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# *On the global existence of real analytic solutions of linear partial differential equations on unbounded domain*

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## § 0. Introduction

In this article we extend Kawai's method [12] giving a global real analytic solution of the equation $P(D)u=f$ with constant coefficients to the case of unbounded domains.

As is well known, the complicated structure of the topological vector space $\mathscr{A}(\Omega)$ of real analytic functions on an open set $\Omega\subset\boldsymbol{R}^n$ long prevented the study of the surjectivity of the operator $P(D):\mathscr{A}(\Omega)\to\mathscr{A}(\Omega)$ by the duality method even for a convex $\Omega$, and it is rather recently that this difficulty has been overcome by the introduction of some additional techniques, especially of Hörmander's Phragmén-Lindelöf type principle [5] (see the survey article by Cattabriga [3] for further references on this line).

On the other hand, Kawai's idea, which we retake here, is to abandon the topological vector space structure and to give the renaissance to the classical representation formula for the solutions by way of the fundamental solution: For illustration let $P(D)$ be elliptic, and take $f(x)\in\mathscr{A}(\Omega)$ on a *bounded* $\Omega$. Then we can choose an extension $\tilde{f}$ of $f$ as a hyperfunction with support in $\overline{\Omega}$. Let $E(x)$ be a fundamental solution of $P(D)$. Then

$$u=\tilde{f}*E \tag{0.1}$$

will satisfy $P(D)u=\tilde{f}$ on $\boldsymbol{R}^n$, hence $P(D)u|_\Omega=f$ on $\Omega$. Besides, the analytic singularity of $\tilde{f}$, which is limited in $\partial\Omega$ by the construction, will remain there even after the convolution by $E$ by virtue of the regularity of the latter. Thus $u|_\Omega$ will be a required global real analytic solution. By the introduction of the consideration of S.S. (singular spectrum) in place of the mere analytic singular support, this argument can be refined to give a global existence theorem for locally hyperbolic operators on bounded domains with some geometric condition.

Here we try to consider an *unbounded* domain $\Omega$. Let again $P(D)$ be elliptic, and take $f\in\mathscr{A}(\Omega)$. This time we can extend $f$ to an exponentially decreasing Fourier hyperfunction $\tilde{f}$, that is, we can find a set of defining functions $\{F_j(z)\}_{j=1}^N$ for $f$:

$$f(x)=\sum_{j=1}^{N} F_j(x+i\Gamma_j 0) \tag{0.2}$$

such that $F_j(z)=O(e^{-\delta|\mathrm{Re}\,z|})$ locally uniformly in $\mathrm{Im}\,z$ on some wedge $\boldsymbol{R}^n+i\Gamma_j$. The exponential type $\delta$ can be arbitrarily preassigned. Thus if we choose $\delta$ greater than the exponential type of the growth order of the fundamental solution $E(x)$, then we can again calculate the convolution (0.1). Notice that the integration for the hyperfunctions is performed on a path deformed into the complex domain and the decay condition for $F_j(z)$ in (0.2) is assured only on such a path: We may choose $F_j(z)$ analytically prolongeable to the real points where $f(x)$ itself is real analytic, but we cannot assert the decay of $F_j(x)$ on the real axis.

In order to refine this argument we need to introduce some generalized classes of Fourier hyperfunctions and study their S.S. at infinity. We list up in § 1 the definitions and the necessary properties of these new classes of Fourier hyperfunctions. The outline of the justification of the whole theory of Fourier hyperfunctions and the proofs to the extended part are given in the Appendix. Employing them we give in § 2 a global existence theorem for locally hyperbolic operators which extends Kawai's main result to the case of unbounded domains. In § 3 we also refine the method of Kawai, and improve his result even for bounded domains. Namely, for a locally hyperbolic operator $P(D)$, we show $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$ under "pointwise" geometric condition about the local propagation cones at the boundary points of $\Omega$, whereas Kawai assumed it "uniformly" in some sense (see Theorem 3.5).

A short announcement of the main results in § 2 has been given in [10]. We extract here the schematic description of the principle of micro-localization of the given "global" problem:

(0.3) $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$.

$\Uparrow$ localization

(0.4) $P(D)\{u\in\mathscr{B}(\boldsymbol{R}^n);\ u|_\Omega\in\mathscr{A}(\Omega)\}\supset\{f\in\mathscr{B}[\bar{\Omega}];\ f|_\Omega\in\mathscr{A}(\Omega)\}$.

$\Uparrow$ micro-localization

(0.5) For any $\xi\in\boldsymbol{S}^{n-1}$ there exists a neighborhood $\Delta\ni\xi$ such that

$$P(D)\{u\in\mathscr{B}(\boldsymbol{R}^n);\ u|_\Omega\in\mathscr{A}(\Omega)\}+\mathscr{A}(\boldsymbol{R}^n)\supset\{f\in\mathscr{B}(\boldsymbol{R}^n);\ \mathrm{S.S.}\,f\subset\partial\Omega\times\Delta\}.$$

$\Uparrow$ stratification

(0.6) There exists a stratification $S^{n-1}=\Xi_0\sqcup\cdots\sqcup\Xi_m$ such that

$$P(D)\{u\in\mathscr{B}(\boldsymbol{R}^n);\ u|_\Omega\in\mathscr{A}(\Omega)\}+\{u\in\mathscr{B}(\boldsymbol{R}^n);\ \mathrm{S.S.}\,f\subset\partial\Omega\times(\Xi_{k+1}\cap\Delta)\}$$
$$\supset\{f\in\mathscr{B}(\boldsymbol{R}^n);\ \mathrm{S.S.}\,f\subset\partial\Omega\times(\Xi_k\cap\Delta)\}\,.$$

The final step has been newly introduced in this article (see § 3).

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The author dedicates this article to Prof. S. L. Sobolev because he was to talk about this subject at the symposium held in honour of Prof. Sobolev in October 1983 but could not attend it by an accident.

## § 1. Fourier hyperfunctions with growth or decay of type $[s, \delta]$.

As in the classical theory of Fourier hyperfunctions (see Kawai [11]), we introduce the directional compactification of $\boldsymbol{R}^n$ by the points at infinity:

$$\boldsymbol{D}^n=\boldsymbol{R}^n\sqcup\boldsymbol{S}^{n-1}_\infty\,,$$

and its complex neighborhood $\boldsymbol{D}^n+i\boldsymbol{R}^n$. Recall that by the definition of the topology of $\boldsymbol{D}^n$ a (real) neighborhood of a point $a\infty\in\boldsymbol{D}^n$ at infinity contains a truncated part of a cone $\Gamma$ containing the direction $a$ and the corresponding points at infinity:

$$(\Gamma\cap\{|x|>R\})\sqcup\{x\infty\,;\,x\in\Gamma\setminus\{0\}\}\,.$$

(A complex neighborhood of $a\infty$ contains the direct product of such a set with $i\{|y|<\varepsilon\}$ for some $\varepsilon>0$.)

Definition 1.1. Let $s>0$, $\delta\in R$ be constants. For an open set $U\subset \boldsymbol{D}^n+i\boldsymbol{R}^n$ we put

$$\text{(1.1)}\quad \tilde{\mathcal{O}}^{s,\delta}(U)=\{F(z)\in\mathcal{O}(U\cap\boldsymbol{C}^n);\ {}^\forall K\Subset U,\ {}^\forall\varepsilon>0,\ \sup_{z\in K\cap\boldsymbol{C}^n}|F(z)|e^{-(\delta+\varepsilon)|\mathrm{Re}\,z|^s}<+\infty\},$$

and let $\tilde{\mathcal{O}}^{s,\delta}$ denote the corresponding sheaf on $\boldsymbol{D}^n+i\boldsymbol{R}^n$. (Here $K\Subset U$ implies that the closure of $K$ in $\boldsymbol{D}^n$, which is compact but not necessarily in $\boldsymbol{C}^n$, is contained in $U$. Note that the growth (or decay if $\delta<0$) condition is meaningful only if $U$ contains a point at infinity, hence $\tilde{\mathcal{O}}^{s,\delta}|_{\boldsymbol{C}^n}=\mathcal{O}$.)

It is clear that (1.1) itself is the section module of $\tilde{\mathcal{O}}^{s,\delta}$ on $U$. Concerning this sheaf we prepare an analogue of the Oka-Cartan cohomology vanishing theorem.

DEFINITION 1.2. We say that an open set $U\subset \boldsymbol{D}^n+i\boldsymbol{R}^n$ is *pseudo-convex* if there exists a continuous plurisubharmonic function $\varphi(z)$ on $U\cap \boldsymbol{C}^n$ satisfying the following properties:

1) For any $K\Subset U$, $\varphi(z)$ is bounded on $K\cap \boldsymbol{C}^n$;
2) For any $c>0$ the set $\{z\in U\cap \boldsymbol{C}^n;\ \varphi(z)<c\}$ is relatively compact in $U$.

THEOREM 1.3. *Let $U\subset \boldsymbol{D}^n+i\boldsymbol{R}^n$ be pseudo-convex in the sense of the above definition. Assume further that $U\cap \boldsymbol{C}^n$ is contained in a set of the form*

$$C_s(C_1, C_2):=\{z\in \boldsymbol{C}^n;\ |\mathrm{Im}\, z|\leq C_1|\mathrm{Re}\, z|+C_2\} \tag{1.2}$$

*with some positive constants $C_1<\min\{1, \tan^{-1}(\pi/4s)\}$ and $C_2$. Then we have*

$$H^p(U, \tilde{\mathcal{O}}^{s,\delta})=0 \qquad \text{for } p\geq 1\,. \tag{1.3}$$

REMARK. In this article it suffices to consider the open set $U$ whose trace to $\boldsymbol{C}^n$ is contained in $C_s(0, C_2)=\{|\mathrm{Im}\, z|\leq C_2\}$. We need the above general case when we treat the *modified* Fourier hyperfunctions.

