for sufficiently small $\rho>0$. Using Lemma 4.3, we have

$$H(\rho) \leqq C_{13}\rho^{\alpha} \qquad \text{for some } \alpha>0.$$

Then, for sufficiently small x,

$$|du(x)|^2 \leqq C_{14}|x|^{-2+\alpha},$$

holds. The Sobolev imbedding theorem : $H^{1,p} \to C^0$ implies that u is continuous in $B(x_0, \rho_0)$. Q.E.D.

REMARK 5.2. It seems that the condition (3) in the above proposition is connected with a property of the Green function of the Laplacian of (M, g).

We next describe the regularity properties of energy minimizing maps. A map u of $\mathfrak{M}(M, N)$ is called an *energy minimizing* map if u satisfies

$$E(u) \leqq E(v) \quad \text{for any } v \in \mathfrak{M}(M, N) \text{ with } v=u \text{ on } \partial M.$$

Obviously, an energy minimizing map is a stationary harmonic map. Hereafter, we assume that M and N are compact. Schoen-Uhlenbeck [11] proved the following regularity theorem for energy minimizing maps.

THEOREM 5.3. *There exists $\varepsilon_1>0$ depending only on m, g, N such that if an energy minimizing map $u \in \mathfrak{M}(M, N)$ satisfies*

$$\rho^{2-m}\int_{B(x,\rho)} |du|^2 dV \leqq \varepsilon_1,$$

in a ball $B(x, \rho)$, then, u is Hölder continuous in $B(x, \rho/2)$.

As a corollary of this theorem we have the following result.

PROPOSITION 5.4. *Let $u \in \mathfrak{M}(M, N)$ be an energy minimizing map and let $\mathcal{S}=\mathcal{S}(u)$ be the set of singular points of u. Then,*

(1) *if the integral $\int_M |du|^m dV$ is finite, then the set is empty, that is, u is a smooth harmonic map,*

(2) *if the integral $\int_M |du|^p dV$ is finite for some $p<m$, then, $H_{m-p}(\mathcal{S})$*

$=0$ *where H_s is the s-dimensional Hausdorff measure.*

PROOF. (1) From the Hölder inequality, we have

$$\rho^{2-m}\int_{B(x,\rho)}|du|^2 dV \leqq c(m)\Big(\int_{B(x,\rho)}|du|^m dV\Big)^{2/m}.$$

Thus, for any x we may choose $\rho>0$ small enough to satisfy

$$\rho^{2-m}\int_{B(x,\rho)}|du|^2 dV \leqq \varepsilon_1.$$

Using Theorem 5.2, we obtain the desired result.

(2) Theorem 5.2 implies that for any $x \in \mathcal{S}$ and $\rho>0$

$$\varepsilon_1<\rho^{2-m}\int_{B(x,\rho)}|du|^2 dV \leqq c(m, p)\rho^{2-2m/p}\Big(\int_{B(x,\rho)}|du|^p dV\Big)^{2/p}.$$

For each $\delta>0$, the family $P_\delta=\{\bar{B}(x, \rho): x \in \mathcal{S},\ 0<\rho \leqq \delta\}$ is a covering of $\mathcal{S}$, where $\bar{B}(x, \rho)$ denotes the closed geodesic ball with the center x and the radius ρ. From the Besicovitch covering theorem (see [2]), there exist pairwise disjoint subfamilies $P_1, \cdots, P_l$ $(l=l(m))$ such that $Q_\delta=P_1 \cup \cdots \cup P_l$ covers $\mathcal{S}$. Hence, we have

$$\sum_{\bar{B}(x,\rho)\in P_j} \rho^{m-p} \leqq c(m, p)\varepsilon_1^{-p}\int_{\cup P_j}|du|^p dV \leqq c(m, p)\varepsilon_1^{-p}\int_M |du|^p dV.$$

$$\sum_{\bar{B}(x,\rho)\in Q_\delta} \rho^{m-p} \leqq c(m, p)\varepsilon_1^{-p}\int_{\cup Q_\delta}|du|^p dV \leqq c(m, p) l \varepsilon_1^{-p}\int_M |du|^p dV. \tag{5.2}$$

