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Splitting singular fibers in good torus fibrations

By Masaaki UE

Torus fibrations defined in [M1] form a class of 4-manifolds which includes elliptic surfaces. A good torus fibration is defined as a torus fibration f: $M \to B$ such that at each singular point in M, the germ f is smoothly $(+)$-equivalent to the germ at 0 of the function $z_1^m z_2^n$ or $\bar{z}_1^m z_2^n$: $C^2 \to C$, with $m+n>1$ ([M2]). The diffeomorphism type of a good torus fibration f: $M \to S^2$ with at least one singular fiber such that every singular fiber is of type $I_1^{\pm}$ or a twin is determined in [M2] in the case when $\sigma(M) \neq 0$ and in [I] in the case when $\sigma(M)=0$. Moishezon showed (Theorem 8a, Lemma 6 in [Mo]) that every minimal analytic singular fiber is diffeomorphic (in fact deformable) to a sum of fibers of type I_1^+. This fact can be proved topologically when the number of divisors in the fiber is not so large (Lemma 1-8). Moishezon used this fact to give the classification of the diffeomorphism types of the elliptic surfaces over CP^1 ([K1], [Mo]). Therefore as a first step to determine the diffeomorphism types of other good torus fibrations over S^2, it is natural to consider whether non-analytic fibers can be blown down to sums of simpler fibers or not. In this paper we will prove that every singular fiber is blown down to a sum of fibers of type $I_1^{\pm}$ and Twins after performing a connected sum with at most 2 copies of CP^2 or $-CP^2$. In fact every good singular fiber except for twins can be transformed by blowing-ups and downs to some analytic fiber. Furthermore non-analytic fibers of type $\tilde{D}$ can be transformed (after performing blowing-ups and downs) to extra sums of fibers of type $I_1^{\pm}$, which are essentially non-analytic. However we will show that some of non-analytic fibers of type $\tilde{E}$ and $\tilde{D}$ themselves cannot be blown down to sums of fibers of type $I_1^{\pm}$ and Twins. This implies that a non-analytic version of Moishezon's theorem for analytic singular fibers ([Mo]) does not necessarily hold. But we can prove that if M is a 1-connected good torus fibration over S^2 without multiple fibers and non-analytic fibers of type $\tilde{D}$, then a connected sum of M and at most one copy of CP^2 or $-CP^2$ is diffeomorphic to a connected sum of some copies of $\pm CP^2$ (Theorem 2).

In this paper we will consider all the manifolds and maps in the smooth category. We will denote CP^2 and $-CP^2$ by P and Q respectively. All the diffeomorphisms will be assumed to be orientation-preserving.

§ 1. Classification of reduced good singular fibers

In [M2] good singular fibers are classified into classes of type $\tilde{A}$, $\tilde{E}$, and $\tilde{D}$. The simplest fiber is the one of type $I_1^{\pm}$ which is an immersed S^2 with one self-intersection with intersection number ± 1. The second one is called a twin singular fiber which consists of two embedded S^2's intersecting each other twice such that one of the intersection number is $+1$ and the other is -1 ([M2]). In this paper all the twin singular fibers will be assumed to be non-multiple and denoted by the common symbol Tw.

DEFINITION 1. If a regular neighborhood of a singular fiber F is diffeomorphic to a connected-sum of a regular neighborhood of another singular fiber F' and some P's, Q's, and/or $S^2 \times S^2$'s, then we say that F *is blown down to* F'.

DEFINITION 2. If a regular neighborhood of a singular fiber F is diffeomorphic to a torus fibration over D^2 which contains singular fibers of type $I_1^{\pm}$ (and Tw's) and contains no other singular fibers, then we say that F *is splittable to a sum of fibers of type* $I_1^{\pm}$ (and Tw's). If the numbers of fibers of type I_1^+, I_1^-, and Tw's are a, b, and c respectively, then we denote the sum of such fibers by $aI_1^+ + bI_1^+ + cTw$. (More detailed description will be given in § 2.)

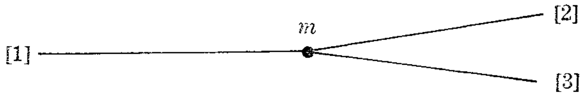
REMARK 1. The blowing down precesses defined above includes a natural blowing down of a divisor of a singular fiber with self-intersection 0 or ± 1. But we need later more general ones for non-analytic singular fibers in the proof of Theorem 1.

REMARK 2. It is shown in [Mo] that any relatively minimal analytic singular fiber is deformable to a sum of fibers of type I_1^+'s. However, in the cases of non-analytic fibers of type $\tilde{E}$ and $\tilde{D}$ with which we will be concerned, it seems difficult to formulate "deformations". The boundary of the regular neighborhood of such fiber is a T^2-bundle over S^1 with a trace of monodromy $\neq 2$, which contains only one non-separating incompressible torus up to isotopy (a fiber). Hence if we replace a regular neighborhood of a fiber of type $\tilde{E}$ or $\tilde{D}$ in a torus fibration by another fibration with the same boundary, then the resulting manifold still admits a torus fibering structure since any gluing map is up to isotopy a fiber map with respect to the T^2-bundle structures induced by the two pieces.

DEFINITION 3. A good singular fiber is said to be *reduced* if it contains no divisor of self-intersection number 0 or ± 1 which intersects other divisors at most 2 points. (It may be possible to perform extra blowing down processes, but then the resulting fiber is no longer a good one.)

PROPOSITION 1. *A reduced good singular fiber of type $\tilde{E}$ or $\tilde{D}$ is represented by one of the following diagrams. (The number on the vertex denotes the multiplicity of the corresponding divisor and the sign on the edge denotes the intersection number of the divisors represented by the vertices adjacent to the edge.*

Type $\tilde{E}_k$:



where [i] denotes the i-th linear branch, m is the multiplicity of the divisor corresponding to the vertex of valency 3 and $m=3$, 4, and 6 for $k=6$, 7, and 8 respectively.

In the cases of type $\tilde{E}$ below, the vertices on the right ends of the linear branches coincide with the vertices of valency 3 in the above diagram.

Type $\tilde{E}_6^+$: Each linear branch is either $1 \overset{+}{—} 2 \overset{+}{—} 3$ *(type 1) or* $1 \overset{-}{—} 3$ *(type 2).*

Type $\tilde{E}_6^-$: Each linear branch is either $1 \overset{-}{—} 2 \overset{-}{—} 3$ *(type 1) or* $1 \overset{+}{—} 3$ *(type 2). Type $\tilde{E}_6^-$ is dual (all the signs reversed) to type $\tilde{E}_6^+$.*

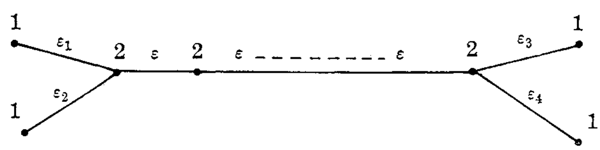
Type $\tilde{E}_7^+$: The first linear branch is either $2 \overset{+}{—} 4$ *(type 1) or* $2 \overset{-}{—} 4$ *(type 2), the other ones are either* $1 \overset{+}{—} 2 \overset{+}{—} 3 \overset{+}{—} 4$ *(type 1) or* $1 \overset{-}{—} 4$ *(type 2).*

Type $\tilde{E}_7^-$: dual to $\tilde{E}_7^+$ as in $\tilde{E}_6$.

Type $\tilde{E}_8^+$: The first linear branch is either $1 \overset{+}{—} 2 \overset{+}{—} 3 \overset{+}{—} 4 \overset{+}{—} 5 \overset{+}{—} 6$ *(type 1) or* $1 \overset{-}{—} 6$ *(type 2), the second one is either* $2 \overset{+}{—} 4 \overset{+}{—} 6$ *(type 1) or* $2 \overset{-}{—} 6$ *(type 2), and the third one is either* $3 \overset{+}{—} 6$ *(type 1) or* $3 \overset{-}{—} 6$ *(type 2).*

Type $\tilde{E}_8^-$: dual to $\tilde{E}_8^+$.

Type $\tilde{D}_k^\varepsilon$: A graph of the form



where ε, ε_1, ε_2, ε_3, $\varepsilon_4=\pm$, and the number of the edges with sign ε is k.

PROOF. Case of $\tilde{E}_k^{\pm}$. By Theorem 3.1 in [M2], one of the linear branches is of the form

$$\overset{m_\nu}{\bullet}\overset{\varepsilon_\nu}{\longrightarrow}\overset{m_{\nu-1}}{\bullet}\cdots\cdots\overset{m_1}{\bullet}\overset{\varepsilon_1}{\longrightarrow}\overset{n}{\bullet} \qquad (n=3,\ 4,\ 6)$$

such that

$(*)$ $1=(n, m_1)=(m_1, m_2)=\cdots=(m_{\nu-1}, m_\nu)=m_\nu$,

$(**)$ $-\Theta_1^2=(\varepsilon_1 n+\varepsilon_2 m_2)/m_1$, $-\Theta_2^2=(\varepsilon_2 m_1+\varepsilon_3 m_3)/m_2, \cdots,$ $-\Theta_\nu^2=\varepsilon_\nu m_{\nu-1}/m_\nu$ are all integers with absolute value ≥ 2 since the graph is reduced. Then it follows that

$$n+m_2\geq 2m_1,\ m_1+m_3\geq 2m_2,\ \cdots,\ m_{\nu-2}+m_\nu\geq 2m_{\nu-1},\ m_{\nu-1}\geq 2m_\nu=2\,,$$

and hence $m_1\geq m_2\geq\cdots\geq m_\nu$, $n\geq m_1+m_\nu=m_1+1$. Then in the case of $\tilde{E}_6$, each branch must be of the form $\overset{1}{\bullet}\overset{\varepsilon}{—}\overset{2}{\bullet}\overset{\varepsilon'}{—}\overset{3}{\bullet}$ or $\overset{1}{\bullet}\overset{\varepsilon}{—}\overset{3}{\bullet}$ and in the first case $\varepsilon=\varepsilon'$ by $(*)$ and $(**)$. In the case of $n=6$ $(\tilde{E}_8)$, the linear edge given above must satisfy $6\geq m_1\geq\cdots\geq m_{\nu-1}\geq 2>m_\nu=1$. By considering $(*)$ and $(**)$, we can see that the branch must be of type 1 or 2 given above. The proof of the cases of other linear branches are easier. Then by the classification given in Theorem 3.1 in [M2] the above cases give all the possible combinations of linear branches for type $\tilde{E}$ (It suffices to consider the integrability conditions for the self-intersection numbers of the divisors coresponding to the vertices of valency 3.). The cases of type $\tilde{D}$ are proved similarly.

REMARK 3. A reduced non-multiple singular fiber of type $\tilde{A}$ is either a Kodaira's singularity of type ${}_1I_b$ ([Ko]) or ${}_1I_b$ with opposite orientation, or a twin (Theorem 6.3 in [M2]).

Notations. (1) We will denote the singular fiber of type $\tilde{E}_k^{\varepsilon}$ in which the first linear branch is of type i_1, the second one is of type i_2, and the third one is of type i_3 by $\tilde{E}_k^{\varepsilon}(i_1, i_2, i_3)$ where $k=6, 7, 8$, $\varepsilon=\pm$, and $i_s\,(s=1, 2, 3)=1$ or 2 as in Proposition 1.

(2) We will denote the singular fiber of type $\tilde{D}_k^{\varepsilon}$ for $k\geq 1$ by

$\tilde{D}_k^+(1)$ if $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=\varepsilon=1$, $\tilde{D}_k^-(1)$ if $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=\varepsilon=-1$

$\tilde{D}_k^+(2)$ if $\varepsilon=1$, one of ε_i's is -1, and the others are 1,

$\tilde{D}_k^-(2)$ if $\varepsilon=-1$, one of ε_i's is 1, and the others are -1,

$\tilde{D}^+(3)$ if $\varepsilon=1$, one of ε_i's is 1, and the others are -1,

$\tilde{D}_k^-(3)$ if $\varepsilon=-1$, one of ε_i's is -1, and the others are 1,

$\tilde{D}_k^+(4)$ if $\varepsilon=1$, $\varepsilon_1=\varepsilon_3=1$, $\varepsilon_2=\varepsilon_4=-1$, or $\varepsilon_1=\varepsilon_3=-1$, $\varepsilon_2=\varepsilon_4=1$,

$\tilde{D}_k^-(4)$ if $\varepsilon=-1$, $\varepsilon_1=\varepsilon_3=1$, $\varepsilon_2=\varepsilon_4=-1$, or $\varepsilon_1=\varepsilon_3=-1$, $\varepsilon_2=\varepsilon_4=1$,
$\tilde{D}_k^+(5)$ if $\varepsilon=1$, $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=-1$, $\tilde{D}_k^-(5)$ if $\varepsilon=-1$, $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=1$,
$\tilde{D}_k^+(6)$ if $\varepsilon=1$, $\varepsilon_1=\varepsilon_2=1$, $\varepsilon_3=\varepsilon_4=-1$, or $\varepsilon_1=\varepsilon_2=-1$, $\varepsilon_3=\varepsilon_4=1$,
$\tilde{D}_k^-(6)$ if $\varepsilon=-1$, $\varepsilon_1=\varepsilon_2=1$, $\varepsilon_3=\varepsilon_4=-1$, or $\varepsilon_1=\varepsilon_2=-1$, $\varepsilon_3=\varepsilon_4=1$.

If $k=0$, $\tilde{D}_0^+(j)$ and $\tilde{D}_0^-(j)$ in the above representations are identical and denoted by $\tilde{D}_0(j)$. $\tilde{D}_0(6)$ is identical to $\tilde{D}_0(4)$.

