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Existence of the wave operators in long range scattering: The case of parabolic operators

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Abstract. Existence of the wave operators is proved for operators of the form $h_0(P)+W_s+W_L(Q,P)$ on $L^2(R^n)$ where h_0 is a smooth real valued function on R^n, W_s is a short range perturbation and $W_L(Q,P)$ a smooth long range perturbation. Our case includes $h_0(\xi)=\xi_1^2+\xi_2$ for ξ in R^2. Note that $\det h_0''(\xi)=0$ for all ξ.

§ 1. Introduction.

One of the ways of analysing spectral properties of differential operators on $L^2(R^n)$ is by doing scattering theory [1, 2]. In scattering theory the new geometric method of Enss is well established; see [3, 4] and references therein. In [5] we developed Enss' Theory for a class of simply characteristic operators with short range potentials. In [6] a stationary theory was already established for simply characteristic operators with short range potentials. We want to extend Enss' Theory to simply characteristic operators with long range potentials. To the best of our knowledge the existence of the wave operators for this case is not known. The aim of this article is to fill this gap.

Existence of the wave operators in long rang scattering is known [1, 2, 7, 8, 9, 10, 11]. The authors of [1, 2, 7, 8, 9, 10] exclusively deal with the operator $P^2+W_s+W_L(Q,P)$. [11] treats general operators of the form $h_0(P)+W_s+W_L(Q,P)$ with the condition $\{\xi:\det h_0''(\xi)=0\}\subsetneqq R^n$. While this condition on $H_0=h_0(P)$ is good enough for elliptic and hyperbolic operators it excludes the simplest parabolic operator $P_1^2+P_2$ on $L^2(R^2)$.

In this article the main ideas are taken from [11, 12].

§ 2. Statement of the result.

For the free and total Hamiltonians H_0 and H on $L^2(R^n)$ we impose the following assumptions A1, $\cdots$, A4.

A1: $h_0: R^n \to R$ is a C^∞ function such that h_0 and all its derivatives are of atmost polynomial growth.

Let Q, P denote the position and momentum operators on $L^2(R^n)$ given by $Q=(Q_1, \cdots, Q_n)$, $P=(P_1, \cdots, P_n)$, $(Q_j f)(x)=x_j f(x)$, $P_j=-iD_j$, $D_j=\partial/\partial x_j$. Put $H_0=h_0(P)$. Note that by the assumption A1 the operator H_0 maps $\mathcal{S}$ into itself where, $\mathcal{S}=\mathcal{S}(R^n)$ is the Schwartz space of rapidly decreasing functions.

A2: (The short range condition) W_s is an operator with $\mathcal{S}\subset \mathrm{Dom}\, W_s$ and for some $\varepsilon_0>0$ the operator $W_s\varphi(P)(1+Q^2)^{(1+\varepsilon_0)/2}$ is bounded for each φ in $C_0^\infty(R^n)$. Here $C_0^\infty(R^n)$ is the space of all infinitely differentiable functions on R^n with compact support.

A3: (The long range condition) $W_L: R^n \times R^n \to R$ is a C^∞ function and there exists some δ in $(0, 1]$ such that

$$\sup_{\xi \text{ in } B}|D_x^\alpha D_\xi^\beta W_L(x, \xi)| \leq K(B, \alpha, \beta)(1+|x|)^{-|\alpha|-\delta}$$

holds for any bounded subset B of R^n. Here and hereafter the letter K denotes generic constants. Define the operator $W_L(Q, P)$ by

$$[W_L(Q, P)f](q)=(2\pi)^{-n/2}\int d\xi \hat{f}(\xi) W_L(q, \xi) \exp[iq\cdot\xi]$$

where $\hat{f}=\mathcal{F}f$, the Fourier transform of f, is given by

$$\hat{f}(\xi)=(2\pi)^{-n/2}\int dy\, f(y) \exp[-iy\cdot\xi].$$

Also $\mathrm{Dom}\, W_L \supset \mathcal{S}$. [This may impose some growth restrictions on ξ of $W_L(x, \xi)$].

