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Remarks on necessary conditions for the existence of global real analytic solutions of linear partial differential equations on a compact set

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§ 0. Introduction

Let $P(D)$ denote a linear partial differential operator with constant coefficients and $\mathscr{A}$ the sheaf of germs of real analytic functions on $\boldsymbol{R}^n$. The problem of finding a necessary and sufficient condition for the global solvability $P(D)\mathscr{A}(K)=\mathscr{A}(K)$ on a compact set $K\subset\boldsymbol{R}^n$ seems to be open. Though it is much easier than the problem of the global solvability $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$ on an open set $\Omega\subset\boldsymbol{R}^n$, it requires some non-trivial necessary condition unlike the case of C^∞-solutions. Since we can apply the closed range theorem to the space $\mathscr{A}(K)$, the abstract necessary and sufficient condition is that ${}^tP(D)\mathscr{B}[K]$ is closed in $\mathscr{B}[K]$, where $\mathscr{B}[K]$ denotes the dual space of $\mathscr{A}(K)$ and is in fact the space of hyperfunctions with supports in K. A sufficient condition for this is that we have

$$(0.1)\qquad f\in\mathscr{B}_*(\boldsymbol{R}^n),\quad \operatorname{supp}{}^tP(D)f\subset K \;\Rightarrow\; \operatorname{supp}f\subset K,$$

and in particular a convex compact set satisfies this condition for any $P(D)$ (see e.g. Komatsu [10]). However, as far as we know we do not know yet if this is also necessary for the closedness of ${}^tP(D)\mathscr{B}[K]$ in $\mathscr{B}[K]$.

In these notes we derive some concrete necessary conditions for $P(D)\mathscr{A}(K)=\mathscr{A}(K)$ directly from our results on continuation of real analytic solutions. (See Theorems 1.1-1.3. See also Theorem 2.1 for a result concerning systems.) We believe that our method is new and so are some of the results itself. It is Prof. G. Bratti who first taught us the possibility of such applications of the theorems on continuation of real analytic solutions (see Bratti and Trevisan [1]).

§ 1. Study for single operators

We first show the following theorem which is an analogue to the corresponding result for $C^\infty(\Omega)$ (see e.g. Hörmander [3], Corollary 3.7.1). It is very plausible that it is already written somewhere though we could not find it.

THEOREM 1.1. *If $P(D)\mathscr{A}(K)=\mathscr{A}(K)$ for any K, then $P(D)$ must be must be elliptic.*

Note that the converse assertion of this theorem follows from the sufficient condition (0.1) and can also be proved directly with use of the fundamental solution.

Theorem 1.1 follows from the following more concrete result:

LEMMA 1.2. *If $P(D)\mathscr{A}(K)=\mathscr{A}(K)$ for $K=\{1\leq|x|\leq 2\}$, then $P(D)$ must be elliptic.*

PROOF OF LEMMA. As is easily seen we have $P(D)\mathscr{A}(K)=\mathscr{A}(K)$ if and only if we have $q(D)\mathscr{A}(K)=\mathscr{A}(K)$ for every irreducible factor q of P. Therefore we may suppose that P is irreducible. As a special element of $\mathscr{A}(K)$ we choose the standard fundamental solution $E(x)$ of the Laplacian Δ on $\boldsymbol{R}^n$. Assume that

$$P(D)u=E \text{ on } K.$$

Put $L=\{|x|\leq 2\}$. Let $[u]\in B(L)$ be any extension of u as a hyperfunction to L. Then we have

$$P(D)[u]=E+v,$$

where v is a hyperfunction with support in $\{|x|<1\}$, hence

$$\Delta P(D)[u]=P(D)\Delta[u]=\delta+\Delta v.$$

On the other hand, on K, Δu is a real analytic solution of $P(D)(\Delta u)=0$. Therefore if $P(D)$ is not elliptic, then by Theorem 3 of Kaneko [5] on continuation of real analytic solutions, Δu can be continued to an element $f\in A(L)$. Thus $w=\Delta[u]-f$ becomes a hyperfunction with support in $|x|<1$ satisfying

$$P(D)w=\delta+\Delta v.$$

If we apply the Fourier transformation to both sides, then we obtain an identity between entire functions (of certain growth characterized by the Paley-Wiener type theorem for hyperfunctions with compact support which we do not need for the moment):

$$P(\zeta)\hat{w}(\zeta)=1-(\zeta_1^2+\cdots+\zeta_n^2)\hat{v}(\zeta). \tag{1.1}$$