§ 2. Framed link pictures of the sums of $I_1^\pm$'s

The monodromy of a singular fiber F is represented by the matrix A (up to conjugate) such that the boundary of the regular neighborhood F is diffeomorphic to $T^2\times[0, 1]/(x, 1)\sim(h(x), 0)$ and $h(S^1\times\{*\}, \{*\}\times S^1)=(S^1\times\{*\}, \{*\}\times S^1)A$. This description for the monodromy is identical to the one given in [M2]. A regular neighborhood of a singular fiber of type $I_1^\pm$ is diffeomorphic to $T^2\times D^2$ with a 2-handle of framing ∓ 1 (a vanishing cycle) attached along a simple loop on $T^2\times_*\subset T^2\times\partial D^2$ as in Figure 1 ([K2], [H]). Hence a sum of singular fibers of type $I_1^\pm$ is diffeomorphic to $T^2\times D^2$ with the corresponding vanishing cycles attached. If their attaching curves are represented by $a_1m+c_1l, \cdots, a_km+c_kl$, where a_im+c_il corresponds to $I_1^{\varepsilon_i}$, and they are put in numerical order from the front to the back of Figure 1, then the total monodromy is represented by the matrix

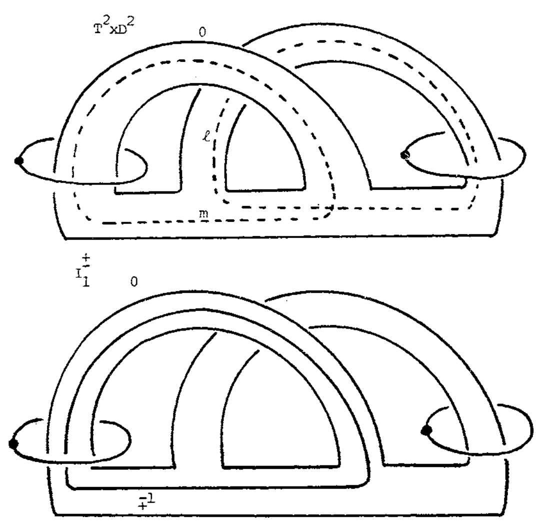


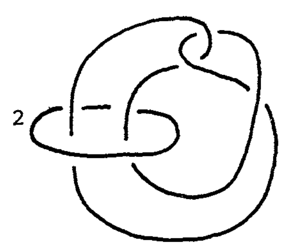
Figure 1

$A_1\begin{pmatrix}1 & \varepsilon_1\\0 & 1\end{pmatrix}A_1^{-1}A_2\begin{pmatrix}1 & \varepsilon_2\\0 & 1\end{pmatrix}A_2^{-1}\cdots A_k\begin{pmatrix}1 & \varepsilon_k\\0 & 1\end{pmatrix}A_k^{-1}$ where $A_i=\begin{pmatrix}a_i & b_i\\c_i & d_i\end{pmatrix}$ with $a_id_i-b_ic_i=1$. It depends only on $\pm(a_i, c_i)$ since $A\begin{pmatrix}1 & \pm 1\\0 & 1\end{pmatrix}A^{-1}=\begin{pmatrix}1\mp ac & \pm a^2\\ \mp c^2 & 1\pm ac\end{pmatrix}$ for $A=\begin{pmatrix}a & b\\c & d\end{pmatrix}$ with $ad-bc=1$. The framing of the curve $am+cl$ corresponding to I_1^ε in Figure 1 is $-ac-\varepsilon$. We will denote the above sum of $I_1^{\pm}$'s by $((a_1, c_1)_{\varepsilon_1}, \cdots, (a_k, c_k)_{\varepsilon_k})$. This representation determines the diffeomorphism types of the sum of $I_1^{\pm}$. (But the representation itself is not unique.)

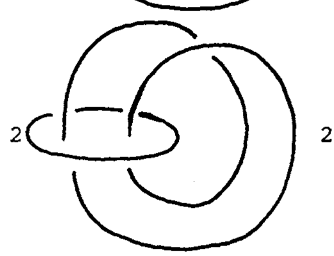
LEMMA 1. *The sum $2I_1^-$ with monodromy conjugate to $\begin{pmatrix}0 & -1\\1 & 1\end{pmatrix}$ is diffeomorphic to D^4 with a 2-handle attached along a left-handed trefoil knot in ∂D^4 with framing* 0 *and is represented as* $((1, 0)_-, (0, 1)_-)$.

LEMMA 2. $3I_1^-$ *with monodromy conjugate to $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ is represented by the following*:

$((1, 0)_-, (1, 0)_-, (0, 1)_-)$ *which is represented by the following link picture,*

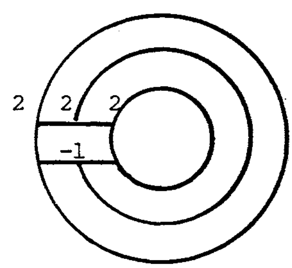


or equivalently,

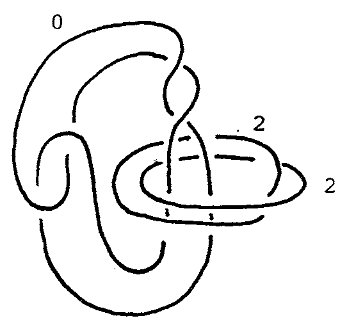


PROOFS OF LEMMAS 1, 2. Suppose that the monodromy of $3I_1^-$ of type $((a, c)_-, (a', c')_-, (a'', c'')_-)$ is conjugate to $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$. Then $\begin{pmatrix}1+ac & -a^2\\c^2 & 1-ac\end{pmatrix}\times\begin{pmatrix}1+a'c' & -a'^2\\c'^2 & 1-a'c'\end{pmatrix}=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1-a''c'' & a''^2\\-c''^2 & 1+a''c''\end{pmatrix}$. Compare the trace of the left side with the one of the right side to obtain $2-(ac'-a'c)^2=a''^2+c''^2$. Therefore we see that $ac'-a'c=\pm 1$ and $(a'', c'')=(\pm 1, 0)$ or $(0, \pm 1)$, or $ac'=a'c$, $a''=\pm 1$, and $c''=\pm 1$. Then elementary calculus shows that the type is either $((1,0)_-, (1,1)_-, (0,1)_-)$ or $((1,0)_-, (0,1)_-, (1,0)_-)$ up to the changes of the basis (including the cyclic permutations of the orders of the terms). Furthermore we can replace $(a,c)_-(0,1)_-$ by $(0,1)_-(a,c-a)_-$ by an elementary transformation since $\begin{pmatrix} 1+ac & -a^2 \\ c^2 & 1-ac \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1+a(c-a) & -a^2 \\ (c-a)^2 & 1-a(c-a) \end{pmatrix}$. Hence the above two representations are equivalent. The case of $2I_1^-$ is proved similarly. The link pictures are easily derived by Kirby calculus [Ki]. In fact the second picture in Lemma 2 is transformed to the first one by a handle-sliding.

LEMMA 3. *Suppose that $4I_1^-$ with representation $((1,0)_-, (a,c)_-, (a',c')_-, (0,1)_-)$ has the monodromy conjugate to $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Then $4I_1^-$ is diffeomorphic to the following*: $((a,c),(a',c')) = ((0,1),(1,0))$ *and*

$4I_1^- =$ 

or equivalently,

$4I_1^- =$ 

PROOF. The proof is similar to the proofs of Lemmas 1 and 2, so we omit it.

REMARK 4. There are "dual" statements for $2I_1^+$ with monodromy conjugate to $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $3I_1^+$ with monodromy conjugate to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $4I_1^+$

with monodromy conjugate to $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ corresponding to Lemma 1, 2, and 3. The singularities in Lemma 1, (case 2 of) Lemma 2, and (case 1 of) Lemma 3 are Kodaira's singularities of type II, III, and IV with opposite orientation. We will denote them (and their regular neighborhoods) by II_-, III_-, IV_- respectively.

§ 3. Main Theorem

We will prove that every good singular fiber is stably blown down (i.e., blown down after connected sum with some P's or Q's) to a sum of $I_1^\pm$'s except for type $\tilde{A}$. The cases of type $\tilde{A}$ are already treated in [M2] so we omit them. The singular fiber with all the signs on the edges $+$ (resp. $-$) is *analytic* (resp. *anti-analytic*) each of which is blown down to a sum of I_1^+'s (resp. I_1^-'s) by [Mo]. The other fibers are called *non-analytic*. We denote the analytic fibers by Kodaira's descriptions in [Ko]. The anti-analytic ones are represented by the corresponding symbols for the analytic ones with $-$ added.

THEOREM 1. *Reduced good singular fibers of type $\tilde{E}$ and $\tilde{D}$ are stably blown down to sums of $I_1^\pm$'s as follows. (We only describe the cases of $\tilde{E}^+$ and $\tilde{D}^+$. The dual statements for $\tilde{E}^-$ and $\tilde{D}^-$ are obtained by exchanging all the signs and replacing all the P's (Q's) by Q's (P's). The corresponding framed links are obtained by changing all the crossings and the signs of the framings.)*

(I) $\tilde{E}$.

(1) $\tilde{E}_6^+(1, 1, 1) = IV^* = 8I_1^+$,

(2) $\tilde{E}_6^+(1, 1, 2)\# P = IV_-\# 4Q$, (2′) $\tilde{E}_6^+(1, 1, 2)\# 2Q = IV^*\# P$,

(3) $\tilde{E}_6^+(1, 2, 2) = IV_-\# 2Q$,

(4) $\tilde{E}_6^+(2, 2, 2) = IV_-\# P$,

where $IV_- = 4I_1^-$.

(5) $\tilde{E}_7^+(1, 1, 1) = III^* = 9I_1^+$,

(6) $\tilde{E}_7^+(2, 1, 1) = III_-\# 6Q$,

(7) $\tilde{E}_7^+(1, 2, 1) = III_-\# 4Q$,

(8) $\tilde{E}_7^+(2, 2, 1)\# P = III_-\# 2P\# 3Q$, $\tilde{E}_7^+(2, 2, 1)\# Q = III_-\# P\# 4Q$,

(8′) $\tilde{E}_7^+(2, 2, 1)\# 4Q = III^*\# 2P$,

(9) $\tilde{E}_7^+(1, 2, 2) = III_-\# S^2\times S^2$,

(10) $\tilde{E}_7^+(2, 2, 2) = III_-\# 2P$,

where $III_{-}=3I_1^{-}$.

(11) $\tilde{E}_8^+(1, 1, 1)=II^*=10I_1^+$,

(12) $\tilde{E}_8^+(1, 1, 2)=II_{-}\#P\#7Q$,

(13) $\tilde{E}_8^+(1, 2, 1)=II_{-}\#P\#6Q$,

(14) $\tilde{E}_8^+(1, 2, 2)=II_{-}\#2P\#5Q$,

(15) $\tilde{E}_8^+(2, 1, 1)=II_{-}\#P\#3Q$,

(16) $\tilde{E}_8^+(2, 1, 2)=II_{-}\#2S^2\times S^2$,

(17) $\tilde{E}_8^+(2, 2, 1)=II_{-}\#2P\#Q$,

(18) $\tilde{E}_8^+(2, 2, 2)=II_{-}\#3P$,

where $II_{-}=2I_1^{-}$.

(II) $\tilde{D}$.

$\tilde{D}_k$ *for* $k\geq 1$.

(19) $\tilde{D}_k^+(1)=I_k^*=(6+k)I_1^+$,

(20) $\tilde{D}_k^+(2)\#2P=N_k\#4Q$, (20′) $\tilde{D}_k^+(2)\#Q=I_k^*\#P$,

(21) $\tilde{D}_k^+(3)=N_k\#2Q$, (21′) $\tilde{D}_k^+(3)\#3Q=I_k^*\#3P$,

(22) $\tilde{D}_k^+(4)\#P=N_k\#3Q$, (22′) $\tilde{D}_k^+(4)\#2Q=I_k^*\#2P$,

(23) $\tilde{D}_k^+(5)\#P=N_k\#2P\#Q$, $\tilde{D}_k^+(5)\#Q=N_k\#P\#2Q$,

(23′) $\tilde{D}_k^+(5)\#4Q=I_k^*\#4P$,

(24) $\tilde{D}_k^+(6)\#P=N_k\#3Q$, (24′) $\tilde{D}_k^+(6)\#2Q=I_k^*\#2P$,

where $N_k=(k-1)I_1^+ +5I_1^-$ *which is represented by the framed link in Figure* 2.

$\tilde{D}_k$ *for* $k=0$

$(19)_0$ $\tilde{D}_0(1)=I_0^*=6I_1^+$.

$(20)_0$ $\tilde{D}_0(2)\#3P=I_{0-}^*\#3Q$, $(20')_0$ $\tilde{D}_0(2)\#Q=I_0^*\#P$,

$(21)_0$ $\tilde{D}_0(3)\#P=I_{0-}^*\#Q$, $(21')_0$ $\tilde{D}_0(3)\#3Q=I_0^*\#3P$,

$(22)_0$ $\tilde{D}_0(4)\#2P=I_{0-}^*\#2P$, $(22')_0$ $\tilde{D}_0(4)\#2Q=I_0^*\#2P$,

$(23)_0$ $\tilde{D}_0(5)=I_{0-}^*=6I_1^-$.

