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On the existence of weak solutions of a non-linear mixed problem for the Navier-Stokes equations in a time dependent domain

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Summary—We prove the existence of a weak solution of a non-linear mixed problem for the Navier-Stokes equations in a non-cylindrical domain in $R^3\times(0, T)$.

§ 1. Introduction

In this paper we study a non linear mixed problem for the Navier-Stokes equations in a non-cylindrical domain in $R^3\times(0, T)$. We deal with the flow of a fluid in a tube with time dependent initial and final sections where we give physically expressive boundary conditions (see [6], [10], [11]).

In the last years, many authors considered initial-boundary value problems for the Navier-Stokes equations in a non-cylindrical (x, t)-domain (see [1], [3], [5], [7], [9]). In particular in [1] H. Fujita and N. Sauer proved the existence of weak solutions for problems of this type using a kind of penalty method. However this method presents serious difficulties for non linear mixed problems.

We shall prove the existence of weak solutions of the problem of § 2 via an elliptic approximation. The essential point of our proof is the estimate of a time difference quotient.

The paper is composed of two sections. In § 2 we describe the problem, introduce particular functional spaces and give the definition of weak solutions. In § 3 we give an existence proof.

§ 2. Statement of the problem and notation

Let $\Omega(t)$ be an open bounded set of R^3 depending on $t\in(0, T)$, T is a finite positive number. As t increases over $(0, T)$, $\Omega(t)$ generates a (x, t)-domain $\hat{\Omega}$ and the boundary $\Gamma(t)$ of $\Omega(t)$ generates a (x, t)-hypersurface $\hat{\Gamma}$. We assume $\hat{\Gamma}$ is a C^1-hypersurface and $\Gamma(t)=\Gamma_1(t)\cup\Gamma_2(t)\cup\Gamma_3$ (Γ_3 is independent of t) with mes. $\Gamma_3 \neq 0$. Then we can represent $\Gamma_1(t)$ and $\Gamma_2(t)$ by $x_1 = \psi_1(x_2, x_3, t)$ and $x_1 = \psi_2(x_2, x_3, t)$ respectively in terms of C^1-functions ψ_1, ψ_2 (in each patch of a finite covering of $\Gamma_1(t)$ and $\Gamma_2(t)$).

The motion in $\Omega(t)$ of an incompressible fluid of viscosity and density 1 subject to the external force $f = f(x, t) = (f_1(x, t), f_2(x, t), f_3(x, t))$ is governed by the equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \Delta u + \nabla p = f \qquad \nabla \cdot u = 0 \tag{2.1}$$

where $u = u(t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ denotes the velocity and $p = p(x, t)$ the pressure. In (2.1) $u \cdot \nabla u = \sum_{1i}^{3} u_i \partial u / \partial x_i$.

Denoting by ν_t the outside normal to $\Gamma(t)$, we shall consider the boundary and initial conditions (we denote $\partial \psi_i / \partial t \cos(\nu_t, x_1)$ by $\widetilde{\partial \psi_i / \partial t}$)

$$\begin{aligned} &\frac{1}{2}|u|^2 \nu_t - \frac{1}{2} u \widetilde{\frac{\partial \psi_1}{\partial t}} + p\nu_t - \frac{\partial u}{\partial \nu_t} = -\alpha_1 && \text{on } \hat{\Gamma}_1 \\ &\frac{1}{2}|u|^2 \nu_t - \frac{1}{2} u \widetilde{\frac{\partial \psi_2}{\partial t}} + p\nu_t - \frac{\partial u}{\partial \nu_t} = -\alpha_2 && \text{on } \hat{\Gamma}_2 \\ &u = 0 && \text{on } \Gamma_3 \times (0, T) \\ &u(x, 0) = u_0 && \text{in } \Omega(0) . \end{aligned} \tag{2.2}$$

The first two relations in (2.2) determine the value of the density of the energy flux of the fluid on $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ respectively. The other conditions in (2.2) are standard.