Since $P(\zeta)$ is by the assumption a non-elliptic polynomial, we must have $n\geq 2$, hence the set of the common zeros

$$\{\zeta\in \boldsymbol{C}^n;\ P(\zeta)=\zeta_1^2+\cdots+\zeta_n^2=0\}$$

is non-void. But (1.1) gives a contradiction on any point of this set. Thus P must be elliptic. q. e. d.

Next we consider an operator $P(D)$ whose principal part is the partial Laplacian:

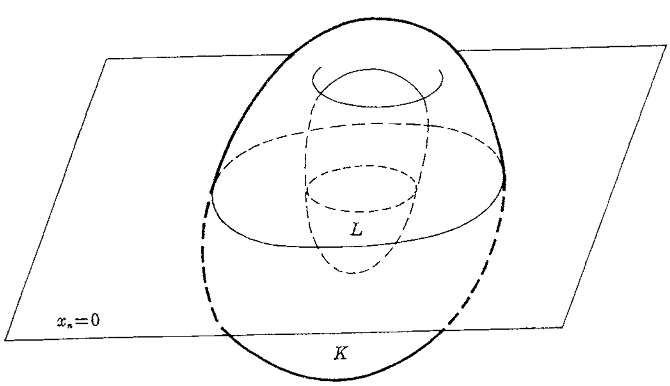
$$P(D)=D_1^2+\cdots+D_{n-1}^2+\text{lower order terms}. \tag{1.2}$$

It contains e. g. the heat equation with the time variable x_n. Let $n\geq 3$.

THEOREM 1.3. *If $P(D)\mathscr{A}(K)=\mathscr{A}(K)$ for $P(D)$ in* (1.2), *then K must be convex with respect to the hyperplane $x_n=const$. More precisely, for any c*

$$(\boldsymbol{R}^n\setminus K)\cap\{x^n<c\}\ \textit{and}\ (\boldsymbol{R}^n\setminus K)\cap\{x_n>c\}$$

cannot contain a bounded connected component (as in the figure).



Figure

PROOF. Assume that there exists a bounded component L of, say, $(\boldsymbol{R}^n\setminus K)\cap\{x_n<0\}$ as is described in the theorem. As in the preceding paragraph, it suffices to show a contradiction assuming that there exists a real analytic solution u of $P(D)u=E$ on K, where E is now a solution of the second order elliptic equation

$$q(D)E:=(\lambda^2D_1^2+D_2^2+\cdots+D_n^2)E=\delta(x-a)$$

with $\lambda>0$ specified later and $a\in L$. Without loss of generality we can assume that $a=(0,\cdots,0,-1)$. By Theorem 2.12 of Kaneko [7] on continuation of real analytic solutions which is analogous to the Hartogs extension theorem for holomorphic functions of several variables, we can find a real analytic extension $f(x)$ of $q(D)u$ on $(K\cup L)\cap\{x_n<0\}$. (See Corollary 2.14 there. Though the calculus is made for the heat equation there, it is obviously valid for any operator with the same principal part. The assumption of convexity of the set L to which we continue the solution is removed by Kaneko [8].) Thus denoting by $[u]$ a hyperfunction extension of u to $(K\cup L)\cap\{x_n<0\}$, we obtain this time

$$P(D)w=\delta(x-a)+q(D)v,$$

where $v=P(D)[u]-E$ and $w=q(D)[u]-f$ are hyperfunctions on $x_n<0$ with supports in L. Now choose an extension $[v]$ resp. $[w]$ with minimal support of v resp. w to a hyperfunction on the whole space $\boldsymbol{R}^n$. Then we have