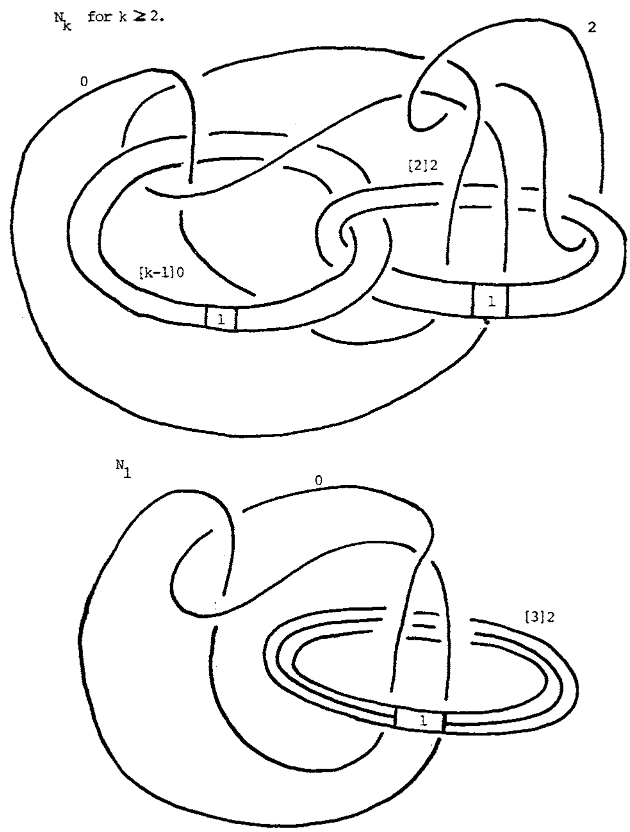
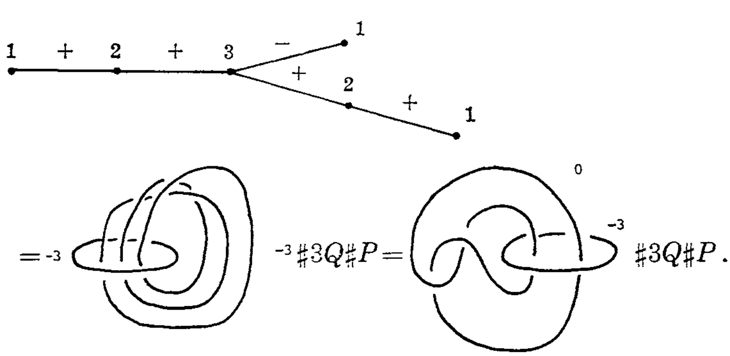


Figure 2

Proof. It suffices to prove for $\tilde{E}^+$ and $\tilde{D}^+$.

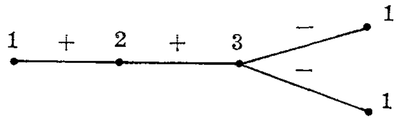
$\tilde{E}_6^+$.

(2)



We denote the last framed link by $F6$.

(3)

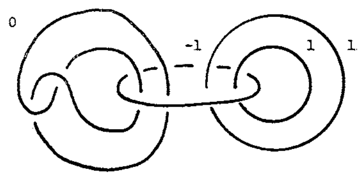


$=F6\sharp 2P\sharp Q.$

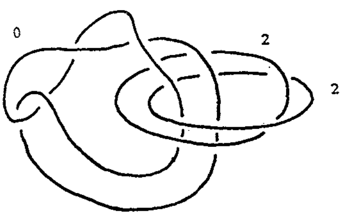
On the other hand we have:

LEMMA 4. $F6\sharp 2P=IV_{-}\sharp Q.$

PROOF. (blow down)

$F6\sharp 2P=$ 

$=$



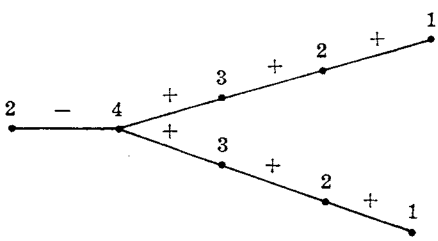
$\sharp Q=IV_{-}\sharp Q$ (Lemma 3).

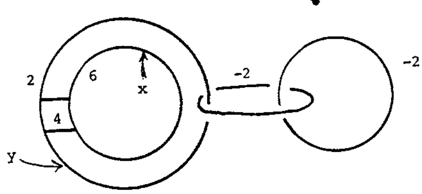
Thus we obtain (2) and (3).

Case (1) and case (4) (dual to (1)) are proved by [Mo].

$\tilde{E}_7^+$.

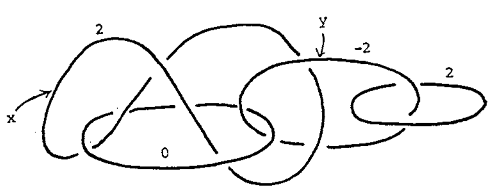
(6)



$=$ 

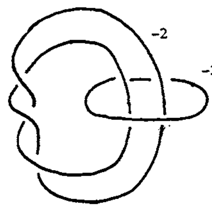
$\sharp 4Q$ $(x \longrightarrow x-y)$

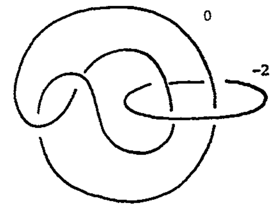
$=$



$\sharp 4Q$ $(x \longrightarrow x-2y)$

$=$

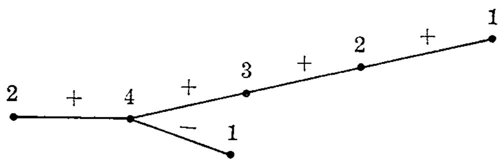


$\sharp S^2\times S^2\sharp 4Q=$


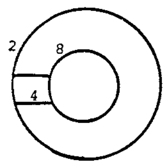
$\sharp P\sharp 5Q.$

We denote the last framed link by $F7$.

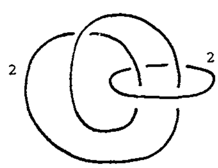
(7)



(handle-sliding)

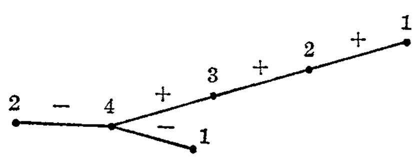
$=$


$\sharp 4Q \qquad =$



$\sharp 4Q=III_{-}\sharp 4Q$ (Lemma 2).

(8)

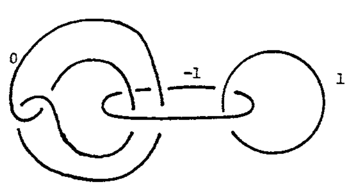


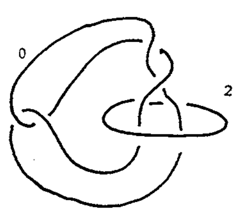
$=F7\sharp 2S^2\times S^2.$

On the other hand we have:

Lemma 5. $F7\sharp P=III_{-}\sharp Q.$

Proof.

$F7\sharp P=$


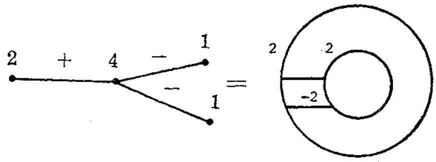
$=$


$\sharp Q$

$=III_{-}\sharp Q$ (Lemma 2).

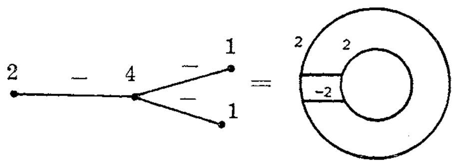
Thus we obtain (6)-(8).

(9)



$\#S^2\times S^2=III_-\#S^2\times S^2$.

(10)

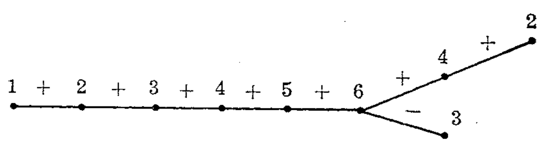


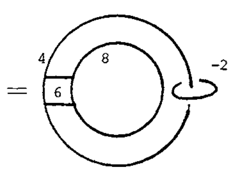
$\#2P=III_-\#2P$.

(The cases (5) and (10) are proved by [Mo].)

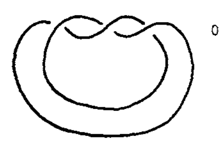
$\tilde{E}_8^+$.

(12)





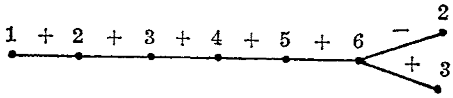
$\#6Q$ handle sliding $=$

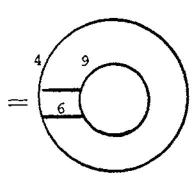


$\#S^2\times S^2\#6Q$

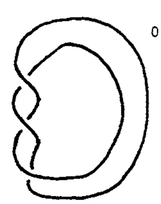
$=II_-\#P\#7Q$ (Lemma 2).

(13)





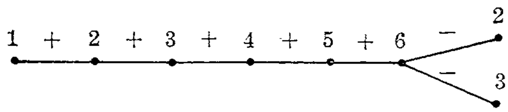
$\#6Q$ handle sliding $=$

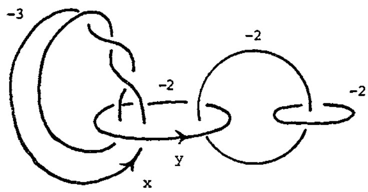


$\#6Q\#P$

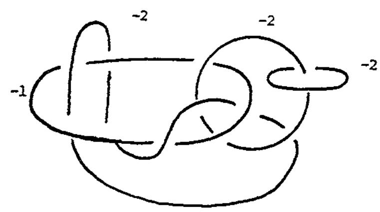
$=II_-\#6Q\#P$ (Lemma 2).

(14)

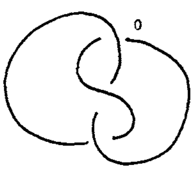


$=$ 

$\#2S^2\times S^2$ $\quad x \underset{=}{\longrightarrow} x+y$

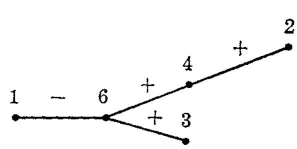
$=$ 

$\#2S^2\times S^2$ $\quad$ blow-down 3 times $=$



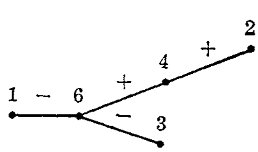
$\#2S^2\times S^2\#3Q=II_-\#2P\#5Q$ (Lemma 2).

(15)



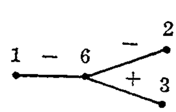
$=II_-\#P\#3Q$.

(16)



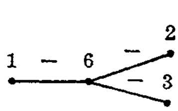
$=II_-\#2S^2\times S^2$.

(17)



$=II_-\#2P\#Q$.

(18)

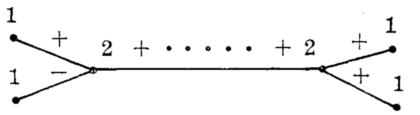


$=II_-\#3P$.

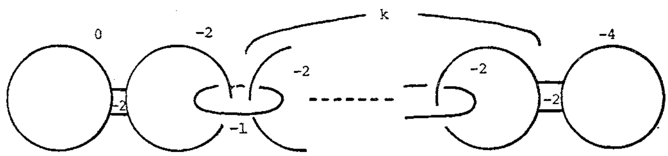
∎ (The cases (11) and (18) are proved by [Mo].)

$\tilde{D}_k$ for $k\geq 2$.

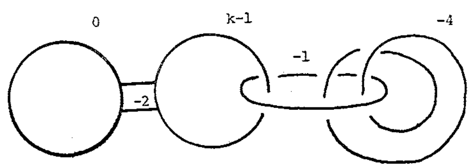
(20)



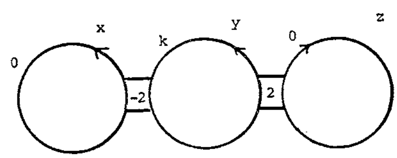
slidehandles twice
$=$



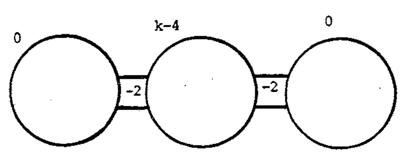
blow-down
$k+1$-times
$=$



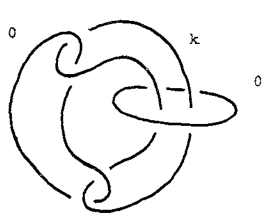
$\sharp(k+1)Q$ blow-down
$=$



$\sharp(k+2)Q$ $y \longrightarrow y-z$
$=$



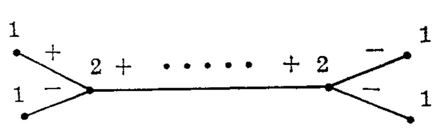
$\sharp(k+2)Q$ $z \longrightarrow z-x$
in the 1'st picture
$=$



$\sharp(k+2)Q$.

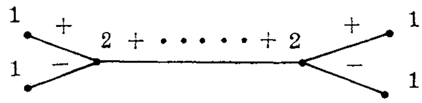
We denote the last framed link by FDk (including the case $k=1$).

(21)



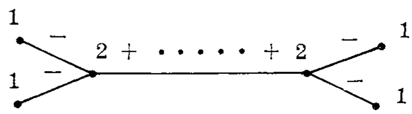
$=FDk\sharp 2P\sharp kQ$.

