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Existence of singular solutions and null solutions for linear partial differential operators

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Let $P(z, \partial)$ be a linear partial differential operator with coefficients holomorphic in $U, U \subset C^{n+1}$, and $K=\{\Phi(z)=0\}$ be a nonsingular hypersurface in U. In the present paper we study the equation

$$(0.1) \qquad P(z, \partial)u(z)=f(z),$$

where $f(z)$ is holomorphic on $U-K$.

One of our two main results in this paper is that there is a solution $u(z)$ to (0.1) with singularity on K and we get a bound of the growth order of $u(z)$ near K. We obtain it, by using the localization $P_{\mathrm{loc},K,0}$ and the characteristic indices $\sigma_1, \sigma_{1,1}$ introduced in [14], [15] (Theorems 1.4 and 1.5).

The other one is the existence of null solutions for $P(x, \partial)u(x)=0$ in the real domain (Theorem 1.8), which is an application of the first result. Most parts of the contents of this paper have been announced in [14].

§ 1. Notations and summary.

Let C^{n+1} be the $(n+1)$-dimensional complex space; $z=(z_0, z_1, \cdots, z_n)=(z_0, z_1, z'')=(z_0, z')$ denotes its point while $\xi=(\xi_0, \xi_1, \cdots, \xi_n)=(\xi_0, \xi')$ is the variable dual to ξ. We shall use the notation $\partial=(\partial_0, \partial_1, \cdots, \partial_n)=(\partial_0, \partial')$, $\partial_i=\partial/\partial_{z_i}$. The multi-index $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_n)=(\alpha_0, \alpha')$ is an $(n+1)$-tuple of nonnegative integers, $|\alpha|=\alpha_0+\alpha_1+\cdots+\alpha_n$, and $\partial^\alpha=(\partial_0)^{\alpha_0}(\partial_1)^{\alpha_1}\cdots(\partial_n)^{\alpha_n}=(\partial_0)^{\alpha_0}(\partial')^{\alpha'}$. For a real number a, $[a]$ means its integral part. N is the set of natural numbers and Z is the set of integers. For a linear partial differential operator $A(z, \partial)$, $A(z, \xi)$ denotes its total symbol.

Let $P(z, \partial)$ be a linear partial differential operator of order m whose coefficients are holomorphic in an open set U of C^{n+1} containing $z=0$. Let $K=\{\Phi(z)=0\}$ be a nonsingular complex hypersurface through $z=0$, that is, $\Phi(0)=0$, $d\Phi(0)\neq 0$. First let us define $\sigma_1, \sigma_{1,1}$ and $P_{\mathrm{loc},K,0}$, introduced in [14], [15]. For simplicity, we may choose the coordinate so that $\Phi(z)=z_0$, hence $K=\{z_0=0\}$. Put

$$\begin{cases} P(z,\partial)=\sum_{k=0}^{m} P_k(z,\partial), \\ P_k(z,\partial)=\sum_{l=0}^{k} A_{k,l}(z,\partial')(\partial_0)^{k-l}, \end{cases} \tag{1.1}$$

where $P_k(z,\xi)$ $(A_{k,l}(z,\xi'))$ is homogeneous in ξ (resp. ξ') with degree k (resp. l). We develop $A_{k,l}(z,\xi')$ with respect to z_0 at $z_0=0$:

$$A_{k,l}(z,\xi')=\sum_{j=0}^{\infty} A_{k,l,j}(z',\xi')z_0^j. \tag{1.2}$$

Put

$$\begin{cases} d_k=\min\{l+j\,;\, A_{k,l,j}(z',\xi')\not\equiv 0\}, \\ J_k=\min\{j\,;\, l+j=d_k,\ A_{k,l,j}(z',\xi')\not\equiv 0\}, \\ L_k=d_k-J_k,\ \ J=J_m,\ \ L=L_m. \end{cases} \tag{1.3}$$

If $P_k(z,\xi)\equiv 0$, $d_k=J_k=-L_k=+\infty$. Now we give

DEFINITION 1.1. We call $A_{m,L,J}(z',\partial')$ *the localization on* K of $P(z,\partial)$, which is denoted by $P_{\mathrm{loc},K,0}$.

We put

$$\sigma_1=\max\{1,\ (d_m-d_k)/(m-k)\,;\ 0\le k\le m-1\} \tag{1.4}$$

and, moreover, if $\sigma_1=1$,

$$\begin{cases} \sigma_{1,1}=\max\{1,\ (J_m-J_k)/(m-k)\,;\ k\in B\}, \\ B=\{k\,;\ d_m-d_k=m-k,\ k\neq m\}. \end{cases} \tag{1.5}$$

DEFINITION 1.2. We call σ_1 *the first characteristic index of* K, and $\sigma_{1,1}$ *the* (1.1)*-characteristic index of* K for $P(z,\partial)$.

REMARK 1.3. Other characteristic indices and localizations have been defined in [13], [14], [15]. $\sigma_{1,1}$ is called the first subcharacteristic index in [14].

We prepare some function spaces:

$$(1.6)\quad \begin{cases} \tilde{\mathcal{O}}(U-K)=\{\text{the space of all holomorphic functions on the universal covering space of } U-K\}, \\ \tilde{\mathcal{O}}_{\gamma}(U-K)=\{f(z)\in\tilde{\mathcal{O}}(U-K)\,;\ \text{for any } \alpha,\beta \text{ there are constants } A_{\alpha,\beta} \\ \quad \text{and } c \text{ such that for } z\in U-K \text{ with } \alpha<\arg z_0<\beta, \\ \quad |f(z)|\leq A_{\alpha,\beta}\exp(c|z_0|^{-\gamma})\}, \\ \tilde{\mathcal{O}}_{(0,\delta)}(U-K)=\{f(z)\in\tilde{\mathcal{O}}(U-K)\,;\ \text{for any } \alpha,\beta \text{ there are constants} \\ \quad A_{\alpha,\beta} \text{ and } c \text{ such that for } z\in U-K \text{ with } \alpha<\arg z_0<\beta, \\ \quad |f(z)|\leq A_{\alpha,\beta}\exp(c|\log z_0|^{\delta})\}. \end{cases}$$

Now let us consider

$$(1.7)\qquad P(z,\partial)u(z)=f(z),$$

where $f(z)\in\tilde{\mathcal{O}}(U-K)$. Let us assume

$$(1.8)\qquad P_{\mathrm{loc},K,0}(0,\xi')=A_{m,L,J}(0,\xi')\not\equiv 0.$$

We may assume by coordinate transformation,

$$(1.8)'\qquad A_{m,L,J}(0,\xi')|_{\xi'=(1,0,\cdots,0)}\neq 0.$$

Thus we deal with the problem

$$(1.9)\qquad \begin{cases} P(z,\partial)u(z)=f(z), \\ (\partial_1)^h u(z_0,0,z'')=\varphi_h(z_0,z'') \quad \text{for } 0\leq h\leq L-1, \end{cases}$$

where $f(z)\in\tilde{\mathcal{O}}(U-K)$, $\varphi_h(z_0,z'')\in\tilde{\mathcal{O}}((U-K)\cap\{z_1=0\})$.

We have:

THEOREM 1.4. *Under condition* (1.8)′, *there is a solution* $u(z)\in\tilde{\mathcal{O}}(\Omega-K)$ *of* (1.9), *where* Ω *is a neighbourhood of* $z=0$ *which is independent of* $f(z)$ *and* $\varphi_h(z_0,0,z'')$.

Moreover, let $\varphi_h(z_0,0,z'')=0$ *for* $0\leq h\leq L-1$. *Then*
(a) *if* $\sigma_1>1$ *and* $f(z)\in\tilde{\mathcal{O}}_{\sigma_1-1}(U-K)$, $u(z)$ *is actually in* $\tilde{\mathcal{O}}_{\sigma_1-1}(\Omega-K)$,
(b) *if* $\sigma_1=1$ *and* $f(z)\in\tilde{\mathcal{O}}_{(0,\sigma_{1,1})}(U-K)$, $u(z)$ *is actually in* $\tilde{\mathcal{O}}_{(0,\sigma_{1,1})}(\Omega-K)$.

Thus we have

THEOREM 1.5. *If* $P_{\mathrm{loc},K,0}(z',\xi')$ *is noncharacteristic for some* ξ' *at* $z'=0$ ((1.8) *holds*), *then there is a solution* $u(z)\in\tilde{\mathcal{O}}(\Omega-K)$ *to* (1.7), *and* (a) *and* (b) *in Theorem* 1.4 *hold.*

REMARK 1.6. We have defined $P_{\mathrm{loc},K,0}$, σ_1, $\sigma_{1,1}$ in [15] by another method without using special coordinates. These definitions are invariant under coordinate transformations. The condition (1.8) is equivalent to that the principal symbol of $P_{\mathrm{loc},K,0}$ does not identically vanish as a function in the cotangent-space at $z=0$.

REMARK 1.7. Equations (1.7) and (1.9) were investigated by many authors. The existence of $u(z)$ was studied in [2], [3], [4], [5], [20] for operators with constant multiple characteristics. For less restrictive operators, we refer to [7], [17], [18], [19]. But the condition (1.8) is much weaker than those assumed in above-mentioned papers. For example, Theorem 1.4 is applicable to operators with the principal symbol vanishing on K, such as

$$P(z,\partial)=(z_0)^2(\partial_0)^3(\partial_1)^2+a(z)(\partial_1)^5+\text{lower order terms}. \tag{1.10}$$

The behaviour near K of homogeneous solutions with singularity on K was investigated in [11], [12].

We also show the existence of null solutions in real domains. Let x denote the point in R^n. Let $K_R=\{x\in U_R\subset R^{n+1};\ \Phi(x)=0\}$ be a nonsingular real analytic hypersurface in R^{n+1} which is characteristic for $P(x,\partial)$, where $\Phi(x)$ is a real valued analytic function in an open set U_R containing $x=0$, with $\Phi(0)=0$ and $d\Phi(0)\neq 0$. So $\Phi(x)$ is holomorphically extended in a complex neighbourhood of the origin. We can assume $K_R=\{x_0=0\}$ by means of a coordinate transformation. We have:

THEOREM 1.8. *Assume that* (1.8)′ *holds and* $\mathrm{ord.}\,A_{m,L,J}(x',\partial')=L\geq 1$. *Then there is a function* $u(x)$, *which is* C^∞ *in a neighbourhood* ω *of* $x=0$ *and analytic except on* K_R, *such that*

$$\begin{cases} P(x,\partial)u(x)=0 \\ \mathrm{supp.}\ u(x)\subset\{x\in\omega;\ x_0\geq 0\}, \\ \mathrm{supp.}\ u(x)\ni\{x=0\}. \end{cases} \tag{1.11}$$

REMARK 1.9. The existence of null solutions was shown for operators with simple characteristics in [6], [9], [10]. Null solutions were constructed for operators with constant multiple characteristics in [1], [8]. The previous results were generalized in [16], [19]. But Theorem 1.8 covers all the above-mentioned results.

Although we shall show Theorems 1.4 and 1.8 in the following sections, let us here sketch the outline of proofs. First we note that

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(e^{\zeta},z')}{(\zeta-\log z_0)}d\zeta \tag{1.12}$$

and

$$(\zeta-\log z_0)^{-1}=\frac{1}{2\pi i}\int_{C(\theta)}\exp(-\lambda\zeta)(z_0)^{\lambda}(\log\lambda)d\lambda. \tag{1.13}$$

Here Γ is a path enclosing once $\zeta=\log z_0$, and $C(\theta)$ is a path which starts from $\infty\exp(i\theta)$ and goes around $\lambda=0$ once and ends at $\infty\exp(i(\theta+2\pi))$. θ depends on $\arg(\zeta-\log z_0)$. So

$$f(z)=(2\pi i)^{-2}\int_{\Gamma}d\zeta\int_{C(\theta)}(z_0)^{\lambda}f(e^{\zeta},z')\exp(-\lambda\zeta)\log\lambda d\lambda. \tag{1.14}$$

Hence in order to solve (1.9) we proceed to the construction of $v(z,\lambda,\zeta)$ satisfying formally

$$\begin{cases} P(z,\partial)v(z,\lambda,\zeta)=(z_0)^{\lambda+d_m-m}f(e^{\zeta},z'), \\ (\partial_1)^h v(z_0,0,z'',\lambda,\zeta)=(z_0)^{\lambda}\varphi_h(e^{\zeta},z'') \quad \text{for } 0\leq h\leq L-1, \end{cases} \tag{1.15}$$

where constant d_m-m appears so that we may choose $S(0)=0$ in (1.16) below. This is not essential, for we may replace $f(z)$ by $(z_0)^{m-d_m}f(z)$.

In §2 we construct $v(z,\lambda,\zeta)$ in the form

$$\begin{cases} v(z,\lambda,\zeta)=\sum_{n=0}^{\infty}\sum_{s=-\infty}^{S(n)}v_{n,s}(z',\lambda,\zeta)(z_0)^{\lambda+s}, \\ v_{n,s}(z',\lambda,\zeta)=\sum_{\delta\in\Delta(n,s)}v_{\delta}(z',\lambda,\zeta), \end{cases} \tag{1.16}$$

where $v_{\delta}(z',\lambda,\zeta)$ is a meromorphic function in λ and $\Delta(n,s)$ is a finite set determined by n and s. In §3 we investigate the finite set $\Delta(n,s)$. The set $\Delta(n,s)$ gives us information of the distribution and multiplicity of poles of $v_{\delta}(z',\lambda,\zeta)$. It is used for estimation. In §4 we estimate $v_{\delta}(z',\lambda,\zeta)$.

We construct a solution $u(z)$ of (1.9) by integrating $v(z,\lambda,\zeta)$ in λ and ζ. Put

$$v_{\delta}^{*}(z,\zeta)=\frac{1}{2\pi i}\int_{C(\theta)}\exp(-\lambda\zeta)(z_0)^{\lambda+s}v_{\delta}(z',\lambda,\zeta)\log\lambda\,d\lambda, \tag{1.17}$$

$$v_{n,s}^{*}(z,\zeta)=\sum_{\delta\in\Delta(n,s)}v_{\delta}^{*}(z,\zeta), \tag{1.18}$$

$$V(z,\zeta)=\sum_{n=0}^{\infty}\sum_{s=-\infty}^{S(n)}v_{n,s}^{*}(z,\zeta). \tag{1.19}$$

Then we have

$$u(z)=\frac{1}{2\pi i}\int_{\Gamma}V(z,\zeta)\,d\zeta. \tag{1.20}$$

In §5 we integrate $v_{\delta}(z',\lambda,\zeta)$ in λ, get $v_{\delta}^{*}(z,\zeta)$ and estimate it. In §6 we show convergence of $V(z,\zeta)$ in (1.19). In §7 we perform integration in λ, show that $u(z)$ defined by (1.20) is a solution of (1.9) and complete the proofs of Theorems 1.4 and 1.8.

Finally, we note that different constants appearing in the following will be denoted by the same symbols, $A, B, C, \cdots$.

§2. Construction of a formal solution.

In §2 we construct $v(z,\lambda,\zeta)$ satisfying formally

$$\begin{cases} P(z,\partial)\,v(z,\lambda,\zeta)=(z_0)^{\lambda+d_m-m}f(e^{\zeta},z'), \\ (\partial_1)^{h}v(z_0,0,z'',\lambda,\zeta)=(z_0)^{\lambda}\varphi_h(e^{\zeta},z'') \quad \text{for } 0\le h\le L-1 \end{cases} \tag{2.1}$$

(see (1.15)). A solution $u(z)$ of (1.9) will be obtained after we integrate $v(z,\lambda,\zeta)$ in λ and ζ, multiplying functions in λ and ζ. Let us recall (1.8)$'$, namely, that $P_{\mathrm{loc},K,0}$ is noncharacteristic with respect to ∂_1 at $z'=0$.