We can calculate these cohomology groups by way of a $\bar{\partial}$-complex of section modules of some fine sheaves with the same growth or decay property as above. Similarly we can translate the Martineau-Harvey duality and the Malgrange theorem to our case:

THEOREM 1.4. *Let $K\subset \boldsymbol{D}^n+i\boldsymbol{R}^n$ be a compact set admitting a fundamental system of neighborhoods of open sets as described in the above theorem. Then for a neighborhood $U$ of $K$ we have*

$$H^p_K(U, \tilde{\mathcal{O}}^{s,\delta})=0 \qquad \text{for } p\neq n\,, \tag{1.4}$$

$$H^n_K(U, \tilde{\mathcal{O}}^{s,\delta})=\mathcal{Q}^{s,\delta}(K)'\,. \tag{1.5}$$

*Here $\mathcal{Q}^{s,\delta}$ denotes the sheaf on $\boldsymbol{D}^n+i\boldsymbol{R}^n$ defined by the following section modules:*

$$\mathcal{Q}^{s,\delta}(U)=\{F(z)\in \mathcal{O}(U\cap \boldsymbol{C}^n);\ \forall K\Subset U, \exists\varepsilon>0 \text{ such that } \sup_{z\in K\cap \boldsymbol{C}^n}|F(z)|e^{(\delta+\varepsilon)|\mathrm{Re}\, z|^s}<+\infty\}\,. \tag{1.6}$$

*The duality* (1.5) *holds in the sense of an (FS)-(DFS) space pair by the naturally induced topologies.*

THEOREM 1.5. *Let $U\subset \boldsymbol{D}^n+i\boldsymbol{R}^n$ be any open set such that $U\cap \boldsymbol{C}^n$ is con-*

*tained in* $C_s$ *of* (1.2). *Then we have*

$$H^n(U, \tilde{\mathcal{O}}^{s,\delta})=0 .$$

For the proof of these theorems see the Appendix. Thus by the standard argument for the foundation of the theory of hyperfunctions (see the sketch given in the Appendix; for more in detail see e.g. Komatsu [13] or Kaneko [9] for expository references), we can deduce from these results the following

THEOREM AND DEFINITION 1.6. $\boldsymbol{D}^n$ *becomes a purely n-codimensional subset of* $\boldsymbol{D}^n+i\boldsymbol{R}^n$ *with respect to* $\tilde{\mathcal{O}}^{s,\delta}$ *(that is,* $\mathcal{H}^p_{\boldsymbol{D}^n}(\tilde{\mathcal{O}}^{s,\delta})=0$ *for* $p\neq n$*). We put* $\mathcal{Q}^{s,\delta}=\mathcal{H}^n_{\boldsymbol{D}^n}(\tilde{\mathcal{O}}^{s,\delta})$ *and call it the sheaf of Fourier hyperfunctions of the growth (or decay) type* $O(e^{\delta+|x|^s})$, *or "type* $[s, \delta]$*" imitating the notation of entire function theory. It is a flabby sheaf on* $\boldsymbol{D}^n$.

Because $\tilde{\mathcal{O}}^{s,\delta}|_{\boldsymbol{C}^n}=\mathcal{O}$, $\mathcal{Q}^{s,\delta}|_{\boldsymbol{R}^n}$ agrees with the sheaf $\mathcal{B}$ of usual hyperfunctions. Thus the flabbiness of $\mathcal{Q}^{s,\delta}$ implies in particular the following

COROLLARY 1.7. *The natural restriction mapping from* $\mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$ *to* $\mathcal{B}(\boldsymbol{R}^n)$ *(or more generally to* $\mathcal{B}(\Omega)$ *for any open subset* $\Omega\subset\boldsymbol{R}^n$*) is surjective. That is, an element* $f\in\mathcal{B}(\boldsymbol{R}^n)$ *can always be extended to an element* $\tilde{f}\in\mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$.

Recall that by the general property of flabby sheaves we can always choose the extension such that $\operatorname{supp}\tilde{f}\subset\overline{\overline{\operatorname{supp} f}}$. (Here and in the sequel the double bar denotes the closure in $\boldsymbol{D}^n$ of a subset of $\boldsymbol{R}^n$ in order to be distinguished from the usual closure in $\boldsymbol{R}^n$, denoted by the simple bar.)

REMARK. The classical Fourier hyperfunction introduced by M. Sato and discussed by T. Kawai [11] corresponds to $s=1$, $\delta=0$. The case $s=1$ and $\delta<0$ is called rapidly decreasing Fourier hyperfunctions and effectively employed by the author (e.g. in [8]) after M. Sato's suggestion. The case $s=1$, $\delta>0$ is contained in the theory of Fourier ultra-hyperfunctions of Park-Morimoto [17]; Up to this growth type the Fourier image can be interpreted by their theory. For the growth type with $s>1$, $\delta\geq 0$ we have not yet an established theory of Fourier transformation. In the sequel we will employ them only in relation with the convolution operation.

For later use we need the representation of the space $\mathcal{Q}^{s,\delta}(\Omega)=H^n_\Omega(\boldsymbol{C}^n, \tilde{\mathcal{O}}^{s,\delta})$ for $\Omega\subset\boldsymbol{D}^n$ by boundary values. First we give an intuitive explanation for the reader not familiar with the theory of hyperfunctions: $f(x)\in \boldsymbol{Q}^{s,\delta}(\Omega)$ means that $f(x)$ has a boundary value expression

$$f(x)=\sum_{j=1}^N F_j(x+i\Gamma_j 0) . \tag{1.7}$$

Here $\Gamma_j$ is an open convex cone with vertex at 0 and $F_j(z)\in\tilde{\mathcal{O}}^{s,\delta}(\Omega+i\Gamma_j0)$, that is, a section of $\tilde{\mathcal{O}}^{s,\delta}$ on an infinitesimal wedge of the type $\Omega+i\Gamma_j0$: Note that by an infinitesimal wedge of the type $\Omega+i\Gamma_j0$ we mean an open subset $U\subset\Omega+i\Gamma_j$ of $\boldsymbol{D}^n+i\boldsymbol{R}^n$ such that for any $K\Subset\Omega$ and any proper subcone $\Delta_j\Subset\Gamma_j$ there exists $\varepsilon>0$ such that

$$U\supset(K+i\Delta_j)\cap(\boldsymbol{D}^n+i\{|y|<\varepsilon\});$$

By a proper subcone $\Delta_j\Subset\Gamma_j$ we mean another open convex cone $\Delta_j$ with vertex at 0 such that $\Delta_j\cap\{|y|=1\}\Subset\Gamma_j\cap\{|y|=1\}$. Note that such $U$ conserves a fixed "breadth" in the variables $y=\operatorname{Im} z$ up to the points at infinity contained in $\Omega$. This is one of the main differences between Fourier hyperfunctions and ordinary hyperfunctions: If we forget the growth (decay) condition of $F_j(z)$ in (1.7), then we obtain an ordinary hyperfunction on $\Omega\cap\boldsymbol{R}^n$ which is, by definition, equal to the restriction $f(x)|_{\Omega\cap\boldsymbol{R}^n}$. Note however, that in a boundary value expression of an ordinary hyperfunction on $\Omega\cap\boldsymbol{R}^n$, the defining functions $F_j(z)$ may grow arbitrarily when $|\operatorname{Re} z|\to\infty$ and their domains of definition may diminish the "breadth" in the variables $\operatorname{Im} z$ as $|\operatorname{Re} z|\to\infty$. Thus the extendibility in Corollary 1.6 means that we can improve the boundary value expression to make satisfy these two additional properties.

We should well notice the rather surprising fact that any $f\in\mathcal{B}(\boldsymbol{R}^n)$ (e.g. $e^x\in\mathcal{B}(\boldsymbol{R})$) can be extended to a "rapidly" decreasing $\tilde{f}$ (e.g. $e^x=F_+(x+i0)-F_-(x-i0)$ with exponentially decreasing $F_\pm(z)$). Note however that the defining functions may decrease only along the level surface $\operatorname{Im} z=\text{const.}\neq0$; Even if all the $F_j(z)$, hence $f$, may be continued up to the real axis, they may have an arbitrarily given large growth there.

Now we give a precise mathematical interpretation to the boundary value representation (1.7). We put

$$\tilde{\mathcal{O}}^{s,\delta}(\Omega+i\Gamma_0)=\varinjlim_{U}\tilde{\mathcal{O}}^{s,\delta}(U)\,,$$

where $U$ runs all the infinitesimal wedges of the type $\Omega+i\Gamma0$. Then we introduce the direct sum of $\boldsymbol{C}$-linear spaces

$$X=\bigoplus_{\Gamma}\tilde{\mathcal{O}}^{s,\delta}(\Omega+i\Gamma0)\,,$$

where $\Gamma$ runs all the open convex cones (including half spaces or even $\boldsymbol{R}^n$). As usual we naturally identify an element $F(z)\in\tilde{\mathcal{O}}^{s,\delta}(\Omega+i\Gamma0)$ with the corresponding one in $X$. Then we consider the $\boldsymbol{C}$-linear subspace $Y$ of $X$ generated by all the elements of the form

$$F_1(z)+F_2(z)-F_3(z); \quad F_j(z)\in\tilde{\mathcal{O}}^{s,\delta}(\Omega+i\Gamma_j0) \quad (1\leq j\leq 3)$$

such that $\Gamma_1\cap\Gamma_2\supset\Gamma_3$ and that $F_1(z)+F_2(z)=F_3(z)$ holds on a common domain of definition. Finally we let

$$\mathcal{Q}^{s,\delta}(\Omega)=X/Y$$

be our definition of the space of Fourier hyperfunctions of the type $[s,\delta]$ on $\Omega$ by way of the boundary value representations. We denote by $F(x+i\Gamma0)$ the element of $\mathcal{Q}^{s,\delta}(\Omega)$ defined by $F(z)\in X$.