(22)

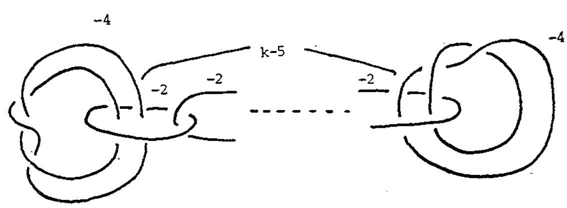


$= FDk\sharp P\sharp(k+1)Q\,.$

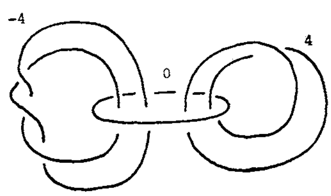
(23)



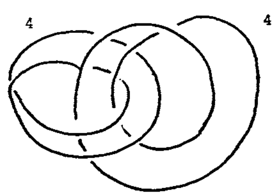
$=$



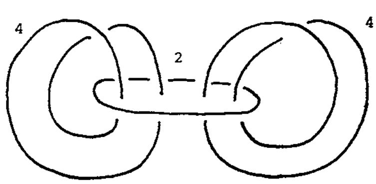
$\sharp 4S^2\times S^2 \qquad (k\geq 5)$



$\sharp 3S^2\times S^2 \qquad (k=4)$



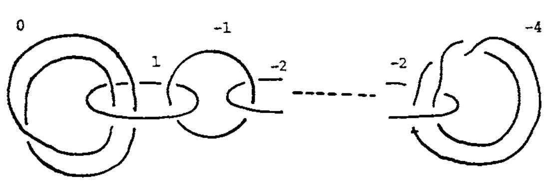
$\sharp 3S^2\times S^2 \qquad (k=3)$



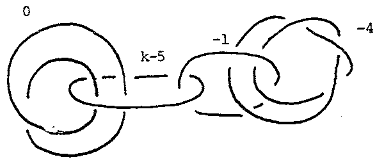
$\sharp 2S^2\times S^2 \qquad (k=2)\,.$

Hence

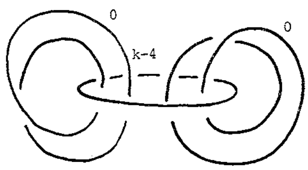
$\tilde{D}_k^+(5)\sharp P=$



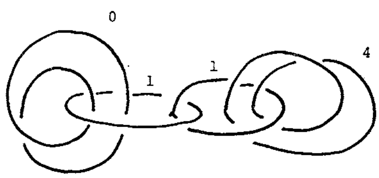
$\sharp 4P\sharp 4Q$

$=$ 

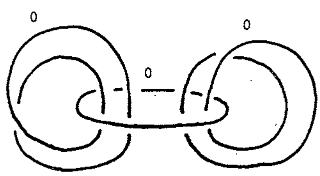
$\sharp(k-2)Q\sharp 4P$

$=$ 

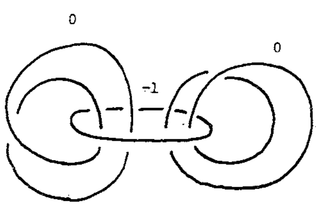
$\sharp(k-1)Q\sharp 4P = FDk\sharp(k-1)Q\sharp 4P$.
$(k \geq 5)$

$\tilde{D}_4^+(5)\sharp P =$ 

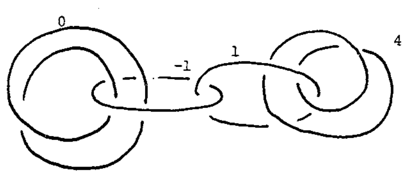
$\sharp 3P\sharp 3Q$

$=$ 

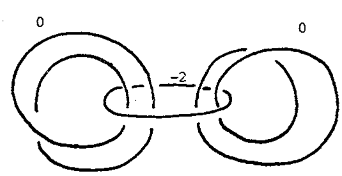
$\sharp 4P\sharp 3Q = FD4\sharp 4P\sharp 3Q$.

$\tilde{D}_3(5)\sharp P =$ 

$\sharp 4P\sharp 2Q = FD3\sharp 4P\sharp 2Q$.

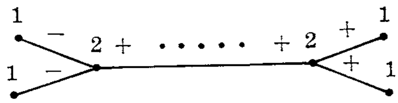
$\tilde{D}_2(5)\sharp P =$ 

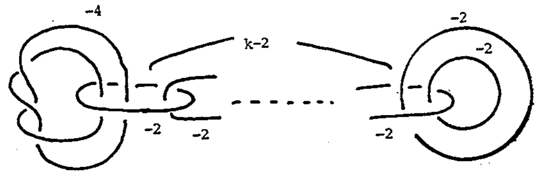
$\sharp 3P\sharp Q$

$=$ 

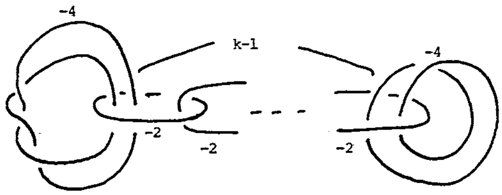
$\sharp 4P\sharp Q = FD2\sharp 4P\sharp Q$.

Similarly we also have $\tilde{D}_k^+(5)\sharp Q = FDk\sharp 3P\sharp kQ$.

(24) 

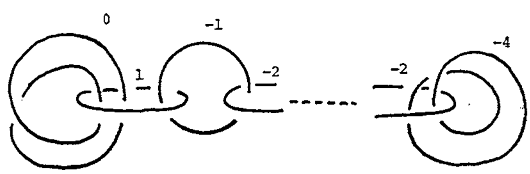
$=$ 

$\# 2S^2 \times S^2$

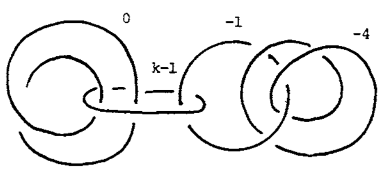
$=$ 

$\# 2S^2 \times S^2$.

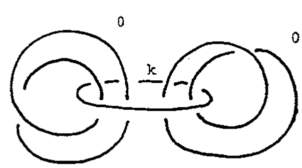
Hence

$\tilde{D}_k^+(6) \# P =$ 

$\# 2P \# 2Q$

$=$ 

$\# 2P \# kQ$

$=$ 

$\# 2P \# (k+1)Q = FDk \# 2P \# (k+1)Q$.

On the other hand, we can see the following:

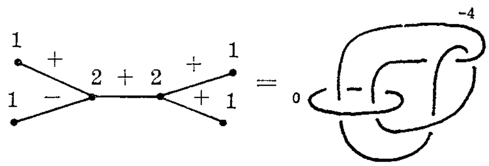
LEMMA 6. *$FDk \# (k-1)Q \# 2P = N_k \# Q$ for $k \geq 1$, where $N_k = (k-1)I_1^+ + 5I_1^-$ is represented as* $([k-1](1, 0)_+, (1, 1)_-, (1, 2)_-, 3(0, 1)_-)$ *or equivalently* $([k-1] \times (1, -1)_+, (1, 0)_-, (1, 1)_-, 3(0, 1)_-)$, *which is transformed by elementary transformations to* $([k-1](1, 0)_+, (0, 1)_-, (1, 0)_-, (0, 1)_-, (1, 0)_-, (0, 1)_-)$. $([k-1](1, 0)_+$ *means* $\overbrace{(1, 0)_-, (1, 0)_-, \cdots, (1, 0)_-}^{k-1}.)$

The proof will be given in § 5.

This proves the cases (20)-(24) for $k \geq 2$. The case (19) is proved by [Mo].

(19)-(24) for $k=1$.

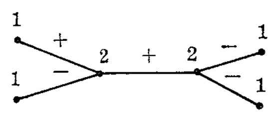
(20)



$\# P \# 3Q$.

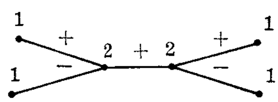
We denote the framed link in the right side by $FD1'$.

(21)



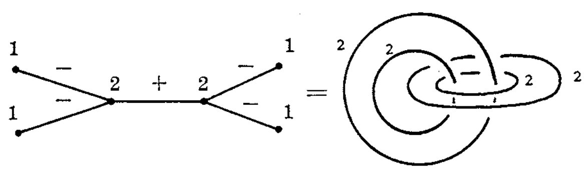
$= FD1' \# 3P \# Q$.

(22)



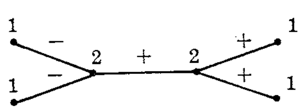
$= FD1' \# 2P \# 2Q$.

(23)



$\# S^2 \times S^2$.

(24)



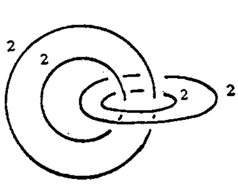
$= FD1' \# 2S^2 \times S^2$.

On the other hand, we can see the following:

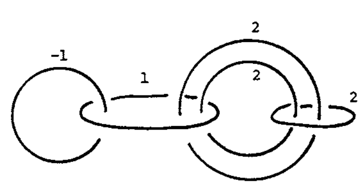
LEMMA 7. $FD1' \# 3P = N_1 \# Q$.

This follows from Lemma 6 since $FD1 = FD1' \# P$ (blow-down).

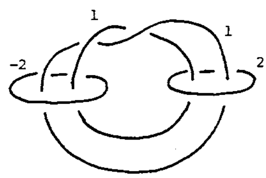
For (23) we see that

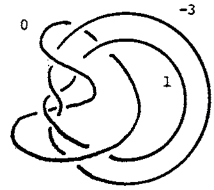


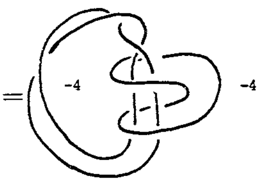
$\# S^2 \times S^2 \# P =$



$\# 2P$

= 

$\sharp 3P=$ 

$\sharp 4P=$ 

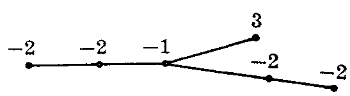
$\sharp 5P$

$=FD1'\sharp 5P$.

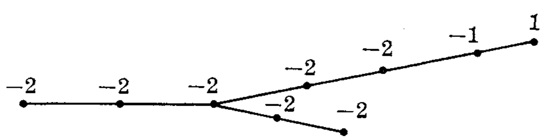
Similarly we have $D_1^+(5)\sharp Q=FD1'\sharp 4P\sharp Q$.

This proves (20)-(24) for $k=1$. (19) is proved by [Mo].

PROOF OF (2′), (8′), (20′)-(24′). In this paragraph the weights on the vertices denote the self-intersection numbers of the corresponding divisors.

$\tilde{E}_6^+(1, 1, 2)\sharp 2Q=$ 

$\sharp 2Q$ $\overset{\text{blow-up}}{=}$

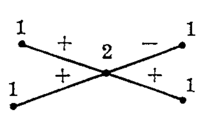


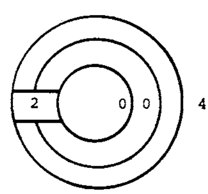
$\overset{\text{blow-down}}{=}$ $IV^*\sharp P$.

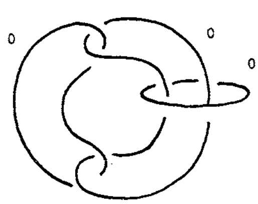
The proofs are similar for the rest cases. First perform blowing-ups by Q's to get the linear branches of the form $\underline{-2\cdots-1\ \ 1}$ and blow down the P's on the right side. Every case is reduced to some analytic one which splits into the sum of I_1^+'s by [Mo]. This also proves $(20)_0$-$(22)_0$ as the dual of $(20')_0$-$(22')_0$.

For type $\tilde{D}_0$ we give the alternative proofs using Lemma 8 below, which give the topological proof of Moishzon's result ([Mo]) for I_0^*.

$(20)_0$



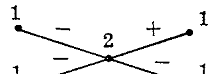
= 

$\sharp 2Q \cong$ 

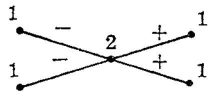
$\sharp 2Q$.

We denote the last framed link by $FD0$.

$(21)_0$



$=FD0\sharp 2P$.

(22)$_0$ 

$=FD0\sharp S^2\times S^2$.

For these cases we have:

LEMMA 8. *$FD0\sharp 3P=I_{0-}^*\sharp Q$ where $I_{0-}^*=((1,\ 0)_-,(0,\ 1)_-,(1,\ 0)_-,(0,\ 1)_-,$ $(1,0)_-,(0,1)_-)=((1,0)_-,(1,1)_-,(1,2)_-,3(0,1)_-)$.*

The proof will be given in § 5.

This proves $(20)_0$-$(22)_0$. $(19)_0$ and $(23)_0$ are proved by [Mo].

The statements in Theorem 1 for the cases of $\tilde{E}$ and $\tilde{D}_0$ give locally best possible results in the following sense.