$$P(D)[w]=\delta(x-a)+q(D)[v]+g,$$

where g is a hyperfunction with support in $\bar{L}\cap\{x_n=0\}$, which we assume to be contained in the $(n-1)$-dimensional ball $|x'|<A$. Apply Fourier transformation to both sides and put $P(\zeta)=q(\zeta)=0$. Then we obtain

$$0=e^{i\zeta_n}+\hat{g}(\zeta).$$

As for $\hat{g}(\zeta)$, in view of the Paley-Wiener type theorem we have the following estimate: For any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|\hat{g}(\zeta)|\leq C_\varepsilon e^{\varepsilon|\zeta|+A|\mathrm{Im}\,\zeta'|}\quad\text{for }\zeta\in\boldsymbol{C}^n,$$

where $\zeta'=(\zeta_1,\cdots,\zeta_{n-1})$. Thus we obtain

$$\text{(1.3)}\qquad \mathrm{Im}\,\zeta_n\geq-\varepsilon|\zeta|-A|\mathrm{Im}\,\zeta'|-C_\varepsilon'\quad\text{for}\quad P(\zeta)=q(\zeta)=0.$$

Recall here that we have

$$P(\zeta)=\zeta_1^2+\cdots+\zeta_{n-1}^2+\text{lower order terms.}$$

Note that for ζ satisfying $q(\zeta)=\lambda^2\zeta_1^2+\zeta_2^2+\cdots+\zeta_n^2=0$ these lower order terms are in any case of $O(|\zeta_1|+\cdots+|\zeta_{n-1}|)$. Thus we can choose a one parameter family of common complex zeros of $P(\zeta)$, $q(\zeta)$ such that

$$\zeta_1\sim\rho,\ \zeta_2\sim i\rho,\ \zeta_3=o(\rho),\cdots,\ \zeta_{n-1}=o(\rho),\ \zeta_n\sim -i\sqrt{\lambda^2-1}\rho$$

when $\rho\to+\infty$. Now if we choose $\sqrt{\lambda^2-1}>A$, then for these common zeros the estimate (1.3) gives a contradiction as $\rho\to+\infty$. q. e. d.

§ 2. Study for systems

Here we give a system version of Theorem 1.1, or rather of the lemma attached to it. It seems less known as a result.

THEOREM 2.1. *Let $K=\{1\leq|x|\leq 2\}$ and let $P(D)$ be an $s_1\times s_0$ matrix of linear differential operators with constant coefficients. Let $\mathcal{M}$ denote the $C[D]$-module* Coker ${}^tP(D)$ *and let $Q(D)$ be a system of compatibility conditions for $P(D)$. Then*

$$\mathcal{A}(K)^{s_0}\xrightarrow{P(D)}\mathcal{A}(K)^{s_1}\xrightarrow{Q(D)}\mathcal{A}(K)^{s_2}\tag{2.1}$$

is exact if and only if the following two conditions are satisfied:

1) *The $C[D]$-module* $\mathrm{Ext}^1(\mathcal{M}, C[D])$ *is either elliptic or* 0.
2) $\mathrm{Ext}^2(\mathcal{M}, C[D])$ *has no elliptic component.*

For the comparison we first examine the solvability in hyperfunctions. In Palamodov [13] similar consideration is done only for modules of free dimension ≤ 1.

LEMMA 2.2. *Let K and $P(D)$ be as in Theorem 2.1. Then the sequence*

$$\mathcal{B}(K)^{s_0}\xrightarrow{P(D)}\mathcal{B}(K)^{s_1}\xrightarrow{Q(D)}\mathcal{B}(K)^{s_2}$$

is exact if and only if $\mathrm{Ext}^2(\mathcal{M}, C[D])=0$.