We shall give the definition of weak solutions of the problem (2.1), (2.2). Let us begin by giving some definitions and basic notations.

Let Ω be a bounded open set in R^3 with boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We will need the following function spaces.

$$D(\Omega) = \{\varphi \mid \varphi \in (C^\infty(\bar{\Omega}))^3, \ \varphi = 0 \text{ on } \Gamma_3, \ \nabla \cdot \varphi = 0\}$$

$$H(\Omega) = \{\text{the completion of } D(\Omega) \text{ under the } (L^2(\Omega))^3\text{-norm}\}$$

$$V(\Omega) = \{\text{the completion of } D(\Omega) \text{ under the } (H^1(\Omega))^3\text{-norm}\} .$$

We let

$$(u, v)_\Omega = \sum_1^3{}_i \int_\Omega u_i v_i dx; \quad |u|_\Omega^2 = (u, u)_\Omega; \quad (u, v)_\Gamma = \sum_1^3{}_i \int_\Gamma u_i v_i d\Gamma;$$

$$((u, v))_\Omega = \sum_1^3{}_i \int_\Omega \nabla u_i \nabla v_i dx; \quad \|u\|_\Omega^2 = ((u, u))_\Omega; \quad \|u\|_{\mathscr{L}} = \text{norm in } \mathscr{L}.$$

Let $\hat{\Omega}$ be the (x, t)-domain $\cup_t \Omega(t)\times\{t\}$. For functions u defined in $\hat{\Omega}$ we define $\beta(u)$ by

$$\beta(u)^2=\int_0^T \|u\|^2_{\Omega(t)}\,dt$$

whenever the integral makes sense. Then we introduce

$$D(\hat{\Omega})=\{\varphi \mid \varphi\in(C^\infty(\bar{\hat{\Omega}}))^3,\ \varphi=0 \text{ on } \Gamma_3,\ \nabla\cdot\varphi=0\}$$

$$H(\hat{\Omega})=\{\text{the completion of } D(\hat{\Omega}) \text{ under the } (L^2(\hat{\Omega}))^3\text{-norm}\}$$

$$V(\hat{\Omega})=\{\text{the completion of } D(\hat{\Omega}) \text{ under the norm } \beta(u)\}\,.$$

By $U(\hat{\Omega})$ we mean the set of all $u\in V(\hat{\Omega})$ such that $\sup_t|u|_{\Omega(t)}$ over $(0, T)$ is finite. We set the definition of weak solutions.

u will be a weak solution of the problem (2.1), (2.2) if one has

i) $u\in U(\hat{\Omega})$;

ii) $\forall\,\varphi\in D(\hat{\Omega})$ with $\varphi(T)=0$

$$\int_0^T\Big\{\Big(u,\frac{\partial\varphi}{\partial t}\Big)_{\Omega(t)}-(u\cdot\nabla u,\varphi)_{\Omega(t)}+\frac{1}{2}\sum_1^2{}_i\Big(u,\varphi\widetilde{\frac{\partial\psi_i}{\partial t}}\Big)_{\Gamma_i(t)}$$
$$+\frac{1}{2}\sum_1^2{}_i(|u|^2\nu_t,\varphi)_{\Gamma_i(t)}+\sum_1^2{}_i(\alpha_i,\varphi)_{\Gamma_i(t)}+(f,\varphi)_{\Omega(t)}-((u,\varphi))_{\Omega(t)}\Big\}dt$$
$$=-(u_0,\varphi(0))_{\Omega(0)}\,. \tag{2.3}$$

(2.3) is obtained from the first of (2.1) multiplying it by a test function φ integrating over $\hat{\Omega}$ and bearing in mind the three conditions of (2.2). The following theorem holds.