Let us rewrite $P(z,\partial)$ in another form. Put

$$\begin{cases} M(0)=\{(k,l,j)\,;\,d_m-(l+j)-m+k=0,\ A_{k,l,j}(z',\xi')\not\equiv 0,\ k\neq m\}, \\ M(r)=\{(k,l,j)\,;\,d_m-(l+j)-m+k=-r,\ A_{k,l,j}(z',\xi')\not\equiv 0\}, \quad (r\neq 0), \end{cases} \tag{2.2}$$

$$\begin{cases} A=A(z,\partial)=\displaystyle\sum_{l+j=d_m}A_{m,l,j}(z',\partial')(z_0)^{j}(\partial_0)^{m-l}, \\ B_r=B_r(z,\partial)=\displaystyle\sum_{(k,l,j)\in M(r)}A_{k,l,j}(z',\partial')(z_0)^{j}(\partial_0)^{k-l}, \end{cases} \tag{2.3}$$

and

$$B=B(z,\partial)=A+\sum_{r<0}B_r. \tag{2.4}$$

Then we have

$$P=P(z,\partial)=B+\sum_{r=0}^{\infty}B_r. \tag{2.5}$$

So we construct $v(z, \lambda, \zeta)=\sum_{n=0}^{\infty} v_n(z, \lambda, \zeta)$ in the following way:

$$(2.6)_0 \qquad \begin{cases} Bv_0=(z_0)^{\lambda+d_m-m}f(e^{\zeta}, z'), \\ (\partial_1)^h v_0|_{z_1=0}=(z_0)^{\lambda}\varphi_h(e^{\zeta}, z'') \quad \text{for } 0\leq h\leq L-1, \end{cases}$$

$$(2.6)_{n+1} \qquad \begin{cases} Bv_{n+1}+B_0v_n+\cdots+B_rv_{n-r}+\cdots+B_nv_0=0, \\ (\partial_1)^h v_{n+1}|_{z_1=0}=0 \quad \text{for } 0\leq h\leq L-1. \end{cases}$$

If such $v_n(z, \lambda, \zeta)$ $(n=0, 1, \cdots)$ are found, then $v(z, \lambda, \zeta)=\sum_{n=0}^{\infty} v_n(z, \lambda, \zeta)$ formally satisfies (2.1).

Now we try to construct $v_n(z, \lambda, \zeta)$ in the form

$$(2.7) \qquad v_n(z, \lambda, \zeta)=\sum_{s=-\infty}^{S(n)} (z_0)^{\lambda+s}v_{n,s}(z', \lambda, \zeta).$$

Let us derive equations to be satisfied by $v_{n,s}(z', \lambda, \zeta)$ $(n\geq 0, -\infty< s\leq S(n))$. First we claim

LEMMA 2.1. *The following relations hold true*:

$$(2.8) \qquad B_r((z_0)^{\lambda}w(z'))=(z_0)^{\lambda+d_m-m+r}B_r(\lambda)w(z'),$$

$$(2.9) \qquad A((z_0)^{\lambda}w(z'))=(z_0)^{\lambda+d_m-m}F(m-L;\lambda)A(\lambda)w(z'),$$

where

$$(2.10) \qquad \begin{cases} F(p;\lambda)=\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-p+1), \qquad F(0;\lambda)=1, \\ B_r(\lambda)=\sum_{(k,l,j)\in M(r)} F(k-l;\lambda)A_{k,l,j}(z', \partial'), \\ A(\lambda)=\sum_{l+j=d_m} F(L-l;\lambda-(m-L))A_{m,l,j}(z', \partial'). \end{cases}$$

PROOF. We have

$$(z_0)^j A_{k,l,j}(z', \partial')(\partial_0)^{k-l}((z_0)^{\lambda}w(z'))=F(k-l;\lambda)(z_0)^{\lambda-k+l+j}A_{k,l,j}(z', \partial')w(z').$$

Hence we get (2.8) and (2.9). $\square$

Substituting v_n to (2.6) and using Lemma 2.1, we see that $v_{n,s}$ should satisfy the following:

$$(2.11)\quad \begin{cases} F(m-L\,;\lambda+s)A(\lambda+s)v_{n+1,s}+\sum_{r<0} B_r(\lambda+s-r)v_{n+1,s-r} \\ \qquad +\sum_{r=0}^{n} B_r(\lambda+s-r)v_{n-r,s-r}=\delta_{n+1,0}\delta_{s,0}f(e^{\zeta},z'), \\ (\partial_1)^h v_{n+1,s}(0,z'',\lambda,\zeta)=\delta_{n+1,0}\delta_{s,0}\varphi_h(e^{\zeta},z'') \\ \text{for } 0\leq h\leq L-1, \text{ where } \delta_{i,j} \text{ is Kronecker's delta.} \end{cases}$$

Thus we have equations for $v_{n,s}(z',\lambda,\zeta)$.

It follows from (1.8)′ that $v_{n,s}(z',\lambda,\zeta)$ are successively determined from (2.11) in a neighbourhood Ω' of $z'=0$ which is independent of n and s, and $S(n)=\max(0,n-1)$. Obviously $v_{n,s}(z',\lambda,\zeta)$ is a meromorphic function in λ.

Let us represent $v_{n,s}$ concretely. We introduce some notations. Let $G(\lambda)g$ denote the solution $w(z',\lambda,\zeta)$ to the equation

$$(2.12)\quad \begin{cases} A(\lambda)w(z',\lambda,\zeta)=g(z',\lambda,\zeta), \\ (\partial_1)^h w(0,z'',\lambda,\zeta)=0 \quad \text{for } 0\leq h\leq L-1. \end{cases}$$

For linear operators C_i $(1\leq i\leq p)$, $\prod_{i=1}^{p} C_i$ means $C_1C_2\cdots C_p$. Put

$$(2.13)\quad \begin{cases} M^*(s,p)=\Big\{\delta=((k_1,l_1,j_1),\ (k_2,l_2,j_2),\ \cdots,(k_p,l_p,j_p))\,; \\ \qquad \sum_{i=1}^{p}(-d_m+l_i+j_i+m-k_i)=s,\ A_{k_i,l_i,j_i}(z',\xi')\not\equiv 0 \quad \text{for } 1\leq i\leq p\Big\}, \\ M^*(s)=\bigcup_{p} M^*(s,p). \end{cases}$$

δ is an element in $(N\cup\{0\})^{3p}$. $(k_i,l_i,j_i)\in\delta$ means that the triple (k_i,l_i,j_i) appears in δ as $\delta=((k_1,l_1,j_1),\cdots,(k_i,l_i,j_i),\cdots,(k_p,l_p,j_p))$.

We have from (2.11), if $(n+1,s)\neq(0,0)$,

$$(2.14)\quad v_{n+1,s}=-F(m-L\,;\lambda+s)^{-1}G(\lambda+s)\Big\{\sum_{r<0} B_r(\lambda+s-r)v_{n+1,s-r} + \sum_{r=0}^{n} B_r(\lambda+s-r)v_{n-r,s-r}\Big\},$$

which yields the following:

Proposition 2.2. *$v_{n,s}$ has a representation in the form*

$$(2.15)\quad v_{n,s}=\sum_{\delta\in\Delta(n,s)}(-1)^p v_\delta, \quad (n,s)\neq(0,0),$$

where

$$
(2.16)\qquad \begin{cases} v_\delta = \dfrac{\prod_{i=1}^{p} F(k_i-l_i\,;\,\lambda+s-(r_1+r_2+\cdots+r_i))}{\prod_{i=0}^{p} F(m-L\,;\lambda+s-(r_0+r_1+\cdots+r_i))} w_\delta(z',\lambda,\zeta), \\ w_\delta = \prod_{i=1}^{p} C_{k_i,l_i,j_i}(\lambda+s-(r_0+r_1+\cdots+r_{i-1}))w_0, \\ C_{k,l,j}(\lambda)=G(\lambda)A_{k,l,j}(z',\partial'),\ w_0=F(m-L\,;\,\lambda)v_{0,0}, \end{cases}
$$

$r_0=0$, $r_i=-d_m+l_i+j_i+m-k_i$ for $i\geq 1$ and $\Delta(n,s)$ is a finite subset of $M^*(s)$.

PROOF. Let $n=0$. Then we have from (2.10) and (2.14)

$$
\begin{aligned}
v_{0,s} &= -F(m-L\,;\,\lambda+s)^{-1}\sum_{r_1<0}\Big\{\sum_{(k_1,l_1,j_1)\in M(r_1)} F(k_1-l_1\,;\,\lambda+s-r_1)C_{k_1,l_1,j_1}(\lambda+s)\Big\}v_{0,s-r_1} \\
&= F(m-L\,;\,\lambda+s)^{-1}\sum_{r_1<0}\Big[\Big\{\sum_{(k_1,l_1,j_1)\in M(r_1)} F(k_1-l_1\,;\,\lambda+s-r_1)C_{k_1,l_1,j_1}(\lambda+s)\Big\} \\
&\quad\times F(m-L\,;\,\lambda+s-r_1)^{-1} \\
&\quad\times\Big\{\sum_{r_2<0}\Big(\sum_{(k_2,l_2,j_2)\in M(r_2)} F(k_2-l_2\,;\,\lambda+s-r_1-r_2)C_{k_2,l_2,j_2}(\lambda+s-r_1)\Big)v_{0,s-r_1-r_2}\Big\}\Big] \\
&= \sum_{r_1,r_2} F(m-L\,;\,\lambda+s)^{-1}F(m-L\,;\,\lambda+s-r_1)^{-1} \\
&\quad\times\Big\{\sum_{\substack{(k_1,l_1,j_1)\in M(r_1)\\(k_2,l_2,j_2)\in M(r_2)}} F(k_1-l_1\,;\,\lambda+s-r_1)F(k_2-l_2\,;\,\lambda+s-r_1-r_2) \\
&\quad\times C_{k_1,l_1,j_1}(\lambda+s)C_{k_2,l_2,j_2}(\lambda+s-r_1)v_{0,s-r_1-r_2}\Big\}.
\end{aligned}
$$

By repeating this argument, we have (2.15) and (2.16) for $n=0$. Assume that $v_{n,s}$ is represented as in (2.15) and (2.16), if $n\leq N$ or if $n=N+1$ and $s\geq S+1$. Then it follows from (2.14) that $v_{N+1,S}$ is also represented in the form (2.15) and (2.16). □

The set $\Delta(n,s)$ is investigated in §3. We note that $w_0(z',\lambda,\zeta)$ is the solution to the equations

$$
(2.17)\qquad \begin{cases} A(\lambda)w_0(z',\lambda,\zeta)=f(e^\zeta,z'), \\ (\partial_1)^h w_0(0,z'',\lambda,\zeta)=F(m-L\,;\,\lambda)\varphi_h(e^\zeta,z'') \quad \text{for } 0\leq h\leq L-1. \end{cases}
$$

Now let us derive another representation of v_δ from Proposition 2.2 to use in later sections. Put for $\delta\in\Delta(n,s)$

$$(2.18)\qquad \begin{cases} B(+,\delta)=\{i\,;(k_i,l_i,j_i)\in\delta \text{ and } k_i-l_i>m-L\}, \\ B(*,\delta)=\{i\,;(k_i,l_i,j_i)\in\delta,\ k_i-l_i<m-L \text{ and } r_i\geq 1\}, \\ B(0,\delta)=\{i\,;(k_i,l_i,j_i)\in\delta,\ k_i-l_i<m-L \text{ and } r_i=0\}, \\ B(-,\delta)=\{i\,;(k_i,l_i,j_i)\in\delta \text{ and } k_i-l_i<m-L\}. \end{cases}$$

Define

$$(2.19)\qquad \begin{cases} F_1(\lambda)=\prod_{i\in B(+,\delta)} F(k_i-l_i-m+L\,;\,\lambda-m+L-(r_1+\cdots+r_i)), \\ F_2(\lambda)=\prod_{i\in B(*,\delta)} F(m-L-k_i+l_i\,;\,\lambda-k_i+l_i-(r_1+\cdots+r_i)), \\ F_3(\lambda)=\prod_{i\in B(0,\delta)} F(m-L-k_i+l_i\,;\,\lambda-k_i+l_i-(r_1+\cdots+r_i)), \end{cases}$$

$$(2.20)\qquad H_\delta(\lambda)=\prod_{i=0}^{p} F(m-L\,;\,\lambda-(r_0+r_1+\cdots+r_i))$$

and

$$(2.21)\qquad w_\delta^*(z',\lambda,\zeta)=\left\{\prod_{i=1}^{p} F(k_i-l_i\,;\,\lambda+s-(r_1+r_2+\cdots+r_i))\right\}w_\delta(z',\lambda,\zeta).$$

We note that if $\sigma_1=1$, then $r_i\geq 0$ and $B(-,\delta)=B(0,\delta)\cup B(*,\delta)$.
Thus we have:

Proposition 2.3. *$v_\delta(z',\lambda,\zeta)$ is represented in the form*

$$(2.22)\qquad v_\delta(z',\lambda,\zeta)=w_\delta^*(z',\lambda,\zeta)/H_\delta(\lambda+s),$$

and, moreover, if $\sigma_1=1$,

$$(2.23)\qquad v_\delta(z',\lambda,\zeta)=\frac{F_1(\lambda+s)}{F(m-L\,;\,\lambda+s)F_2(\lambda+s)F_3(\lambda+s)}w_\delta(z',\lambda,\zeta).$$

§ 3. The Set $\Delta(n,s)$.

In this section we investigate the set $\Delta(n,s)$ appearing in the summation (2.15) in Proposition 2.2. Let $\#A$ denote the cardinal number of a set A. Now recall the definition of the set $M^*(s,p)$ ((2.13)). We put

$$(3.1)\qquad \Delta(n,s,p)=\Delta(n,s)\cap M^*(s,p),$$

and for $\delta\in M^*(s)$

$$(3.2)\qquad \begin{cases} A(+,\delta)=\{i\,;(k_i,l_i,j_i)\in\delta \text{ and } r_i>0\}, \\ A(0,\delta)=\{i\,;(k_i,l_i,j_i)\in\delta \text{ and } r_i=0\}, \\ A(-,\delta)=\{i\,;(k_i,l_i,j_i)\in\delta \text{ and } r_i<0\}, \end{cases}$$

where $r_i=-d_m+l_i+j_i+m-k_i$ $(i\geq 1)$ and $r_0=0$. Let us define

$$(3.3)\qquad a_+=\sum_{i\in A(+,\delta)} r_i,\ a_-=-\sum_{i\in A(-,\delta)} r_i,$$

$$(3.4)\qquad b=\sharp A(+,\delta),\quad c=\sharp A(0,\delta),$$

$$(3.5)\qquad \mu_i=\sum_{j=0}^{i} r_j,\quad \mu^*=\max_{0\leq i\leq p}\mu_i,\quad \mu_*=\min_{0\leq i\leq p}\mu_i,\quad \kappa=\max_{0\leq i\leq p}|\mu_i|.$$

First we prepare the following:

LEMMA 3.1. *For* $\delta\in\Delta(n,s,p)$ $((n,s)\neq(0,0))$,

$$(3.6)\qquad a_+-a_-=s,$$

$$(3.7)\qquad a_++b+c=n.$$

PROOF. The equality (3.6) is obvious. Let us show (3.7) by induction on n and s. We note that $n\geq s$ and recall (2.14). If $n=0$, then $s<0$ and $r_i<0$. So we have (3.7) for $n=0$. Assume that (3.7) is valid for $0\leq n\leq N$ and for $n=N+1$ and $s\geq S+1$. It follows from (2.14) that (3.7) is also valid for $n=N+1$ and $s=S$. □

We also have:

PROPOSITION 3.2. *For* $\delta\in\Delta(n,s,p)$, *the following holds*;

(i) $0\geq\mu_*\geq -a_-=-(n-s)+b+c$, *and if* $s\leq 0$, *then* $\mu_*\leq s$.

(ii) $n-(b+c)=a_+\geq\mu^*\geq 0$, *and if* $s\geq 0$, *then* $\mu^*\geq s$.

(iii) $\kappa\leq\max(a_+,a_-)\leq\max(n,n-s)$, $0\leq\mu_i-\mu_*\leq 2n-s$ *and* $0\leq\mu^*-\mu_i\leq 2n-s$.

(iv) $p\leq n-s$, $b+c\leq\min(n,n-s)$ *and* $b\leq\min(n/2,n-s)$.

(v) *There is a constant* A *such that*

$$j_1+j_2+\cdots+j_p\leq A(n+|s|).$$

PROOF. The first equality in (i) follows from (3.5), (3.6) and (3.7). In view of the fact that $\mu_p=a_+-a_-=s$, we have $\mu_*\leq s$ if $s\leq 0$. We also get (ii) similarly. Since $n\geq a_+$ and $n-s\geq a_-$, we have (iii).