PROOF. Put $L=\{|x|\leq 2\}$. Let $\mathcal{B}_Q(K)$ etc. denote the space of hyperfunction solutions of $Q(D)u=0$ on K etc. First recall the theorem on continuation of hyperfunction solutions by Komatsu (see [11], Theorem

4.4) according to which every $f\in\mathcal{B}_Q(K)$ can be continued to an element $\tilde{f}\in\mathcal{B}_Q(L)$ if and only if $\mathrm{Ext}^1(\mathrm{Coker}\ {}^tQ, C[D])=0$. Note also that by definition we have $\mathrm{Ext}^1(\mathrm{Coker}\ {}^tQ, C[D])\cong\mathrm{Ext}^2(\mathcal{M}, C[D])$. Therefore if the latter vanishes we can extend $f\in\mathcal{B}_Q(K)$ to $\tilde{f}\in\mathcal{B}_Q(L)$ and then solve the equation $P(D)u=\tilde{f}$ on the compact convex set L by means of Komatsu's existence theorem ([10], Theorem 3). Then $u|_K$ is a required solution. Conversely, suppose that every $f\in\mathcal{B}_Q(K)$ can be written as $f=P(D)u$ with $u\in\mathcal{B}(K)^{s_0}$. Choose a hyperfunction extension $\tilde{u}\in\mathcal{B}(L)^{s_0}$ of u. Then $\tilde{f}=P(D)\tilde{u}$ satisfies $Q(D)\tilde{f}=0$ on L, hence serves as an extension of f as a hyperfunction solution to L. Thus $\mathrm{Ext}^2(\mathcal{M}, C[D])$ must be zero. q. e. d.

The following lemma is our essential tool as well as the continuation theorem of real analytic solutions.

LEMMA 2.3. *Let Ω be a convex open set. Then $(\mathcal{B}/\mathcal{A})_*(\Omega)$ is a flat $C[D]$-module. Here $(\mathcal{B}/\mathcal{A})_*(\Omega)$ denotes the space of sections of $\mathcal{B}/\mathcal{A}$ with supports compact in Ω.*

PROOF. Let

$$C[D]^{s_0}\xrightarrow{P(D)}C[D]^{s_1}\xrightarrow{Q(D)}C[D]^{s_2}$$

be exact. We must show that

$$(\mathcal{B}/\mathcal{A})_*(\Omega)^{s_0}\xrightarrow{P(D)}(\mathcal{B}/\mathcal{A})_*(\Omega)^{s_1}\xrightarrow{Q(D)}(\mathcal{B}/\mathcal{A})_*(\Omega)^{s_2}$$

is also exact. Note that

$$(\mathcal{B}/\mathcal{A})_*(\Omega)\cong(\mathcal{Q}/\underset{\approx}{\mathcal{O}})_*(\Omega)\cong(\mathcal{Q}(\boldsymbol{D}^n)\cap\underset{\approx}{\mathcal{O}}(\boldsymbol{D}^n\setminus\Omega))/\underset{\approx}{\mathcal{O}}(\boldsymbol{D}^n),$$

where $\mathcal{Q}$ denotes the sheaf of Fourier hyperfunctions and $\underset{\approx}{\mathcal{O}}$ the sheaf of rapidly decreasing real analytic functions of modified type (see Kawai [9], pp. 494-495). In the last term $\mathcal{Q}(\boldsymbol{D}^n)\cap\underset{\approx}{\mathcal{O}}(\boldsymbol{D}^n\setminus\Omega)$ is the abbreviation of $\{f\in\mathcal{Q}(\boldsymbol{D}^n); f|_{\boldsymbol{D}^n\setminus\Omega}\in\underset{\approx}{\mathcal{O}}(\boldsymbol{D}^n\setminus\Omega)\}$. The final equality follows from the cohomology vanishing theorem for $\underset{\approx}{\mathcal{O}}$ whose proof may be found e.g. in Saburi [14] (see Theorem 3.1.5). On account of Lemma 5.1.2 in [9], the Fourier transforms of the elements of the numerator of this quotient space constitute the space of holomorphic functions on a conical neighborhood $|\mathrm{Im}\,\zeta|<\delta(|\mathrm{Re}\,\zeta|+1)$ of the real axis with the estimate: for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$(2.2)\qquad |\hat{f}(\zeta)|\le C_\varepsilon \exp(\varepsilon|\zeta|+H_{\bar{\Omega}}(\operatorname{Im}\zeta)-\delta|\operatorname{Im}\zeta|),$$