THEOREM 1. *We assume*

$$u_0\in H(\Omega(0));\quad f\in L^2(0, T; H(\Omega(t));\quad \alpha_1\in L^2(\hat{\Gamma}_1),\quad \alpha_2\in L^2(\hat{\Gamma}_2).$$

Then there exists a weak solution u of the problem (2.1), (2.2).

§ 3. Proof of the Theorem 1

Let us begin by considering the following approximating problem.

3.1. Auxiliary problem

We look for u such that $\forall\varphi\in(H^1(\hat{\Omega}))^3\cap H(\hat{\Omega})$.

$$\int_0^T \left\{ \frac{1}{m}\left(\frac{\partial u^m}{\partial t}, \frac{\partial \varphi}{\partial t}\right)_{\Omega(t)} + ((u^m, \varphi))_{\Omega(t)} + (u^m \cdot \nabla u^m, \varphi)_{\Omega(t)} - \left(u^m, \frac{\partial \varphi}{\partial t}\right)_{\Omega(t)} \right.$$
$$- \frac{1}{2}\sum_1^2{}_i \left(u^m, \varphi \widetilde{\frac{\partial \psi_i}{\partial t}}\right)_{\Gamma_i(t)} - \frac{1}{2}\sum_1^2{}_i (|u^m|^2 \nu_t, \varphi)_{\Gamma_i(t)} - \sum_1^2{}_i (\alpha_i, \varphi)_{\Gamma_i(t)}$$
$$\left. - (f, \varphi)_{\Omega(t)} \right\} dt = (u_0^m, \varphi(0))_{\Omega(0)} - (u^m(T), \varphi(T))_{\Omega(T)} . \tag{3.11}$$

$$u^m \in (H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})\,; \quad u_0^m \longrightarrow u_0 \quad \text{in} \quad H(\Omega(0)) . \tag{3.12}$$

We set

$$a(u^m, \varphi) = \int_0^T \left\{ \frac{1}{m}\left(\frac{\partial u^m}{\partial t}, \frac{\partial \varphi}{\partial t}\right)_{\Omega(t)} + ((u^m, \varphi))_{\Omega(t)} + (u^m \cdot \nabla u^m, \varphi)_{\Omega(t)} \right.$$
$$\left. - \left(u^m, \frac{\partial \varphi}{\partial t}\right)_{\Omega(t)} - \frac{1}{2}\sum_1^2{}_i (|u^m|^2 \nu_t, \varphi)_{\Gamma_i(t)} - \frac{1}{2}\sum_1^2{}_i \left(u^m, \varphi \widetilde{\frac{\partial \psi_i}{\partial t}}\right)_{\Gamma_i(t)} \right\} dt$$
$$+ (u^m(T), \varphi(T))_{\Omega(T)}\,;$$
$$\langle L, \varphi \rangle = \int_0^T \left\{ (f, \varphi)_{\Omega(t)} + \sum_1^2{}_i (\alpha_i, \varphi)_{\Gamma_i(t)} \right\} dt + (u_0^m, \varphi(0))_{\Omega(0)} .$$

By the following well known theorem (see for example [2] page 106 or [4]) one obtains the existence of a solution in $(H^1(\hat{\Omega}))^3$ of the equation

$$a(u^m, \varphi) = \langle L, \varphi \rangle \tag{3.13}$$

(for convenience we denote different constants by the same symbol).

THEOREM 2. *If*

i) *there exists a constant $c > 0$ such that*

$$a(u^m, u^m) \geq c \| u^m \|^2_{(H^1(\hat{\Omega}))^3}$$

ii) *the form $u^m \to a(u^m, \varphi)$ is weakly continuous in $(H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$ i.e. $u_n^m \longrightarrow u^m$ weakly in $(H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$ implies $\lim_{n\to\infty} a(u_n^m, \varphi) = a(u^m, \varphi)$.*

Then (3.13) *has a solution in $(H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$.*

The condition ii) is obvious. The condition i) can be easily proved; in fact

$$a(u^m, u^m) = \int_0^T \left\{ \frac{1}{m} \left| \frac{\partial u^m}{\partial t} \right|^2_{\Omega(t)} + \| u^m \|^2_{\Omega(t)} \right\} dt + \frac{1}{2} | u^m(T) |^2_{\Omega(T)}$$

$$+\frac{1}{2}|u^m(0)|^2_{\Omega(0)} \geqslant c\|u^m\|^2_{(H^1(\hat{\Omega}))^3}.$$

Then there exists a solution in $(H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$ of (3.13).