Let us show (iv) and (v). We have $p=\sharp A(+,\delta)+\sharp A(0,\delta)+\sharp A(-,\delta)$ $=b+c+\sharp A(-,\delta)=n-a_++\sharp A(-,\delta)\leq n-a_++a_-=n-s$. Since $b+c=n-a_+$

$=n-s-a_-$, $b+c\leq \min(n, n-s)$ and $b\leq n-s$. We also have $2b\leq a_+ +b+c=n$. Hence $b\leq n/2$. Finally, $j_1+j_2+\cdots+j_p=s+\sum_{i=1}^{p}(d_m-l_i-m+k_i)\leq s+Cp\leq s+C(n-s)\leq A(n+|s|)$. □

PROPOSITION 3.3. *There are constants A and B independent of n and s such that* $\sharp\Delta(n, s)\leq AB^{n+|s|}$.

PROOF. In view of (iv) in Proposition 3.2, we have $\Delta(n, s)=\bigcup_{p=1}^{n-s}\Delta(n, s, p)$. We note that $s=r_1+r_2+\cdots+r_p$, $r_i\geq -(\sigma_1-1)(m-k_i)$ and for any r the set $\{(k, l, j)\,;\ -d_m+(l+j)+m-k=r,\ A_{k,l,j}(z', \xi')\not\equiv 0\}$ is finite and its cardinal number is bounded by a constant independent of r. So $\sharp\Delta(n, s, p)\leq AC^p$. Hence $\sharp\Delta(n, s)\leq AB^{n+|s|}$. □

PROPOSITION 3.4. *Let* $\delta=\prod_{i=1}^{p}(k_i, l_i, j_i)\in\Delta(n, s, p)$. *If* $\sigma_1>1$, then

$$\sum_{i=1}^{p}(m-k_i)\geq a_-/(\sigma_1-1)+c. \tag{3.8}$$

If $\sigma_1=1$, *then*

$$m(n-s)\geq d=\sum_{i\in A(0,\delta)}(m-k_i)\geq \max\{0, n/2-s\}. \tag{3.9}$$

PROOF. Let $\sigma_1>1$. We have $a_-\leq(\sigma_1-1)\Big\{\sum_{i\in A(-,\delta)}(m-k_i)\Big\}$. If $i\in A(0, \delta)$, then $m>k_i$. Hence $\sum_{i=1}^{p}(m-k_i)\geq\sum_{i\in A(-,\delta)}(m-k_i)+\sum_{i\in A(0,\delta)}(m-k_i)\geq a_-/(\sigma_1-1)+c$.

Let $\sigma_1=1$. We have $m(\sharp A(0, \delta))\geq\sum_{i\in A(0,\delta)}(m-k_i)\geq\sharp A(0, \delta)$. Since $b+c=n-s$ and $b\leq n/2$, we have $n/2-s\leq c=\sharp A(0, \delta)\leq n-s$. Hence we obtain (3.9). □

Next we study the function $H_\delta(\mu)$, which determines the distribution of poles of $v_\delta(z', \lambda, \zeta)$ (see (2.20) and (2.22)). Put

$$h_\delta(\mu)=\mu\prod_{i=1}^{p}(\mu-\mu_i)=\prod(\mu-\tilde{\mu}_i)^{\alpha_i}\,, \tag{3.10}$$

where $\tilde{\mu}_i\neq\tilde{\mu}_j$ if $i\neq j$. We have $H_\delta(\mu)=\prod_{j=0}^{m-L-1}h_\delta(\mu-j)$.

Put

$$q_1=\sharp\{\tilde{\mu}_i\,;\ \alpha_i=1\},\quad q_2=\sharp\{\tilde{\mu}_i\,;\ \alpha_i\geq 2\} \tag{3.11}$$

and

$$N(\delta)=p+1-(q_1+2q_2). \tag{3.12}$$

Then we have:

PROPOSITION 3.5. *The following holds*;

$$N(\delta)\leq 2\{(n-s)+\mu_*\}-(2b+c), \tag{3.13}$$

$$N(\delta)\leq 2\{n-(b+c)-\mu^*\}+c. \tag{3.14}$$

Proposition 3.5 will be used in § 5 in order to estimate the function $v_\delta^*(z,\zeta)$ defined by (1.17), which is given in terms of $v_\delta(z',\lambda,\zeta)$.

To show Proposition 3.5, we introduce another notion: *doubly counting number of finite sequences.* Let A be a finite sequence $\{a_i\}$ $(0\leq i\leq q)$. Put

$$f(x)=\prod_{i=0}^{q}(x-a_i)=\prod_{i=1}^{q'}(x-\tilde{a}_i)\prod_{i=q'+1}^{q'+q''}(x-\tilde{a}_i)^{\alpha_i},\quad \tilde{a}_i\neq\tilde{a}_j\ (i\neq j),\ \alpha_i\geq 2.$$

Then we define $N_d(A)=q'+2q''$, which we call *the doubly counting number of* A.

LEMMA 3.6. *Let* $a_0=0$, $a_i=c_1+c_2+\cdots+c_i$ *for* $1\leq i\leq q$, *where* $c_i\in Z$ *and* $A=\{a_i\}$ $(0\leq i\leq q)$. *Then*

$$N_d(A)\geq -2a_*+q-2c_*-c+1, \tag{3.15}$$

$$N_d(A)\geq 2a^*+q-2c^*-c+1, \tag{3.16}$$

where

$$a^*=\max_{0\leq i\leq q}a_i,\ a_*=\min_{0\leq i\leq q}a_i,\ c_*=-\sum_{c_j<0}c_j,\ c^*=\sum_{c_j>0}c_j \text{ and } c=\#\{i\,;c_i=0\}.$$

PROOF. Let us show (3.15). First we assume that $c_i\neq 0$ for $1\leq i\leq q$. If all $c_i=1$ or -1, then $N_d(A)\geq -2a_*+a_q+1$. If $c_i=n_i$ (or $-n_i$), $n_i\in N$, put $a_{i,j}=a_{i-1}+j$ (resp. $a_{i-1}-j$) for $1\leq j\leq n_i-1$ and $\tilde{A}=\{a_0, a_{1,1}, a_{1,2},\cdots, a_{i,j},\cdots, a_q\}$. Then $N_d(\tilde{A})\geq -2a_*+a_q+1$. Since $\sum_{i=1}^{q}(n_i-1)=c^*+c_*-q$, we have $N_d(A)+c^*+c_*-q\geq N_d(\tilde{A})\geq -2a_*+a_q+1$. Hence $N_d(A)\geq -2a_*+q-2c_*+1$. If $c\neq 0$, we replace q by $q-c$. Thus $N_d(A)\geq -2a_*+q-2c_*-c+1$. We can also get (3.16) in a similar way. □

PROOF OF PROPOSITION 3.5. Put $A(\delta)=\{\mu_i\}$ $(0\leq i\leq p)$. Let us show (3.13). By Lemma 3.6, we have

(3.17) $$N_d(A(\delta))\geq -2\mu_*+p-2a_- -c+1.$$

Therefore $N(\delta)=p+1-N_d(A(\delta))\leq 2(a_+-s+\mu_*)+c=2\{(n-s)+\mu_*\}-(2b+c)$. In the same way we have (3.14) from (3.16). $\square$

What we have shown in this section will be used in the following sections.

§ 4. Estimates.

Here in § 4 we give estimates of $w_\delta(z',\lambda,\zeta)$. The operator that we consider contain a parameter λ. So we need estimates with λ. We employ the majorants used in Hamada [4], Hamada, Leray and Wagschal [5], Komatsu [8] and Wagschal [20]. First we summarize the properties of those majorants without proof. For details we refer to the above papers, in particular, [8]. The same symbols $A, B, C, \cdots$ etc. denote various constants indifferently like in other sections.

From now on, we always assume that $0<r<R'<R$. Unlike other sections, in this section z means $z=(z_1, z_2, \cdots, z_n)\in C^n$ and $z'=(z_2, z_3, \cdots, z_n)$. For formal power series $a(z)$ and $b(z)$, $a(z)\ll b(z)$ means that each Taylor coefficient of $b(z)$ bounds the absolute value of the corresponding coefficient of $a(z)$.

PROPOSITION 4.1 (Wagschal). *Let $\Theta(t)$ be a formal power series in one variable t such that $\Theta(t)\gg 0$ and*

(4.1) $$(R'-t)\Theta(t)\gg 0.$$

Then for derivatives, $\Theta^{(j)}(t)=(d/dt)^j\Theta(t)$ for $j=0, 1, \cdots$, we have

(4.2) $$\Theta^{(j)}(t)\ll R'\Theta^{(j+1)}(t)$$

and

(4.3) $$(R-t)^{-1}\Theta^{(j)}(t)\ll (R-R')^{-1}\Theta^{(j)}(t).$$

In the sequel we put $t=\rho z_1+z_2+\cdots+z_n$ with a constant $\rho\geq 1$ to be determined later. We assume that $\Theta(t)$ satisfies the conditions in Proposition 4.1 and that all coefficients are holomorphic on $\{z\in C^n; |z_i|\leq R\}$.

PROPOSITION 4.2 (Wagschal). *Let*

(4.4) $$A(z,\partial)=\sum_{|\alpha|\leq m,\alpha_1\leq m_1} a_\alpha(z)\partial^\alpha$$

be a linear partial differential operator. Then there is a constant A independent of $\Theta(t)$ and $\rho\geq 1$ such that if

$$u(z)\ll\Theta^{(j)}(t), \tag{4.5}$$

then

$$A(z,\partial)u(z)\ll A\rho^{m_1}\Theta^{(j+m)}(t). \tag{4.6}$$

Put

$$\begin{cases} \theta^{(k)}(t)=k!/(r-t)^{k+1} & \text{for } k\geq 0, \\ \theta^{(k)}(k)=\int_0^t \theta^{(k+1)}(s)ds & \text{for } k<0. \end{cases} \tag{4.7}$$

We note that

$$(d/dt)^l\theta^{(k)}(t)=\theta^{(k+l)}(t) \tag{4.8}$$

and that if $k\geq 0$, then $\theta^{(k)}(t)$ satisfies the conditions in Proposition 4.1. But if $k<0, \theta^{(k)}(t)$ does not satisfy them. Let us employ, according to [4],

$$\Theta_k(t)=\frac{R'}{(R'-t)}\theta^{(k)}(t), \tag{4.9}$$

which satisfies the conditions in Proposition 4.1. We have:

Proposition 4.3. (a) *If $k<l$, then*

$$\Theta_l^{(j)}(t)\ll\Theta_k^{(j-k+l)}(t). \tag{4.10}$$

(b) *If $k\geq 0$, then*

$$\theta^{(j+k)}(t)\ll\Theta_k^{(j)}(t)\ll\frac{R'}{(R'-r)}\theta^{(j+k)}(t). \tag{4.11}$$

(c) *If $k<0$ and $R'>2r$, then*

$$\theta^{(j+k)}(t)\ll\Theta_k^{(j)}(t)\ll\frac{2^{|k|}}{(R'-2r)}\theta^{(j+k)}(t). \tag{4.12}$$

For the proof of Propositions 4.1-4.3 we refer to [8]. Now we investigate estimates of functions with a parameter λ.

Proposition 4.4. *Let $L(z,\partial)$ be a linear partial differential operator* *of order m and assume that the order with respect to ∂_1 is less than or equal to m_1. Suppose that $u(z,\lambda)\ll\exp(c|\lambda|z_1)\Theta(t)$. Then there is a constant A such that*

$$L(z,\partial)u(z,\lambda)\ll A\rho^{m_1}\exp(c|\lambda|z_1)\Big\{\sum_{p=0}^{m}(c|\lambda|)^p\Theta^{(m-p)}(t)\Big\}. \tag{4.13}$$

PROOF. In view of Proposition 4.1, we have

$$a_\alpha(z)\partial^\alpha u(z,\lambda)\ll C\rho^{\alpha_1}\exp(c|\lambda|z_1)\Big\{\sum_{p+q=|\alpha|}(c|\lambda|)^p\Theta^{(q)}(t)\Big\}.$$

So we have

$$\begin{aligned}L(z,\partial)u(z,\lambda)&\ll C\rho^{m_1}\exp(c|\lambda|z_1)\sum_{|\alpha|=0}^{m}\Big\{\sum_{p+q=|\alpha|}(c|\lambda|)^p\Theta^{(q)}(t)\Big\}\\&\ll A\rho^{m_1}\exp(c|\lambda|z_1)\Big\{\sum_{p=0}^{m}(c|\lambda|)^p\Theta^{(m-p)}(t)\Big\}. \quad\square\end{aligned}$$

Let $L(\lambda;z,\partial)$ be an operator with a parameter λ of the form

$$L(\lambda;z,\partial)=\sum_{j=0}^{m}\lambda^jL_{m-j}(z,\partial), \tag{4.14}$$

where $\mathrm{ord}.L_{m-j}(z,\partial)\le m-j$, $L_m(z,\partial)=(\partial_1)^m+\sum_{j=1}^{m}A_j(z,\partial')(\partial_1)^{m-j}$ and $\mathrm{ord}.A_j(z,\partial')\le j$. We have:

PROPOSITION 4.5. *There are constants C,γ and ρ independent of $\Theta(t)$ such that if*

$$\begin{cases}L(\lambda;z,\partial)u(z,\lambda)\ll\exp(\gamma|\lambda|z_1)\Theta^{(m)}(t),\\(\partial_1)^hu(z,\lambda)|_{z_1=0}=0\quad\text{for } 0\le h\le m-1,\end{cases} \tag{4.15}$$

then

$$u(z,\lambda)\ll C\exp(\gamma|\lambda|z_1)\Theta(t). \tag{4.16}$$

PROOF. Let $L^*_{m-j}(z,\partial)$ and $A^*_j(z,\partial')$ be majorant differential operators of $L_{m-j}(z,\partial)$ and $A_j(z,\partial')$ respectively, that is, each coefficient of $L^*_{m-j}(z,\partial)$ $(A^*_j(z,\partial'))$ is a majorant of the corresponding coefficient of $L_{m-j}(z,\partial)$ (resp. $A_j(z,\partial')$).

Let us consider a differential inequality

$$\begin{aligned}(\partial_1)^mU(z,\lambda)\gg&\sum_{j=1}^{m}|\lambda|^jL^*_{n-j}(z,\partial)U(z,\lambda)\\&+\sum_{j=1}^{m}A^*_j(z,\partial')(\partial_1)^{m-j}U(z,\lambda)+\exp(\gamma|\lambda|z_1)\Theta^{(m)}(t).\end{aligned} \tag{4.17}$$

It is our purpose to find out a solution $U(z,\lambda)=C\exp(\gamma|\lambda|z_1)\Theta(t)$ to (4.17) for suitable C,γ and ρ. If $U(z,\lambda)$ exists, then $u(z,\lambda)\ll U(z,\lambda)$.