By the definition of the Hausdorff measure, we obtain

$$H_{m-p}(\mathcal{S}) \leqq c(m, p) l \varepsilon_1^{-p}\int_M |du|^p dV<\infty.$$

Similarly, we have

$$H_m(\cup Q_\delta) \leqq c(m, p) l \varepsilon_1^{-p}\delta^p\int_M |du|^p dV.$$

The dominated convergence theorem implies

$$\lim_{\delta\to 0}\int_{\cup Q_\delta}|du|^p dV=0.$$

Using this in (5.2), we shows that $H_{m-p}(\mathcal{S})=0$. Q.E.D.

Furthermore, we apply the same argument as in [11, § 5] to get the following.

PROPOSITION 5.5. *If $u \in \mathfrak{M}(M, N)$ is an energy minimizing map whose gradient du is in L^p for some $p<m$, then the Hausdorff dimension of the singular set $\mathcal{S}$ of u is at least $m-[p]-1$. In particular, if $p=m-1$, then every singular point of u is isolated.*

Appendix.

On the equator map

Here we present an example of a stationary harmonic map with isolated singularity. We consider the *equator map*

$$u_e : B=\{x \in \boldsymbol{R}^m : |x|<1\} \longrightarrow S^m \subset \boldsymbol{R}^{m+1}$$

defined by $u_e=(x/|x|, 0)$. Hildebrandt [5] showed that u_e is a weakly harmonic map if $m \geqq 3$. Obviously, u_e has isolated singularity at the origin. We state the following result.

PROPOSITION A.1. *The equator map u_e of B into S^m is a stationary harmonic map for $m \geqq 3$.*

PROOF. By the definition of stationary harmonic maps and Proposition 3.1, it is sufficient to show that for every C^1 vector field $X =\sum_i X^i(\partial/\partial x_i)$ on B with compact support, the following first variation formula holds.

$$\int_B \left(|du_e|^2 \operatorname{div} X - 2 \sum_{i,j=1}^m D_i u_e D_j u_e D_i X^j\right) dx = 0. \tag{A.1}$$

By direct calculation, the left hand side of (A.1) is

$$\int_B \left((m-3)|x|^{-2} \operatorname{div} X + 2|x|^{-4} \sum_{i,j=1}^m x_i x_j D_i X^j\right) dx.$$

Applying the Stokes theorem on $B-B(\varepsilon)$ $(0<\varepsilon<1)$, we obtain

$$\begin{aligned} &\int_{B-B(\varepsilon)} \left((m-3)|x|^{-2} \operatorname{div} X + 2|x|^{-4} \sum_{i,j=1}^m x_i x_j D_i X^j\right) dx \\ &= -(m-1) \int_{\partial B(\varepsilon)} |x|^{-2}(X - X(0)) \cdot \boldsymbol{n} dS, \end{aligned}$$

where $\boldsymbol{n}$ is the outward unit normal vector of $\partial B(\varepsilon)$. We estimate the right hand side of this equality as

$$\left|\int_{\partial B(\varepsilon)}|x|^{-2}(X-X(0))\cdot \boldsymbol{n}dS\right| \leqq \mathrm{vol}(S^{m-1})\varepsilon^{m-3} \sup_{\partial B(\varepsilon)}|X-X(0)|.$$

Letting ε tend to 0, we show the formula (A.1). Q.E.D.

Furthermore, Jäger-Kaul [6] proved the following result, which is concerned with the stability of u_e as a critical point of the energy functional E.

PROPOSITION A.2. (1) *The equator map* u_e *is an unstable critical point of* E *for* $3 \leqq m \leqq 6$.

(2) *For* $m \geqq 7$, u_e *attains the absolute minimum of the energy functional* E *on the class* $\{u \in H^{1,2}(B, S^m) : u = u_e \text{ on } \partial B\}$.

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