and the elements of the denominator with the estimate

$$(2.3)\qquad |\hat{f}(\zeta)|\le Ce^{-\delta|\zeta|},$$

where $\delta>0$ varies from elements to elements in both cases. Then the problem reduces to show for any $\hat{f}(\zeta)$ with the former estimate the latter estimate assuming it for $Q(\zeta)\hat{f}(\zeta)$. This is a consequence of Malgrange's inequality on division for matrices, which is a weaker variant of the Fundamental Principle of Ehrenpreis-Palamodov. (See Hörmander [4], Proposition 7.6.5 and Theorem 7.6.11. We can apply the proof of the latter after adjusting a holomorphic weight function of the type $\exp(-\varepsilon\sqrt{\zeta^2+1})$ and discussing in the domain $|\operatorname{Im}\zeta|<\delta(|\operatorname{Re}\zeta|+1)$ instead of $\boldsymbol{C}^n$.) q. e. d.

LEMMA 2.4. *If $\mathcal{M}$ is an elliptic $\boldsymbol{C}[D]$-module, then $\mathcal{M}\otimes(\mathcal{B}/\mathcal{A})_*(\Omega)$ is zero. Here the tensor product is over the ring $\boldsymbol{C}[D]$.*

PROOF. Let $\mathcal{M}=\operatorname{Coker}{}^tP(D)$ be a representation of $\mathcal{M}$. In view of Lemma 2.3 we have the exact sequence

$$0\leftarrow\mathcal{M}\otimes(\mathcal{B}/\mathcal{A})_*(\Omega)\leftarrow(\mathcal{B}/\mathcal{A})_*(\Omega)^{s_0}\xleftarrow{{}^tP(D)}(\mathcal{B}/\mathcal{A})_*(\Omega)^{s_1}.$$

Hence it suffices to show the surjectivity of ${}^tP(D)$ in this sequence. By a theorem of Lech [12], we can find an elliptic single differential operator $q(D)$ such that $q(D)\mathcal{M}=0$. This implies that there exists an $s_1\times s_0$ matrix $R(D)$ of differential operators such that

$$q(D)I_{s_0}={}^tP(D)R(D),$$

where I_{s_0} denotes the identity matrix of size s_0. Hence if $E(x)$ is a fundamental solution of $q(D)$ which is real analytic outside the origin, then $R(D)E(x)I_{s_0}$ serves as a fundamental solution of ${}^tP(D)$ in a sense similar to Ehrenpreis [2] (Section 2 to Chapter VI), which is real analytic outside the origin. Employing the convolution by this fundamental solution we can solve ${}^tP(D)u=f$ in the space $(\mathcal{B}/\mathcal{A})_*(\Omega)$. Indeed, any element of $(\mathcal{B}/\mathcal{A})_*(\Omega)^{s_0}$ may be assumed to be defined by $f\in\mathcal{B}(\boldsymbol{R}^n)^{s_0}$ with $\operatorname{sing\,supp} f$ compact, say, contained in the ball $|x|\le r$. If we choose $R>0$ large, denote by χ_R the characteristic function of the ball $|x|\le R$ and put

$$u=R(D)EI_{s_0}*(\chi_R(x)f(x)),$$

then this is real analytic in $\{|x|<R-r\}\setminus \operatorname{sing\,supp} f$, hence it defines a solution of ${}^tP(D)u=f$ in $(\mathcal{B}/\mathcal{A})_*(\Omega)$. q. e. d.

Lemma 2.4 is obvious if we assume the Fundamental Principle. In fact, via the Fourier transformation the space $\mathcal{M}\otimes(\mathcal{B}/\mathcal{A})_*(\Omega)$ becomes isomorphic to the quotient of the two spaces of holomorphic functions on the variety $N(\mathcal{M})\cap\{|\operatorname{Im}\zeta|<\delta(|\operatorname{Re}\zeta|+1)\}$ with the estimate (2.2) resp. (2.3). Since this variety reduces to a bounded set for sufficiently small $\delta>0$ in case when $\mathcal{M}$ is elliptic, this quotient space becomes trivial. Though the Fundamental Principle of this type can be proved in a way analogous to the case of $\mathcal{B}[K]$ in Kaneko [6], it requires much more spaces when detailed out. Therefore we preferred here an easier way.