First we have, from Proposition 4.4,

$$L^*_{m-j}(z,\partial)\exp(\gamma|\lambda|z_1)\Theta(t)\ll A\exp(\gamma|\lambda|z_1)\rho^{m-j}\left\{\sum_{k=0}^{m-j}(\gamma|\lambda|)^k\Theta^{(m-j-k)}(t)\right\}.$$

Hence

$$\begin{aligned}(4.18)\qquad &\sum_{j=1}^{m}|\lambda|^j L^*_{m-j}(z,\partial)\{\exp(\gamma|\lambda|z_1)\Theta(t)\}\\ &\ll A\exp(\gamma|\lambda|z_1)\sum_{p=1}^{m}\left(\sum_{j=1}^{p}\gamma^{p-j}\rho^{m-j}\right)|\lambda|^p\Theta^{(m-p)}(t).\end{aligned}$$

In the same way, we have

$$\begin{aligned}(4.19)\qquad &\sum_{j=1}^{m}A_j^*(z,\partial')(\partial_1)^{m-j}\{\exp(\gamma|\lambda|z_1)\Theta(t)\}\\ &\ll A\exp(\gamma|\lambda|z_1)\sum_{p=0}^{m-1}\left(\sum_{j=1}^{m-p}\rho^{m-j-p}\right)(\gamma|\lambda|)^p\Theta^{(m-p)}(t)\end{aligned}$$

and

$$(4.20)\qquad (\partial_1)^m\left\{\exp(\gamma|\lambda|z_1)\Theta(t)\right\}=\exp(\gamma|\lambda|z_1)\left\{\sum_{p=0}^{m}\binom{m}{p}\rho^{m-p}(\gamma|\lambda|)^p\Theta^{(m-p)}(t)\right\}.$$

So we have only to show the existence of C,γ and ρ such that

$$\begin{aligned}(4.21)\qquad C\binom{m}{p}\Theta^{(m-p)}(t)\gamma^p\rho^{m-p}\gg{}&CA\left(\sum_{j=1}^{p}\gamma^{p-j}\rho^{m-j}\right)\Theta^{(m-p)}(t)\\ &+CA\left(\sum_{j=1}^{m-p}\rho^{m-j-p}\right)\gamma^p\Theta^{(m-p)}(t)+\delta_{p,0}\Theta^{(m)}(t).\end{aligned}$$

Let $\rho>1$ and $\gamma\rho>1$. Then $\sum_{j=1}^{p}(\gamma\rho)^{-j}<(\gamma\rho-1)^{-1}$ and $\sum_{j=1}^{m-p}\rho^{-j}<(\rho-1)^{-1}$. In order to make (4.21) hold, we choose C,γ and ρ so that

$$(4.22)\qquad C\binom{m}{p}\rho^{m-p}\geq CA\,\rho^m/(\gamma\rho-1)+CA\,\rho^{m-p}/(\rho-1)+\delta_{p,0}.$$

First we take ρ so that $\binom{m}{p}\rho^{m-p}/2>A\rho^{m-p}/(\rho-1)$ and fix ρ. Next we take C and γ so that $C\binom{m}{p}\rho^{m-p}/2>CA\,\rho^m/(\gamma\rho-1)+\delta_{p,0}$. Thus (4.21) is valid for these C,γ and ρ. This completes the proof. □

In the same way we have

COROLLARY 4.6. *There are constants* C, γ *and* ρ *independent of* $\Theta(t)$ *and* $\tilde{\kappa}$ $(\tilde{\kappa}>0)$ *such that if* $|\eta|\leq\tilde{\kappa}$ *and*

$$\left\{\begin{array}{l} L(\lambda+\eta;\, z, \partial)u(z, \lambda) \ll \exp(\gamma(|\lambda|+\tilde{\kappa})z_1)\Theta^{(m)}(t), \\ (\partial_1)^h u(z, \lambda)|_{z_1=0}=0 \quad \text{for } 0\leq h\leq m-1, \end{array}\right. \tag{4.23}$$

then

$$u(z, \lambda) \ll C\exp(\gamma(|\lambda|+\tilde{\kappa})z_1)\Theta(t). \tag{4.24}$$

We apply Propositions 4.4, 4.5 and Corollary 4.6 to w_δ in the representation of $v_{n,s}$ (see (2.15)-(2.16)). Let us recall that z denotes the point $(z_1, z')=(z_1, z_2, \cdots, z_n)\in C^n$ in this section. Let us assume that in $(2.6)_0$ and (2.17)

$$\left\{\begin{array}{l} f(e^\zeta, z) \ll m(\zeta)\Theta_0^{(L)}(t), \\ \varphi_h(e^\zeta, z') \ll m(\zeta)\Theta_0(t)|_{z_1=0} \quad \text{for } 0\leq h\leq L-1. \end{array}\right. \tag{4.25}$$

LEMMA 4.7. *Under the condition* (4.25), *the solution* w_0 *to* (2.17) *satisfies*

$$w_0(z, \lambda, \zeta) \ll Am(\zeta)(1+|\lambda|)^{m-L}\exp(\gamma|\lambda|z_1)\Theta_0(t), \tag{4.26}$$

where γ *is a constant with the property stated in Proposition* 4.5 *for* $L(\lambda;\, z, \partial)=A(\lambda)$.

PROOF. By putting $\tilde{w}_0=w_0-F(m-L;\, \lambda)\left\{\sum_{h=0}^{L-1}(z_1)^h\varphi_h/h!\right\}$,

we have

$$\left\{\begin{array}{l} A(\lambda)\tilde{w}_0=f(e^\zeta, z)+F(m-L;\, \lambda)\tilde{f}(e^\zeta, z), \\ (\partial_1)^h w_0|_{z_1=0}=0 \quad \text{for } 0\leq h\leq L-1. \end{array}\right. \tag{4.27}$$

Hence from Proposition 4.5, $\tilde{w}_0 \ll Am(\zeta)(1+|\lambda|)^{m-L}\exp(\gamma|\lambda|z_1)\Theta_0(t)$. So we have (4.26). □

For $w_\delta(z, \lambda, \zeta)$ in (2.16), we have:

PROPOSITION 4.8. *Assume* (4.25). *Then the following estimate holds:*

$$\begin{aligned} w_\delta(z, \lambda, \zeta) &\ll Am(\zeta)B^{n+|s|}(1+|\lambda|)^{m-L} \\ &\quad\times \exp(\gamma(|\lambda+s|+\kappa)z_1)\left\{\sum_{r=0}^{l(\delta)}(|\lambda+s|+\kappa)^r\Theta_{-pL}^{(l(\delta)-r)}(t)\right\}, \end{aligned} \tag{4.28}$$

where $l(\delta)=l_1+l_2+\cdots+l_p$ *and* A, B, γ *and* ρ *are independent of* n, s, λ, δ *and* $\kappa=\max_{0\le i\le p}|r_0+r_1+\cdots+r_i|$ *(see* (3.5)*)*.

PROOF. First we note $|\lambda+s-(r_0+r_1+\cdots+r_{i-1})|\le|\lambda+s|+\kappa$. Put

$$w_{\delta,k}=\left\{\prod_{i=k}^{p} C_{k_i,l_i,j_i}(\lambda+s-(r_0+r_1+\cdots+r_{i-1}))\right\}w_0. \tag{4.29}$$

Then we want to show

$$\begin{aligned} w_{\delta,k}\ll & Am(\zeta)C^{p-k}B^{j_k+j_{k+1}+\cdots+j_p}(1+|\lambda|)^{m-L} \\ & \times\exp(\gamma(|\lambda+s|+\kappa)z_1)\left\{\sum_{r=0}^{l_k+l_{k+1}+\cdots+l_p}(|\lambda+s|+\kappa)^r\Theta_{-L(p+1-k)}^{(l_k+\cdots+l_p-r)}(t)\right\}. \end{aligned} \tag{4.30}$$

Let $k=p$ in (4.29). By Lemmas 4.1, 4.7 and Proposition 4.4 we have

$$\begin{aligned} A_{k_p,l_p,j_p}w_0 &\ll Am(\zeta)B^{j_p}(1+|\lambda|)^{m-L}\exp(\gamma(|\lambda+s|+\kappa)z_1)\left\{\sum_{r=0}^{l_p}(|\lambda+s|+\kappa)^r\Theta_0^{(l_p-r)}(t)\right\} \\ &\ll Am(\zeta)B^{j_p}(1+|\lambda|)^{m-L}\exp(\gamma(|\lambda+s|+\kappa)z_1)\left\{\sum_{r=0}^{l_p}(|\lambda+s|+\kappa)^r\Theta_{-L}^{(l_p-r+L)}(t)\right\}. \end{aligned}$$

Hence we have, by Corollary 4.6,

$$\begin{aligned} & C_{k_p,l_p,j_p}(\lambda+s-(r_0+r_1+\cdots+r_{p-1}))w_0 \\ & \ll Am(\zeta)B^{j_p}(1+|\lambda|)^{m-L}\exp(\gamma(|\lambda+s|+\kappa)z_1)\left(\sum_{r=0}^{l_p}(|\lambda+s|+\kappa)^r\Theta_{-L}^{(l_p-r)}(t)\right). \end{aligned}$$

Thus we have (4.30) for $k=p$. If (4.29) is valid for $k=K+1$, then we can also show that (4.30) is valid for $k=K$. Therefore (4.30) holds for $k=1$. In view of (v) in Proposition 3.2 this implies that (4.28) holds. □

PROPOSITION 4.9. *Assume* (4.25). *Then, for a small neighbourhood* Ω' *of* $z=0$ *we have*

$$\begin{aligned} |w_\delta(z,\lambda,\zeta)|\le & Am(\zeta)B^{n+|s|}(1+|\lambda|)^{m-L}\exp(\gamma|\lambda+s||z_1|) \\ & \times((pL)!)^{-1}\left\{\sum_{r=0}^{l(\delta)}(|\lambda+s|+\kappa)^r(l(\delta)-r)!\right\}. \end{aligned} \tag{4.31}$$

PROOF. It follows from Proposition 4.3 that for $0\le t\le r/2$, the inequalities

$$\Theta_{-pL}^{(l(\delta)-r)}(t)\le A^{p-r+1}\theta^{(l(\delta)-r-pL)}(t)\le A^{p+1}(l(\delta)-r)!/(pL)!$$

hold. In view of (iii) in Proposition 3.2, we have (4.31) from (4.28). □

For $w_\delta^*(z, \lambda, \zeta)$ defined by (2.21), we have

PROPOSITION 4.10. *Assume* (4.25). *Then, for a small neighbourhood* Ω' *of* $z=0$, *it holds that*

$$|w_\delta^*(z, \lambda, \zeta)| \le Am(\zeta) B^{n+|s|}(1+|\lambda|)^{m-L} \exp(\gamma|\lambda+s||z_1|)((pL)!)^{-1} \times \left\{ \sum_{r=0}^{l(\delta)} (|\lambda+s|+\kappa)^{r+k(\delta)-l(\delta)}(l(\delta)-r)! \right\}, \tag{4.32}$$

where $k(\delta)=\sum_{i=1}^{p} k_i$.

§ 5. Integration I.

We proceed to constructing a solution $u(z)$ (or $u(x)$) in Theorem 1.4 (resp. Theorem 1.8). As stated in § 1, we will construct it by integrating $v(z, \lambda, \zeta)$ in λ and ζ (see (1.17)-(1.20)). Firstly in this section we perform integration in λ:

$$\begin{aligned} v_\delta^*(z, \zeta) &= \frac{1}{2\pi i} \int_{C-s} (z_0)^{\lambda+s} \exp(-\lambda\zeta) v_\delta(z', \lambda, \zeta) \log \lambda \, d\lambda \\ &= \frac{\exp(s\zeta)}{2\pi i} \int_C \exp(-(\zeta - \log z_0)\mu) w_\delta^*(z', \mu-s, \zeta) \\ &\quad \times \log(\mu-s)/H_\delta(\mu)\, d\mu \quad (\mu=\lambda+s), \end{aligned} \tag{5.1}$$

(see (1.17)), where the path C will be determined later and $C-s$ means its translation by s. The purpose of this section is to study the behaviour of $v_\delta^*(z, \zeta)$ as a function of $t=\zeta-\log z_0$. In particular we shall obtain estimates of $v_\delta^*(z, \zeta)$ as $t=\zeta-\log z_0 \to \infty$ in some sector.

We will study integration of $v_\delta^*(z, \zeta)$ with respect to ζ in § 7.

Now we define the paths $C(\theta)$, $C(+)$ and $C(-)$: Put

$$\begin{cases} C_1(\theta) = \{\mu = |\mu| \exp(i\theta)\,;\, R \le |\mu| < +\infty\}, \\ C_2(\theta) = \{\mu = R\exp(i\rho)\,;\, \theta \le \rho \le \theta + 2\pi\}, \\ C_3(\theta) = \{\mu = |\mu| \exp(i(\theta+2\pi))\,;\, R \le |\mu| < +\infty\}, \end{cases} \tag{5.2}$$

and $C(\theta)=C_1(\theta) \cup C_2(\theta) \cup C_3(\theta)$, $C(\theta)$ is a path which starts at $\infty \exp(i\theta)$ on $C_1(\theta)$, goes around the origin on $C_2(\theta)$ and ends at $\infty \exp(i(\theta+2\pi))$. R will be suitably chosen (see Figure 5.1). Put

$$(5.3)\qquad \begin{cases} C_1(-)=\{\mu=\xi-i\,;\ -\infty<\xi\leq 1\}, \\ C_2(-)=\{\mu=1+\tau i\,;\ -1\leq\tau\leq 1\}, \\ C_3(-)=\{\mu=\xi+i\,;\ -\infty<\xi\leq 1\}, \end{cases}$$

and

$$(5.4)\qquad \begin{cases} C_1(+)=\{\mu=\xi+i\,;\ -1\leq\xi<+\infty\}, \\ C_2(+)=\{\mu=-1+\tau i\,;\ -1\leq\tau\leq 1\}, \\ C_3(+)=\{\mu=\xi-i\,;\ -1\leq\xi<+\infty\}. \end{cases}$$

$C(-)=C_1(-)\cup C_2(-)\cup C_3(-)$ ($C(+)=C_1(+)\cup C_2(+)\cup C_3(+)$) is a path which starts at $-\infty$ on $C_1(-)$ (resp. $+\infty$ on $C_1(+)$), passes along $C_2(-)$ (resp. along $C_2(+)$) and ends at $-\infty$ on $C_3(-)$ (resp. $+\infty$ on $C_3(+)$) (see Figure 5.2).

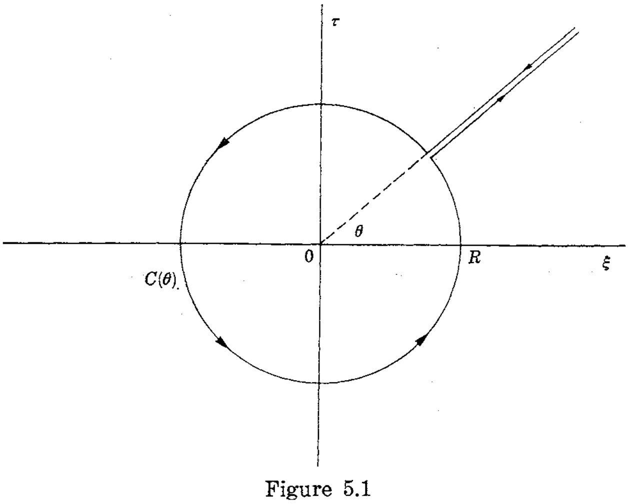

Figure 5.1

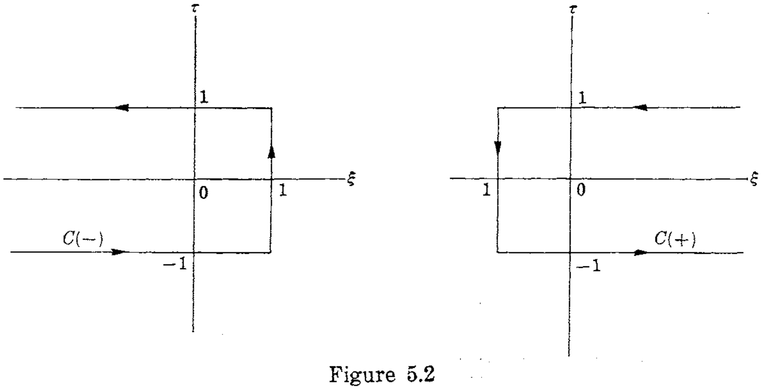

Figure 5.2

Putting $C=C(\theta)$, $R>\kappa=\max_{0\leq i\leq p}|\mu_i|$ (see (3.5)) in (5.1) and (5.2) and $t=\zeta-\log z_0$, we have

$$v_\delta^*(z,\zeta)=\frac{\exp(s\zeta)}{2\pi i}\int_{C(\theta)}\exp(-t\mu)w_\delta^*(\mu)\log(\mu-s)/H_\delta(\mu)\,d\mu, \tag{5.5}$$

where

$$w_\delta^*(\mu)=w_\delta^*(z',\mu-s,\zeta). \tag{5.6}$$

$v_\delta^*(z,\zeta)$ will be also denoted by $v_\delta^*(t)$.