COMPLEMENT TO THEOREM 1. (i) *Neither $\tilde{E}_6^+(1,1,2)\sharp Q$, $\tilde{E}_6^-(1,1,2)\sharp P$, nor $\tilde{E}_7^\pm(2,2,1)$ can be blown down to a sum of $I_1^\pm$'s and Tw's.* (ii) *$\tilde{D}_0(2)\sharp 2P$, $\tilde{D}_0(3)\sharp 2Q$, $\tilde{D}_0(4)\sharp P$, and $\tilde{D}_0(4)\sharp Q$ cannot be blown down to sums of $I_1^\pm$'s and Tw's.*

PROOF. (i) It suffices to consider $\tilde{E}^+$ cases. We consider the euler number e and Matsumoto's signature function σ of the singular fiber ([M3]). $\sigma(F)$ for the singular fiber F is defined as $\sigma(F)=-\phi(\beta)+\mathrm{Sign}(N)$ where $\phi(\beta)$ is Meyer's function for the monodromy β of F ([Me]) and $\mathrm{Sign}(N)$ is the signature of the regular neighborhood N of F ([M3]). We see that $e(\tilde{E}_6^+(1,1,2))=7$ and $\sigma(\tilde{E}_6^+(1,1,2))=2/3-1+1-3=-7/3$ since the monodromy of $\tilde{E}_6^+$ is $\begin{pmatrix}-1&-1\\1&0\end{pmatrix}$ and $\phi\left(\begin{pmatrix}-1&-1\\1&0\end{pmatrix}\right)=-2/3$. Suppose that $\tilde{E}_6^+(1,1,2)\sharp Q=N\sharp aP\sharp bQ$ where $N=\alpha I_1^++\beta I_1^-+\gamma Tw$. Then $e(N)=\alpha+\beta+2\gamma=8-a-b$, and $\sigma(N)=2(\beta-\alpha)/3=-10/3+b-a$. Glue $IV=4I_1^+$ and N together to construct a closed torus fibration M over S^2 in which every singular fiber is of type $I_1^\pm$ or Tw. Then $e(M)=12-a-b$, and $\sigma(M)=-8/3-10/3+b-a=-6+b-a$. If $\sigma(M)\neq 0$, then $e(M)\geq 12$ and $\sigma(M)$ is divisible by 8 ([M4]). Therefore $(a,b)=(0,0)$ and hence $\sigma(M)=-6$, a contradiction. Therefore $\sigma(M)=0$. It follows that $(a,b)=(0,6)$ since $0\leq a+b<8$, $a\geq 0$, and $b\geq 0$ (Note that $e(N)>0$ since the monodromy of N is non-trivial.). Thus $\alpha+\beta+2\gamma=2$ and $2(\beta-\alpha)/3=8/3$. But there are no non-negative integers α, β, and γ which satisfy the above equations. Then $\tilde{E}_6^+\sharp Q$ cannot be blown down to a sum of $I_1^\pm$'s and Tw's. Next suppose that $\tilde{E}_7^+(2,2,1)=N\sharp aS^2\times S^2$ ($\tilde{E}_7^+(2,2,1)$ is spin) where $N=\alpha I_1^+ +\beta I_1^-+\gamma Tw$. $e(N)=\alpha+\beta+2\gamma=7-2a$ and $\sigma(N)=2(\beta-\alpha)/3=0$ $\left(\phi\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)=-1\right)$. Therefore $\beta=\alpha$ and hence $e(N)=2(\alpha+\gamma+a)=7$, which is a contradiction. This proves the case of $\tilde{E}_7^+(2,2,1)$. (ii) Suppose that $\tilde{D}_0(2)\sharp 2P = N\sharp aP\sharp bQ$ where $N=\alpha I_1^+ + \beta I_1^- + \gamma Tw$. Then $e(N)=8-a-b$ and $\sigma(N)=-2+2-a+b\left(\phi\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right)=0\right)$. Add $6I_1^+$ to N to construct a torus fibration M over S^2 with $e(M)=14-a-b$ and $\sigma(M)=-4-a+b$. If $\sigma(M)\neq 0$, then $a+b \leq 2$ and $\sigma(M)\equiv 0$ (mod. 8) by [M2]. But there are no such non-negative integers a and b. Therefore $\sigma(M)=0$ and hence $(a,b)=(0,4)$ or $(1,5)$ (Note that $e(N)>0$.). In the first case we have $\alpha+\beta+2\gamma=4$ and $2(\beta-\alpha)/3=4$ and in the second case we have $\alpha+\beta+2\gamma=2$ and $2(\beta-\alpha)/3=4$. But there are no such non-negative integers α, β, and γ in either case. This proves the case for $\tilde{D}_0(2)$. The cases of $\tilde{D}_0(3)$ and $\tilde{D}_0(4)$ are proved similarly so we omit them.

We do not know whether the results in Theorem 1 in the cases of $\tilde{D}_k$'s for large k are locally best-possible in general.

§ 4. Stable diffeomorphism types of good torus fibrations

We consider a (stable) diffeomorphism type of a good torus fibration M over S^2 without multiple fibers which contains at least one non-analytic fiber of type $\tilde{E}$ or $\tilde{D}$. It suffices to consider the case when every singular fiber of M is reduced. M itself may permit extra blowing-down processes, but we do not consider the minimal form of M here. For, there is no canonical minimal form for a non-analytic good singular fiber in general since some of such fibers cannot be blown down to sums of $I_1^\pm$'s and/or Tw's and in these cases the blowing down processes cannot be determined canonically (Complement to Theorem 1 in § 3). If all the singular fibers in M can be replaced by the sum of fibers of type $I_1^\pm$ and Tw by Theorem 1, then we say that M is reducible. The diffeomorphisms given in Theorem 1 induce fiber-preserving diffeomorphisms on the boundary of the regular neighborhoods of the fibers up to isotopy (Remark 2 in § 3), and hence the replaced new structures are still torus fibrations. There are many reducible and also many non-reducible examples (Complement to Theorem 1).

THEOREM 2. *Let M be a 1-connected good torus fibration over S^2 without multiple fibers which contains at least one singular fiber. Furthermore suppose that M has no non-analytic singular fiber of type $\tilde{D}$. Then a connected-sum of M and at most one copy of P or Q is diffeomorphic to the manifold of the form $aP\sharp bQ$. More precisely the following ones are of such forms*:

(1) *$M\sharp P$ (resp. $N\sharp Q$) if every fiber is analytic (resp. anti-analytic).*

(2) *Case when M has non-analytic fibers.*

(i) *$M\sharp P$ and $M\sharp 2Q$ (resp. $M\sharp Q$ and $M\sharp 2P$) if every non-analytic fiber is of type $\tilde{E}_6^+(1, 1, 2)$ (resp. $\tilde{E}_6^-(1, 1, 2)$),*

(ii) *$M\sharp P$ and $M\sharp Q$ otherwise.*

PROOF. We may assume that all the fibers are reduced. Case 1 is proved by Theorem 11 in [Ma]. By Theorem 1 the non-splittable fibers in M are of type $\tilde{E}_6^{\pm}(1, 1, 2)$ and $\tilde{E}_7^{\pm}(2, 2, 1)$ only. Suppose that M contains at least one such fiber. Let $N\subset M$ be the torus fibration over D^2 which contains all the non-splittable fibers in M and is represented by $a\tilde{E}_6^+(1, 1, 2)+a'\tilde{E}_6^-(1, 1, 2)+b\tilde{E}_7^+(2, 2, 1)+b'\tilde{E}_7^-(2, 2, 1)$. By successive applications of (2) and (2′) in Theorem 1 we have

(i) $(2k-1)\tilde{E}_6^+(1, 1, 2)\sharp P=4kI_1^-+8(k-1)I_1^+\sharp(2k+2)Q,$

$(2k-1)\tilde{E}_6^+(1, 1, 2)\sharp 2Q=4(k-1)I_1^-+8kI_1^+\sharp 2(k-1)Q\sharp P,$

(ii) $2k\tilde{E}_6^+(1, 1, 2)\sharp P=4kI_1^-+8kI_1^+\sharp 2kQ\sharp P,$

$2k\tilde{E}_6^+(1, 1, 2)\sharp 2Q=4kI_1^-+8kI_1^+\sharp(2k+2)Q.$

Also by (8) we have

(iii) $b\tilde{E}_7^+(2, 2, 1)\sharp P=3bI_1^-\sharp(b+1)P\sharp 3bQ,$

$b\tilde{E}_7^+(2, 2, 1)\sharp Q=3bI_1^-\sharp bP\sharp(3b+1)Q.$

(We have dual statements for $\tilde{E}_6^-(1, 1, 2)$ and $\tilde{E}_7^-(2, 2, 1)$ by replacing all the P's (Q's) by Q's (P's) and changing all the signs in (i)-(iii).) This proves case (2)-(i). If $(a, a')=(0, 0)$ or $(b, b')=(0, 0)$, $a\neq 0$, $a'\neq 0$, then both $N\sharp P$ and $N\sharp Q$ split. If $(b, b')\neq(0, 0)$, then first split $\tilde{E}_7^{\pm}(2, 2, 1)$'s by the blowing-up to obtain both of new copies of P and Q, which appear by the blowing-down processes. Then use them to split the other $\tilde{E}_6^{\pm}$'s, and hence both $N\sharp P$ and $N\sharp Q$ split. If $a'=b=b'=0$ or $a=b=b'=0$ and M has another non-analytic fiber, then M itself has the factor P or Q. Therefore in case (2)-(ii) both $M\sharp P$ and $M\sharp Q$ are of the form $\alpha I_1^+ +\beta I_1^- +\gamma Tw\sharp aP\sharp bQ$. Then by [M2], [M4], [I], and Theorem 11 in [Ma] M is diffeomorphic to a connected sum of some copies of P and Q except for the following cases: (iv) $M=\alpha I_1^+\sharp bQ$, (iv′) $M=\beta I_1^-\sharp aP$. However by Theorem 1 every non-analytic fiber is (stably) splittable to a manifold of the form $sI_1^+ +tI_1^-\sharp uP\sharp vQ$ such that $u>0$ if $t=0$ and $v>0$ if $s=0$. Hence neither (iv) nor (iv′) can occur. This proves Theorem 2.

The diffeomorphism types of reducible good torus fibrations over S^2 without multiple fibers do not depend on the choices of the gluing maps of the regular neighborhoods of the singular fibers except for twins obtained by blowing-downs of fibers of type $\tilde{A}$ and there are no obstructions to constructing cross sections (euler class) for such singular fiberings. This follows from the corresponding fact about the torus fibrations in which every singular fiber is either $I_1^{\pm}$ or Tw ([M2]).

The (stable) diffeomorphism types of good torus fibrations for the general cases will be discussed elsewhere.

Examples. Let $s_1=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $s_2=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$.

1. Let $M=\tilde{E}_6^-(1, 1, 2)$ with monodromy $s_1s_2s_1s_2+\tilde{E}_8^-(1, 2, 2)$ with monodromy $s_1s_2+\tilde{D}_0(4)$ with monodromy $(s_1s_2)^6$. Then $M=2I_1^+\#2Q\#5P+\tilde{E}_6^-(1, 1, 2)+\tilde{D}_0(4)=6I_1^+\#Q\#9P+\tilde{D}_0(4)=6I_1^++6I_1^-\#3Q\#7P=8Q\#12P$.

2. Let $M=\tilde{E}_8^-(2, 2, 1)$ with monodromy $s_1s_2+\tilde{D}_{6m-1}^+(5)$ with monodromy $(s_1s_2)^3s_1^{6m-1}+(6m+5)I_1^+$ with monodromies

$$s_1^{-6m+2}s_2s_1^{6m-2},\ \cdots,\ s_1^{-1}s_2s_1,\ s_2,\ \overbrace{s_1,\ s_2,\ \cdots,\ s_1,\ s_2}^{6}$$

respectively. Then $M=2I_1^+\#2Q\#P+(6m+5)I_1^++\tilde{D}_{6m-1}^+(5)=(12m+5)I_1^++5I_1^-\#2P\#3Q=(2m-1)P\#(10m-1)Q\#7P\#8Q=(2m+6)P\#(10m+7)Q$.

3. Let $M=\tilde{E}_6^-(1, 1, 2)$ with monodromy $(s_1s_2)^2+8I_1^+$ with monodromies

$$\overbrace{s_1,\ s_2,\ \cdots,\ s_1,\ s_2}^{8}$$

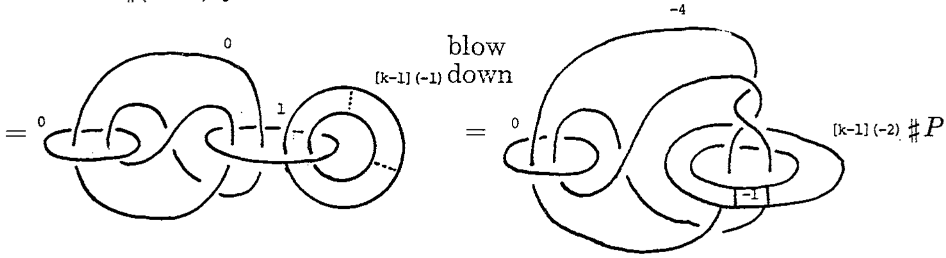
respectively. Then M is homeomorphic to $5P\#8Q$ by Freedman's Theorem ([F]). But the diffeomorphism type of M is not determined by Theorem 1 only. It seems that more arguments are necessary to determine the diffeomorphism types of non-reducible cases.