A4: (The self adjointness). The operator $H=H_0+W_s+W_L(Q, P)$ which is defined at least on $\mathcal{S}$ is symmetric and has a self adjoint extension denoted by the same letter H.

Now denote by U_t and V_t the free and total evolutions given by

$$U_t=\exp[-itH_0], \quad V_t=\exp[-itH].$$

Define $G=\{\xi \text{ in } R^n : \nabla h_0(\xi) \neq 0\}$.

Now we have

THEOREM 2.1. *Let the above conditions be satisfied. Then there exists a C^∞ function $X: R\times R^n \to R$ so that for f in $\mathcal{F}^{-1} L^2(G)$ we have*

(i) $\Omega_{\pm}f = \text{s-}\lim_{t\to\pm\infty} V_t^* \exp[-iX(t, P)]f$ *exists*

(ii) $\|\Omega_{\pm}f\| = \|f\|$

(iii) $V_s\Omega_{\pm}f = \Omega_{\pm}U_s f$ *for all real* s.

(iv) $\Omega_{\pm}f \in \mathcal{H}_{ac}(H)$, *the absolutely continuous subspace for* H.

We prove the theorem in § 3 using the techniques of (i) non-stationary phase (Theorem XI. 14 of [1] or Lemma A.1 of [11]) and (ii) oscillatory integerals [13].

§ 3. Proof of Theorem 2.1.

Following [12] we make a time dependent cut off on the long range potential W_L. Choose χ_0 in $C^\infty(R^n)$, a real valued function, such that $\chi_0 = 0$ for $|x| \le 1$ and 1 for $|x| \ge 2$. Define $W(t, x, \xi)$ by

$$W(t, x, \xi) = \chi_0(\log\langle t\rangle x/\langle t\rangle) W_L(x, \xi) \tag{1}$$

where $\langle t\rangle = (1+t^2)^{1/2}$. From the assumption A3 it is easy to see that for δ_0 in $(0, \delta)$

$$\sup_{\xi \text{ in } B} |D_x^\alpha D_\xi^\beta W(t, x, \xi)| \le K(B, \alpha, \beta)\langle t\rangle^{-|\alpha|-\delta_0} \tag{2}$$

for bounded sets B of R^n. Now without loss of generality we can assume that $\delta_0 \notin \{1, 1/2, 1/3, \cdots\}$.
Choose the positive integer m_0 such that

$$m_0\delta_0 < 1 < (m_0+1)\delta_0. \tag{3}$$

Now define $Y(m, t, \xi)$ for $m = 1, \cdots, m_0$, $t \ge 0$ by

$$\begin{cases} Y(0, t, \xi) = 0 \\ Y(m, t, \xi) = \int_0^t d\tau W(\tau, \tau h_0'(\xi) + Y_\xi'(m-1, \tau, \xi), \xi) \\ X(t, \xi) = th_0(\xi) + Y(m_0, t, \xi). \end{cases} \tag{4}$$

Lemma 3.1.

(i) *For any bounded subset* B *of* R^n *we have*
$\sup_{\xi \text{ in } B} |D_\xi^\alpha Y(m, t, \xi)| \le K(B, m, \alpha)\langle t\rangle^{1-\delta_0}$.

(ii) $\lim_{t\to\infty} Y(m_0, t+s, \xi) - Y(m_0, t, \xi) = 0$ *for each* s *in* R, ξ *in* R^n.

Proof. (i) The proof is by induction on m. For $m=0$ it is clear. Assume the result for $m-1$. Now when $|\alpha|=0$ the result is obvious for m. So assume $|\alpha|\geq 1$. Put

$$Z=Z(\tau, \xi)=\tau h_0'(\xi)+Y_\xi'(m-1, \tau, \xi)$$

so that

$$Y(m, t, \xi)=\int_0^t d\tau\, W(\tau, Z, \xi). \tag{5}$$

Now by the induction hypothesis we have, for ξ in B,

$$|D_\xi^\alpha Z(\tau, \xi)|\leq K(\alpha)\langle\tau\rangle. \tag{6}$$

By induction on $|\alpha|$ we can easily prove that $D_\xi^\alpha\{W(\tau, Z(\tau, \xi), \xi)\}$ is a finite linear combination of terms of the form $W_{i,j}(\tau, Z, \xi)\, D_\xi^{\alpha_1}Z\cdots D_\xi^{\alpha_k}Z$ where

(a) $W_{i,j}(\tau, x, \xi)=D_\xi^j D_x^i W(\tau, x, \xi)$.