LEMMA 1.8. *We have $X/Y\cong H^n_\Omega(\boldsymbol{D}^n+i\boldsymbol{R}^n,\tilde{\mathcal{O}}^{s,\delta})$. The isomorphism h is given as follows: Given $F(x+i\Gamma0)\in X$ we choose $\eta^0,\eta^1,\cdots,\eta^n\in\boldsymbol{R}^n$ such that, putting $E^{\eta^j}=\{y\in\boldsymbol{R}^n;\langle y,\eta^j\rangle>0\}$ we have*

$$\bigcup_{j=0}^n E^{\eta^j}=\boldsymbol{R}^n\setminus\{0\}, \qquad \bigcap_{j=1}^n E^{\eta^j}\Subset\Gamma,$$

*and that $\eta^1,\cdots,\eta^n$ have positive orientation. Then we choose a complex neighborhood U of $\Omega$, put $U_j=U\cap(\Omega+iE^{\eta^j})$, $j=0,\cdots,n$, and finally put*

$$h(F(x+i\Gamma0))=F(z)U\wedge U_1\wedge\cdots\wedge U_n,$$

*where the right hand side represents the cohomology class of $H^n_\Omega(U,\tilde{\mathcal{O}}^{s,\delta})$ defined by $F(z)\in\tilde{\mathcal{O}}^{s,\delta}(U\cap U_1\cap\cdots\cap U_n)$ via the n-cocycle of the relative covering $(\{U,U_0,\cdots,U_n\},\{U_0,\cdots,U_n\})$ of the pair $(U,U\setminus\Omega)$. The inverse mapping of h is given as follows: Choose $\eta^0,\eta^1,\cdots,\eta^n\in\boldsymbol{R}^n$ satisfying $\bigcup_{j=0}^n E^{\eta^j}=\boldsymbol{R}^n\setminus\{0\}$. Choose a neighborhood U of $\Omega$ which is pseudo-convex in the sense of Definition* 1.2 *and put $U_j$ as above. Then we let an n-cocycle*

$$\sum_{j=0}^n F_j(z)U\wedge U_0\wedge\cdots\wedge\hat{U}_j\wedge\cdots\wedge U_n$$

*correspond to*

$$\sum_{j=0}^n F_j(x+i\Gamma_j0), \quad \textit{where} \quad \Gamma_j=\bigcap_{k\neq j}E^{\eta^k}.$$

The proof of this lemma can be literally obtained from the corresponding assertion for ordinary hyperfunctions (see e.g. Kaneko [9bis], Theorem 7.1.7).

The sheaf of real analytic functions in our sense should be by definition $\mathcal{P}^{s,\delta}:=\tilde{\mathcal{O}}^{s,\delta}|_{\boldsymbol{D}^n}$. We have

THEOREM 1.9. *We have the natural injective sheaf homomorphism $\mathcal{P}^{s,\delta}\hookrightarrow\mathcal{Q}^{s,\delta}$.*

By the intuitive expression, $f(x)\in\mathscr{P}^{s,\delta}(\Omega)$ corresponds to the element $f(x+i\Gamma 0)\in\mathscr{Q}^{s,\delta}(\Omega)$ with any $\Gamma$. It is clear that the imbedding is compatible with the ordinary one $\mathscr{A}\hookrightarrow\mathscr{B}$ on $\boldsymbol{R}^n$. The injectivity follows also from that of the latter.

THEOREM 1.10. *We have* $H^1(\Omega,\mathscr{P}^{s,\delta})=0$ *for any open subset* $\Omega\subset\boldsymbol{D}^n$. *Hence the quotient sheaf* $\mathscr{Q}^{s,\delta}/\mathscr{P}^{s,\delta}$ *is flabby and we have* $(\mathscr{Q}^{s,\delta}/\mathscr{P}^{s,\delta})(\Omega)=\mathscr{Q}^{s,\delta}(\Omega)/\mathscr{P}^{s,\delta}(\Omega)$.

This theorem follows from Theorem 1.3 and the variant of the Grauert theorem on the construction of a fundamental system of pseudoconvex neighborhoods of a real open set (see Kawai [11], Theorem 2.1.6). From theorem 1.10 by the routine argument follows the possibility of decomposition of the singular support:

COROLLARY 1.11. *Let* $f(x)\in\mathscr{Q}^{s,\delta}(\Omega)$ *and assume that* sing supp $f$ *is covered by a finite number of closed subsets* $X^j\subset\Omega$, $j=1,\cdots,N$. *Then we can find* $f^j(x)\in\mathscr{Q}^{s,\delta}(\Omega)$, $j=1,\cdots,N$ *such that*

$$f=f_1+\cdots+f_N;\qquad \operatorname{sing\,supp} f_j\subset X^j\,.$$

Here sing supp $f$ naturally means the support of $f \bmod \mathscr{P}^{s,\delta}$ as a section of the quotient sheaf $\mathscr{Q}^{s,\delta}/\mathscr{P}^{s,\delta}$ on $\Omega$.

Next we extend the notion of micro-analyticity to these classes of Fourier hyperfunctions.

DEFINITION 1.12. We say that $f(x)\in\mathscr{Q}^{s,\delta}(\Omega)$ is *micro-analytic* at $(a,\xi)\in\Omega\times\boldsymbol{S}^{n-1}$ if we can choose a local boundary value expression (1.7) of $f$ on a neighborhood of $a$ such that $\Gamma_j\cap\{\langle y,\xi\rangle<0\}\neq\emptyset$ for all $j$. The subset of $\Omega\times\boldsymbol{S}^{n-1}$ of the points where $f$ is not micro-analytic is called the *singular spectrum* (S.S.) of $f$ and denoted by $\mathrm{S.S.}^{s,\delta}f$ (or simply by S.S. $f$ if there may be no confusion).

$\mathrm{S.S.}^{s,\delta}f$ is by definition a closed subset of $\Omega\times\boldsymbol{S}^{n-1}$. Note that if $a\in\boldsymbol{R}^n$, the growth condition on $F_j(z)$ becomes void and the above definition is compatible with the usual notion of micro-analyticity or S.S. for ordinary hyperfunctions. As usual we write also $(a, i\xi dx\infty)$ in place of $(a,\xi)$ in order to specify the character of the vector $\xi$. If, however, $a$ is at infinity, the available local coordinate transformations are not so ample.

Note also that S.S. depends on the growth type considered: Already the "real analyticity" at infinity contained the growth condition. Consider for example the "Fourier hyperfunction" $e^x$ of one variable. For the choice $s>1$, $\delta\geq 0$ or $s=1$, $\delta\geq 1$ (especially as a Fourier ultra-hyperfunction) it is naturally considered as an element of $\mathscr{Q}^{s,\delta}(D)$ and even of $\mathscr{P}^{s,\delta}(D)$ with the unique defining function $e^z$. Thus it contains no S.S. For the other classes (and especially as a classical Fourier hyperfunction), however, it requires a suitable interpretation (regularization as a generalized function) at $+\infty$, and whatever this interpretation may be, its boundary value expression (or singular spectral decomposition) at $+\infty$ requires two terms $F_{\pm}(z)$ and the S.S. contains the two points $\{(+\infty, \pm idx\infty)\}$.

Though the definition of S.S. employs a local boundary value expression, we can always obtain a global boundary value expression (or singular spectral decomposition) corresponding to the decomposition of S.S. The following is a little weakened variant of the assertion of this type:

THEOREM 1.13. *Let $f$ be a section of $\mathscr{Q}^{s,\delta}$ defined on a neighborhood of a compact set $K\subset \boldsymbol{D}^n$. Assume that*

$$\text{S.S.}^{s,\delta}f\subset \boldsymbol{D}^n\times i\,\mathrm{Int}(\Gamma_1^\circ\cup\cdots\cup\Gamma_N^\circ)dx\infty\,. \tag{1.8}$$

*Then we can find $F_j(z)\in\tilde{\mathcal{O}}^{s,\delta}(K+i\Gamma_j0)$ such that*

$$f(x)=\sum_{j=1}^N F_j(x+i\Gamma_j0)$$

*and that each $F_j(z)$ can be extended as a section of $\tilde{\mathcal{O}}^{s,\delta}$ to a neighborhood of every point $x$ for which*

$$\text{S.S.}^{s,\delta}f\cap\{x\}\times i\Gamma_j^\circ dx\infty=\varnothing \tag{1.9}$$

*(hence a fortiori if $f$ is real analytic at $x$). Here*

$$\Gamma_j^\circ=\{\xi;\,\langle y,\xi\rangle\geq 0 \textit{ for } {}^\forall y\in\Gamma_j\}$$

*denotes the dual cone of $\Gamma_j$ (and also represents its intersection with the unit sphere)*; Int *denotes the interior, and $F_j(z)\in\tilde{\mathcal{O}}^{s,\delta}(K+i\Gamma_j0)$ implies that $F_j(z)\in\tilde{\mathcal{O}}^{s,\delta}(\Omega_j+i\Gamma_j0)$ for some neighborhood $\Omega_j\supset K$.*

This theorem can be deduced by means of the following variant of the curvilinear Radon decomposition of the delta function:

$$\begin{aligned}&\delta(x)=\int_{S^{n-1}}W_s(x,\omega)d\omega;\\ &W_s(x,\omega)=\frac{(n-1)!}{(-2\pi i)^n}\,\frac{(1-ix\omega)^{n-1}-(1-ix\omega)^{n-2}(x^2-(x\omega)^2)}{(x\omega+i(x^2-(x\omega)^2)+i0)^n}\,e^{1-(1+x^2)^s}\,.\end{aligned} \tag{1.10}$$

This is simply the product of the usual decomposition formula (a particular case of Example 1.2.5 of Chapter III in S-K-K [18]) by the damping factor

$e^{1-(1+x^2)^s}$ which takes the value 1 at 0. The integral for $\omega$ converges by the (FS)-topology of $\mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$ in the variable $x$. Further, for an open convex cone $\Delta$ we put

$$W_s(z, \Delta^\circ)=\int_{\Delta^\circ \cap S^{n-1}} W_s(z, \omega)d\omega . \tag{1.11}$$

This becomes an element of $\tilde{\mathcal{O}}^{s,-\delta}(\boldsymbol{D}^n+i\Delta 0)$ for any $\delta>0$, hence its boundary value defines an element of $\mathscr{Q}^{s,-\delta}(\boldsymbol{D}^n)$ which will be denoted by $W_s(x, \Delta^\circ)$. For $f \in \mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$ we can define the "convolution" $f * W_s(x, \Delta^\circ) \in \mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$ directly by way of the defining function as the boundary value of $\langle f(x), W_s(z-x, \Delta^\circ)\rangle_x \in \tilde{\mathcal{O}}^{s,\delta}(\boldsymbol{D}^n+i\Delta 0)$, where $\langle\ ,\ \rangle$ denotes the inner product of the duality (1.5). Choosing for $\Delta^\circ$ the components of a decomposition of $\boldsymbol{S}^{n-1}$ by closed convex pyramids we can thus obtain the corresponding singular spectral decomposition of $f$. Further, we can show that this "convolution" can be calculated locally by way of the local boundary value expression of $f$ at least modulo the real analytic ambiguity (mod $\mathscr{P}^{s,\delta}$), hence follows the additional regularity property for $F_j(z)$ in Theorem 1.13. (For further details see the Appendix.)