Now in order to study $v_\delta^*(z,\zeta)$ $(=v_\delta^*(t))$, we investigate a function

$$v(t)=\int_{C(\theta)}\exp(-t\mu)w(\mu)\Big(\prod_{i=1}^{M}(\mu-a_i)\Big)^{-1}\log(\mu-a)\,d\mu, \tag{5.7}$$

where $w(\mu)$ is an entire function of μ subject to

$$|w(\mu)|\leq A(|\mu|+d)^N\exp(\varepsilon|\mu|), \tag{5.8}$$

where $a, a_i\in R$, N is a positive integer and R in $C(\theta)$ is chosen so that

$$R>\kappa^*=\max\{|a_1|,|a_2|,\cdots,|a_M|,|a|\}.$$

Put

$$S(R_1,R_2,\alpha,\beta)=\{t\in C^1;\ R_1\leq|t|\leq R_2,\ \alpha<\arg t<\beta\}. \tag{5.9}$$

For $v(t)$ we have the following:

PROPOSITION 5.1. (i) $v(t)\in\tilde{\mathcal{O}}$ $(|t|>\varepsilon)$.
(ii) *For $t\in S(R_1,R_2,\alpha,\beta)$ $(R_1>\varepsilon)$, there are constants A and B such that*

$$|v(t)|\leq AB^{M+N+\kappa^*}e^dN!/M!, \tag{5.10}$$

where A depends on $S(R_1,R_2,\alpha,\beta)$ and B depends on R_1 and R_2.

PROOF. By varying θ in $C(\theta)$, we have (i). Let us show (ii). We divide $v(t)$ into two parts:

$$v(t)=\int_{C_2(\theta)}+\int_{C_1(\theta)\cup C_3(\theta)}=v_0(t)+v_1(t).$$

Assume that $|\arg t|<\pi/2$ and put $\theta=0$ and $R=M+\kappa^*+1$ in $C(\theta)$. We have

$$\begin{aligned}|v_0(t)|&\leq A\int_{C_2(0)}\exp((R_2+\varepsilon)R)\Big\{(R+d)^NR\Big(\prod_{i=1}^{M}|\mu-a_i|\Big)^{-1}|d\mu|\Big\}\\&\leq Ae^dC^{M+\kappa^*}N!/M!.\end{aligned}$$

Here we make use of $x^N \leq \exp(x)N!$ for $x \geq 0$ and $|\log(\mu-a)| \leq AR$ and of

$$\max_{|\mu|=R} \left|\left(\prod_{i=1}^{M}(\mu-a_i)\right)\right|^{-1} \leq (M!)^{-1}.$$

For $v_1(t)$, if $\operatorname{Re} t \geq R_1 > \varepsilon$, then we have

$$|v_1(t)| \leq A\int_R^{\infty} \exp(-(\operatorname{Re} t)\mu)|w(\mu)| \left|\prod_{i=1}^{M}(\mu-a_i)\right|^{-1} d\mu.$$

Hence

$$\begin{aligned}|v_1(t)| &\leq A(M!)^{-1}\int_0^{\infty} \exp(-(R_1-\varepsilon)\mu)(\mu+d)^N\, d\mu \\ &\leq A(M!)^{-1}2^N\int_0^{\infty} \exp(-(R_1-\varepsilon)\mu)(\mu^N+d^N)\, d\mu \\ &\leq AB^N e^d N!/M!.\end{aligned}$$

Thus it has been shown that (5.10) is valid for $\alpha=-\beta=\pi/2$. By varying θ in $C(\theta)$ and by the same method, we can show that (5.10) is also valid for any α and β. $\square$

Let us apply Proposition 5.1 to $v_\delta^*(t)=v_\delta^*(z,\zeta)$. Put for small $\varepsilon>0$

$$\Omega_\varepsilon=\{z=(z_0,z')\,;\,\gamma|z_1|<\varepsilon,\ z'\in\Omega'\}, \tag{5.11}$$

where γ and Ω' are the same as in Proposition 4.10. We have from Proposition 4.10 the following:

PROPOSITION 5.2. (i) *$v_\delta^*(t)$ is holomorphic in $\{(z,\zeta)\,;\,z\in\Omega_\varepsilon,\ |\zeta-\log z_0| >\varepsilon\}$.*
(ii) *For $\zeta-\log z_0\in S(R_1,R_2,\alpha,\beta)$ $(R_1>\varepsilon)$ and $z\in\Omega_\varepsilon$, we have*

$$|v_\delta^*(z,\zeta)| \leq Am(\zeta)\,|\exp(s\zeta)|B^{n+|s|}\Big/\left(\sum_{i=1}^{p}(m-k_i)\right)!, \tag{5.12}$$

where A depends on $S(R_1,R_2,\alpha,\beta)$ and B depends only on R_1 and R_2.

PROOF. (i) follows from (i) in Proposition 5.1. Let us show (ii). It follows from (5.5) and Proposition 4.10 that by putting $\kappa^*=d=\kappa$ and $M=(m-L)(p+1)$, which is the degree of $H_\delta(\mu)$, we have

$$\begin{aligned}|v_\delta^*(t)| &\leq Am(\zeta)\,|\exp(s\zeta)|B^{n+|s|}((pL)!M!)^{-1}C^{M+k(\delta)+\kappa} \\ &\quad\times\left\{\sum_{r=0}^{l(\delta)}(r+k(\delta)-l(\delta))!(l(\delta)-r)!\right\}.\end{aligned}$$

Since $M+k(\delta)+\kappa\leq C(n+|s|)$ and $k(\delta)!/(pL)!M!\leq C^{p+1}/(pm-k(\delta))!$, we have (5.12). $\square$

We obtain estimates of $v_\delta^*(t)=v_\delta^*(z,\zeta)$ in bounded sets in t-space by Proposition 5.2. We need information on the behaviour of $v_\delta^*(t)$ as $t\to\infty$. Put for $R_1>\varepsilon$ and $n\in Z$,

$$
(5.13)\quad \begin{cases} \Omega_+^\varepsilon(R_1,n)=\{(z,\zeta)\,;\ z\in\Omega_\varepsilon,\ \mathrm{Re}\,t\geq R_1,\ |\arg t-2n\pi|<\pi/2\}, \\ \Omega_-^\varepsilon(R_1,n)=\{(z,\zeta)\,;\ z\in\Omega_\varepsilon,\ \mathrm{Re}\,t\leq -R_1,\ |\arg t-(2n+1)\pi|<\pi/2\}, \end{cases}
$$

where $t=\zeta-\log z_0$.

Let us obtain estimates for $v_\delta^*(t)$ in $\Omega_\pm^\varepsilon(R_1,n)$. We need the information of poles of $H_\delta(\mu)$. We return to (5.5) and (3.10). Put

$$
(5.14)\quad \begin{cases} h_\delta^1(\mu)=\prod_{\{i;\alpha_i=1\}}(\mu-\tilde{\mu}_i)\prod_{\{i;\alpha_i\geq 2\}}(\mu-\tilde{\mu}_i)^2, \\ h_\delta^2(\mu)=\prod_{\{i;\alpha_i>2\}}(\mu-\tilde{\mu}_i)^{\alpha_i-2}, \end{cases}
$$

and

$$
(5.15)\quad \begin{cases} H_\delta^1(\mu)=h_\delta^1(\mu)h_\delta^1(\mu-1)\cdots h_\delta^1(\mu-(m-L)+1), \\ H_\delta^2(\mu)=h_\delta^2(\mu)h_\delta^2(\mu-1)\cdots h_\delta^2(\mu-(m-L)+1). \end{cases}
$$

We have $H_\delta(\mu)=H_\delta^1(\mu)H_\delta^2(\mu)$.

First let us deduce estimates for $v_\delta^*(t)$ in $\Omega_+^\varepsilon(R_1,n)$. Define

$$
(5.16)\quad I_+(t)=\frac{1}{2\pi i}\int_{C(+)}\exp(-t\mu)w_\delta^*(\mu+\mu_*)H_\delta(\mu+\mu_*)^{-1}\log(\mu-s+\mu_*)\,d\mu
$$

and

$$
(5.17)\quad J_+(t)=\frac{1}{2\pi i}\int_{C(+)}\exp(-t\mu)w_\delta^*(\mu+\mu_*)H_\delta^1(\mu+\mu_*)^{-1}\log(\mu-s+\mu_*)\,d\mu,
$$

where $\mu_*=\min_{0\leq i\leq p}\mu_i$ (see (3.5)). We note that

$$
(5.18)\quad v_\delta^*(t)=v_\delta^*(z,\zeta)=\exp(s\zeta-t\mu_*)I_+(t)
$$

and

$$
(5.19)\quad H_\delta^2(-\partial/\partial_t+\mu_*)I_+(t)=J_+(t).
$$

First we estimate $J_+(t)$ and next $I_+(t)$ by using (5.19). So we have an estimate of $v_\delta^*(t)$ in $\Omega_+^\varepsilon(R_1, n)$ from (5.18).

PROPOSITION 5.3. *Suppose that* $\operatorname{Re} t \geq \tilde{\gamma} > \varepsilon$ *and* $|\arg t - 2n\pi| < \pi/2$ $(n \in Z)$. *Then for* $z \in \Omega_\varepsilon$,

$$(5.20) \qquad |J_+(t)| \leq Am(\zeta) B^{n+|s|} \exp(|t|) k(\delta)!/(pL)! \{(q_1 + 2q_2)!\}^{m-L}$$

holds, where A *depends on* n *and* $\tilde{\gamma}$, B *depends on* $\tilde{\gamma}$, *and* q_1, q_2 *are those in* (3.11).

PROOF. From Proposition 4.10, and by deforming the path $C(\theta)$, $\theta = -2n\pi$, to $C(+)$, we have

$$(5.21) \qquad |J_+(t)| \leq Am(\zeta) B^{n+|s|}((pL)!)^{-1} \Big\{ \sum_{r=0}^{l(\delta)} (l(\delta) - r)! \int_{C(+)} \exp(-\operatorname{Re}(t\mu) + |\varepsilon\mu|)$$
$$\times (|\mu| + |\mu_*| + \kappa)^{r + k(\delta) - l(\delta) + m - L + 1} / |H_\delta^1(\mu + \mu_*)| \, |d\mu| \Big\},$$

where we have made use of $|\log(\mu - s + \mu_*)| \leq A_1(|\mu| + |\mu_*| + \kappa)$ and A_1 depends on $n \in Z$. If $\operatorname{Re} t \geq \tilde{\gamma}$, we get

$$(5.22) \qquad \int_{C(+)} \exp(-\operatorname{Re}(t\mu) + \varepsilon|\mu|) \{(|\mu| + |\mu_*| + \kappa)^{r + k(\delta) - l(\delta) + (m-L) + 1} / H_\delta^1(\mu + \mu_*)\} |d\mu|$$
$$\leq CD^{n+|s|} \exp(|t|)(r + k(\delta) - l(\delta) + m - L + 1)! \sup_{\mu \in C(+)} |H_\delta^1(\mu + \mu_*)|^{-1}.$$

Since on $C(+)$ it holds that

$$|H_\delta^1(\mu + \mu_*)|^{-1} \leq CD^{\tilde{q}(m-L)} (\tilde{q}!)^{-m+L}, \quad \tilde{q} = q_1 + 2q_2,$$

and $\tilde{q} \leq C(n + |s|)$, we have (5.20) from (5.21) and (5.22). $\square$

Next let us estimate $I_+(t)$. We put

$$(5.23) \qquad H_\delta^2(\mu + \mu_*) = \prod_{i=1}^{l} (\mu - \tau_i),$$

where $l = \sum_{\{i; \alpha_i > 2\}} (\alpha_i - 2)$ and $0 \leq \tau_i \leq 2n - s$ by Proposition 3.2. By (5.19), we have

$$(5.24) \qquad H_\delta^2(-\partial/\partial_t + \mu_*) I_+(t) = (-1)^l \Big\{ \prod_{i=1}^{l} (\partial/\partial_t + \tau_i) \Big\} I_+(t) = J_+(t).$$

By integrating (5.24), we have $I_+(t) = \sum_{j=0}^{l} I_{+,j}(t)$, where

$$(5.25)\qquad I_{+,0}(t)=(-1)^{l}\int_{\tilde{\gamma}}^{t}\exp(-\tau_{l}(t-t_{l-1}))dt_{l-1}$$
$$\cdots\int_{\tilde{\gamma}}^{t_2}\exp(-\tau_{2}(t_2-t_1))\,dt_1\int_{\tilde{\gamma}}^{t_1}\exp(-\tau_{1}(t_1-t_0))J_{+}(t_0)dt_0,$$

$$(5.26)\qquad I_{+,j}(t)=\int_{\tilde{\gamma}}^{t}\exp(-\tau_{l}(t-t_{l-1}))dt_{l-1}$$
$$\cdots\int_{\tilde{\gamma}}^{t_{j+2}}\exp(-\tau_{j+2}(t_{j+2}-t_{j+1}))dt_{j+1}$$
$$\times\int_{\tilde{\gamma}}^{t_{j+1}}\exp(-\tau_{j+1}(t_{j+1}-t_j)-\tau_j(t_j-\tilde{\gamma}))dt_j\Big\{\prod_{i=j+1}^{l}(\partial/\partial_t+\tau_i)\Big\}I_{+}(\tilde{\gamma})$$

for $1\leq j\leq l-1$,
and

$$(5.27)\qquad I_{+,l}(t)=\exp(-\tau_{l}(t-\tilde{\gamma}))I_{+}(\tilde{\gamma}),$$

where $\tilde{\gamma}>\varepsilon$.

Now we have the following:

Proposition 5.4. *For* $(z,\zeta)\in\Omega_{+}^{\varepsilon}(\tilde{\gamma},n)$

$$(5.28)\qquad |v_{\delta}^{*}(z,\zeta)|\leq Am(\zeta)B^{n+|s|}\,|\exp(|t|+(s-\mu_{*})\zeta)|$$
$$\times|z_0|^{\mu_*}|t|^{(m-L)N(\delta)}\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!$$

holds. A depends on $\tilde{\gamma}$ *and* n, *and B depends only on* $\tilde{\gamma}$.

Proof. It follows from (5.24)-(5.27) that for $(z,\zeta)\in\Omega_{+}^{\varepsilon}(\tilde{\gamma},n)$,

$$(5.29)\qquad |I_{+}(t)|\leq\sum_{j=1}^{l-1}|t-\tilde{\gamma}|^{l-j}\Big|\Big\{\prod_{i=j+1}^{l}(\partial/\partial_t+\tau_i)\Big\}I_{+}(\tilde{\gamma})\Big|\Big/(l-j)!$$
$$+|I_{+}(\tilde{\gamma})|+|I_{+,0}(t)|.$$

In view of Proposition 5.2 and (5.18), we have

$$(5.30)\qquad \Big|\Big\{\prod_{i=j+1}^{l}(\partial/\partial_t+\tau_i)\Big\}I_{+}(\tilde{\gamma})\Big|\leq Am(\zeta)B^{n+|s|}(l-j)!\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!\,.$$

Noting $l=(m-L)N(\delta)$ and $N(\delta)=p+1-q_1-2q_2$, we have from Proposition 5.3, for $(z,\zeta)\in\Omega_{+}^{\varepsilon}(\tilde{\gamma},n)$

$$(5.31)\qquad |I_{+,0}(t)|\leq Am(\zeta)B^{n+|s|}|t-\tilde{\gamma}|^{l}\exp(|t|)k(\delta)!/((m-L)N(\delta))!(pL)!$$
$$\times\{(p+1-N(\delta))!\}^{m-L}$$
$$\leq Am(\zeta)B^{n+|s|}|t-\tilde{\gamma}|^{(m-L)N(\delta)}\exp(|t|)\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!.$$

Thus we have from (5.29)-(5.31)

$$|I_+(t)|\leq Am(\zeta)B^{n+|s|}(1+|t-\tilde{\gamma}|)^{(m-L)N(\delta)}\exp(|t|)\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!. \tag{5.32}$$

We have (5.28) from (5.18). □

In order to get an estimate of $v_\delta^*(t)=v_\delta^*(z,\zeta)$ in $\Omega_-^\varepsilon(\tilde{\gamma},n)$, we put

$$I_-(t)=\frac{1}{2\pi i}\int_{C(-)}\exp(-t\mu)w_\delta^*(\mu+\mu^*)H_\delta(\mu+\mu^*)^{-1}\log(\mu-s+\mu^*)d\mu, \tag{5.33}$$

$$J_-(t)=\frac{1}{2\pi i}\int_{C(-)}\exp(-t\mu)w_\delta^*(\mu+\mu^*)H_\delta^1(\mu+\mu^*)^{-1}\log(\mu-s+\mu^*)d\mu. \tag{5.34}$$

Then we have

$$v_\delta^*(t)=v_\delta^*(z,\zeta)=\exp(s\zeta-t\mu^*)I_-(t) \tag{5.35}$$

and

$$H^2(-\partial/\partial_t+\mu^*)I_-(t)=J_-(t). \tag{5.36}$$

By a method similar to that used for $I_+(t)$ and $J_+(t)$, we have:

PROPOSITION 5.5. *For* $(z,\zeta)\in\Omega_-^\varepsilon(\tilde{\gamma},n)$

$$\begin{aligned}|v_\delta^*(z,\zeta)|\leq{}& Am(\zeta)B^{n+|s|}|\exp(|t|+(s-\mu^*)\zeta)|\\&\times|z_0|^{\mu^*}|t|^{(m-L)N(\delta)}\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!\end{aligned} \tag{5.37}$$

holds. A depends on $\tilde{\gamma}$ *and* n, *and B depends on* $\tilde{\gamma}$.