(b) $1\leq|i+j|\leq|\alpha|$.

(c) The product $D_\xi^{\alpha_1}Z\cdots D_\xi^{\alpha_k}Z$ may or may not appear. If it appears, then

(d) $k\leq|i|$

(e) $1\leq|\alpha_1|, \cdots, |\alpha_k|\leq|\alpha|$.

Now for any typical term $W_{i,j}(\tau, Z, \xi)D_\xi^{\alpha_1}Z\cdots D_\xi^{\alpha_k}Z$ we have using (2) and (6)

$$\sup_{\xi \text{ in } B} |W_{i,j}(\tau, Z, \xi)\cdot D_\xi^{\alpha_1}Z\cdots D_\xi^{\alpha_k}Z|\leq K\langle\tau\rangle^{-\delta_0}. \tag{7}$$

Now the result follows from (5) and (7).

(ii) Clearly for ξ in R^n we get

$$\begin{aligned}&|Y(m_0, t+s, \xi)-Y(m_0, t, \xi)|\\ &=\left|\int_t^{t+s} d\tau\, W(\tau, \tau h_0'(\xi)+Y_\xi'(m_0-1, \tau, \xi), \xi)\right|\leq K(\xi)\int_t^{t+s} d\tau\langle\tau\rangle^{-\delta_0}.\end{aligned}$$

The result easily follows since $\lim_{t\to\infty}\int_t^{t+s} d\tau\langle\tau\rangle^{-\delta_0}=0$. Q.E.D.

LEMMA 3.2.

(i) *If B is any bounded subset of R^n and $m\geq 1$ then*

$$\sup_{\xi \text{ in } B} |D_\xi^\alpha\{Y(m, t, \xi)-Y(m-1, t, \xi)\}|\leq K(B, m, \alpha)\langle t\rangle^{1-m\delta_0}.$$

(ii) *For B as in* (i) *we have*

$$\sup_{\xi \text{ in } B} |W(t, X'_\xi(t, \xi), \xi) - W(t, th'_0(\xi) + Y'_\xi(m_0 - 1, t, \xi), \xi)| \le K(B)\langle t\rangle^{-(m_0+1)\delta_0}.$$

(iii) *For any given* $b>0$ *choose* $t_0 = t_0(b) \ge 0$ *such that* $2/\log\langle t_0\rangle = b$. *Then for* $t \ge t_0$ *and any* g *in* $L^2(R^n)$ *we get, with* $F(M)$ *denoting the indicator function of the set* M,

$$\|[W_L(Q, P) - W(t, Q, P)]g\| \le \{1 + \|\chi_0\|_\infty\} \|F(|Q| \le b\langle t\rangle) W_L(Q, P) g\|.$$

PROOF. (i) For $m=1$ the result follows by Lemma 3.1 (i). For $m \ge 2$, clearly

$$\begin{aligned}
&Y(m, t, \xi) - Y(m-1, t, \xi)\\
&= \int_0^t d\tau \int_0^1 d\rho [(\nabla_x W)(\tau, \tau h'_0(\xi) + \rho Y'_\xi(m-1, \tau, \xi) + (1-\rho) Y'_\xi(m-2, \tau, \xi), \xi)]\\
&\quad \times [Y'_\xi(m-1, \tau, \xi) - Y'_\xi(m-2, \tau, \xi)].
\end{aligned}$$