Employing the boundary value expression given by Theorem 1.13, we can define the convolution operation between two general Fourier hyperfunctions satisfying suitable growth and decay conditions:

DEFINITION AND LEMMA 1.14. *Let $f(x) \in \mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$, $g(x) \in \mathscr{Q}^{s,-\delta'}(\boldsymbol{D}^n)$ and assume that $\delta'>\delta\geq 0$. Then, choosing boundary value expression $f(x)=\sum_{j=1}^{M} F_j(x+i\Gamma_j 0)$, $g(x)=\sum_{k=1}^{N} G_k(x+i\Delta_k 0)$ for each, we can define $f*g$ by*

$$(f*g)(x)=\sum_{j,k}\left[\int_{\operatorname{Im}\zeta=\eta_k} F_j(z-\zeta)G_k(\zeta)d\zeta\right]_{z\to x+i(\Gamma_j+\Delta_k)0} . \tag{1.12}$$

*Here $\eta_k \in \Delta_k$ is such that $G_k(\zeta)$ is defined on $\operatorname{Im}\zeta=\eta_k$. The result becomes an element of $\mathscr{Q}^{s,\delta''}(\boldsymbol{D}^n)$, where*

$$\begin{aligned} &\delta''=\delta && \text{if } s\leq 1 ,\\ &\delta''=\frac{\delta\delta'}{(\delta'^{1/(s-1)}-\delta^{1/(s-1)})^{s-1}} && \text{if } s>1 . \end{aligned} \tag{1.13}$$

*It does not depend on the choice of the paths of integration nor the boundary value representations employed.*

The integral in the bracket converges for $z \in \boldsymbol{D}^n+i\eta_k+i\Gamma_j 0$ in the ordinary sense and does not depend on the choice of $\eta_k$. Hence by the analytic continuation it becomes an element of $\tilde{\mathcal{O}}^{s,\delta''}(\boldsymbol{D}^n+i(\Gamma_j+i\Delta_k)0)$. The verification of the growth condition follows from the following elementary calculation:

$$(1.14)\quad \begin{aligned}(\delta+\varepsilon)|z-w|^s-(\delta'-\varepsilon)|w|^s &\leq -\varepsilon|w|^s+\frac{(\delta+\varepsilon)(\delta'-2\varepsilon)}{((\delta'-2\varepsilon)^{1/(s-1)}-(\delta+\varepsilon)^{1/(s-1)})^{s-1}}|z|^s\\ &\leq -\varepsilon|w|^s+\left[\frac{\delta\delta'}{(\delta'^{1/(s-1)}-\delta^{1/(s-1)})^{s-1}}+O(\varepsilon)\right]|z|^s.\end{aligned}$$

Note that the constant $\delta''$ for $s>1$ also approaches $\delta$ as $\delta'\to\infty$. The independence of the total result $f*g$ from the boundary value expressions follows immediately from the following weakened variant of the edge of the wedge theorem of Martineau's type (see the Appendix for the proof):

LEMMA 1.15. *Assume that*

$$\sum_{j=1}^{N} F_j(x+i\Gamma_j 0)=0$$

*as an element of $\mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$. Then for any choice of $\Delta_j\Subset\Gamma_j$ we can find sections $H_{jk}(z)\in\tilde{\mathcal{O}}^{s,\delta}(\boldsymbol{D}^n+i(\Delta_j+\Delta_k)0)$, anti-symmetric in $j$, $k=1,\cdots,N$, such that*

$$(1.15)\qquad F_j(z)=\sum_{k=1}^{N}H_{jk}(z)\quad in\ \tilde{\mathcal{O}}^{s,\delta}(\boldsymbol{D}^n+i\Delta_j 0),\quad j=1,\cdots,N.$$

The S.S. of the result of the convolution can be estimated as follows:

THEOREM 1.16. *Let $f\in\mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$, $g\in\mathcal{Q}^{s,-\delta'}(\boldsymbol{D}^n)$ be as above. Then we have*

$$(1.16)\quad \mathrm{S.S.}^{s,\delta''}f*g\subset\{(x+y,\xi)\in\boldsymbol{D}^n\times\boldsymbol{S}^{n-1};\,(x,\xi)\in\mathrm{S.S.}^{s,\delta}f,\,(y,\xi)\in\mathrm{S.S.}^{s,-\delta'}g\}.$$

*Here for $x, y\in\boldsymbol{D}^n$ the sum $x+y$ represents the following:*

1) *$x+y$ in the usual sense if $x, y\in\boldsymbol{R}^n$;*
2) *$x\infty+y=x\infty$ if $y\in\boldsymbol{R}^n$;*
3) *$x\infty+y\infty=\{(tx+(1-t)y)\infty;\ 0\leq t\leq 1\}$ if $x$ and $-y$ are not of the same direction;*
4) *$x\infty+(-x)\infty=\boldsymbol{D}^n$.*

To prove this theorem we can apply the elementary proof employing the defining functions of the usual estimation formula for S.S. $f*g$ for a pair of hyperfunctions one of which has compact support (see e.g. Kaneko [9], pp. 72–73). The outline is the following: Every term of the right hand side of (1.12) obviously has the S.S. in $\boldsymbol{D}^n\times i(\Gamma_j+\Delta_k)^\circ dx\infty=\boldsymbol{D}^n\times i(\Gamma_j^\circ\cap\Delta_k^\circ)dx\infty$. Moreover, if we employ there as the boundary value representations of $f$, $g$ those supplied by Theorem 1.13, then $F_j(z)$ resp. $G_k(z)$ can also be extended to a neighborhood of $x$ as a section of $\mathscr{P}^{s,\delta}$ resp. $\mathscr{P}^{s,-\delta'}$ if $\mathrm{S.S.}^{s,\delta}f\cap\{x\}+i\Gamma_j^\circ dx\infty=\emptyset$ resp. $\mathrm{S.S.}^{s,-\delta'}g\cap\{x\}+i\Delta_k^\circ dx\infty=\emptyset$. Let us then deform the path of this integral beyond the real axis to the imaginary direction $-\Gamma_j$ on a neighborhood of such points as the latter condition holds. Let $x\in \boldsymbol{D}^n$ be a point which satisfy $\mathrm{S.S.}^{s,\delta}f\cap\{x-w\}+i\Gamma_j^\circ dx\infty=\emptyset$ every time if $\mathrm{S.S.}^{s,-\delta'}g\cap\{w\}+i\Delta_k^\circ dx\infty\neq\emptyset$ for each fixed $w\in \boldsymbol{D}^n$. Then if $w$ runs along the above deformed path the integrand $F_j(z-w)G_k(w)$ becomes holomorphic on a complex neighborhood of $x$ and there satisfies uniformly an estimate like (1.14). Thus we see that

$$\begin{aligned}\mathrm{S.S.}^{s,\delta''}F_j(x+i\Gamma_j0)*G_k(x+i\Delta_k0)\subset\{(x+y,\ i\xi dx\infty);\ \xi\in\Gamma_j^\circ\cap\Delta_k^\circ,\\ \mathrm{S.S.}^{s,\delta}f\cap\{x\}+i\Gamma_j^\circ dx\infty\neq\emptyset,\quad \mathrm{S.S.}^{s,-\delta'}g\cap\{y\}+i\Delta_k^\circ dx\infty\neq\emptyset\}\,.\end{aligned}$$

Refining the singular spectral decomposition we finally obtain the above estimate (1.16).

Finally we introduce the sheaf of Fourier microfunctions on $\boldsymbol{D}^n\times \boldsymbol{S}^{n-1}$ associated with our class.

DEFINITION 1.17. We define the sheaf $\mathscr{R}^{s,\delta}$ of *Fourier microfunctions* of the growth (decay) type $[s,\delta]$ on $\boldsymbol{D}^n\times \boldsymbol{S}^{n-1}$ as the one associated with the presheaf

$$\boldsymbol{D}^n\times \boldsymbol{S}^{n-1}\supset\Omega\times\Delta\to\mathscr{Q}^{s,\delta}(\Omega)/\{f\in\mathscr{Q}^{s,\delta}(\Omega);\ \mathrm{S.S.}^{s,\delta}f\cap\Omega\times\Delta=\emptyset\}\,. \tag{1.17}$$

We can easily verify that the section modules (1.15) give rise to a presheaf on $\boldsymbol{D}^n\times \boldsymbol{S}^{n-1}$ which satisfies the axiom (F I) of sheaves (that is, the localizability of the uniqueness of a section). However, the axiom (F II) (that is, the localizability of the construction of a section) is far less obvious. Anyway we can prove the following

THEOREM 1.18. *$\mathscr{R}^{s,\delta}$ becomes a flabby sheaf. If $\Delta$ is connected, then* (1.17) *gives the section module $\mathscr{R}^{s,\delta}(\Omega\times\Delta)$ itself.*

From the flabbiness of $\mathscr{R}^{s,\delta}$ we can obtain the following valuable consequence:

COROLLARY 1.19. *Let $f(x)\in\mathscr{Q}^{s,\delta}(\Omega)$ and suppose that $\mathrm{S.S.}^{s,\delta}f\subset\bigcup_{j=1}^N X^j$, where $X^j$ are closed subsets of $\Omega\times \boldsymbol{S}^{n-1}$. Then we can find $f_j\in\mathscr{Q}^{s,\delta}(\Omega)$ such that*

$$f=f_1+\cdots+f_N;\quad \mathrm{S.S.}^{s,\delta}f_j\subset X^j\quad (j=1,\cdots,N)\,.$$

Combining this decomposition with Theorem 1.13 (which is already accurate in itself if $N=1$ because of the uniqueness of the single defining function), we can strengthen the latter as follows:

COROLLARY 1.20. *Let* $f(x)\in\mathscr{Q}^{s,\delta}(\Omega)$ *and suppose that*

$$\text{S.S.}^{s,\delta}f\subset\Omega\times i(\Gamma_1^\circ\cup\cdots\cup\Gamma_N^\circ)dx\infty\,. \tag{1.18}$$

*Then we can find* $F_j(z)\in\tilde{\mathscr{O}}^{s,\delta}(\Omega+i\Gamma_j0)$ *such that*

$$f(x)=\sum_{j=1}^{N}F_j(x+i\Gamma_j0)$$

*and that each* $F_j(z)$ *satisfies the same regularity condition as described in Theorem* 1.13.[1]

The proof of Theorem 1.18 requires a good preparation on the theory of Fourier hyperfunctions with hyperfunction parameters $\omega$. Since we do not need the exact flabbiness of $\mathscr{R}^{s,\delta}$ at infinity for the application in this article, we send it to a forthcoming paper and adopt instead the following substitute which means the flabbiness of $\mathscr{C}=\mathscr{R}^{s,\delta}|_{\boldsymbol{R}^n\times\boldsymbol{S}^{n-1}}$ and the softness of $\Gamma_{\boldsymbol{S}_\infty^{n-1}\times\boldsymbol{S}^{n-1}}(\mathscr{R}^{s,\delta})$.