PROOF OF THEOREM 2.1. We examine the following three cases:

a) $\operatorname{Ext}^1(\mathcal{M}, \boldsymbol{C}[D])$ is neither elliptic nor zero.

b) $\operatorname{Ext}^2(\mathcal{M}, \boldsymbol{C}[D])$ contains an elliptic component.

c) $\operatorname{Ext}^1(\mathcal{M}, \boldsymbol{C}[D])$ is either elliptic or zero and $\operatorname{Ext}^2(\mathcal{M}, \boldsymbol{C}[D])$ contains no elliptic component.

We shall show that in the cases a) and b) (2.1) is not exact, while in the case c) it is exact. It is the case a) which corresponds to the discussion of the single operator.

a) By the assumption there exists a non-elliptic non-zero primary submodule $P'(D)\boldsymbol{C}[D]^{s'}\supset P(D)\boldsymbol{C}[D]^{s_0}$ of $\operatorname{Ker} Q(D)$ appearing in the primary decomposition of $P(D)\boldsymbol{C}[D]^{s_0}\subset\operatorname{Ker} Q(D)$ (which is equivalent to the primary decomposition of $\operatorname{Ext}^1(\mathcal{M}, \boldsymbol{C}[D])\cong\operatorname{Ker} Q(D)/P(D)\boldsymbol{C}[D]^{s_0}$). For this system $P'(D)$, $Q(D)$ as well serves as the compatibility system because $\operatorname{Ker} {}^tP'(D)\subset \operatorname{Ker} {}^tP(D)$ by the duality. We have hence

$$\operatorname{Ext}^1(\operatorname{Coker} {}^tP', \boldsymbol{C}[D])\cong\operatorname{Ker} Q(D)/P'(D)\boldsymbol{C}[D]^{s'},$$

and this is a non-elliptic primary module. Let $q(D)$ be an element of $\operatorname{Ker} Q(D)\setminus P'(D)\boldsymbol{C}[D]^{s'}$. We can assume without loss of generality that

$$q(D)\notin P'(D)\boldsymbol{C}[D]^{s'}+\Delta\boldsymbol{C}[D]^{s_1}. \tag{2.4}$$

If this is not the case, we may replace Δ by $\Delta+\mu$ with a suitable $\mu\in\boldsymbol{C}$ satisfying this property. In fact, if

$$\operatorname{Ker} Q(D)\subset P'(D)\boldsymbol{C}[D]^{s'}+(\Delta+\mu)\boldsymbol{C}[D]^{s_1},$$

then by applying $Q(D)$ to both sides we see that

$$\operatorname{Ker} Q(D)\subset P'(D)\boldsymbol{C}[D]^{s'}+(\Delta+\mu)\operatorname{Ker} Q(D),$$

hence by dividing by $P'(D)C[D]^{s'}$ that

$$\mathrm{Ext}^1(\mathrm{Coker}\,{}^tP', C[D]) \subset (\Delta+\mu)\,\mathrm{Ext}^1(\mathrm{Coker}\,{}^tP', C[D]).$$

This implies that

$$\mathrm{supp}\,\mathrm{Ext}^1(\mathrm{Coker}\,{}^tP', C[D]) \cap \mathrm{supp}\, C[D]/(\Delta+\mu)C[D] = \varnothing,$$

where supp denotes the support of the $C[D]$-module, i. e. the set of complex zeros of the associated prime ideals. We can, however, always choose $\mu \in C$ so that the above intersection is non-void.