Now the proof follows by induction on m by using the techniques of the proof of Lemma 3.1 (i).
(ii) Follows from (i) by writing

$$\begin{aligned}
&W(t, th'_0(\xi) + Y'_\xi(m_0, t, \xi), \xi) - W(t, th'_0(\xi) + Y'_\xi(m_0 - 1, t, \xi), \xi)\\
&= \int_0^1 d\rho [(\nabla_x W)(t, th'_0(\xi) + \rho Y'_\xi(m_0, t, \xi) + (1-\rho) Y'_\xi(m_0 - 1, t, \xi), \xi)]\\
&\quad \times [Y'_\xi(m_0, t, \xi) - Y'_\xi(m_0 - 1, t, \xi)].
\end{aligned}$$

(iii) From the equation (1) we see that

$$W_L(Q, P) - W(t, Q, P) = [1 - \chi_0(Q \log\langle t\rangle / \langle t\rangle)] W_L(Q, P).$$

Now, using the support property of χ_0, we conclude the proof.

Q. E. D.

Now we introduce an evolution concentrated in a fixed compact subset in the momentum representation. For any real valued C^∞ function ϕ: $R^n \to R$ define

$$(8) \quad \begin{cases}
\phi(\xi) = h_0(\xi)\varphi(\xi)\\
W(\varphi, t, x, \xi) = W(t, x, \xi)\varphi(\xi)\\
Y(\varphi, 0, t, \xi) = 0\\
Y(\varphi, m, t, \xi) = \int_0^t d\tau\ W(\varphi, \tau, \tau\phi'(\xi) + Y'_\xi(\varphi, m-1, \tau, \xi), \xi)\\
X(\varphi, t, \xi) = t\phi(\xi) + Y(\varphi, m_0, t, \xi).
\end{cases}$$

Note that if $\varphi = 1$ then $X(\varphi, t, \xi) = X(t, \xi)$. Now we restrict our attention when $\varphi \in C_0^\infty(R^n)$.

LEMMA 3.3. *Let* $\varphi \in C_0^\infty(R^n)$ *be real valued,* ϕ, $W(\varphi, \cdots)$, $Y(\varphi, \cdots)$, $X(\varphi, \cdots)$ *be as above. Then*

(i) $\sup_\xi |D_\xi^\alpha Y(\varphi, m, t, \xi)| \leq K(m, \alpha)\langle t\rangle^{1-\delta_0}$ *for* $m=1, \cdots, m_0$

(ii) $\sup_\xi |D_\xi^\alpha [Y(\varphi, m, t, \xi) - Y(\varphi, m-1, t, \xi)]| \leq K(m, \alpha)\langle t\rangle^{1-m\delta_0}$ *for* $m=1, \cdots, m_0$

(iii) $\sup_\xi |W(\varphi, t, X_\xi'(\varphi, t, \xi), \xi) - W(\varphi, t, t\phi'(\xi) + Y_\xi'(\varphi, m_0-1, t, \xi), \xi)| \leq K\langle t\rangle^{-(m_0+1)\delta_0}$

(iv) $\sup_\xi |D_\xi^\alpha X(\varphi, t, \xi)| \leq K(\alpha)\langle t\rangle$.

(v) *Suppose B is an open set such that $\varphi=1$ on B. Then* $X(\varphi, t, \xi) = X(t, \xi)$ *for ξ in B.*

PROOF. For (i), (ii), and (iii) use the techniques in the proofs of Lemma 3.1, 3.2. The proof of (iv) is clear by (i). For (v) clearly it suffices to show

$$D_\xi^\alpha Y(\varphi, m, t, \xi) = D_\xi^\alpha Y(m, t, \xi) \quad \text{for } \xi \text{ in } B,\ |\alpha| \geq 0 \text{ and each } m.$$

For this we use induction on m. For $m=0$ it is clear. Assume the result to be true for $m-1$. Then

$$\begin{aligned} Y(\varphi, m, t, \xi) &= \int_0^t d\tau\ W(\tau, \tau\phi'(\xi) + Y_\xi'(\varphi, m-1, \tau, \xi), \xi)\varphi(\xi) \\ &= \int_0^t d\tau\ W(\tau, \tau h_0'(\xi) + Y_\xi'(\varphi, m-1, \tau, \xi), \xi) \quad \text{for } \xi \text{ in } B \\ &= \int_0^t d\tau\ W(\tau, \tau h_0'(\xi) + Y_\xi'(m-1, \tau, \xi), \xi) \quad \text{for } \xi \text{ in } B. \text{ Here we have used the induction hypothesis} \\ &= Y(m, t, \xi) \quad \text{for } \xi \text{ in } B. \end{aligned}$$

So clearly $D_\xi^\alpha Y(\varphi, m, t, \xi) = D_\xi^\alpha Y(m, t, \xi)$ for ξ in B. Q.E.D.