PROPOSITION 1.21. *Let* $f(x)\in\mathscr{Q}^{s,\delta}(\Omega)$ *and suppose that* $\text{S.S.}^{s,\delta}f\subset\bigcup_{j=1}^N X^j$, *where* $X^j$ *are closed subsets of* $\Omega\times\boldsymbol{S}^{n-1}$. *Then for any neighborhood* $W^j$ *of* $X^j\cap(\boldsymbol{S}_\infty^{n-1}\times\boldsymbol{S}^{n-1})$ *in* $\boldsymbol{S}_\infty^{n-1}\times\boldsymbol{S}^{n-1}$ *we can find* $f_j\in\mathscr{Q}^{s,\delta}(\Omega)$ *such that*

$$f=f_1+\cdots+f_N;\qquad \text{S.S.}^{s,\delta}f_j\subset(X^j\cap(\boldsymbol{R}^n\times\boldsymbol{S}^{n-1}))\sqcup W^j\,.$$

This proposition can be proved rather easily by means of the known flabbiness of the sheaf $\mathscr{C}$ of ordinary microfunctions and the Radon decomposition (1.10). We do it in detail in the Appendix.

REMARK. The notion of S.S. of Fourier hyperfunctions at infinity was first introduced by Kaneko [7]. There an analogue for Sato's fundamental theorem on the micro-analyticity of solutions to a noncharacteristic direction is discussed. On the other hand, Lieutenant [15] considered S.S. at infinity for the sheaf extension of $\mathscr{B}$ to $\boldsymbol{D}^n$ without growth condition. It corresponds to consider in the boundary value expression (1.7) only the uniformity of the "breadth" for the domain of definition of $F_j(z)$ up to the points at infinity of $\Omega$, and abandon the growth condition on $F_j(z)$. Here the growth condition is important for us because our main purpose is to give an interpretation to the classical representation formula (0.1) for

1) This assertion can be proved in a way as elementary as Theorem 1.13 by virtue of de Roever's subtle argument [25] without relying on Corollary 1.19.

solutions. It is possible, however, that the discussion without growth condition may be more adapted to the original problem of the global existence of real analytic solutions. In fact, as the study of Hörmander [5] reveals, the essential ponit of this problem lies in what extent we can enlarge the complex neighborhood (to each direction $i\xi dx\infty$). Cf. also Lieutenant [14] in this respect. The study in this line will require a fairly different tool (e.g. the semiglobal existence of holomorphic solutions in an infinitesimal wedge for pseudo-differential operators with constant coefficients obtained as the factors of $P(D)$).

## § 2. Global existence of real analytic solutions on unbounded domains for locally hyperbolic operators.

Now we enter into the main subject. To illustrate our basic idea we first give an abstract type theorem on the global existence of real analytic solutions which is extracted from the argument of Kawai [12] and adapted to the case of unbounded domains.

THEOREM 2.1. *Assume that $P(D)$ admits a set of fundamental solutions $E^j(x) \in \mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$, $j=1, \cdots, N$ for some $s$, $\delta$ satisfying the following estimate:*

$$\text{S.S.}^{s,\delta}E^j(x) \subset \{0\} \times \boldsymbol{S}^{n-1} \cup \bigcup_{\xi \in N(P_m) \cap \boldsymbol{S}^{n-1}} \overline{\overline{K^j_\xi}} \times \{\xi\}\,, \tag{2.1}$$

*where $P_m$ is the principal part of $P$, $N(P_m)=\{\xi \in \boldsymbol{R}^n;\ P_m(\xi)=0\}$, $K^j_\xi \subset \boldsymbol{R}^n$ is a closed cone with vertex at 0 depending on $\xi$ in an upper semi-continuous way (see Remark below) and $\overline{\overline{K^j_\xi}}$ denotes its closure in $\boldsymbol{D}^n$. Let $\Omega \subset \boldsymbol{R}^n$ be a connected open set and let $\bar{\partial}\Omega$ denote its boundary in $\boldsymbol{D}^n$ (that is, there contained also the accumulation points of $\Omega$ at infinity). Assume that we have the covering by closed subsets (or "decomposition" not necessarily disjoint to each other as more fitted to the intuition):*

$$\bar{\partial}\Omega \times \boldsymbol{S}^{n-1} = \bigcup_{j=1}^{N} X^j \tag{2.2}$$

*such that*

$$(a, \xi) \in X^j \text{ implies either } P_m(\xi) \neq 0 \text{ or } (\{a\} + \overline{\overline{K^j_\xi}}) \cap \Omega = \varnothing\,. \tag{2.3}$$

(Here the meaning of the vector sum in $\boldsymbol{D}^n$ is as in Theorem 1.16.) Then we have $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$.

REMARK. If we employ the convention that $K^j_\xi=\{0\}$ for $\xi$ satisfying $P_m(\xi)\neq 0$ (which represents the fact that $P(D)$ is microlocally elliptic there and is well concordant with the notation), then the condition (2.3) can be simplified as follows:

(2.3)′ $$(a,\xi)\in X^j \text{ implies } (\{a\}+K_\xi^j)\cap\Omega=\emptyset\ .$$

In the sequel we shall mainly employ this simplified form for our geometric condition.

Proof of Theorem 2.1. Take $f(x)\in\mathscr{A}(\Omega)$. Choose an extension $\tilde f\in\mathscr{Q}^{s,-\delta'}(\boldsymbol{D}^n)$ of $f$ with support in $\overline{\overline{\Omega}}$ and with $\delta'>\delta$. In view of Corollary 1.19 (see Remark below) we can decompose $\tilde f$ in a way

(2.4) $$\tilde f=\sum_{j=1}^N \tilde f^j$$

such that $\mathrm{S.S.}^{s,-\delta'}\tilde f^j\subset X^j$. Put

(2.5) $$u=\sum_{j=1}^N \tilde f^j * E^j\ .$$

By virtue of the assumed estimate for S.S. $E^j$ and the assumption on $X^j$, we can easily see by Theorem 1.16 that $\tilde f^j * E^j$ is real analytic in $\Omega$ for each $j$. Thus $u|_\Omega$ is a required real analytic solution of $P(D)u=f$ in $\Omega$. q.e.d.

Remark. If we employ Proposition 1.21 instead of Corollary 1.19, we employ the additional condition of upper semi-continuity of $\xi\mapsto K_\xi^j$; Its exact meaning is as follows:

(2.6) *For each fixed $\xi^0\in\boldsymbol{S}^{n-1}$, given any $\varepsilon>0$ there exists $\delta>0$ such that if $|\xi-\xi^0|<\delta$, $K_\xi^j\cap\{|x|=1\}$ is contained in the $\varepsilon$-neighborhood of $K_{\xi^0}^j\cap\{|x|=1\}$.*

In fact in this case we can only utilize a decomposition (2.4) with

$$\mathrm{S.S.}^{s,-\delta'}\tilde f^j\subset(X^j\cap(\boldsymbol{R}^n\times\boldsymbol{S}^{n-1}))\sqcup W^j\ ,$$

where $W^j$ is any given neighborhood of $X^j\cap(\boldsymbol{S}_\infty^{n-1}\times\boldsymbol{S}^{n-1})$ in $\boldsymbol{S}_\infty^{n-1}\times\boldsymbol{S}^{n-1}$. Note, however, that for a point $a\infty$ at infinity the set $\{a\infty\}+\overline{\overline{K_\xi^j}}$ can intersect $\Omega\subset\boldsymbol{R}^n$ if and only if $a\in -K_\xi^j$. Thus the condition $(\{a\infty\}+\overline{\overline{K_\xi^j}})\cap\Omega=\emptyset$ is stable by a small perturbation of $a\infty\in\boldsymbol{S}_\infty^{n-1}$ or the cone $K_\xi^j$. Therefore under the above condition (2.6) we can conclude as well that $\tilde f^j * E^j$ is real analytic in $\Omega$. The condition (2.6) is satisfied by all the applications below (and in fact it will be no essential restriction in view of the closedness of the set S.S. $E^j$).

Below in the main theorem in the concrete application for locally hyperbolic operators, we have simply $N=2$ and then $K_\xi^1$, $K_\xi^2$ are simply written $\pm K_\xi$ respectively. Nevertheless in the above we have adopted a little generalized formulation with the aim to refine the existence theorem for operators with reducible localization (see e.g. Example 2.4 below). In this line we can further localize with respect to $\xi$ the assumption of the existence of such a set of fundamental solutions in the following way:

THEOREM 2.1′. *Assume that for every point $\xi^0 \in \boldsymbol{S}^{n-1}$ there exists its neighborhood $\Delta$ in $\boldsymbol{S}^{n-1}$ and a set of micro-local fundamental solutions $E^{\Delta,j}(x) \in \mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$ $j=1, \cdots, N_\Delta$ for some $s$, $\delta$ such that*

1) *$P(D)E^{\Delta,j}(x)-\delta(x)$ is micro-analytic in $\boldsymbol{D}^n \times \Delta$;*
2) $\mathrm{S.S.}^{s,\delta}E^{\Delta,j}(x) \subset \{0\}\times\bar{\Delta} \cup \boldsymbol{D}^n \times \partial\Delta \cup \bigcup_{\xi\in\Delta} \overline{\overline{K_\xi^{\Delta,j}}} \times \{\xi\}$,

*where $K_\xi^{\Delta,j} \subset \boldsymbol{R}^n$ is a closed cone with vertex at 0 depending on $\xi \in \Delta$ in an upper semi-continuous way. Let $\Omega \subset \boldsymbol{R}^n$ be as above and assume that for every $\Delta$ mentioned above there exists a decomposition (covering) by relatively closed subsets:*

$$\bar{\bar{\partial}}\Omega \times \Delta = \bigcup_{j=1}^{N_\Delta} X^{\Delta,j} \tag{2.7}$$