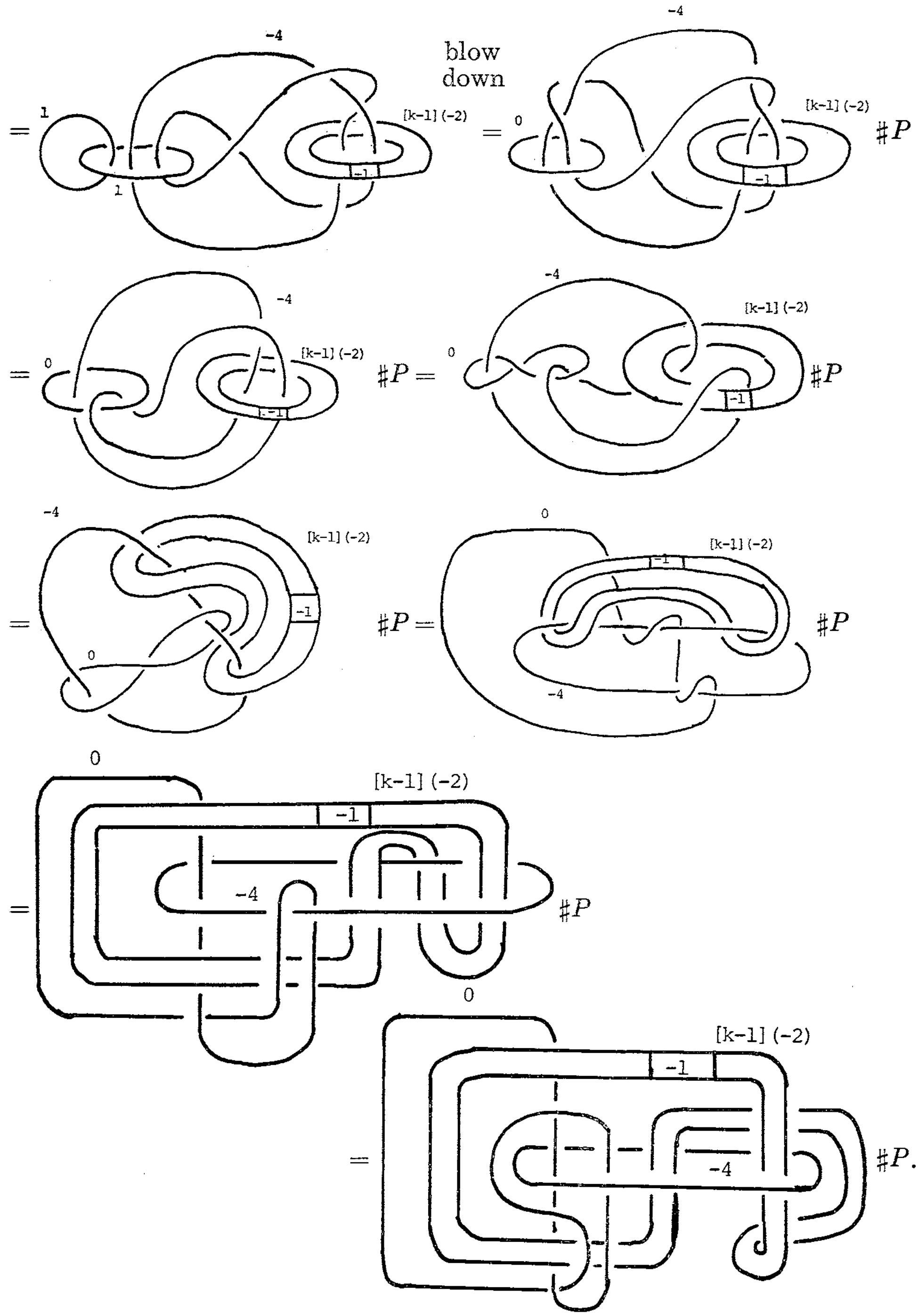
§ 5. Proofs of Lemmas

Proof of Lemma 6.

(I)-1.

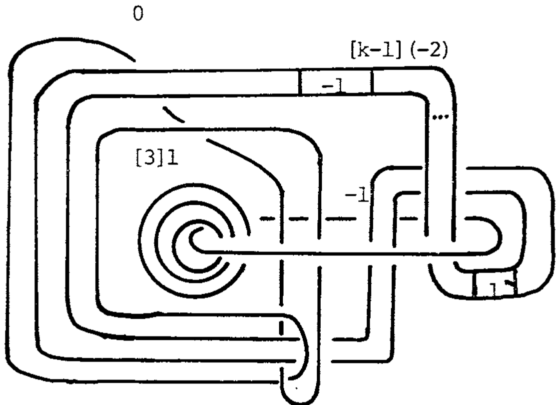
$FDK\#(k-1)Q$




blow
down
-4
1
0
[k-1] (-2)
-1
-4
#P
#P.

(I)-2. $FDk\sharp(k-1)Q\sharp 2P$ = the last framed link $\sharp 3P$

=



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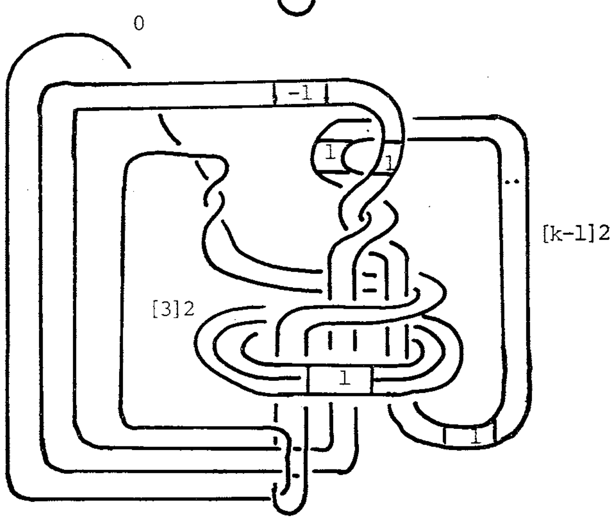
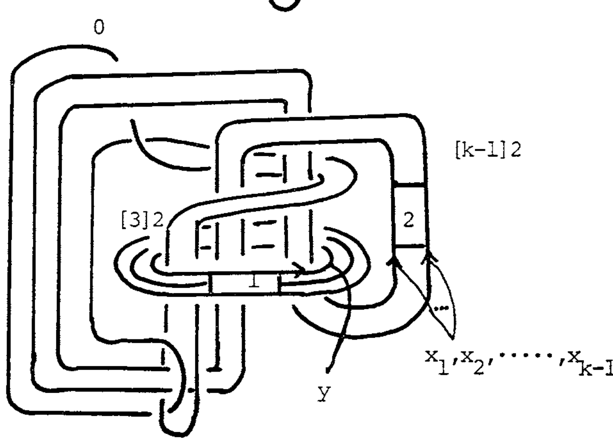


$\sharp Q$

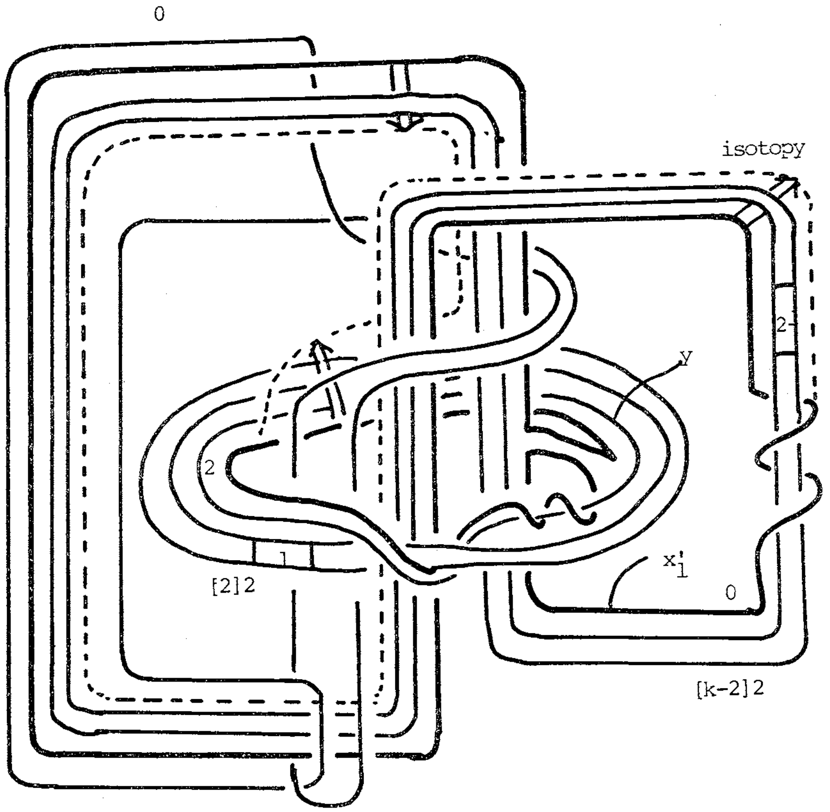
=



$\sharp Q$.

(I)-3. In the last framed link, slide a handle x_i to $x'_i = x_i - y$ inductively ($i = 1, 2, \cdots, k-1$).

The last framed link $\overset{x_1 \longrightarrow x'_1 = x_1 - y}{=}$



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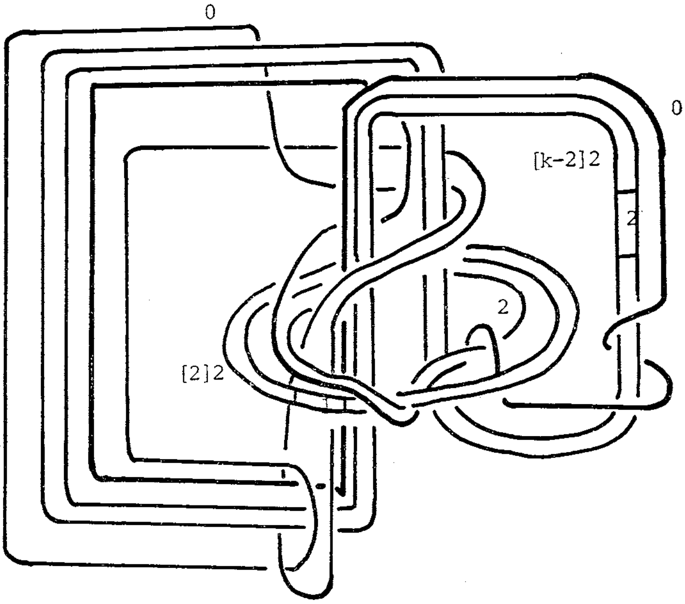
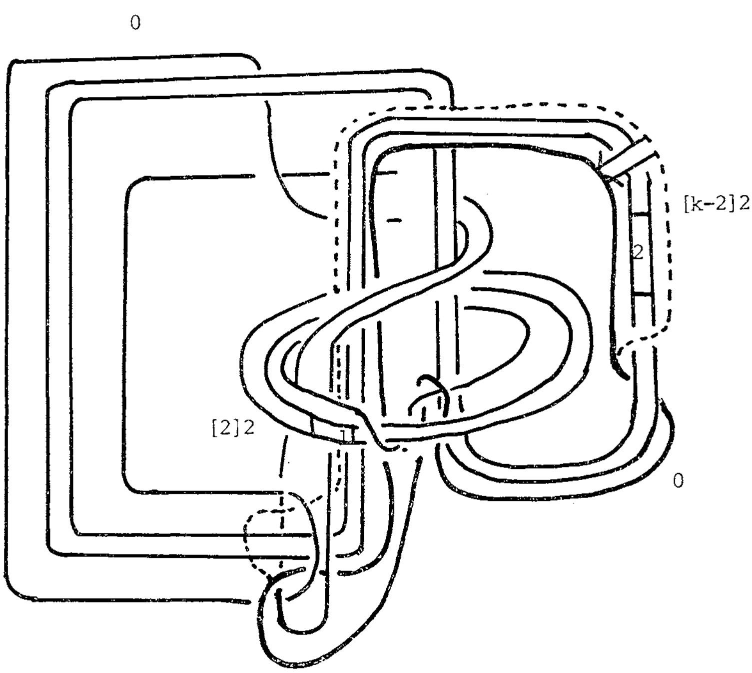
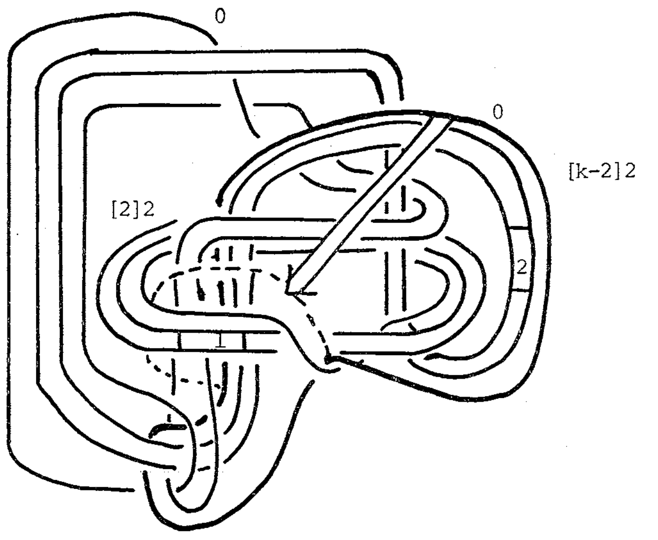