Now define the two evolutions Z_t, $Z_{\varphi,t}$ by

$$(9) \quad \begin{cases} Z_{\varphi,t}(\xi) = \exp[-iX(\varphi, t, \xi)], & Z_t(\xi) = \exp[-iX(t, \xi)] \\ Z_{\varphi,t} = Z_{\varphi,t}(P), & Z_t = Z_t(P). \end{cases}$$

LEMMA 3.4. *Let* $f \in \mathcal{S}$ *be such that* $\hat{f} \in C_0^\infty(G)$. *Choose a real valued* φ *in* $C_0^\infty(G)$ *such that* $\varphi=1$ *on some open neighborhood of* $\operatorname{supp} \hat{f}$. *Then*

(i) $\operatorname{s-lim}_{t\to\infty} V_t^* Z_t f$ *exists iff* $\operatorname{s-lim}_{t\to\infty} V_t^* Z_{\varphi,t} f$ *exists*

(ii) *if either limit exists then* $\operatorname{s-lim}_{t\to\infty} V_t^* Z_t f = \operatorname{s-lim}_{t\to\infty} V_t^* Z_{\varphi,t} f$.

PROOF. Follows from Lemma 3.3 (v) since $Z_t f = Z_{\varphi,t} f$. Q. E. D.

LEMMA 3.5. *Let* f, φ *be as in Lemma* 3.4. *Then*

$$\int_1^\infty dt \left\| \frac{d}{dt} V_t^* Z_{\varphi,t} f \right\| < \infty.$$

PROOF. Note that

$$\varphi(P) f = f, \quad Z_{\varphi,t} f = Z_t f. \tag{10}$$

Clearly $(Z_{\varphi,t} f)^\wedge \in C_0^\infty(G)$ and so $Z_{\varphi,t} f \in \mathcal{S} \subset \mathrm{Dom}\, H$.
Thus we get

$$\begin{aligned}
&-iV_t \frac{d}{dt}\{V_t^* Z_{\varphi,t} f\} \\
&= \{H_0 + W_s + W_L(Q, P)\} Z_{\varphi,t} f \\
&\quad - \{H_0 \varphi(P) + W(\varphi, t, t\phi'(P) + Y_P'(\varphi, m_0 - 1, t, P), P)\} Z_{\varphi,t} f \\
&= W_s Z_t f + [W_L(Q, P) - W(t, Q, P)] Z_t f \\
&\quad + [W(t, Q, P) - W(t, X_P'(\varphi, t, P), P)] Z_{\varphi,t} f \\
&\quad + Z_t [W(\varphi, t, X_P'(\varphi, t, P), P) - W(\varphi, t, t\phi'(P) + Y_P'(\varphi, m_0 - 1, t, P), P)] f.
\end{aligned}$$

In the last step we have used (10).

Noting $(m_0 + 1)\delta_0 > 1$ and using Lemma 3.3 (iii) we get

$$\begin{aligned}\int_1^\infty dt \| Z_t [W(\varphi, t, X_P'(\varphi, t, P), P) \\ - W(\varphi, t, t\phi'(P) + Y_P'(\varphi, m_0 - 1, t, P), P)] f \| < \infty.\end{aligned} \tag{12}$$