*such that*

$$(a, \xi) \in X^{\Delta,j} \text{ implies } (\{a\}+\overline{\overline{K_\xi^{\Delta,j}}}) \cap \Omega = \emptyset . \tag{2.8}$$

*Then $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$.*

To prove this variant we first decompose $\tilde{f}$ into a sum of the form

$$\tilde{f} = \sum_{\Delta\in\mathscr{D}} \tilde{f}^\Delta; \qquad \mathrm{S.S.}^{s,-\delta'}\tilde{f}^\Delta \Subset \mathrm{S.S.}^{s,-\delta'}\tilde{f} \cap \boldsymbol{D}^n \times i\Delta dx\infty ,$$

where $\Delta$ runs in a finite covering $\mathscr{D}$ of $\boldsymbol{S}^{n-1}$ consisting of small neighborhoods described in the assumption of the theorem. (Such a decomposition can be obtained as follows: Since $\mathscr{D}$ is a finite covering of the compact Hausdorff space $\boldsymbol{S}^{n-1}$, for each $\Delta \in \mathscr{D}$ we can choose $\Delta' \Subset \Delta$ such that $\{\bar{\Delta}'\}$ is a closed covering of $S^{n-1}$ without common interior points. Then we put

$$\tilde{f}^\Delta = \tilde{f} * \underset{\sim}{W}_s(x, \bar{\Delta}') ,$$

where $\underset{\sim}{W}_s(x, \bar{\Delta}')$ is defined by (1.11) with $\bar{\Delta}'$ in place of $\Delta^\circ$. The above estimate for S.S. follows from Lemma 1.14.) Next we decompose each $\tilde{f}^\Delta$ to a sum $\sum_{j=1}^{N}\tilde{f}^{\Delta,j}$ according to the decomposition (2.7), and take

$$u = \sum_{\Delta\in\mathscr{D}} \sum_{j=1}^{N_\Delta} \tilde{f}^{\Delta,j} * E^{\Delta,j} \tag{2.9}$$

instead of (2.5). Hereafter the proof is the same except that $P(D)u-\tilde{f}$ is not precisely equal to 0 but becomes an element of $\mathscr{P}^{s,\delta''}(\boldsymbol{D}^n)$. Recall that an element of $\mathscr{P}^{s,\delta''}(\boldsymbol{D}^n)$ is holomorphic at least in a band $|\mathrm{Im}\, z|<\varepsilon$ for some $\varepsilon>0$. Therefore we can apply the global existence theorem of holomorphic solutions on a convex complex domain (see e.g. Malgrange [16]), and find a real analytic function $v\in\mathscr{A}(\boldsymbol{R}^n)$ such that $P(D)v=(P(D)u-\tilde{f})|_{\boldsymbol{R}^n}$. Thus finally $(u-v)|_\Omega$ is a desired solution.

Remark that the same argument implies also the following

LEMMA 2.2. *Assume that for every point $\xi^0\in\boldsymbol{S}^{n-1}$ there exists its neighborhood $\Delta$ in $\boldsymbol{S}^{n-1}$ and a set of micro-local fundamental solutions $E^{\Delta,j}(x)$ as in Theorem 2.1′ but with $N_\Delta=N$ and $K_\xi^{\Delta,j}=K_\xi^j$ independent of $\Delta$. Then we can construct a set of fundamental solutions $E^j$, $j=1,\cdots,N$ as in Theorem 2.1.*

In fact, this time we solve the equation

$$P(D)E^j=\delta(x)$$

employing the micro-local fundamental solutions $E^{\Delta,j}$ as in the above argument: Choose a decomposition $\mathscr{L}$ of $\boldsymbol{S}^{n-1}$ by closed pyramids such that each $L\in\mathscr{L}$ is contained in some $\Delta=\Delta_L$ as in the assumption of the lemma. Then put

$$E^j=\sum_{L\in\mathscr{L}}\underset{\sim}{W}_s(x,L)*E^{\Delta_L,j}\,. \tag{2.10}$$

Then with the common $K_\xi^j=K_\xi^{\Delta_L,j}$, $E^j$ satisfies (2.1). The modification to a true fundamental solution is just as above.

The assumption that $N_\Delta=N$ or $K_\xi^{\Delta,j}=K_\xi^j$ are independent of $\Delta$ is not so restrictive in practice. In fact in general we can take $N=\max_\Delta N_\Delta$, and for $\Delta$ such that $N_\Delta<N$ we can employ some of $K_\xi^{\Delta,j}$ repeatedly provided that we can make a global upper semi-continuous choice among $K_\xi^{\Delta,j}$'s for different $\Delta$. However, the last assumption is not a fortiori obvious even if we limit our consideration to concrete locally hyperbolic operators as we discuss it below. Therefore the way of presentation of Theorem 2.1′ may be substantially more powerful than that of Theorem 2.1 though it is less elegant.

As another application of the above technique of patching microlocal objects we can construct a fundamental solution with complicated singular support as in Andersson [1]. In fact, for any choice of $j_L$ $(1\le j_L\le N_{\Delta_L})$

$$E^j=\sum_{L\in\mathscr{L}}\underset{\sim}{W}_s(x,L)*E^{\Delta_L,j_L} \tag{2.11}$$

becomes a fundamental solution of $P(D)$ (modulo an element of $\mathscr{P}^{s,\delta}(\boldsymbol{D}^n)$

which can be cancelled as above). The S.S. of this $E^j$ has an estimate similar to (2.1) with $K_\xi^{\Delta_L, j_L}$ instead of $K_\xi^j$, except for those directions $\xi \in \partial L$ where it may contain $K_\xi^{\Delta_{L'}, j_{L'}} \times \{\xi\}$ for the other elements $L' \in \mathscr{L}$ of which $\xi$ is also in the boundary.

Now we consider a typical concrete class of operators which admit a set of fundamental solutions as described in the above abstract theorems.

DEFINITION 2.3 (Cf. Andersson [1]). We say that an operator $P(D)$ is *locally hyperbolic* at $\xi^0 \in \boldsymbol{S}^{n-1}$ (with the vector $v$) if there exists a neighborhood $\Delta \ni \xi^0$ in $\boldsymbol{S}^{n-1}$, a vector $v \neq 0$ and $\varepsilon_0 > 0$ such that

$$\xi \in \Delta,\ 0<|t|<\varepsilon_0 \textit{ implies } P_m(\xi+itv) \neq 0\,. \tag{2.12}$$

We say that $P(D)$ is locally hyperbolic if it is locally hyperbolic at every $\xi^0 \in \boldsymbol{S}^{n-1}$.

We list up here some properties of locally hyperbolic operators. For the details see e.g. the cited article of Andersson. Let $(P_m)_\xi(\eta)$ denote the localization of $P_m(\xi)$ at $\xi \in N(P_m)$. It is by definition the first non-zero coefficient of the Taylor expansion

$$P_m(\xi+t\eta) = (P_m)_\xi(\eta) t^{\alpha(\xi)} + o(t^{\alpha(\xi)})\,. \tag{2.13}$$

It is easily seen that the local hyperbolicity of $P(D)$ at $\xi$ with the vector $v$ implies the hyperbolicity of the localization $(P_m)_\xi(\eta)$ to the direction $v$. (But the converse is not necessarily true; see Example 3.1.) Further, the vector $v$ in the above definition can move in the "normal cone" of the hyperbolic operator $(P_m)_\xi(\eta)$ in the same connected component.[2] Its dual cone, that is, the propagation cone of $(P_m)_\xi(\eta)$ corresponding to the direction $v$, is denoted by $K_\xi$ and is called the local propagation cone of the original operator $P(D)$. $K_\xi$ is a closed convex proper cone depending on $\xi$. Since in these arguments we can reverse the sign of $v$ etc., $-K_\xi$ is also a local propagation cone of $P(D)$. Further, for those $\xi$ where the localization is the product of several hyperbolic operators, we have in general more variety of choice for $K_\xi$ by taking various combination (the vector sum) among those for the irreducible factors. (However these choices do not always give rise to local propagation cones of the original operator. See Example 3.1.) Anyway, we can locally fix the choice of $K_\xi$ so that the correspondence $\xi \mapsto K_\xi$ is locally upper semi-continuous. This is obvious from the lower semi-continuity of the open

2) Hence in defining a locally hyperbolic operator we can always assume in addition that $P_m(v) \neq 0$ as is done in Andersson [1]. However, it is more convenient for us not to require this condition as is seen below.

cone in which $v$ can move satisfying the condition (2.12) (with various $\Delta$ and $\varepsilon_0$). We do not know, however, if we can globally fix such a choice of $K_\xi$. (This is equivalent to know if there exists a continuous non-vanishing $n$-vector field $\xi \mapsto v(\xi)$ on $N(P_m)$ satisfying (2.12). This question is concerned with the orientability of $N(P_m) \subset S^{n-1}$. In fact, $K_\xi$ is nothing but the "normals" of $N(P_m) \subset S^{n-1}$ in $S^{n-1}$ at $\xi$ in the sense that it is the dual cone of the "tangent cone" of $N(P_m) \subset S^{n-1}$ at $\xi$. Thus it is very plausible that by taking a polynomial approximation of the defining function of the imbedding of Klein's bottle with self intersection, we can give a counter-example (though yet we cannot execute this complicated calculus). For generic $\xi$ satisfying $\mathrm{grad}_\xi P_m(\xi) \neq 0$ (that is, for simply characteristic direction $\xi$), we have the natural choice for $K_\xi$ by putting $K_\xi = R^+ \mathrm{grad}_\xi P_m(\xi)$. Here we content ourselves by giving an example showing that this choice is not always consistent at the multiply characteristic points to give a coherent choice for $K_\xi$.

*Example* 2.4. Consider the following example of locally hyperbolic operator:

$$P(D) = P_m(D) = D_1^3 + D_2(D_3^2 + D_4^2 - D_5^2)\,. \tag{2.14}$$

We have

$$\mathrm{grad}_\xi P_m(\xi) = (3\xi_1^2,\ \xi_3^2 + \xi_4^2 - \xi_5^2,\ 2\xi_2\xi_3,\ 2\xi_2\xi_4,\ -2\xi_2\xi_5)\,,$$

hence the localization of $P_m$ is linear except for the two following cases:

1) If $\xi_1 = \xi_2 = 0$, $\xi_3^2 + \xi_4^2 - \xi_5^2 = 0$ but $(\xi_3, \xi_4, \xi_5) \neq (0, 0, 0)$ then

$$(P_m)_\xi(\eta) = 2\eta_2(\xi_3\eta_3 + \xi_4\eta_4 - \xi_5\eta_5)\,.$$

2) If $\xi_1 = \xi_3 = \xi_4 = \xi_5 = 0$ but $\xi_2 \neq 0$ then $(P_m)_\xi(\eta) = \xi_2(\eta_3^2 + \eta_4^2 - \eta_5^2)$.