To passing to the limit in (3.11) we will need a priori estimates of the approximations u^m.

3.2. Standard a priori estimates

We can replace in (3.11) φ by u^m, it comes

$$\int_0^T \left\{ \frac{1}{m} \left| \frac{\partial u^m}{\partial t} \right|^2_{\Omega(t)} + \|u^m\|^2_{\Omega(t)} - \left(u^m, \frac{\partial u^m}{\partial t}\right)_{\Omega(t)} + (u^m \cdot \nabla u^m, u^m)_{\Omega(t)} \right.$$

$$- \frac{1}{2} \sum_1^2 {}_i (|u^m|^2 \nu_t, u^m)_{\Gamma_i(t)} - \frac{1}{2} \sum_1^2 {}_i \left(u^m, u^m \frac{\partial \widetilde{\psi_i}}{\partial t}\right)_{\Gamma_i(t)}$$

$$\left. - \sum_1^2 {}_i (\alpha_i, u^m)_{\Gamma_i(t)} - (f, u^m)_{\Omega(t)} \right\} dt = |u_0^m|^2_{\Omega(0)} - |u^m(T)|^2_{\Omega(T)}.$$

After some calculations, one has

$$\int_0^T \frac{1}{m} \left| \frac{\partial u^m}{\partial t} \right|^2_{\Omega(t)} dt + \int_0^T \|u^m\|^2_{\Omega(t)} dt + |u^m(T)|^2_{\Omega(T)} + |u^m(0)|^2_{\Omega(0)} \leqslant c$$

hence

$$\frac{1}{m}\int_0^T \left| \frac{\partial u^m}{\partial t} \right|^2_{\Omega(t)} dt \leqslant c\,; \quad \int_0^T \|u^m\|^2_{\Omega(t)}\, dt \leqslant c \tag{3.21}$$

$$|u^m(T)|^2_{\Omega(T)} \leqslant c\,; \qquad |u^m(0)|^2_{\Omega(0)} \leqslant c.$$

It follows

$$\lim_{m\to\infty} u^m \underset{V(\hat{\Omega})}{=} u \qquad \text{in the weak topology.}$$

To passing to the limit in the non linear terms of (3.11) we need the convergence of u^m in a suitable strong topology, for example in $(L^2(\hat{\Omega}))^3$. To do this we will prove appropriate estimates.

3.3. Time difference quotients

We denote by $\bar{u}^m(x, t)$ the extension to R^3 of u^m for every $t \in (0, T)$; moreover we put $\bar{u}^m = 0$ for $t < 0$, $t > T$.
We let

$$u_h^m = \frac{1}{h}\int_{t-h}^{t} \bar{u}^m(x, s)ds \qquad (h>0)\,.$$

We can replace in (3.11) φ by u_h^m and get

$$\int_0^T \frac{1}{m}\left(\frac{\partial u^m}{\partial t},\ \frac{\bar{u}^m(t)-\bar{u}^m(t-h)}{h}\right)_{\Omega(t)} dt - \frac{1}{h}\int_0^T (u^m(t),\ \bar{u}^m(t)-\bar{u}^m(t-h))_{\Omega(t)}dt$$

$$+\int_0^T \Big\{((u^m,\ u_h^m))_{\Omega(t)} + (u^m\cdot\nabla u^m,\ u_h^m)_{\Omega(t)} - \frac{1}{2}\sum_1^2{}_i(|u^m|^2\nu_t,\ u_h^m)_{\Gamma_i(t)}$$