Then let $E(x)$ be the standard fundamental solution of Δ and put $f=q(D)E$. Assume that there exists a solution $u \in \mathcal{A}(K)^{s_0}$ of $P(D)u=f$. Then from the inclusion $P'(D)C[D]^{s'} \supset P(D)C[D]^{s_0}$, we can find a solution $u' \in \mathcal{A}(K)^{s'}$ of $P'(D)u'=f$ by a mere algebraic deformation. We have then

$$P'(D)(\Delta u') = \Delta f = q(D)\Delta E = 0$$

on K. Since for the system $P'(D)$, $\mathrm{Ext}^1(\mathrm{Coker}\,{}^tP', C[D])$ does not contain an elliptic component as remarked above, we can apply Theorem 2.3 of Kaneko [6] to the real analytic solution $\Delta u'$ of $P'(D)(\Delta u')=0$ on K, and thus find an extension $\widetilde{\Delta u'} \in \mathcal{B}_{P'}(L)$ of $\Delta u'$ as a hyperfunction solution to L. Hence denoting by $[u'] \in \mathcal{B}(L)^{s'}$ any extension of u' and putting $w=\Delta[u'] - \widetilde{\Delta u'} \in \mathcal{B}_*(L \setminus K)$, we obtain

$$P'(D)w = \Delta P'(D)[u'] = \Delta f + \Delta v = q(D)\delta + \Delta v.$$

Since $\mathcal{B}_*(\boldsymbol{R}^n)$ is a faithfully flat $C[D]$-module (see Komatsu [11], Theorem 4.4) this implies that

$$q(D) \in P'(D)C[D]^{s'} + \Delta C[D]^{s_1},$$

which contradicts to (2.4). Thus (2.1) is not exact in this case.

b) Assume that for any $f \in \mathcal{A}(K)$ satisfying $Q(D)f=0$ we can find $u \in \mathcal{A}(K)$ such that $P(D)u=f$ on K. Let $\tilde{u} \in \mathcal{B}(L)$ be a hyperfunction extension of u. Then $\tilde{f}=P(D)\tilde{u}$ serves as an extension of f as a hyperfunction solution to L. Thus by the necessary part of the cited theorem of Kaneko [6] applied to the operator Q we conclude that Q contains no elliptic component.

c) Let $f \in \mathcal{A}(K)$ satisfy $Q(D)f=0$. By the assumption we can find an extension $\tilde{f} \in \mathcal{B}(L)$ such that $Q(D)\tilde{f}=0$. Consider the sequence

$$0 \to \mathrm{Ext}^1(\mathcal{M}, C[D]) \to C[D]^{s_1}/P(D)C[D]^{s_0} \xrightarrow{Q(D)} C[D]^{s_2},$$

which is exact by definition. When we take the tensor product with $(\mathcal{B}/\mathcal{A})_*(L\setminus K)$ which is $C[D]$-flat by Lemma 2.3, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\mathcal{M}, C[D]) \otimes (\mathcal{B}/\mathcal{A})_*(L\setminus K)$$
$$\longrightarrow (\mathcal{B}/\mathcal{A})_*(L\setminus K)^{s_1}/P(D)(\mathcal{B}/\mathcal{A})_*(L\setminus K)^{s_0} \xrightarrow{Q(D)} (\mathcal{B}/\mathcal{A})_*(L\setminus K)^{s_2}.$$

Note that $\tilde{f}$ defines an element of the middle term which belongs to $\mathrm{Ker}\, Q(D)$. Thus it comes from an element of the first term. Since $\mathrm{Ext}^1(\mathcal{M}, C[D])$ is itself 0 or elliptic, this first term is in any case equal to zero by Lemma 2.4. This implies that $\tilde{f}$ can be written as $\tilde{f}=P(D)u$ with an element $u \in \mathcal{B}(L)$ whose singular support is contained in $L\setminus K$. Thus $u|_K$ is a required real analytic solution of $P(D)u=f$ on K. q.e.d.

Theorem 2.1 holds as well for a compact set K such that $R^n\setminus K$ has non-void bounded connected components with which K constitutes a compact convex set. The proof is just the same. A system version of Theorem 1.1 itself will require informations on higher extension modules and further results on continuation of solutions which are not yet well studied.

The above examples may seem rather particular. We hope however that such an approach may be refined to give a method to discuss the necessary condition even for $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ for open sets Ω from the standpoint of micro-local analysis.

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