By observing $\hat{f} \in C_0^\infty(G)$, the short range condition A2 and using the techniques of non-stationary phase (Theorem XI.14 of [1], or Lemma A.1 of [11]) we conclude

$$\int_1^\infty dt \, \| W_s Z_t f \| < \infty. \tag{13}$$

Note that $\hat{f} \in C_0^\infty(G)$. Now the method of non-stationary phase together with Lemma 3.2 (iii) gives

$$\int_1^\infty dt \, \| [W_L(Q, P) - W(t, Q, P)] Z_t f \| < \infty. \tag{14}$$

By (11–14) we are reduced to the study of the evolution of

$$[W(t, Q, P) - W(t, X'_P(\varphi, t, P), P)]Z_{\varphi,t}f.$$

Let B be any open set such that

$$\operatorname{supp}\hat{f} \subset B \quad \text{and} \quad \varphi = 1 \text{ on } B.$$

Then $X(\varphi, t, \xi) = X(t, \xi)$ on B. Clearly

$$a = \inf\{|X'_\xi(t, \xi)|\langle t\rangle^{-1} : \xi \in B\} > 0.$$

Now choose a real valued η in $C_0^\infty(R^n)$ such that $\eta = 1$ on $|x| \leq a/2$ and 0 on $|x| \geq a$. Define $f_1(t), f_2(t)$ by

$$\begin{aligned}(15) \qquad f_1(t, q) = (2\pi)^{-n/2}\int d\xi\, W(t, q, \xi)\hat{f}(\xi)\{1 - \eta(t^{-1}[q - X'_\xi(\varphi, t, \xi)])\} \\ \times \exp(i[q \cdot \xi - X(\varphi, t, \xi)])\end{aligned}$$

$$(16) \qquad f_1(t) + f_2(t) = [W(t, Q, P) - W(t, X'_P(\varphi, t, P), P)]Z_{\varphi,t}f.$$

Note that $f_1(t)$ can also be written as

$$\begin{aligned}f_1(t, q) = (2\pi)^{-n/2}\int d\xi\, W(t, q, \xi)\hat{f}(\xi)\{1 - \eta(t^{-1}[q - X'_\xi(t, \xi)])\} \\ \times \exp(i[q \cdot \xi - X(t, \xi)]).\end{aligned}$$

By using the last expression for f_1, the method of non-stationary phase and observing $\hat{f} \in C_0^\infty(G)$ we conclude that

$$(17) \qquad \int_1^\infty dt\, \|f_1(t)\| < \infty.$$

In Lemma 3.7 we show that

$$(18) \qquad \int_1^\infty dt\, \|f_2(t)\| < \infty.$$

Now the result follows by (11–18). Q. E. D.

REMARK 3.6. So far we have used the method of non-stationary phase as in [11]. At this stage the author of [11] assumes that $\{\xi : \det h_0''(\xi) = 0\}$ is a proper subset of R^n; developes the method of stationary phase (Lemma A.4 of [11]); estimates $f_2(t, q)$ and thereby $\|f_2(t)\|$. In other words in [11], $f_2(t)$ is estimated in the *position* representation. We shall estimate $f_2(t)$ in the *momentum* representation i.e. find $\hat{f}_2(t, \xi)$, estimate $|\hat{f}_2(t, \xi)|$ and hence $\|f_2(t)\|$. $f_3(t)$ of (25) is seen to be $f_3(t, \xi) = \hat{f}_2(t, \xi) \exp[iX(\phi, t, \xi)]$. This slightly different approach enables us not to impose any condition on $\det h_0''(\xi)$. Instead of the method of stationary phase, we employ the techniques of oscillatory integrals [13].

LEMMA 3.7. (18) *holds.*

PROOF. Let us remark that only for proving this Lemma we introduced the auxiliary evolution $Z_{\phi,t}$.

From (15) and (16) we easily see that

$$(19)\quad f_2(t,q)=(2\pi)^{-n/2}\int d\xi\hat{f}(\xi)\{W(t,q,\xi)\eta(t^{-1}[q-X_\xi'(\phi,t,\xi)]) \\ -W(t,X_\xi'(\phi,t,\xi),\xi)\}\exp(i[q\cdot\xi-X(\phi,t,\xi)]).$$

Clearly

$$(20)\quad \|f_2(t)\|=\sup\{|\langle f_2(t),g\rangle|:\ g\in\mathcal{S},\ \|g\|\leq 1\}.$$