0
0
[k-2]2
2
2
[2]2

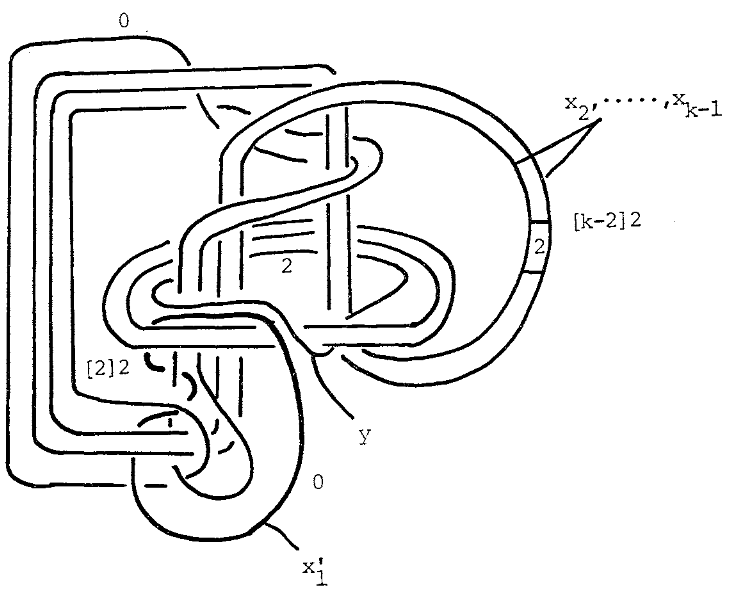
=


0
[k-2]2
2
[2]2
0

=

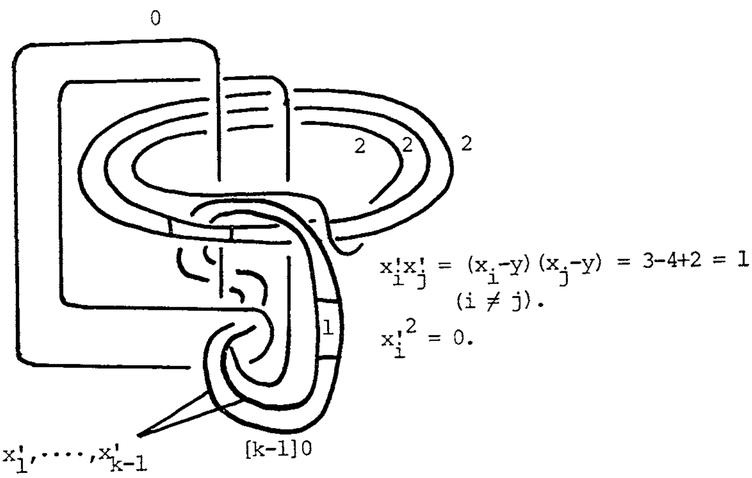


=



successive handle slidings $x_i \to x'_i = x_i - y$

=



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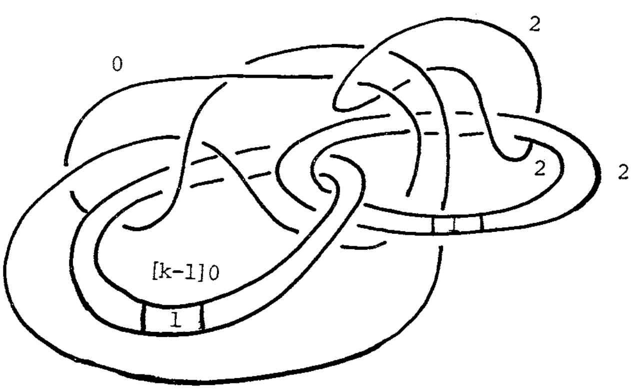
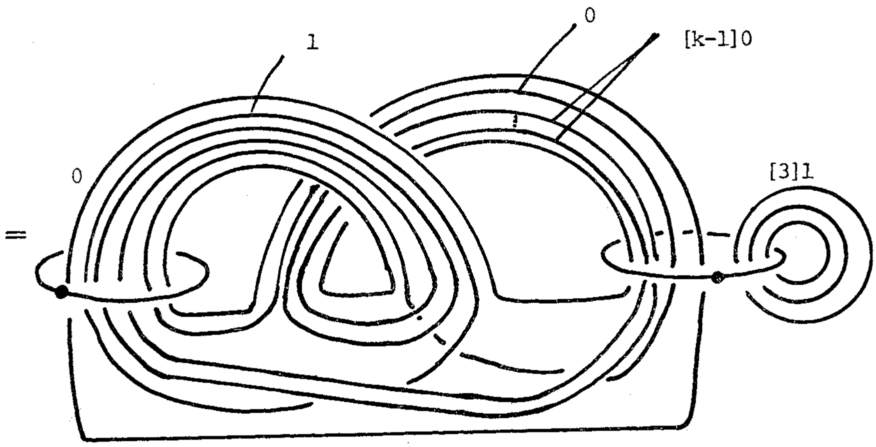
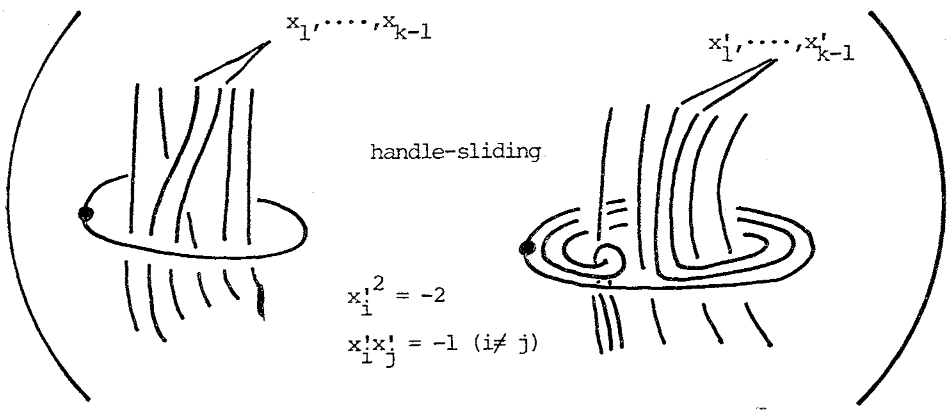


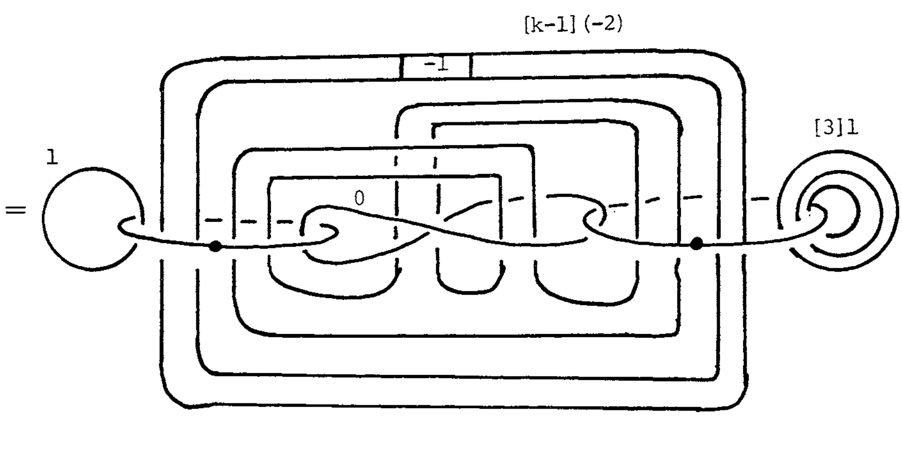
Then $FDk\sharp(k-1)Q\sharp 2P=N_k\sharp Q$.

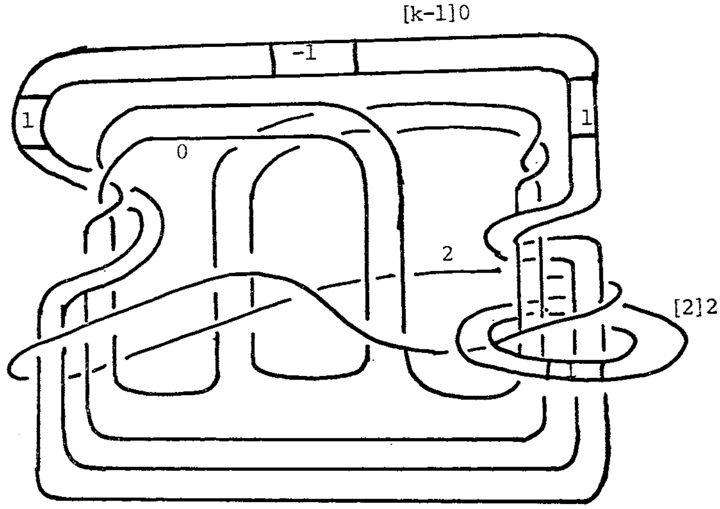
II.

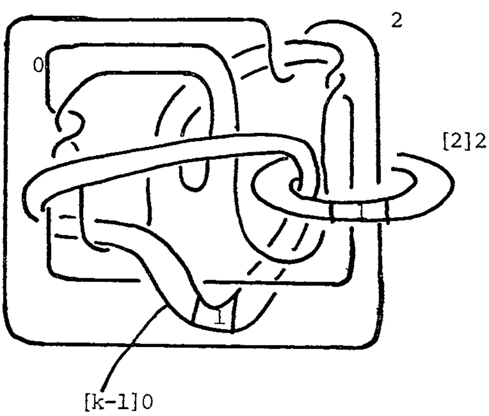
$([k-1](1,-1)_+, (1,0)_-, (1,1)_-, 3(0,1)_-)$

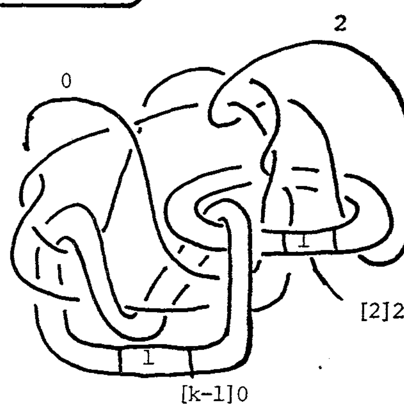


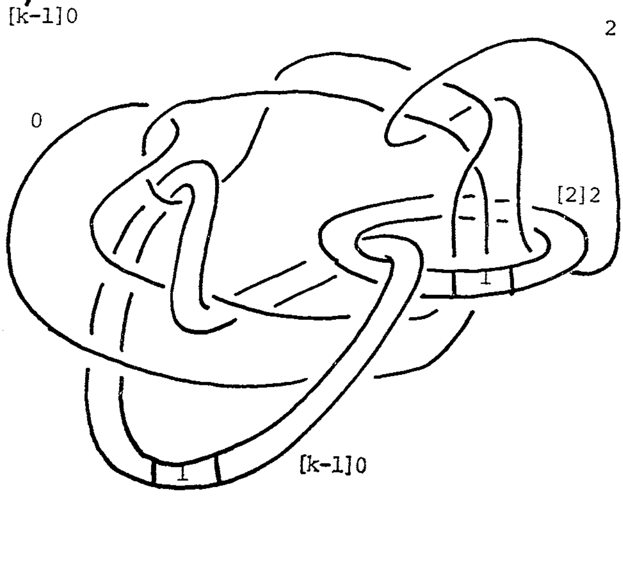


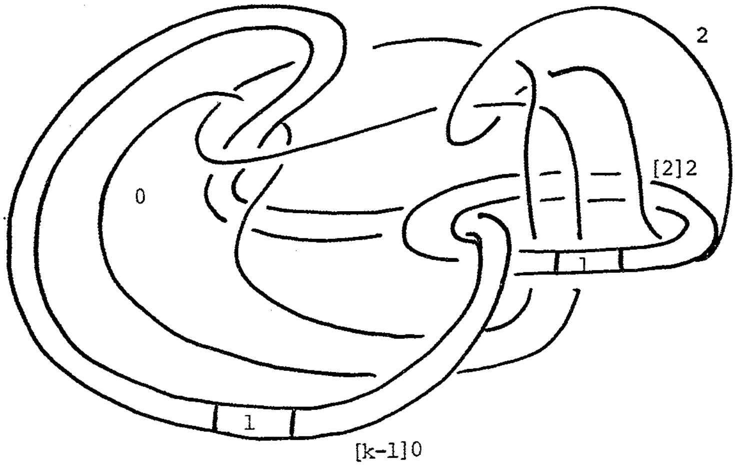


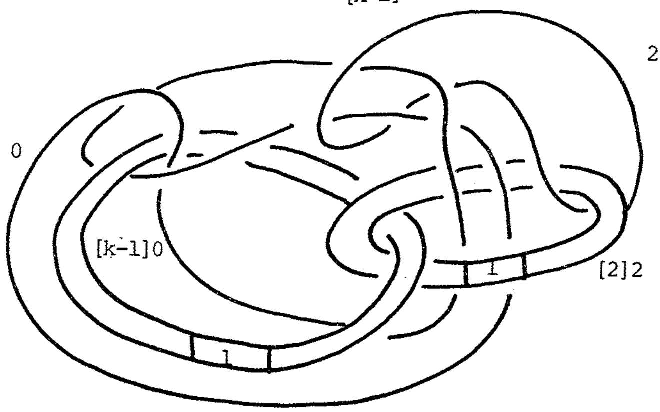
= 
[k-1]0
-1
1
1
0
2
[2]2

= 
2
0
[2]2
1
[k-1]0

= 
2
0
1
[2]2
1
[k-1]0

= 
2
0
[2]2
1
1
[k-1]0

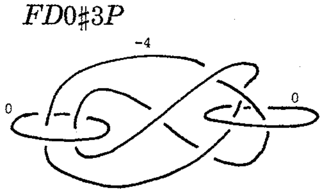
= 

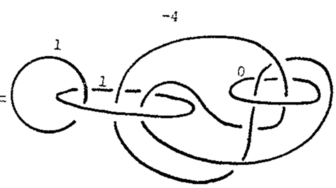
= 

Thus $FDk\#(k-1)Q\#2P=(k-1)I_1^+ + 5I_1^-\#Q$ as stated.

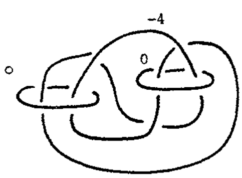
PROOF OF LEMMA 8.