Thus we have, from Propositions 3.4, 3.5, 5.2, 5.4 and 5.5, the following:

THEOREM 5.6. *Let* $\sigma_1>1$. (i) *If* $(z,\zeta)\in\Omega_+^\varepsilon(\tilde{\gamma},n)$,

$$\begin{aligned}|v_\delta^*(z,\zeta)|\leq{}& Am(\zeta)B^{n+|s|}|\exp(|t|+(s-\mu_*)\zeta)||z_0|^{\mu_*}\\&\times|t|^{(m-L)(2(n-s+\mu_*)-(2b+c))}/\Gamma(a_-/(\sigma_1-1)+c+1).\end{aligned} \tag{5.38}$$

(ii) *If* $(z,\zeta)\in\Omega_-^\varepsilon(\tilde{\gamma},n)$,

$$\begin{aligned}|v_\delta^*(z,\zeta)|\leq{}& Am(\zeta)B^{n+|s|}|\exp(|t|+(s-\mu^*)\zeta)||z_0|^{\mu^*}\\&\times|t|^{(m-L)(2(n-b-c-\mu^*)+c)}/\Gamma(a_-/(\sigma_1-1)+c+1).\end{aligned} \tag{5.39}$$

A and B in (5.38) *and* (5.39) *have the same properties as in Propositions*

5.4 *and* 5.5.

(iii) *If* $(z, \zeta) \in \{(z, \zeta)\,;\ z \in \Omega_\varepsilon,\ \zeta - \log z_0 \in S(\tilde{\gamma}, R_2, \alpha, \beta)\}$,

$$|v_\delta^*(z, \zeta)| \le Am(\zeta)|\exp(s\zeta)|B^{n+|s|}/\Gamma(a_-/(\sigma_1 - 1) + c + 1). \tag{5.40}$$

In (5.40) A *depends on* $S(\tilde{\gamma}, R_2, \alpha, \beta)$ *and* B *depends only on* R_2 *and* $\tilde{\gamma}$.

In (i)–(iii) a_-, b *and* c *are those defined by* (3.3) *and* (3.4), $\Gamma(x)$ *is the gamma function and* $m(\zeta)$ *is the same as in* (4.25).

Now let us consider the case $\sigma_1 = 1$. We put $\sigma = \sigma_{1,1}$ and recall (2.23). Put

$$\tilde{w}_\delta(z', \lambda, \zeta) = \frac{F_1(\lambda + s)}{F(m - L\,;\ \lambda + s)F_2(\lambda + s)} w_\delta(z', \lambda, \zeta), \tag{5.41}$$

$$\tilde{w}_\delta(\mu) = \tilde{w}_\delta(z', \mu - s, \zeta). \tag{5.42}$$

We note that $r_i \ge 0$, $\mu_* = 0$ and $\mu^* = s$. Furthermore, we put

$$I_+(t) = \frac{1}{2\pi i}\int_{C(+)} \exp(-t\mu)\tilde{w}_\delta(\mu)F_3(\mu)^{-1}\log(\mu - s)d\mu, \tag{5.43}$$

$$J_+(t) = \frac{1}{2\pi i}\int_{C(+)} \exp(-t\mu)\tilde{w}_\delta(\mu)\log(\mu - s)d\mu, \tag{5.44}$$

$$I_-(t) = \frac{1}{2\pi i}\int_{C(-)} \exp(-t\mu)\tilde{w}_\delta(\mu + s)F_3(\mu + s)^{-1}\log\mu d\mu \tag{5.45}$$

and

$$J_-(t) = \frac{1}{2\pi i}\int_{C(-)} \exp(-t\mu)\tilde{w}_\delta(\mu + s)\log\mu d\mu. \tag{5.46}$$

We have

$$v_\delta^*(t) = v_\delta^*(z, \zeta) = \exp(s\zeta)I_+(t) = (z_0)^s I_-(t), \tag{5.47}$$

$$F_3(-\partial/\partial_t)I_+(t) = J_+(t), \quad F_3(-\partial/\partial_t + s)I_-(t) = J_-(t). \tag{5.48}$$

Now we claim a proposition corresponding to Proposition 5.3:

Proposition 5.7. *Suppose* $(z, \zeta) \in \Omega_+^\varepsilon(\tilde{\gamma}, n)$. *Then*

$$|J_+(t)| \le Am(\zeta)B^{n+|s|}\exp(|t|)l(\delta)!h_1^*!/(pL)!h_2^*!, \tag{5.49}$$

where $h_1^*=\sum_{i\in B(+,\delta)}(k_i-l_i-m+L)$, $h_2^*=\sum_{i\in B(*,\delta)}(m-L-k_i+l_i)$, *A depends on* $\tilde{\gamma}$ *and* n, *and B depends on* $\tilde{\gamma}$.

For $B(+,\delta)$ and $B(*,\delta)$, we refer to §2. The proof is similar to that of Proposition 5.3. So we omit it. Corresponding to Theorem 5.6, we have:

THEOREM 5.8. *Let* $\sigma_1=1$. (i) *For* $(z,\zeta)\in\Omega_+^\varepsilon(\tilde{\gamma},n)$

$$|v_\delta^*(z,\zeta)|\leq Am(\zeta)B^{n+|s|}|\exp(|t|+s\zeta)||t|^{\sigma h}\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!. \tag{5.50}$$

(ii) *For* $(z,\zeta)\in\Omega_-^\varepsilon(\tilde{\gamma},n)$

$$|v_\delta^*(z,\zeta)|\leq Am(\zeta)B^{n+|s|}\exp(|t|)|z_0|^s|t|^{\sigma h}\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!. \tag{5.51}$$

In (5.50) *and* (5.51) $h=\sum_{i\in B(0,\delta)}(m-k_i)$, $\sigma=\sigma_{1,1}$, $t=\zeta-\log z_0$, *A depends on* $\tilde{\gamma}$ *and* n, *and B depends on* $\tilde{\gamma}$.

(iii) *For* $(z,\zeta)\in\{(z,\zeta)\,;\,z\in\Omega_\varepsilon,\ \zeta-\log z_0\in S(\tilde{\gamma},R_2,\alpha,\beta)\}$,

$$|v_\delta^*(z,\zeta)|\leq Am(\zeta)|\exp(s\zeta)|B^{n+s}\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!, \tag{5.52}$$

where A depends on $S(\tilde{\gamma},R_2,\alpha,\beta)$, *B depends on* R_2 *and* $\tilde{\gamma}$. $m(\zeta)$ *is the same as in* (4.25).

PROOF. Let us show (5.50). We have from (5.48) and (5.49) by the method in the preceding part of this section

$$\begin{aligned}|I_+(t)|&\leq Am(\zeta)B^{n+s}\exp(|t|)|t|^{h^*}l(\delta)!h_1^*!/(pL)!h_2^*!h^*!\\&\leq Am(\zeta)B^{n+s}\exp(|t|)|t|^{h^*}\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!,\end{aligned} \tag{5.53}$$

where $h^*=\sum_{i\in B(0,\delta)}(m-L-k_i+l_i)$. It follows from the definition of $\sigma_{1,1}(=\sigma)$ that for $i\in B(0,\delta)$, $m-L-k_i+l_i=J-j_i\leq\sigma_{1,1}(m-k_i)$. So $h^*\leq\sigma_{1,1}h$. Thus we have (5.50). In the same way we have (5.51). The estimate (5.52) follows from Proposition 5.2. □

REMARK 5.9. Let $d=\sum_{i\in A(0,\delta)}(m-k_i)$ (see (3.9)). Then $\sum_{i=1}^{p}(m-k_i)\geq d\geq h$. So we can get inequalities with $|t|^{\sigma h}\Big/\Big(\sum_{i=1}^{p}(m-k_i)\Big)!$ replaced by $|t|^{\sigma d}/d!$ in

(5.50) and (5.51) for $|t|\geq\tilde{\gamma}>\varepsilon$ by choosing other constants A and B.

§ 6. Convergence.

In this section we show that

$$V(z,\zeta)=\sum_{n=0}^{\infty}\sum_{s=-\infty}^{S(n)}v_{n,s}^{*}(z,\zeta) \tag{6.1}$$

converges, where

$$v_{n,s}^{*}(z,\zeta)=\sum_{\delta\in\Delta(n,s)}(-1)^{p}v_{\delta}^{*}(z,\zeta) \tag{6.2}$$

and $v_{\delta}^{*}(z,\zeta)$ is defined by (5.1).

For $R_2>\tilde{\gamma}>\varepsilon>0$ and $\alpha<\beta$, we set

$$\left\{\begin{aligned}\Omega_{+}^{\varepsilon}(\tilde{\gamma},R_2,\alpha,\beta)&=\{(z,\zeta)\,;\,z\in\Omega_{\varepsilon},\,t\in S(\tilde{\gamma},R_2,\alpha,\beta)\}\\&\quad\cup\{(z,\zeta)\,;\,z\in\Omega_{\varepsilon},\ \mathrm{Re}\,t\geq\tilde{\gamma},\ \alpha<\arg t<\beta\},\\\Omega_{-}^{\varepsilon}(\tilde{\gamma},R_2,\alpha,\beta)&=\{(z,\zeta)\,;\,z\in\Omega_{\varepsilon},\ t\in S(\tilde{\gamma},R_2,\alpha,\beta)\}\\&\quad\cup\{(z,\zeta)\,;\,z\in\Omega_{\varepsilon},\ \mathrm{Re}\,t\leq-\tilde{\gamma},\ \alpha<\arg t<\beta\},\end{aligned}\right. \tag{6.3}$$

where $t=\zeta-\log z_0$, $S(\tilde{\gamma},R_2,\alpha,\beta)$ is defined by (5.9) and Ω_{ε} is defined by (5.11) (see Figure 6.1).

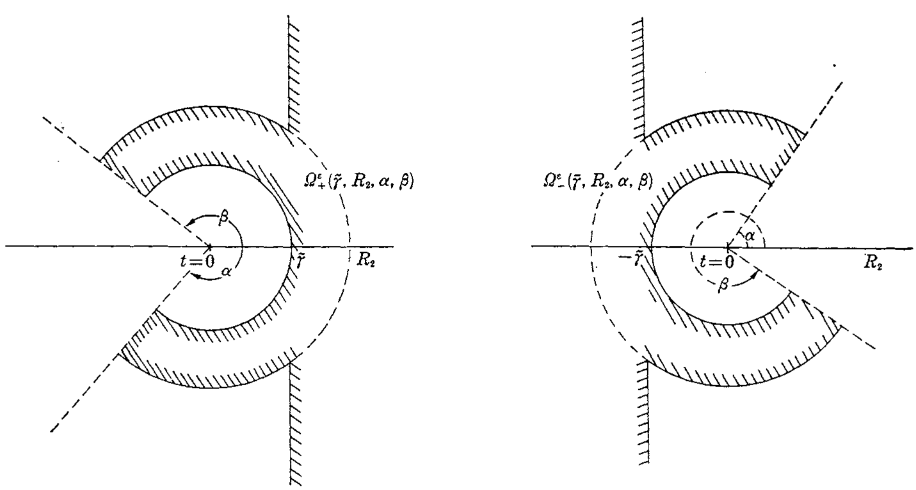

Figure 6.1

Now we assume $\sigma_1>1$. Let us recall Theorem 5.6. Put $\sigma_1=\sigma$. First we study the convergence of $V(z,\zeta)$ in $\Omega_{+}^{\varepsilon}(\tilde{\gamma},R_2,\alpha,\beta)$. We give a lemma.

LEMMA 6.1. *The following inequalities hold:*

(i) $(|z_0|^{-1}|\exp\zeta|)^{-\mu_*}|t|^{2(m-L)(n-s+\mu_*-b-c)}\leq\{(|z_0|^{-1}|\exp\zeta|+|t|^{2(m-L)}\}^{(n-s-b-c)}$,

(ii) *if* $s\leq 0$, *then*

$$(|z_0|^{-1}|\exp\zeta|)^{-\mu_*}|t|^{2(m-L)(n-s+\mu_*-b-c)}$$
$$\leq\{|z_0|^{-1}|\exp\zeta|+|t|^{2(m-L)}\}^{(n-b-c)}(|z_0|^{-1}|\exp\zeta|)^{-s},$$

where μ_*, b *and* c *are the same as defined in* §3 (*see* (3.4) *and* (3.5)).

PROOF. From Proposition 3.2, we have $n-s+\mu_*\geq b+c$. Thus (i) in this lemma holds. $\mu_*\leq s$ if $s<0$. Hence

$$(|z_0|^{-1}|\exp\zeta|)^{-s+s-\mu_*}|t|^{2(m-L)(n-s+\mu_*-b-c)}$$
$$\leq(|z_0|^{-1}|\exp\zeta|)^{-s}\{|z_0|^{-1}|\exp\zeta|+|t|^{2(m-L)}\}^{(n-b-c)}. \quad \square$$

Put $X=|z_0|^{-1}|\exp\zeta|+|t|^{2(m-L)}$, $Y=|t|^{(m-L)}$ and let $(z,\zeta)\in\Omega_+^\varepsilon(\tilde{\gamma},n)$. Then we have from Theorem 5.6 and Lemma 6.1 that if $s>0$, then

$$|v_s^*(z,\zeta)|\leq Am(\zeta)B^{n+s}|\exp(|t|+s\zeta)|X^{n-s-b-c}Y^c/\Gamma(a_-/(\sigma-1)+c+1) \tag{6.4}$$

and that if $s\leq 0$, then

$$|v_s^*(z,\zeta)|\leq Am(\zeta)B^{n+|s|}\exp(|t|)|z_0|^s X^{n-b-c}Y^c/\Gamma(a_-/(\sigma-1)+c+1). \tag{6.5}$$

Thus it follows from (i) and (iv) in Propositions 3.2 and 3.3 that if $s>0$, then

$$\begin{aligned}|v_{n,s}^*(z,\zeta)|\leq{}& Am(\zeta)B^{n+s}|\exp(|t|+s\zeta)|\\ &\times\Big\{\sum_{\substack{b+c\leq n-s\\ b\leq\min(n/2,n-2)}}X^{n-s-b-c}Y^c/\Gamma((n-s-b-c)/(\sigma-1)+1)\Gamma(c+1)\Big\}\end{aligned} \tag{6.6}$$

and that if $s\leq 0$, then

$$\begin{aligned}|v_{n,s}^*(z,\zeta)|\leq{}& Am(\zeta)B^{n+|s|}\exp(|t|)|z_0|^s\\ &\times\Big\{\sum_{\substack{b+c\leq n\\ 2b\leq n}}X^{n-b-c}Y^c/\Gamma((n-s-b-c)/(\sigma-1)+1)\Gamma(c+1)\Big\}.\end{aligned} \tag{6.7}$$

Hence, in order to get convergence of $V(z,\zeta)$ in $\Omega_+^\varepsilon(\tilde{\gamma},n)$, we consider

$$\begin{aligned}I_1=\sum_{n=0}^{\infty}\sum_{s=-\infty}^{-1}|z_0|^s B^{n+|s|}\Big\{&\sum_{\substack{b+c\leq n\\ 2b\leq n}}X^{n-b-c}Y^c/\Gamma((n-s-b-c)/(\sigma-1)+1)\\ &\times\Gamma(c+1)\Big\}\end{aligned} \tag{6.8}$$

and

$$I_2 = \sum_{n=0}^{\infty} \sum_{s=0}^{n} B^{n+s} |\exp(s\zeta)| \times \Big\{ \sum_{\substack{b+c \le n-s \\ b \le \min(n/2, n-s)}} X^{n-s-b-c} Y^c / \Gamma((n-s-b-c)/(\sigma-1)+1)\Gamma(c+1) \Big\}. \tag{6.9}$$

We have the following:

Lemma 6.2. *I_1 converges and*

$$I_1 \le A \exp(BX^{\sigma-1} + CY + D|z_0|^{-(\sigma-1)}) \tag{6.10}$$

holds for $(z, \zeta) \in \Omega_+^{\varepsilon}(\tilde{\gamma}, n)$.