$$-\sum_1^2{}_i(\alpha_i,\ u_h^m)_{\Gamma_i(t)} - \frac{1}{2}\sum_1^2{}_i\left(u^m,\ u_h^m\widetilde{\frac{\partial\psi_i}{\partial t}}\right)_{\Gamma_i(t)} - (f,\ u_h^m)_{\Omega(t)}\Big\}dt$$

$$+(u^m(T),\ u_h^m(T))_{\Omega(T)} = 0\,.$$

By virtue of (3.21), Jensen inequality and the smoothness of $\hat{\Gamma}$ one has

$$\left|\int_0^T \frac{1}{m}\left(\frac{\partial u^m}{\partial t},\ \frac{\bar{u}^m(t)-\bar{u}^m(t-h)}{h}\right)_{\Omega(t)} dt\right|$$

$$\leqslant \frac{1}{m}\int_0^T \left|\frac{\partial u^m}{\partial t}\right|_{\Omega(t)}\cdot\left|\frac{\bar{u}^m(t)-\bar{u}^m(t-h)}{h}\right|_{\Omega(t)} dt \leqslant \frac{c}{m}\int_0^T\left|\frac{\partial u^m}{\partial t}\right|^2_{\Omega(t)} dt \leqslant c\,;$$

$$(3.31)\qquad \left|\int_0^T\left(\left(u^m,\ \frac{1}{h}\int_{t-h}^t \bar{u}^m(x, s)ds\right)\right)_{\Omega(t)} dt\right| \leqslant c\int_0^T \|u^m\|_{\Omega(t)}\left\|\frac{1}{h}\int_{t-h}^t \bar{u}^m(x, s)ds\right\|_{\Omega(t)} dt$$

$$\leqslant c\int_0^T \|u^m\|_{\Omega(t)}\frac{1}{\sqrt{h}}\left(\int_{t-h}^t \|\bar{u}^m(s)\|^2_{R^3}ds\right)^{1/2} dt < c/\sqrt{h}\ ;$$

$$\left|\int_0^T (u^m,\ \nabla u^m,\ u_h^m)_{\Omega(t)}\,dt\right| \leqslant c\int_0^T \|u^m\|_{\Omega(t)}\|u^m\|_{\Omega(t)}\frac{1}{\sqrt{h}}\left(\int_{t-h}^t \|\bar{u}^m\|^2_{R^3}\,ds\right)^{1/2} dt$$

$$\leqslant c/\sqrt{h}\,.$$

Analogously one obtains

$$(3.32)\qquad \begin{aligned} &\left|\sum_1^2{}_i\int_0^T\left(u^m,\ u_h^m\widetilde{\frac{\partial\psi_i}{\partial t}}\right)_{\Gamma_i(t)} dt\right| \leqslant c/\sqrt{h}\ ; && \left|\sum_1^2{}_i\int_0^T(\alpha_i,\ u_h^m)_{\Gamma_i(t)}\,dt\right| < c/\sqrt{h}\ ;\\ &\left|\sum_1^2{}_i\int_0^T(|u^m|^2\nu_t,\ u_h^m)_{\Gamma_i(t)}dt\right| \leqslant c/\sqrt{h}\ ; && \left|\int_0^T(f,\ u_h^m)_{\Omega(t)}\,dt\right| < c/\sqrt{h}\,. \end{aligned}$$