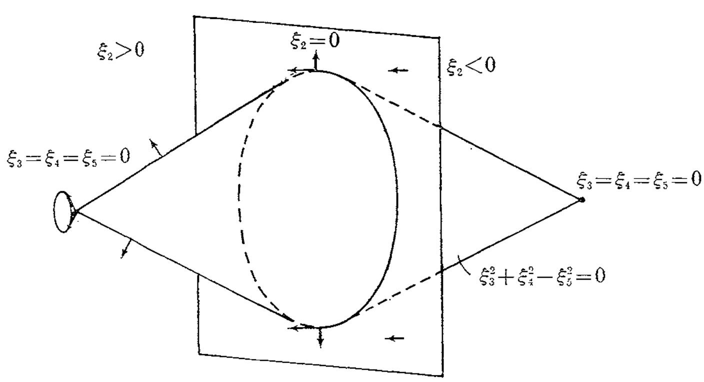


Figure 1. A symbolic figure of $N(P_m) \cap S^{n-1}$ in Example 2.4.

Therefore if we wish $K_\xi$ to be upper semi-continuous in $\xi$, then the choice $K_\xi = \boldsymbol{R}^+\mathrm{grad}_\xi P_m(\xi)$ for $\xi_2>0$ necessarily imposes the choice $K_\xi = -\boldsymbol{R}^+\mathrm{grad}_\xi P_m(\xi)$ for $\xi_2<0$. (On the contrary the choice of the direction of the propagation cone for the factor $\eta_2$ in case 1) is arbitrary, and we have four possible choices of upper semi-continuous correspondence $\xi \mapsto K_\xi$.) Anyway it is true that in this example we can extend this choice to a global upper semi-continuous correspondence $\xi \mapsto K_\xi$.

Now we show

THEOREM 2.5. *Let $P(D)$ be locally hyperbolic in the sense of Definition 2.3 and assume that on $\Delta \subset \boldsymbol{S}^{n-1}$ there exists an upper semi-continuous correspondence $\xi \mapsto K_\xi$ of local propagation cones $K_\xi$. Then we can construct a micro-local fundamental solution $E(x) \in \mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$ with some $s$, $\delta$ such that*

1) *$P(D)E(x)-\delta(x)$ is micro-analytic in $\boldsymbol{D}^n \times \Delta$;*
2) $\mathrm{S.S.}^{s,\delta}E(x) \subset \{0\} \times \bar{\Delta} \cup \boldsymbol{D}^n \times \partial\Delta \cup \bigcup_{\xi \in N(P_m) \cap \Delta} \overline{\overline{K_\xi}} \times \{\xi\}$.

COROLLARY 2.6. *A locally hyperbolic operator always possesses a set of micro-local fundamental solutions $E^{\Delta,j}(x) \in \mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$, $j=1, \cdots, N_\Delta$ as described in Theorem 2.1′, where $K_\xi^{\Delta,j}$, $j=1, \cdots, N_\Delta$ denote all possible upper semi-continuous correspondences of local propagation cones on $\Delta$. (Hence, with $K_\xi^{\Delta,j}$ there exists always $-K_\xi^{\Delta,j}$ among them and for general $\xi$ the set $\{K_\xi^{\Delta,j}\}$ reduces to $\pm K_\xi$.)*

COROLLARY 2.7. *Let $P(D)$ be locally hyperbolic and assume that there exists a global upper semi-continuous correspondence $\xi \mapsto K_\xi$ of local propagation cones on $\boldsymbol{S}^{n-1}$. Then we can construct a pair of fundamental solutions $E^\pm(x) \in \mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$ satisfying*

$$\mathrm{S.S.}^{s,\delta}E^\pm(x) \subset \{0\} \times \boldsymbol{S}^{n-1} \cup \bigcup_{\xi \in N(P_m) \cap S^{n-1}} \pm\overline{\overline{K_\xi}} \times \{\xi\}. \tag{2.15}$$

Kawai proves in [12] the above Corollary 2.7 employing the Neumann series as a perturbation from the case of homogeneous operator $P_m(D)$ to which such a "good" fundamental solution can be calculated directly by the method of plane wave decomposition. Since he considers only bounded domains, he says nothing about the growth condition of $E^\pm(x)$ as $|x|\to\infty$. However, his calculus for the convergence of the Neumann series contains the accurate estimate even in $|x|$ as it tends to $\infty$. Since his method can obviously be micro-localized, the proof of Theorem 2.5, hence of Corollary 2.6, is practically established by him. (It seems however that he did not perceive that a micro-local fundamental solutions suffice to give the same result on the global existence of real analytic solutions.) Here we give another proof based on the inverse Fourier transformation in view of the self containedness and of the eventual possibility to extend the result for more general class of operators.

PROOF OF THEOREM 2.5. On account of the argument employed in Lemma 2.2, to prove these assertions it suffices to construct a micro-local fundamental solution $E(x)$ assuming that $\Delta$ is an arbitrarily small neighborhood of a fixed point $\xi^0 \in \boldsymbol{S}^{n-1}$ (See Remark below.). Let $v$ be any vector satisfying (2.12). Shrinking $\Delta$ and diminishing $\varepsilon_0$ if necessary, we can assume that (2.12) holds even for $\xi$ in $\bar{\Delta}$ and for $0<|t|\leq\varepsilon_0$. Thus

$$|P_m(\xi-itv)|\geq ct^{\mu} \qquad \text{if } \xi\in\bar{\Delta},\ 0<t\leq\varepsilon_0$$

with some constants $c$, $\mu$ ($\mu$ being the maximum of the multiplicity of the real zeros of $P_m(\xi)$ in $\bar{\Delta}$). Thus we have

$$|P_m(\xi-itv)|\geq ct^{\mu}|\xi|^{m-\mu} \qquad \text{if } \xi/|\xi|\in\bar{\Delta},\ 0<t\leq\varepsilon_0|\xi|,$$

hence

$$|P_m(\xi-itv)|\geq K^{\mu}c|\xi|^{m-1} \qquad \text{if } \xi/|\xi|\in\bar{\Delta},\ K|\xi|^{1-1/\mu}\leq t\leq\varepsilon_0|\xi|.$$

Since $P(\xi)$ differs from $P_m(\xi)$ only by lower order terms, for sufficiently large $K$ we can find $R>0$ such that

$$|P(\xi-itv)|\geq\frac{K^{\mu}c}{2}|\xi|^{m-1} \qquad \text{if } \xi/|\xi|\in\bar{\Delta},\ |\xi|\geq R,\ K|\xi|^{1-1/\mu}\leq t\leq\varepsilon_0|\xi|. \tag{2.16}$$

Now introduce the following $n$-chain in $\boldsymbol{C}^n$:

$$D_R=\{\xi-iK|\xi|^{1-1/\mu}v;\ \xi/|\xi|\in\bar{\Delta},\ |\xi|\geq R\}, \tag{2.17}$$

and put

$$E(x)=\frac{1}{(2\pi)^n}\int_{D_R}\frac{e^{ix\zeta}}{P(\zeta)}d\zeta. \tag{2.18}$$

This integral converges even in the sense of ultra-distributions. The meaning of convergence as a hyperfunction is as follows: Replace $x$ by $z=x+iy$ which belongs to the wedge $\boldsymbol{R}^n+i\Gamma$, where $\Gamma$ is the open convex cone determined by $\Gamma^{\circ}\cap\boldsymbol{S}^{n-1}=\bar{\Delta}$. (Without loss of generality we can assume that $\bar{\Delta}$ generates a convex cone.) Then we have the estimate

$$|e^{iz\zeta}|=e^{-\operatorname{Im} z\zeta}=e^{K|\xi|^{1-1/\mu}xv-y\xi}\leq e^{K|\xi|^{1-1/\mu}|x|-\delta(y)|\xi|}, \tag{2.19}$$

where $\delta(y)$ is a positive constant depending on $y$ and may be chosen in common while $y$ moves in a compact subset of $\Gamma$. Thus the integral converges absolutely and locally uniformly to a holomorphic function $E(z)$ on $\boldsymbol{R}^n+i\Gamma$. Then $E(x)$ is nothing but its boundary value $E(x+i\Gamma 0)$. From this interpretation we see at once

$$\text{S.S. } E(x)\subset \boldsymbol{R}^n\times\bar{\Delta}\,. \tag{2.20}$$

We need, however, to interpret $E(x)$ as an element of $\mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$ for some $s$, $\delta$, and improve (2.20) as such. Consider (2.19) again. We have

$$|e^{iz\zeta}|\leq C'e^{-(\delta(y)/2)|\xi|} \qquad \text{if } K|\xi|^{1-1/\mu}|x|\leq\frac{\delta(y)}{2}|\xi|\,,$$

hence the contribution of this part of integral to the defining function $E(z)$ is bounded (with the bound depending on $y$). The remaining part is an integral on the bounded region

$$|\xi|\leq\left(\frac{2K}{\delta(y)}|x|\right)^{\mu},$$

hence it is majorated by

$$C''\left(\frac{2K}{\delta(y)}|x|\right)^{n\mu}e^{K(2K/\delta(y))^{\mu-1}|x|^{\mu}}.$$

This shows that if we choose $s>\mu$, $\delta\geq 0$, then $E(x)\in\mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$ and (2.20) can be replaced by

$$\text{S.S.}^{s,\delta}E(x)\subset D^n\times\bar{\Delta}\,. \tag{2.21}$$

Now we show that $E(x)$ is a micro-local fundamental solution of $P(D)$ in $\boldsymbol{D}^n\times\Delta$ (in the sense of the assertion 1) in our theorem). If we apply $P(D)$ to $E(x)$, we obtain

$$P(D)E(x)=\int_{D_R}e^{ix\zeta}d\zeta\,. \tag{2.22}$$

On the other hand, if we put

$$f(x)=\int_{\Delta_R}e^{ix\xi}d\xi, \quad \text{where} \quad \Delta_R=\{\xi\in\boldsymbol{R}^n;\ \xi/|\xi|\in\bar{\Delta},\ |\xi|\geq R\}\,, \tag{2.23}$$

then $\delta(x)-f(x)=\int_{\boldsymbol{R}^n\setminus\Delta_R}e^{ix\zeta}d\zeta$ becomes micro-analytic in $\boldsymbol{D}^n\times\Delta$. (As usual we can show this by decomposing $\boldsymbol{R}^n\setminus\Delta_R$ by proper cones and estimating the S.S. of each part of integral.) Now by the Cauchy-Poincaré theorem the difference between (2.22) and (2.23) is an integral on the "lateral" $n$-chain $\int_{L_R} e^{ix\zeta}d\zeta$, where

$$L_R=\{\xi-itv;\ ((\xi/|\xi|\in\partial\varDelta,\ |\xi|\geq R)\text{ or }(|\xi|=R,\ \xi/|\xi|\in\varDelta)),\ 0\leq t\leq K|\xi|^{1-1/\mu}\}\,,$$

with the natural orientation induced from that of $\boldsymbol{R}^n$ and the $t$-axis. Hence its S.S. is contained in $\boldsymbol{D}^n\times\partial\varDelta$ as is seen again by the decomposition of the chain. Thus the assertion 1) is proved.