Now for any g in $\mathcal{S}$ we have

$$(21)\quad \langle f_2(t),g\rangle=\langle\hat{f}_2(t),\hat{g}\rangle \\ =\int d\lambda dq d\xi\ \bar{\hat{g}}(\lambda)W(t,q,\xi)\eta(t^{-1}[q-X_\xi'(\varphi,t,\xi)])\hat{f}(\xi) \\ \times\exp[-i\lambda q+iq\xi-iX(\varphi,t,\xi)] \\ -\int d\xi\ \bar{\hat{g}}(\xi)W(t,X_\xi'(\varphi,t,\xi),\xi)\hat{f}(\xi)\exp[-iX(\varphi,t,\xi)].$$

Now we change the variables in the first summand of RHS of (21). For this introduce $Y(t,\lambda,\xi)$ by

$$(22)\quad Y(t,\lambda,\xi)=\int_0^1 d\rho\ X_\xi'(\varphi,t,\rho\lambda+(1-\rho)\xi)$$

so that

$$(23)\quad X(\varphi,t,\lambda)-X(\varphi,t,\xi)=(\lambda-\xi)\cdot Y(t,\lambda,\xi).$$

Now in the first summand of RHS of (21) interchange λ and ξ, then change λ to $\lambda+\xi$ and q to $q+Y(t,\lambda+\xi,\xi)$.

Then we get

$$(24)\quad \langle f_2(t),g\rangle=\int d\xi\ \bar{\hat{g}}(\xi)\exp[-iX(\varphi,t,\xi)]f_3(t,\xi)$$

where $f_3(t)$ is given by

$$f_3(t, \xi)=\int dqd\lambda \exp[iq\cdot\lambda]W(t, q+Y(t, \lambda+\xi, \xi), \lambda+\xi) \times\eta(t^{-1}[q+Y(t, \lambda+\xi, \xi)-X'_\xi(\varphi, t, \lambda+\xi)])\hat{f}(\lambda+\xi) - W(t, X'_\xi(\varphi, t, \xi), \xi)\hat{f}(\xi). \tag{25}$$

Note that by (24) and (20) we get

$$\|f_2(t)\|\leq\|f_3(t)\|. \tag{26}$$

Thus it suffices to analyse $f_3(t)$. For this we use the notion of oscillatory integral, § 6, Chapter 1 of [13]. Note that by Lemma 3.3 (iv) we have

$$\sup_{\lambda,\xi}|D^\alpha_\lambda Y(t, \lambda+\xi, \xi)|+|D^\alpha_\lambda X(\varphi, t, \lambda+\xi)|\leq K_\alpha\langle t\rangle. \tag{27}$$

By using (27) it is easy to see, for each fixed $t\geq 1$ and ξ, that $W(t, q+Y(t, \lambda+\xi, \xi), \lambda+\xi)\eta(t^{-1}[q+Y(t, \lambda+\xi, \xi)-X'_\xi(\varphi, t, \lambda+\xi)])\hat{f}(\lambda+\xi)$ and $W(t, Y(t, \lambda+\xi, \xi), \lambda+\xi)\eta(t^{-1}[Y(t, \lambda+\xi, \xi)-X'_\xi(\varphi, t, \lambda+\xi)])\hat{f}(\lambda+\xi)$ are both in $\mathcal{A}^0_{0,0}$ of Definition 6.1 of [13], as function of q, λ.

So, noting $\eta(0)=1$ we clearly have

$$\begin{aligned} &W(t, X'_\xi(\varphi, t, \xi), \xi)\hat{f}(\xi) \\ &=osc\int dqd\lambda\, e^{iq\lambda}W(t, Y(t, \lambda+\xi, \xi), \lambda+\xi) \\ &\quad\times\eta(t^{-1}[Y(t, \lambda+\xi, \xi)-X'_\xi(\varphi, t, \lambda+\xi)])\hat{f}(\lambda+\xi). \end{aligned} \tag{28}$$