Finally we will estimate

$$-\frac{1}{h}\int_0^T (u^m(t),\ \bar{u}^m(t)-\bar{u}^m(t-h))_{\Omega(t)}dt\,.$$

Bearing in mind the smoothness of $\hat{\Gamma}$ one has

$$-\frac{1}{h}\int_0^T (u^m(t),\, \bar{u}^m(t)-\bar{u}^m(t-h))_{\Omega(t)}dt = -\frac{1}{h}\int_0^T |u^m(t)|^2_{\Omega(t)}\,dt + \frac{1}{2h}\int_0^T |u^m(t)|^2_{\Omega(t)}dt$$

$$+\frac{1}{2h}\int_0^T |\bar{u}^m(t-h)|^2_{\Omega(t)}\,dt - \frac{1}{2h}\int_0^T |u^m(t)-\bar{u}^m(t-h)|^2_{\Omega(t)}dt$$

$$\leqslant -\frac{1}{2h}\int_0^T |u^m(t)|^2_{\Omega(t+h)\cap\Omega(t)}dt - \frac{1}{2h}\int_0^T |u^m(t)|^2_{\Omega(t)\setminus\Omega(t+h)}dt$$

$$+\frac{1}{2h}\int_0^T |\bar{u}^m(t-h)|^2_{\Omega(t)\cap\Omega(t-h)}\,dt + \frac{1}{2h}\int_0^T |\bar{u}^m(t-h)|^2_{\Omega(t)\setminus\Omega(t-h)}dt$$

$$-\frac{1}{2h}\int_h^T |\bar{u}^m(t)-\bar{u}^m(t-h)|^2_{\Omega(t)}dt$$

$$\leqslant -\frac{1}{2h}\int_0^{T-h} |u^m(t)|^2_{\Omega(t)\cap\Omega(t+h)}dt$$

$$+\frac{1}{2h}\int_0^{T-h} |u^m(t)|^2_{\Omega(t+h)\cap\Omega(t)}dt + \frac{c}{\sqrt{h}} - \frac{1}{2h}\int_h^T |\bar{u}^m(t)-\bar{u}^m(t-h)|^2_{\Omega(t)}dt$$

$$\leqslant \frac{c}{\sqrt{h}} - \frac{1}{2h}\int_h^T |\bar{u}^m(t)-\bar{u}^m(t-h)|^2_{\Omega(t)}dt\,.$$

From (3.31), (3.32), (3.33) we conclude

$$\int_h^T |\bar{u}^m(t)-\bar{u}^m(t-h)|^2_{\Omega(t)}dt \leqslant c\sqrt{h}\,. \tag{3.34}$$

By the classical characterization of M. Riesz and A. Kolmogorov of compact sets in $L^2(\hat{\Omega})$ (see [8]) we can prove the set $\{u^m\}$ of u^m satisfying (3.21), (3.34) is relatively compact in $L^2(\hat{\Omega})$. From (3.21) and the relatively compactness of $\{u^m\}$ in $L^2(\hat{\Omega})$ we can choose a subsequence again denoted by u^m such that

$$\lim_{m\to\infty}\int_0^T (u^m\cdot\nabla u^m,\, \varphi)_{\Omega(t)}dt = \int_0^T (u\cdot\nabla u,\, \varphi)_{\Omega(t)}dt \qquad \forall\varphi\in D(\hat{\Omega}).$$

Now

$$\frac{1}{2}\sum_1^2{}_i\int_0^T (|u^m|^2\nu_t,\, \varphi)_{\Gamma_i(t)}dt = \int_0^T (\varphi\cdot\nabla u^m,\, u^m)_{\Omega(t)}dt.$$

By virtue of the strong convergence of u^m in $L^2(\hat{\Omega})$ and (3.21) we have

$$\lim_{m\to\infty}\int_0^T (\varphi\cdot\nabla u^m,\, u^m)_{\Omega(t)}dt = \int_0^T (\varphi\cdot\nabla u,\, u)_{\Omega(t)}dt \qquad \forall\varphi\in D(\hat{\Omega})\,,$$

hence

$$\lim_{m\to\infty}\frac{1}{2}\sum_1^2{}_i\int_0^T(|u^m|^2\nu_t,\ \varphi)_{\Gamma_i(t)}dt=\frac{1}{2}\sum_1^2{}_i\int_0^T(|u|^2\nu_t,\ \varphi)_{\Gamma_i(t)}dt.$$