Proof. We have

$$\begin{cases} I_1 = \sum_{s=1}^{\infty} B^s |z_0|^{-s} I_{1,s}, \\ I_{1,s} = \sum_{n=0}^{\infty} B^n \Big\{ \sum_{\substack{b+c \le n \\ 2b \le n}} X^{n-b-c} Y^c / \Gamma((n+s-b-c)/(\sigma-1)+1)\Gamma(c+1) \Big\}. \end{cases} \tag{6.11}$$

Then

$$\begin{aligned} I_{1,s} &= \sum_{n=0}^{\infty} \sum_{2b \le n} \sum_{c=0}^{n-b} \cdots \le \sum_{b=0}^{\infty} \sum_{n \ge 2b} \sum_{c=0}^{n-b} \cdots \\ &= \sum_{b=0}^{\infty} \sum_{c=0}^{b} \sum_{n=2b}^{\infty} \cdots + \sum_{b=0}^{\infty} \sum_{c=b+1}^{\infty} \sum_{n=b+c}^{\infty} \cdots \\ &\le AE^s \exp(AX^{\sigma-1}) \Gamma(s/(\sigma-1)+1)^{-1} \\ &\quad \times \Big\{ \sum_{b=0}^{\infty} \sum_{c=0}^{b} B^{b+c} X^{b-c} Y^c / \Gamma((b-c)/(\sigma-1)+1)\Gamma(c+1) \\ &\quad + \sum_{b=0}^{\infty} \sum_{c=b+1}^{\infty} B^{b+c} Y^c / \Gamma(c+1) \Big\} \\ &\le AE^s \exp(BX^{\sigma-1} + CY) / \Gamma(s/(\sigma-1)+1). \end{aligned}$$

Hence I_1 converges and (6.10) holds. □

For I_2 we have:

Lemma 6.3. *There is a ζ^* such that for $(z, \zeta) \in \Omega_+^{\varepsilon}(\tilde{\gamma}, n)$ with* $\mathrm{Re}\, \zeta \le \zeta^*$, *$I_2$ converges and*

$$I_2 \le A \exp(BX^{\sigma-1} + CY) \tag{6.12}$$

holds.

PROOF. We have from (6.9)

$$(6.13)\quad \begin{cases} I_2=\sum_{s=0}^{\infty}|\exp(s\zeta)|B^s I_{2,s}, \\ I_{2,s}=\sum_{n=s}^{\infty}B^n\Big\{\sum_{\substack{b+c\leq n-s\\ b\leq\min(n/2,n-s)}} X^{n-s-b-c}Y^c/\Gamma((n-s-b-c)/(\sigma-1)+1)\Gamma(c+1)\Big\}. \end{cases}$$

We have

$$\begin{aligned} I_{2,s}&\leq\Big\{\sum_{n=s}^{2s}\sum_{b+c\leq n-s}\cdots\Big\}+\sum_{n=2s+1}^{\infty}\Big\{\sum_{c=0}^{[n/2]+1-s}\sum_{b=0}^{[n/2]}\cdots+\sum_{c=[n/2]-s}^{n-s}\sum_{b=0}^{n-s-c}\cdots\Big\} \\ &\leq AE^s\exp(BX^{\sigma-1}+CY)+\sum_{n=2s+1}^{\infty}\Big\{\sum_{c=0}^{[n/2]+1-s}X^{[n/2]+1-s-c}Y^c/\Gamma(([n/2] \\ &\quad +1-s-c)/(\sigma-1)+1)\Gamma(c+1)+\sum_{c=[n/2]-s}^{n-s}Y^c/\Gamma(c+1)\Big\}\exp(BX^{\sigma-1}). \end{aligned}$$

Thus $I_{2,s}\leq AE^s\exp(BX^{\sigma-1}+CY)$. Consequently, if $|E\exp\zeta|\leq 1/2$, then I_2 converges and (6.12) holds. □

From Lemmas 6.2 and 6.3, we obtain:

THEOREM 6.4. *For* $(z,\zeta)\in\Omega_+^\varepsilon(\tilde{\gamma},R_2,\alpha,\beta)$ *with* $\operatorname{Re}\zeta\leq\zeta^*$, $V(z,\zeta)$ *converges and*

$$(6.14)\qquad |V(z,\zeta)|\leq Am(\zeta)\exp(BX^{\sigma-1}+CY+D|z_0|^{-(\sigma-1)}+|t|)$$

holds. A depends on $R_2,\tilde{\gamma},\alpha$ and β. ζ^*, B, C and D depend on R_2 and $\tilde{\gamma}$, while they are independent of α and β.

PROOF. It follows from Lemmas 6.2 and 6.3 that $V(z,\zeta)$ converges in $\Omega_+^\varepsilon(\tilde{\gamma},n)$ and (6.14) holds there. We have to show convergence of $V(z,\zeta)$ and its estimate in $S(\tilde{\gamma},R_2,\alpha,\beta)$. It can be done in the same way from (5.40) in Theorem 5.6. Dependence of the constants mentioned follows also from Theorem 5.6. □

Now let us consider convergence of $V(z,\zeta)$ in $\Omega_-^\varepsilon(\tilde{\gamma},R_2,\alpha,\beta)$. We return to (5.39). Instead of Lemma 6.1, we have:

LEMMA 6.5. *The following inequalities hold:*

(i)
$$|z_0\exp(-\zeta)|^{\mu^*}|\zeta-\log(z_0)|^{2(m-L)(n-b-c-\mu^*)}$$
$$\leq\{|z_0\exp(-\zeta)|+|\zeta-\log(z_0)|^{2(m-L)}\}^{(n-b-c)}.$$

(ii) *If* $s\geq 0$,
$$|z_0\exp(-\zeta)|^{\mu^*}|\zeta-\log(z_0)|^{2(m-L)(n-b-c-\mu^*)}$$
$$\leq|z_0\exp(-\zeta)|^{s}(|z_0\exp(-\zeta)|+|\zeta-\log(z_0)|^{2(m-L)})^{(n-s-b-c)}.$$

PROOF. Since $a_+=n-b-c\geq\mu^*\geq 0$ by Proposition 3.2, we have (i). $\mu^*\geq s$ if $s\geq 0$. Hence

$$|z_0\exp(-\zeta)|^{s+\mu^*-s}|\zeta-\log(z_0)|^{2(m-L)(n-b-c-\mu^*)}$$
$$\leq|z_0\exp(-\zeta)|^{s}(|z_0\exp(-\zeta)|+|\zeta-\log(z_0)|^{2(m-L)})^{(n-s-b-c)}. \quad \square$$

By means of Lemma 6.5, we consider the following $\tilde{I}_1$ and $\tilde{I}_2$ instead of I_1 and I_2 defined by (6.8) and (6.9), in order to investigate convergence of $V(z,\zeta)$ in $\Omega_-^{\varepsilon}(\tilde{\gamma},n)$;

$$\tilde{I}_1=\sum_{n=0}^{\infty}\sum_{s=-\infty}^{-1}|\exp(s\zeta)|B^{n+|s|}\times\Big\{\sum_{\substack{b+c\leq n\\ 2b\leq n}}X^{n-b-c}Y^{c}/\Gamma((n-s-b-c)/(\sigma-1)+1)\Gamma(c+1)\Big\}. \tag{6.15}$$

$$\tilde{I}_2=\sum_{n=0}^{\infty}\sum_{s=1}^{n}|z_0|^{s}B^{n+s}\times\Big\{\sum_{\substack{b+c\leq n-s\\ b\leq\min(n/2,n-s)}}X^{n-s-b-c}Y^{c}/\Gamma((n-s-b-c)/(\sigma-1)+1)\Gamma(c+1)\Big\}. \tag{6.16}$$

We have

THEOREM 6.6. *There is an r such that if $|z_0|\leq r$ and $(z,\zeta)\in\Omega_-^{\varepsilon}(\tilde{\gamma},R_2,\alpha,\beta)$, then $V(z,\zeta)$ converges and*

$$|V(z,\zeta)|\leq Am(\zeta)\exp(BX^{\sigma-1}+CY+D|\exp(-\zeta)|^{\sigma-1}+|t|) \tag{6.17}$$

holds. Here A depends on $R_2,\tilde{\gamma},\alpha$ and β. And B, C and D depend on R_2 and $\tilde{\gamma}$ they being independent of α and β.

Convergence of $\tilde{I}_1$ (or $\tilde{I}_2$) and its estimate are obtained in the same way as we get Lemma 6.2 (resp. Lemma 6.3). Hence we get Theorem 6.6.

Next we consider the case $\sigma_1=1$ and put $\sigma=\sigma_{1,1}$. Let us recall Theorem 5.8, Remark 5.9 and Propositions 3.3 and 3.4. It follows from them that for $(z, \zeta) \in \Omega_+^{\varepsilon}(\tilde{\gamma}, n)$

$$(6.18) \qquad |v_{n,s}^*(z, \zeta)| \leq Am(\zeta) B^{n+s} |\exp(|t|+s\zeta)| \Big\{ \sum_{d=\max(0,[n/2]-s)}^{m(n-s)} |t|^{\sigma d} / \Gamma(d+1) \Big\}$$

and for $(z, \zeta) \in \Omega_-^{\varepsilon}(\tilde{\gamma}, n)$

$$(6.19) \qquad |v_{n,s}^*(z, \zeta)| \leq Am(\zeta) B^{n+s} \exp(|t|) |z_0|^s \Big\{ \sum_{d=\max(0,[n/2]-s)}^{m(n-s)} |t|^{\sigma d} / \Gamma(d+1) \Big\}.$$

Put

$$(6.20) \qquad I = \sum_{n=0}^{\infty} \Big\{ \sum_{s=0}^{n} B^{n+s} |\exp(s\zeta)| \Big(\sum_{d=\max(0,[n/2]-s)}^{m(n-s)} |t|^{\sigma d} / \Gamma(d+1) \Big) \Big\}.$$

Let us show convergence of I, which means convergence of $V(z, \zeta)$ in $\Omega_+^{\varepsilon}(\tilde{\gamma}, n)$.

LEMMA 6.7. *There is a ζ^* such that for $(z, \zeta) \in \Omega_+^{\varepsilon}(\tilde{\gamma}, n)$ with* $\mathrm{Re}\,\zeta \leq \zeta^*$ *I converges and*

$$(6.21) \qquad I \leq A \exp(B|t|^{\sigma})$$

holds with constants A and B.

PROOF. We have

$$(6.22) \qquad \begin{cases} I = \sum_{s=0}^{\infty} B^s |\exp(s\zeta)| I_s, \\ I_s = \sum_{n=s}^{\infty} \Big\{ \sum_{d=\max(0,[n/2]-s)}^{m(n-s)} |t|^{\sigma d} / \Gamma(d+1) \Big\}. \end{cases}$$

Then

$$\begin{aligned} I_s &\leq \sum_{d=0}^{\infty} |t|^{\sigma d} / \Gamma(d+1) \Big\{ \sum_{n=[d/m]+s}^{2d+s} B^n \Big\} \\ &\leq AC^s \sum_{d=0}^{\infty} (B|t|^{\sigma})^d / \Gamma(d+1) \leq AC^s \exp(B|t|^{\sigma}). \end{aligned}$$

Hence, if $|C \exp \zeta| \leq 1/2$, I converges and (6.21) holds. $\square$

Finally let us study $V(z, \zeta)$ in $\Omega_-^{\varepsilon}(\tilde{\gamma}, n)$. Put

$$(6.23) \qquad \tilde{I} = \sum_{n=0}^{\infty} \Big\{ \sum_{s=0}^{n} B^{n+s} |z_0|^s \Big(\sum_{d=\max(0,[n/2]-s)}^{m(n-s)} |t|^{\sigma d} / \Gamma(d+1) \Big) \Big\}.$$

We have the following:

LEMMA 6.8. *There is an r such that for $(z,\zeta)\in\Omega^{\varepsilon}_{-}(\tilde{\gamma},n)$ with $|z_0|\le r$, $\tilde{I}$ converges and*

$$\tilde{I}\le A\exp(B|t|^{\sigma}) \tag{6.24}$$

holds with constants A and B.

The proof is the same as that of Lemma 6.7. Thus we obtain

THEOREM 6.9. *Let $\sigma_1=1$. There are ζ^* and r such that for $(z,\zeta)\in\Omega^{\varepsilon}_{+}(\tilde{\gamma},R_2,\alpha,\beta)$ with $\operatorname{Re}\zeta\le\zeta^*$, and for $(z,\zeta)\in\Omega^{\varepsilon}_{-}(\tilde{\gamma},R_2,\alpha,\beta)$ with $|z_0|\le r$, $V(z,\zeta)$ converges and*

$$|V(z,\zeta)|\le Am(\zeta)\exp(B|t|^{\sigma})\quad(\sigma=\sigma_{1,1}) \tag{6.25}$$

holds. A depends on $R_2,\tilde{\gamma},\alpha$ and β. B depends on R_2 and is independent of α and β.

PROOF. Since $|V(z,\zeta)|\le Am(\zeta)\exp(|t|)I$ or $|V(z,\zeta)|\le Am(\zeta)\exp(|t|)\tilde{I}$, we have (6.25). Dependence of constants follows from Theorem 5.8. □

§ 7. Integration II.

In § 7 we perform integration in ζ for $V(z,\zeta)$, construct

$$u(z)=\frac{1}{2\pi i}\int_{\Gamma}V(z,\zeta)d\zeta, \tag{7.1}$$

and complete the proof of Theorems 1.4 and 1.8.

Put $z_0=|z_0|\exp(i\theta)$, $\theta=\arg z_0$, and take R_0 and R_2 so that $\tilde{\gamma}<R_0<R_2$. Let us define paths $\Gamma(+)$ and $\Gamma(-)$ in ζ-space. Firstly we put

$$\begin{cases}\Gamma_1(+)=\{\zeta=\zeta^*+i\eta\,;\text{ if }\theta\ge0,\ 0\le\eta\le\theta,\text{ if }\theta\le0,\ \theta\le\eta\le0\},\\ \Gamma_2(+)=\{\zeta=\xi+i\theta\,;\log|z_0|+R_0\le\xi\le\zeta^*\},\\ \Gamma_3(+)=\{\zeta=\log|z_0|+i\theta+R_0\exp(i\rho)\,;\,0\le\rho\le2\pi\},\end{cases} \tag{7.2}$$

where ζ^* is the same as in Theorems 6.4 and 6.9. $\Gamma(+)$ is a path which starts at ζ^*, goes on $\Gamma_1(+)$ and $\Gamma_2(+)$, goes around on $\Gamma_3(+)$, goes back on $\Gamma_2(+)$ and $\Gamma_1(+)$ and ends at ζ^* (see Figure 7.1).

Similarly we put

$$\begin{cases}\Gamma_1(-)=\{\zeta=\xi+i\alpha\,;\,-\infty<\xi\le\log|z_0|+R_0\},\\ \Gamma_2(-)=\{\zeta=\log z_0+R_0+i\eta\,;\,\alpha\le\eta\le\beta\},\\ \Gamma_3(-)=\{\zeta=\xi+i\beta\,;\,-\infty<\xi\le\log|z_0|+R_0\},\end{cases} \tag{7.3}$$

where $\alpha<\theta<\beta$. $\Gamma(-)$ is a path which starts at $-\infty+i\alpha$ on $\Gamma_1(-)$, goes on $\Gamma_2(-)$ and $\Gamma_3(-)$ and ends at $-\infty+i\beta$ (see Figure 7.1).