Now from (25) and (28) we get

$$\begin{aligned} f_3(t, \xi)&=\int_0^1 d\rho\, osc\int dqd\lambda\, e^{iq\lambda}\hat{f}(\lambda+\xi)\frac{d}{d\rho}\{W(t, \rho q+Y(t, \lambda+\xi, \xi), \lambda+\xi) \\ &\quad\times\eta(t^{-1}[\rho q+Y(t, \lambda+\xi, \xi)-X'_\xi(\varphi, t, \lambda+\xi)])\} \\ &=i\sum_{j=1}^n\int_0^1 d\rho\, osc\int dqd\lambda\, e^{iq\lambda} \quad \text{sum of two terms} \end{aligned} \tag{29}$$

where

first term

$$\begin{aligned} &=D_{\lambda_j}\{W_j(t, \rho q+Y(t, \lambda+\xi, \xi), \lambda+\xi) \\ &\quad\times\eta(t^{-1}[\rho q+Y(t, \lambda+\xi, \xi)-X'_\xi(\varphi, t, \lambda+\xi)])\hat{f}(\lambda+\xi)\} \end{aligned}$$

second term

$$\begin{aligned} &=t^{-1}D_{\lambda_j}\{W(t, \rho q+Y(t, \lambda+\xi, \xi), \lambda+\xi) \\ &\quad\times\eta_j(t^{-1}[\rho q+Y(t, \lambda+\xi, \xi)-X'_\xi(\varphi, t, \lambda+\xi)])\hat{f}(\lambda+\xi)\} \end{aligned}$$

with $W_j(t, x, \xi) = D_{x_j} W(t, x, \xi)$, $\eta_j(x) = D_{x_j}\eta(x)$. In the last step we have used Theorem 6.7, Chapter 1 of [13].

Now apply Theorem 6.4, Chapter 1 of [13] to (29) using (27) to get

$$\sup_{\xi} |f_3(t, \xi)| \leq K(f)\langle t\rangle^{-1-\delta_0} \qquad \text{for } t \geq 1. \tag{30}$$

Note that for any $r = 1, 2, \cdots$

$$\xi_j^{2r} = \sum_{k=0}^{2r} \binom{2r}{k} (\xi_j + \lambda_j)^k \lambda_j^{2r-k} (-1)^k. \tag{31}$$

Use the above relation (31) in (29) apply Theorems 6.7; and 6.4 of Chapter 1 of [13] to see that, remembering (27),

$$\sup_{\xi} |f_3(t, \xi)| (\xi_1^{2r} + \cdot + \xi_n^{2r}) \leq K_r(f)\langle t\rangle^{-1-\delta_0} \qquad \text{for } t \geq 1. \tag{32}$$

From (30) and (32) we get

$$\|f_3(t)\| \leq K(f)\langle t\rangle^{-1-\delta_0} \qquad \text{for } t \geq 1. \tag{33}$$

The result now easily follows from (26) and (33). Q. E. D.

Now we are ready to prove Theorem 2.1. We prove it for the positive sign only. For the other sign it is similar.

PROOF OF THEOREM 2.1 (i). By Lemma 3.4, 3.5 and the denseness of $C_0^\infty(G)$ in $L^2(G)$ the result follows.

PROOF OF THEOREM 2.1 (ii). Easily follows from (i).

PROOF OF THEOREM 2.1 (iii). Follows from (i) and Lemma 3.1 (ii).

PROOF OF THEOREM 2.1 (iv). By Theorem 1 of [14] we have $\mathcal{H}_{ac}(H_0) = \mathcal{F}^{-1}L^2(G)$. Now the result follows by (iii). Q. E. D.

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Note added in proof

REMARK 1. After submitting the article we have learned about [15]. Theorem 30.4.1 of [15] is similar to our Theorem 2.1. The techniques of [15] are similar to ours.

REMARK 2. Taylor's expansion with an estimate for the remainder is obtained for

$$\exp[it\sqrt{(1+p^2)}]W(Q)\exp[-it\sqrt{(1+p^2)}]f$$

in [16] where W is a smooth long range potential and f a "nice" vector. In [16] the proof is purely operator theoretic. By improving the method of [16] we can give a purely operator theoretic proof (without using the notion of oscillatory integrals) for our main theorem 2.1 (i). The details will appear in an addendum.

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