Now passing to the limit $m\to\infty$ in (3.11) we obtain (2.3). Finally we prove

$$\sup_t\cdot|u|_{\Omega(t)}\leqslant c.$$

We let

$$u_{\bar{t}}^m(t)=\begin{cases}u^m(t) & 0\leqslant t\leqslant\bar{t}\\ 0 & t>\bar{t}.\end{cases}$$

We replace in (3.11) φ by $u_{\bar{t}}^m(t)$ and after some calculations obtain

$$|u^m(\bar{t})|_{\Omega(\bar{t})}^2\leqslant\frac{1}{m}\int_0^{\bar{t}}\left|\frac{\partial u^m}{\partial t}\right|_{\Omega(t)}^2dt+\frac{1}{m}\left(\frac{\partial u^m(\bar{t})}{\partial t},\ u^m(\bar{t})\right)_{\Omega(\bar{t})}+\frac{1}{2}|u^m(0)|_{\Omega(0)}^2$$
$$+c\int_0^{\bar{t}}\|u^m\|_{\Omega(t)}^2dt.$$

By virtue of (3.21) and the compactness of $\{u^m\}$ in $L^2(\hat{\Omega})$ one obtains

$$\lim_{m\to\infty}\left|\frac{1}{m}\left(\frac{\partial u^m(\bar{t})}{\partial t},\ u^m(\bar{t})\right)_{\Omega(\bar{t})}\right|\leqslant\lim_{m\to\infty}\frac{1}{\sqrt{m}}\left|\frac{\partial u^m(\bar{t})}{\partial t}\right|_{\Omega(\bar{t})}\frac{1}{\sqrt{m}}|u^m(\bar{t})|_{\Omega(\bar{t})}=0$$

hence

$$|u(\bar{t})|_{\Omega(\bar{t})}\leqslant c.$$

The proof is completed.

References

[1] Fujita, H. and N. Sauer, On existence of weak solutions of the Navier-Stokes equations in regions with moving boundary, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **17** (1970), 403–420.

[2] Giraut, V. and P. A. Raviart, Finite element approximation of the Navier-Stokes, Lecture Notes in Math. vol. 749, Springer Verlag, Berlin, 1979.

[3] Inoue, A. and M. Wakimoto, On existence of solutions of the Navier-Stokes equations in a time dependent domain, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), 303–319.

[4] Leray, J. and J.L. Lions, Quelques resultats de Visik sur les problems elliptiques non lineares par les methodes de Minty-Browder, Bull. Soc. Math. France **93** (1965), 97–107.

[5] Morimoto, H., On existence of periodic weak solutions of the Navier-Stokes equations in regions with periodically moving boundaries, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **18** (1971), 499–524.

[6] Muller, M. and J. Naumann, On evolution inequalities of a modified Navier-

Stokes type, I–II, Apl. Mat. **23** (1978), 174–184 and 397–407.
[7] Myakawa, T. and Y. Teramoto, Existence and periodicity of weak solutions of the Navier-Stokes equations in a time dependent domain, Hiroshima Math. J. **12** (1982), 513–528.
[8] Necas, J., Les méthodes directes en théorie des équations elliptiques, Editions de l'Academie Tchecoslovaque des Sciences, Prague, 1967.
[9] Otani, M. and Y. Yamada, On the Navier-Stokes equations in a non-cylindrical domains: An approach by subdifferential operator theory, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **25** (1978), 185–204.
[10] Prouse, G., On a unilateral problem for the Navier-Stokes equations, I, II, Atti Accad. Naz. Lincei Rend. **52** (1972), 337–342, 467–478.
[11] Salvi, R., Qualche problema unilatero per i fluidi di Bingham, Istit. Lombardo Accad. Sci. Lett. Rend. A **115** (1981), 45–60.

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