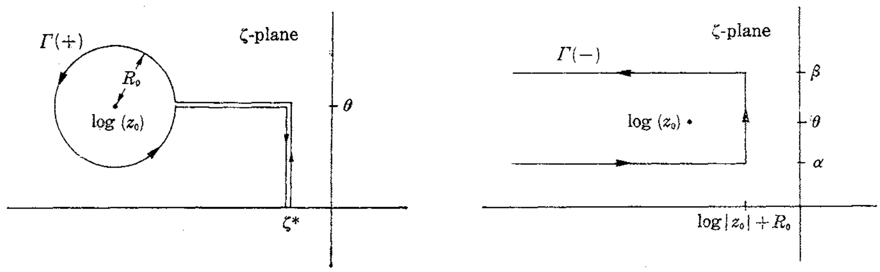

Figure 7.1

Proof of Theorem 1.5. We divide the proof into four steps (I)-(IV).

(I) Let us restrict (z, ζ) in $\Omega_\varepsilon(+)=\Omega^\varepsilon_+(\tilde{\gamma}, R_2, -\pi/2, 5\pi/2)$. Then from Theorems 6.4 and 6.9 we notice that for $(z, \zeta)\in\Omega_\varepsilon(+)$ with $\operatorname{Re}\zeta\leq\zeta^*$,

$$|V(z, \zeta)|\leq Am(\zeta)\exp(\Phi(z_0, \zeta)), \tag{7.4}$$

where

$$\Phi(z_0, \zeta)=\begin{cases} BX^{\sigma_1-1}+CY+D|z_0|^{-(\sigma_1-1)}+|\zeta-\log z_0| & (\sigma_1>1), \\ B|\zeta-\log z_0|^{\sigma_{1,1}} \quad (\sigma_1=1), \end{cases} \tag{7.5}$$

$X=|z_0|^{-1}|\exp\zeta|+|\zeta-\log z_0|^{2(m-L)}$ and $Y=|\zeta-\log z_0|^{m-L}$.

Set $\Gamma=\Gamma(+)$ in (7.1). Then we have

$$u(z)=\frac{1}{2\pi i}\int_{\Gamma(+)}V(z, \zeta)d\zeta=\frac{1}{2\pi i}\int_{\Gamma(+)}\left\{\sum_{n=0}^{\infty}\sum_{s=-\infty}^{S(n)}v^*_{n,s}(z, \zeta)\right\}d\zeta, \tag{7.6}$$

where

$$v^*_{n,s}(z, \zeta)=\frac{1}{2\pi i}\int_C (z_0)^{\lambda+s}\exp(-\lambda\zeta)v_{n,s}(z', \lambda, \zeta)\log\lambda d\lambda. \tag{7.7}$$

By shrinking Ω, we have $u(z)\in\tilde{\mathcal{O}}(\Omega-\{z_0=0\})$.

(II) Let us show $P(z, \partial)u(z)=(z_0)^{d_m-m}f(z)$. Since

$$\begin{aligned} &P(z, \partial)\{(z_0)^{\lambda+s}v_{n,s}(z', \lambda, \zeta)\} \\ &=\left\{F(m-L:\lambda+s)A(\lambda+s)(z_0)^{\lambda+s}+\sum_r B_r(\lambda+s)(z_0)^{\lambda+s+r}\right\}v_{n,s}(z', \lambda, \zeta), \end{aligned} \tag{7.8}$$

(see § 2), we have

$$(7.9)\quad P(z,\partial)u(z)=(2\pi i)^{-2}\int_{\Gamma(+)}d\zeta\sum_{n,s}\int_C \exp(-\lambda\zeta)$$
$$\times\Big\{F(m-L\,;\,\lambda+s)A(\lambda+s)(z_0)^{\lambda+s}+\sum_r B_r(\lambda+s)(z_0)^{\lambda+s+r}\Big\}v_{n,s}\log\lambda d\lambda.$$

Hence

$$(7.10)\quad P(z,\partial)u(z)=(2\pi i)^{-2}\int_{\Gamma(+)}\Big\{\Big(\sum_{n=-1}^{\infty}\phi_*^{n+1}(z,\zeta)\Big)+\Big(\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\phi_r^n(z,\zeta)\Big)\Big\}d\zeta,$$

where

$$(7.11)\quad \phi_*^{n+1}(z,\zeta)=\sum_s\int_C\exp(-\lambda\zeta)\Big\{F(m-L\,;\,\lambda+s)A(\lambda+s)(z_0)^{\lambda+s}$$
$$+\Big(\sum_{r<0}B_r(\lambda+s)(z_0)^{\lambda+s+r}\Big)\Big\}v_{n+1,s}\log\lambda\, d\lambda,$$

$$(7.12)\quad \phi_r^n(z,\zeta)=\sum_s\int_C\exp(-\lambda\zeta)(B_r(\lambda+s)(z_0)^{\lambda+s+r})v_{n,s}\log\lambda\, d\lambda.$$

We have

$$\phi_*^{n+1}(z,\zeta)=\sum_s\int_C\exp(-\lambda\zeta)\Big\{F(m-L\,;\,\lambda+s)A(\lambda+s)v_{n+1,s}$$
$$+\Big(\sum_{r<0}B_r(\lambda+s-r)\,v_{n+1,s-r}\Big)\Big\}(z_0)^{\lambda+s}\log\lambda\, d\lambda,$$

and

$$\phi_r^n(z,\zeta)=\sum_s\int_C\exp(-\lambda\zeta)\{B_r(\lambda+s-r)v_{n,s-r}\}(z_0)^{\lambda+s}\log\lambda\, d\lambda.$$

So in view of the equality

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\phi_r^n(z,\zeta)=\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\phi_r^n(z,\zeta)=\sum_{r=0}^{\infty}\sum_{n-r=0}^{\infty}\phi_r^{n-r}(z,\zeta)=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\phi_r^{n-r}(z,\zeta)$$

and by construction of $v_{n,s}(z',\lambda,\zeta)$ (see (2.11)), we have

$$\sum_{n=-1}^{\infty}\phi_*^{n+1}(z,\zeta)+\sum_{n=0}^{\infty}\sum_{r=0}^{n}\phi_r^{n-r}(z,\zeta)=f(e^\zeta,z')(z_0)^{\lambda+d_m-m}.$$

Thus

$$(7.13)\quad P(z,\partial)u(z)=\frac{1}{2\pi i}(z_0)^{d_m-m}\int_{\Gamma(+)}d\zeta\int_C\exp(-\lambda\zeta)(z_0)^\lambda f(e^\zeta,z')\log\lambda/2\pi i\, d\lambda$$
$$=\frac{1}{2\pi i}(z_0)^{d_m-m}\int_{\Gamma(+)}f(e^\zeta,z')(\zeta-\log z_0)^{-1}d\zeta$$
$$=(z_0)^{d_m-m}f(z_0,z').$$

For $(\partial_1)^h u(z_0, 0, z'')$ $(0 \le h \le L-1)$ we have

$$(7.14)\qquad (\partial_1)^h u(z_0, 0, z'') = \frac{1}{2\pi i}\int_{\Gamma(+)} d\zeta \int_C \exp(-\lambda\zeta)(z_0)^\lambda \varphi_h(e^\zeta, z'') \log \lambda/2\pi i\, d\lambda$$
$$= \frac{1}{2\pi i}\int_{\Gamma(+)} \varphi_h(e^\zeta, z'')(\zeta - \log z_0)^{-1} d\zeta = \varphi_h(z_0, z'').$$

Thus, by noting $f(z_0, z') = (z_0)^{d_m - m}\{(z_0)^{m-d_m} f(z_0, z')\} = (z_0)^{d_m - m}\tilde{f}(z_0, z')$, we have the existence of a solution $u(z)$ to the equation (1.9).

(III) Now let us obtain a bound of $u(z)$. First we state

LEMMA 7.1. *For z_0 with $|z_0| \le R$ and $\alpha < \arg z_0 < \beta$, and $\zeta \in \Gamma(+)$, we have*

$$(7.15)\qquad |\Phi(z_0, \zeta)| \le A_{\alpha,\beta} \exp(B|z_0|^{-(\sigma_1 - 1)})$$

if $\sigma_1 > 1$ and,

$$(7.16)\qquad |\Phi(z_0, \zeta)| \le A_{\alpha,\beta} \exp(B|\log z_0|^{\sigma_{1,1}}),$$

if $\sigma_1 = 1$ with constants $A_{\alpha,\beta}$ and B.

PROOF. Let $\sigma_1 > 1$. We have on $\Gamma_1(+)$ and $\Gamma_2(+)$, $X \le |z_0|^{-1} \exp \zeta^* + |\zeta^* - \log z_0|^{2(m-L)}$ and $Y \le |\zeta^* - \log z_0|^{m-L}$. On $\Gamma_3(+)$, $X \le R_0^{2(m-L)} + \exp R_0$ and $Y \le R_0^{m-L}$. Hence for $\zeta \in \Gamma(+) |\Phi(z_0, \zeta)| \le A_{\alpha,\beta} \exp(B|z_0|^{-(\sigma_1 - 1)})$.

If $\sigma_1 = 1$, then we have (7.16) from (7.5). □

(IV) Now let us proceed to bounds of $u(z)$. Suppose that $\varphi_h(z_0, z'') = 0$ for $0 \le h \le L-1$ and $|f(z)| \le A_{\alpha,\beta} \exp(C|z_0|^{-\gamma})$, $\gamma = \sigma_1 - 1$ and $\sigma_1 > 1$, for $z \in U$ with $\alpha < \arg z_0 < \beta$. Then we can put

$$(7.17)\qquad m(\zeta) = A_{\alpha,\beta} \exp(C \exp(-\gamma \operatorname{Re} \zeta))$$

in (4.25). Then we have from Lemma 7.1.

$$\exp(\Phi(z_0, \zeta)) m(\zeta) \le A_{\alpha,\beta} \exp(B|z_0|^{-\gamma} + C(\exp(-\gamma \operatorname{Re} \zeta)).$$

Hence

$$|u(z)| \le A_{\alpha,\beta} \int_{\Gamma(+)} \exp(\Phi(z_0, \zeta)) m(\zeta) |d\zeta| \le A_{\alpha,\beta} \exp(B|z_0|^{-\gamma}).$$

If $\sigma_1 = 1$ and $|f(z_0, z')| \le A_{\alpha,\beta} \exp(B|\log z_0|^\gamma)$, $\gamma = \sigma_{1,1}$, for $z \in U$ with $\alpha < \arg z_0 < \beta$, then we have from Lemma 7.1 $|u(z)| \le A_{\alpha,\beta} \exp(B|\log z_0|^\gamma)$.

Thus the proof of Theorem 1.4 is completed. □

PROOF OF THEOREM 1.8. Let us return to (2.1). Since $L\geq 1$, we can put

$$(\partial_1)^h v_0(z_0, 0, z'', \lambda, \zeta)=\delta_{h,L-1}(z_0)^\lambda \varphi(e^\zeta, z'') \tag{7.18}$$

and $f(e^\zeta, z')=0$.

Put $\tilde{\gamma}=2\varepsilon$. Let us restrict $(z, \zeta)\in \Omega_\varepsilon(-)=\Omega_-^\varepsilon(2\varepsilon, R_2, -3\pi/2, 3\pi/2)$ and recall Theorems 6.6 and 6.9. First let $\sigma_1>1$. Assume that for $(z_0, z'')\in U\cap\{z_1=0\}$ with $|\arg z_0|\leq\theta_0$

$$|\varphi(z_0, z'')|\leq A\exp(-K|z_0|^{-\gamma}),\ \gamma=\sigma_1-1 \text{ and } K>0. \tag{7.19}$$

Then we can put for ζ with $|\operatorname{Im}\zeta|\leq\theta_0$

$$m(\zeta)=A\exp(-K\exp(-\gamma\operatorname{Re}\zeta)). \tag{7.20}$$

Put $\Gamma=\Gamma(-), \alpha=\theta-\varepsilon, \beta=\theta+\varepsilon$ ($\theta=\arg z_0$) in (7.1). We have

$$u(z)=\frac{1}{2\pi i}\int_{\Gamma(-)} V(z, \zeta)d\zeta. \tag{7.21}$$

LEMMA 7.2. *For any θ_1 with $0<\theta_1<\theta_0$, there are a constant K^* and a neighbourhood Ω of $z=0$, such that if $\varphi(z_0, z'')$ satisfies the estimate* (7.19) *for some $K>K^*$, then*

$$|u(z)|\leq A\exp(-B|z_0|^{-\gamma}) \tag{7.22}$$

holds for $z\in\Omega$ with $|\arg z_0|\leq\theta_1$. Constants A, B and K^* and Ω depend on θ_1, and B is positive.

PROOF. First we take ε so small that $\theta_0-\theta_1>\tilde{\gamma}=2\varepsilon$. Then we choose a small neighbourhood Ω_ε of $z=0$, by noting (6.3) and Theorem 6.6. Let $(z, \zeta)\in\Omega_\varepsilon(-)$. Put $t=\zeta-\log z_0$ and recall (6.17). Put $\Psi(z_0, \zeta)=BX^\gamma+CY+D|\exp(-\gamma\zeta)|+|t|$. We have $|\Psi(z_0, \zeta)|\leq C(|t|^\delta+\exp(\gamma\operatorname{Re} t)+1)+D\exp((-\gamma\operatorname{Re} t)|z_0|^{-\gamma})$, where $\delta=\max(2\gamma(m-L), m-L, 1)$, and $m(t+\log z_0)\leq A\exp(-K\exp(-\gamma\operatorname{Re} t)|z_0|^{-\gamma})$. Hence if $\operatorname{Re} t\leq R_2$, we have

$$\exp(\Psi(z_0, \zeta))m(\zeta)\leq A\exp(C|t|^\delta+(D-K)\exp(-\gamma\operatorname{Re} t)|z_0|^{-\gamma}). \tag{7.23}$$

So if $K>K^*=D+2$, we obtain for a $B>0$

$$|V(z, \zeta)|\leq A\exp(C|t|^\delta-\exp(-\gamma\operatorname{Re} t)|z_0|^{-\gamma}-B|z_0|^{-\gamma}). \tag{7.24}$$

Thus we get from (7.21)

$$|u(z)| \leq A \exp(-B|z_0|^{-\gamma}). \tag{7.25}$$

All constants depend on ε. So they depend on θ_1. □

Next let us consider the case $\sigma_1=1$. Put $\gamma=\sigma_{1,1}$, if $\sigma_{1,1}>1$. If $\sigma_{1,1}=1$, let γ be a constant with $\gamma>1$.

Assume that for $(z_0, z'') \in U \cap \{z_1=0\}$ with $|\arg z_0| \leq \theta_0$,

$$|\varphi(z_0, z'')| \leq A \exp(-K|\log z_0|^{\gamma}), \quad K>0. \tag{7.26}$$

We can put for ζ with $|\operatorname{Im} \zeta| \leq \theta_0$

$$m(\zeta) = A \exp(-K|\zeta|^{\gamma}). \tag{7.27}$$

We can also define $u(z)$ by (7.21). We have

LEMMA 7.3. *For any θ_1 with $0<\theta_1<\theta_0$, there are a K^* ($K^*=0$, if $\sigma_{1,1}=1$) and neighbourhood Ω of $z=0$ so that: If $\varphi(z_0, z'')$ satisfies the estimate* (7.26) *for some $K>K^*$, then*

$$|u(z)| \leq A \exp(-B|\log z_0|^{\gamma}) \tag{7.28}$$

holds for $z \in \Omega$ with $|\arg z_0| \leq \theta_1$. Constants K^*, A and $B>0$, and Ω depend on θ_1.

The proof of Lemma 7.3 is similar to that of Lemma 7.2. We can also show that $\partial^{\alpha} u(z) \to 0$ as $z_0 \to 0$ in the sector $\{z \in \Omega; |\arg z_0| \leq \theta_1\}$ under the condition (7.19) or (7.26). $u(z)$ satisfies $P(z, \partial) u(z)=0$ and $(\partial_1)^{L-1} u(z_0, 0, z'') = \varphi(z_0, z'')$ and it decays rapidly in $\{z \in \Omega; |\arg z_0| \leq \theta_1\}$. Hence

$$u(x) = \begin{cases} \text{the restriction of } u(z) \text{ to } R^n, & x_0 \geq 0 \\ 0, \quad x_0 \leq 0, \end{cases} \tag{7.29}$$

is a desired function.

REMARK 7.4. We can show which class of nonquasianalytic functions $u(x)$ belongs to, by more precise estimates of $u(x)$. That will be discussed in a forthcomming paper.

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