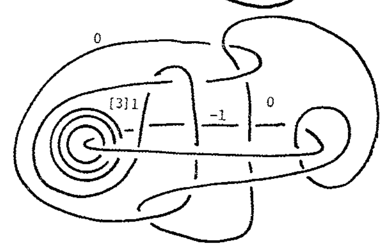
(I) $FD0\#3P$

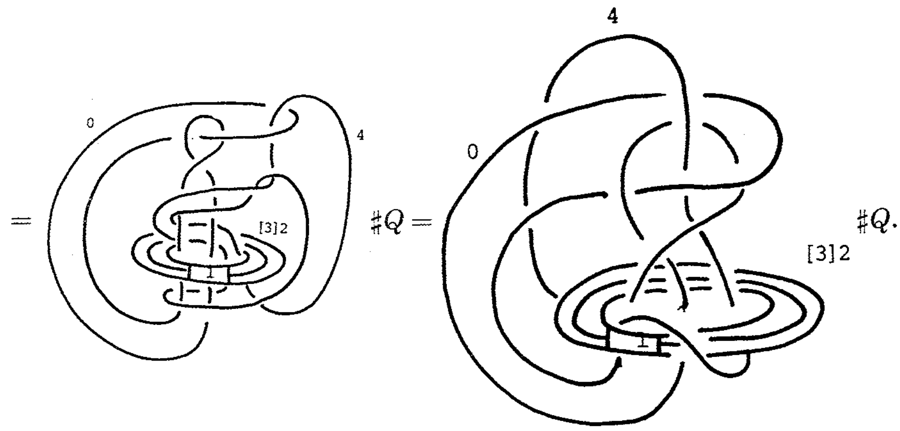
= 

$\#3P=$ 

$\#2P$

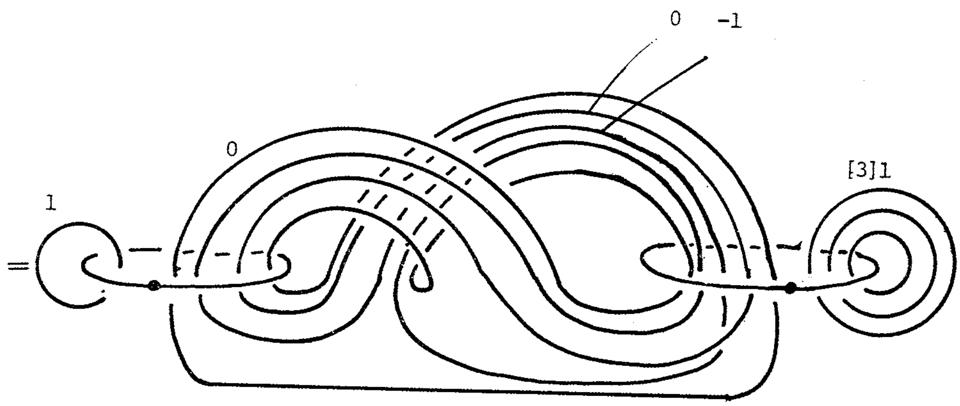
= 

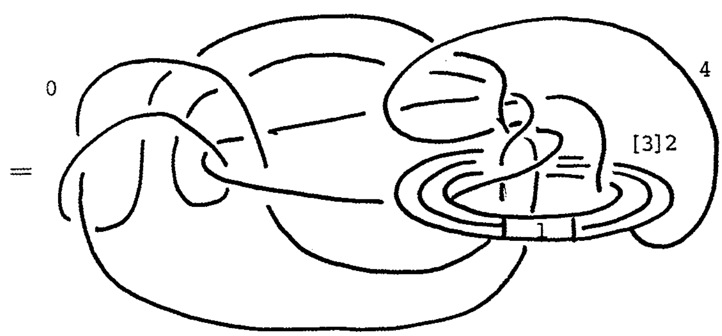
$\#3P=$ 



We denote the last framed link by N_0.

$((1, 0)_-, (1, 1)_-, (1, 2)_-, 3(0, 1)_-)$





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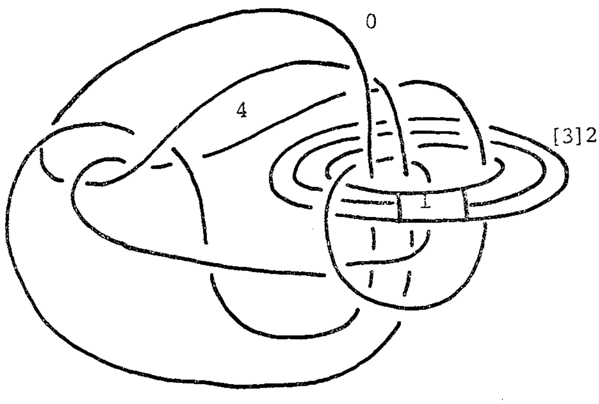

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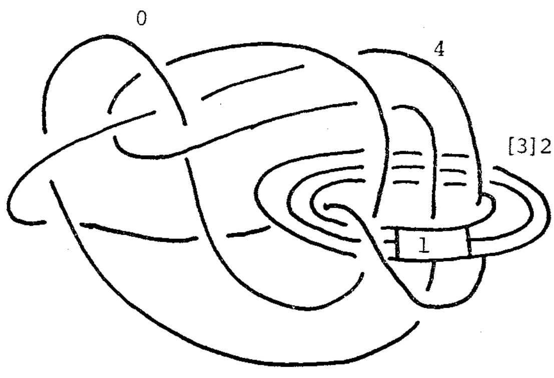
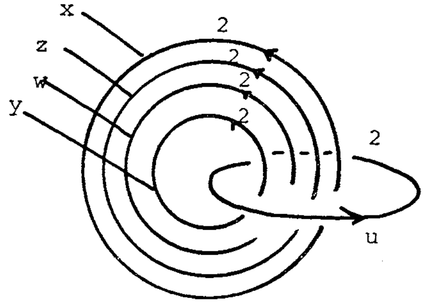


$=N_0$.

(II) We prove that $((1,0)_-, (0,1)_-, (1,0)_-, (0,1)_-, (1,0)_-, (0,1)_-) = I_{0-}^*$.
It suffices to prove that $N_0 = I_{0-}^*$.

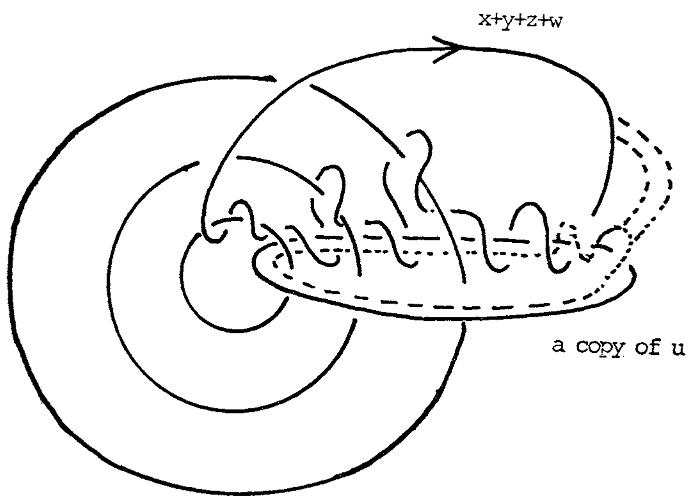
$I_{0-}^* =$



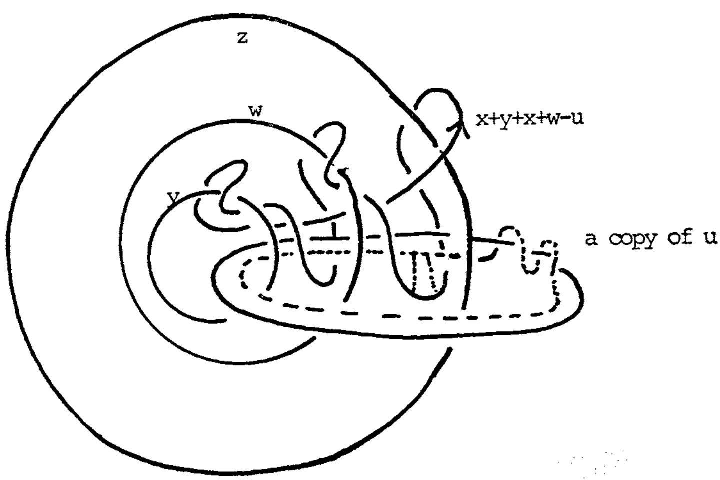
Perform handle-slidings as follows:

$$\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} \longrightarrow \begin{pmatrix} x_0 = x+y+z+w-2u \\ y_0 = y+z-2u \\ z_0 = z-u \\ w_0 = w \\ u_0 = u \end{pmatrix}$$

$I_{0-}^* =$



$=$



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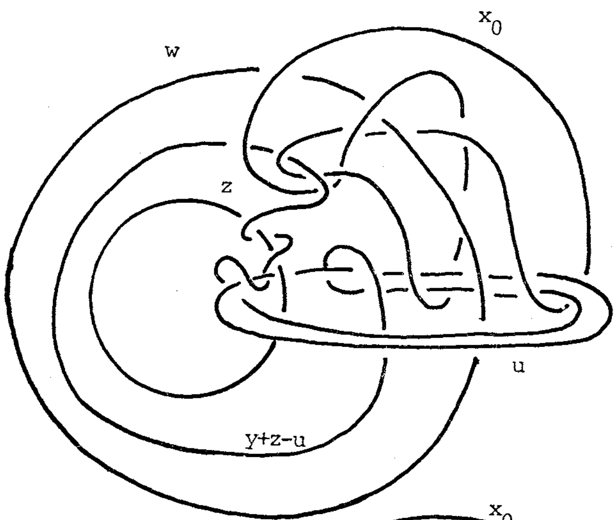
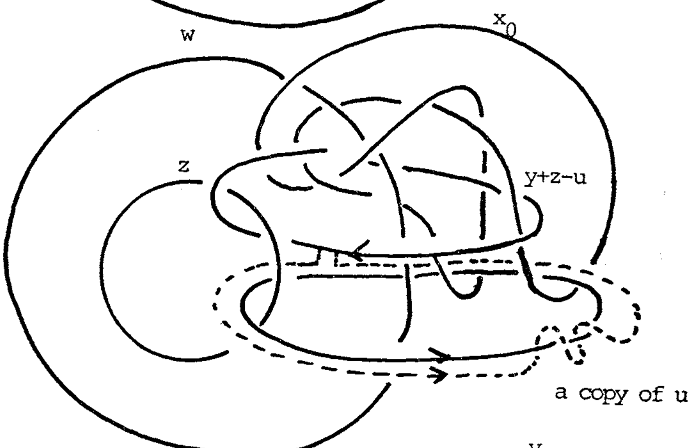
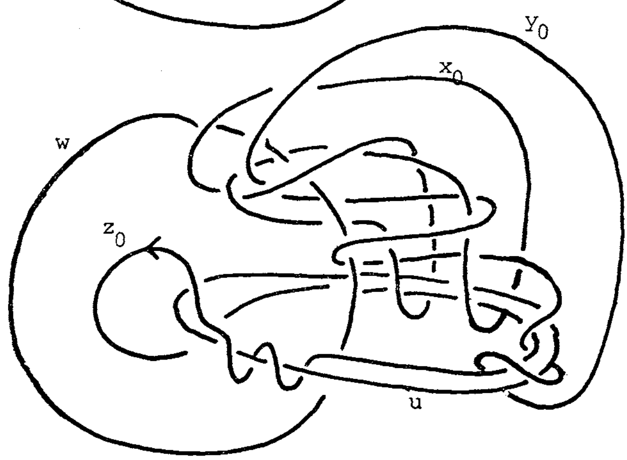

x_0
w
z
u
y+z-u

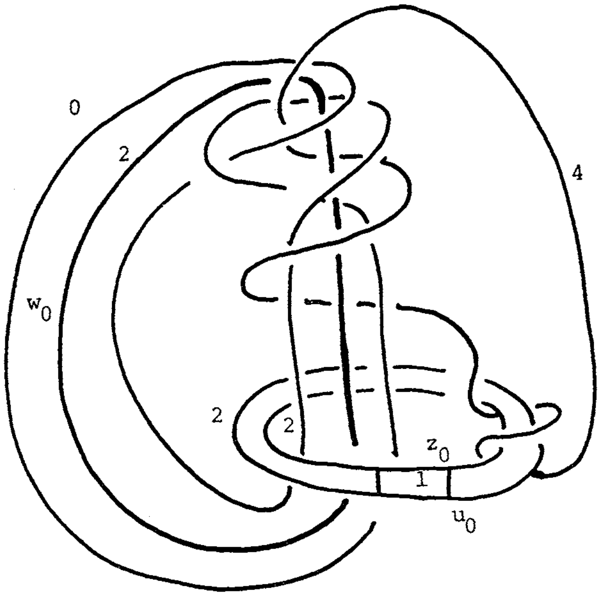
=


x_0
w
z
y+z-u
a copy of u

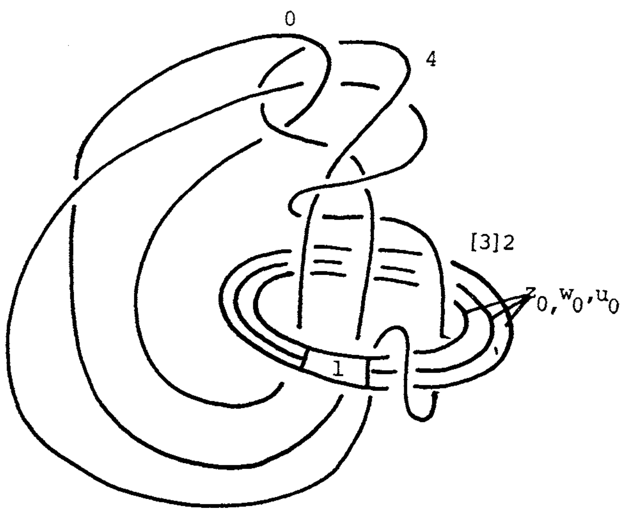
=

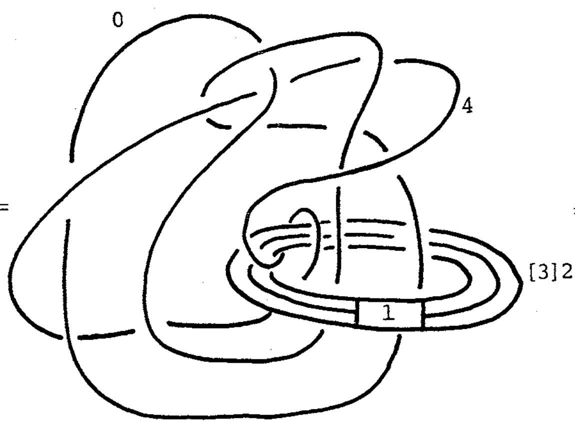

y_0
x_0
w
z_0
u

=



isotopy
=



=  $=N_0$.

This proves Lemma 8.

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