We show finally the fine estimate 2). Let now $\xi^0$ denote any point in $\varDelta$. Let $\gamma>0$ be an arbitrarily small constant and put

$$E_\gamma(x)=\int_{D_R\cap\{|\xi/|\xi|-\xi^0|\leq\gamma\}}\frac{e^{ix\zeta}}{P(\zeta)}d\zeta\,.$$

By the same argument as above we see that $E(x)-E_\gamma(x)$ is micro-analytic to the directions in $\{\xi\in\boldsymbol{S}^{n-1};|\xi-\xi^0|<\gamma\}$. Thus to see the propagation of S.S. with the direction component $\xi^0$, it suffices to examine $E_\gamma(x)$. Let $x$ be limited in the domain

$$\langle x, v\rangle<-\gamma|x|\,. \tag{2.24}$$

(More precisely, let $z=x+iy$ run in $\boldsymbol{R}^n+i\Gamma$ with the above limitation to the real part.) Then for $\zeta=\xi-itv$ we have

$$|e^{iz\zeta}|\leq e^{-t\gamma|x|-y\xi}\,.$$

Thus by the Cauchy-Poincaré theorem we can deform the path of integral from the original one

$$\{\xi-itv;\ t=K|\xi|^{1-1/\mu},\ |\xi/|\xi|-\xi^0|\leq\gamma,\ |\xi|\geq R\}$$

to a new one which goes linearly away from the real axis plus the lateral part

$$\begin{aligned}&\{\xi-itv;\ t=\varepsilon_0|\xi|,\ |\xi/|\xi|-\xi^0|\leq\gamma,\ |\xi|\geq R\}\\ &\cup\{\xi-itv;\ (|\xi/|\xi|-\xi^0|=\gamma,\ |\xi|\geq R)\text{ or }(|\xi/|\xi|-\xi^0|\leq\gamma,\ |\xi|=R)),\\ &\quad K|\xi|^{1-1/\mu}\leq t\leq\varepsilon_0|\xi|\}\,.\end{aligned}$$

Let us denote by $E'_\gamma(z)$ resp. $E''_\gamma(z)$ the part of $E_\gamma(z)$ defined by the integral on the first resp. second component of this new path. As boundary values for $y\to0$ along $\Gamma$, these holomorphic functions obviously define hyperfunctions on the domain (2.24), and even sections of $\mathscr{Q}^{s,\delta}$ on the domain of $\boldsymbol{D}^n$ augmented by the corresponding points at infinity. (Notice however that they do not separately define hyperfunctions globally on $\boldsymbol{R}^n$.)

Now as for the integral for $E'_\gamma(z)$ we can let $y$ run in the set

$$|y|<\varepsilon_0\gamma|x|.$$

Hence $E'_\gamma(z)$ becomes a holomorphic function in such a complex neighborhood of (2.24) and is bounded apart from the finite boundary points. As for $E''_\gamma(z)$, by dividing the path as usual we can see that S.S.$E''_\gamma(x)$ contains only the directions in $\{\xi\in S^{n-1}; |\xi-\xi^0|=\gamma\}$. Thus we have shown

$$\text{S.S.}^{s,\delta}E(x)\cap D^n\times\{\xi^0\}=\text{S.S.}^{s,\delta}E_\gamma(x)\cap D^n\times\{\xi^0\}\subset\overline{\{\langle x, v\rangle\geq-\gamma|x|\}}\times\{\xi^0\}.$$

Since $\gamma>0$ is arbitrary, we obtain

$$\text{S.S.}^{s,\delta}E(x)\cap D^n\times\{\xi^0\}\subset\overline{\{\langle x, v\rangle\geq 0\}}\times\{\xi^0\}. \tag{2.25}$$

Now we let $v$ move in the normal cone $\Gamma_\xi$ of $(P_m)_\xi(\eta)$ containing $v$, that is, the open convex cone whose dual equals to $K_\xi$. Since the correspondence $\xi\mapsto\Gamma_\xi$ is lower semi-continuous, any element of $K_{\xi^0}$ can be attained by a continuous vector field $v(\xi)$ on $\Delta$ which deforms the original constant vector field $v$ continuously on a neighborhood of $\xi^0$. In view of the Cauchy-Poincaré theorem, formula (2.18) remains valid during this deformation. Therefore the above argument applies to the deformed path. Thus we can finally replace (2.25) by

$$\text{S.S.}^{s,\delta}E(x)\cap D^n\times\{\xi^0\}\subset\bigcap_{v\in\Gamma_{\xi^0}}\overline{\{\langle x, v\rangle\geq 0\}}\times\{\xi^0\}=\overline{K_{\xi^0}}\times\{\xi^0\}.$$

Since $\xi^0\in\Delta$ is arbitrary, this is just the estimate 2) of our theorem. q.e.d.

REMARK. Supplementary explanation will be necessary on the point why we can do the construction of the fundamental solution micro-locally: In principle, we first decompose $\Delta$ by $\{\Delta_\lambda\}$ and the component of the $\delta$-function as

$$\underset{\sim}{W}_s(x, \Delta)=\sum_\lambda \underset{\sim}{W}_s(x, \Delta_\lambda)$$

and then put

$$E(x)=\sum_\lambda E_\lambda(x)*\underset{\sim}{W}_s(x, \Delta_\lambda),$$

with micro-local fundamental solutions $E_\lambda(x)$ constructed as above on a neighborhood of $\bar{\Delta}_\lambda$. However, if $\{\Delta_\lambda\}$ is only locally finite, the final sum may be infinite as the one for Fourier hyperfunctions. Since we did not give the detailed theory of Fourier micro-functions $\mathscr{R}^{s,\delta}$, we cannot employ the interpretation as a locally finite sum of its global sections on $\boldsymbol{D}^n \times \Delta$. (It is not obvious if we can choose a global representative in Fourier hyperfunctions of thus interpreted sum.) Therefore we must add a remark that we can always diminish $\{\Delta_\lambda\}$ to a finite covering. This follows from the following fact combined with the compactness argument: At each boundary point of $\Delta$, we write down all the upper semicontinuous correspondence $\xi \mapsto K_\xi$ on its sufficiently small neighborhood $\Delta_\lambda$. Then one of them must coincide on $\Delta \cap \Delta_\lambda$ with the correspondence $\xi \mapsto K_\xi$ given in the assumption of the theorem. There is another way of constructing $E(x)$ on $\boldsymbol{D}^n \times \Delta$: We first construct a continuous vector field $v(\xi)$ from the assumption of the theorem. Then the above proof will give a fundamental solution $E(x)$ directly on $\boldsymbol{D}^n \times \Delta$. The method of construction of such $v(\xi)$ is as follows: The correspondence $\Delta \ni \xi \mapsto \Gamma_\xi$ (the cone in which $v$ can move) being lower semi-continuous, we can choose a sufficiently fine locally finite covering $\{\Delta_\lambda\}$ and a corresponding family of open convex cones $\{\Gamma_\lambda\}$ such that

$$\xi \in \Delta_\lambda \Rightarrow \Gamma_\xi \supset \Gamma_\lambda, \qquad \xi \in \Delta_\lambda \cap \Delta_\mu \Rightarrow \Gamma_\xi \supset \Gamma_\lambda + \Gamma_\mu .$$

Then choose a fixed unit $n$-vector $v_\lambda \in \Gamma_\lambda$ arbitrarily for every $\lambda$, and with a partition of unity $\{\varphi_\lambda(\xi)\}$ associated to $\{\Delta_\lambda\}$ put

$$v(\xi) = \sum_{\lambda \in \Lambda} \varphi_\lambda(\xi) v_\lambda .$$

Anyway this remark is not important in our following applications because we only need the existence of micro-local fundamental solutions in the sequel.

Now we are ready to assert something on the global existence of real analytic solutions for locally hyperbolic operators. First we give the following literal consequences.

THEOREM 2.8. *Let $P(D)$ be locally hyperbolic and let $E^{\Delta,j}(x) \in \mathcal{Q}^{s,\delta}(\boldsymbol{D}^n)$, $j=1, \cdots, N_\Delta$ be a set of micro-local fundamental solutions as is given by Corollary 2.6. Let $\Omega \subset \boldsymbol{R}^n$ be a connected open set satisfying the following geometric condition: For any $\Delta$ appearing in the definition of the set $\{E^{\Delta,j}(x)\}$ we can find a decomposition by closed subsets*

$$\overline{\partial \Omega} \times \Delta = \bigcup_{j=1}^{N_\Delta} X^{\Delta,j} \tag{2.26}$$

*such that*

$$(a, \xi) \in X^j \textit{ implies } (\{a\} + \overline{K_\xi^{\Delta,j}}) \cap \Omega = \varnothing . \tag{2.27}$$

*(Here $\overline{\overline{\partial\Omega}}$ is, by definition, the closure in $\boldsymbol{D}^n$ of the boundary $\partial\Omega$ of $\Omega$ in $\boldsymbol{R}^n$.) Then we have $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$.*

To prove this from Theorem 2.1′ it suffices to show that the assumption of the latter at the points at infinity in $\bar{\partial}\Omega\setminus\overline{\overline{\partial\Omega}}$ is automatically fulfilled in our present case. Choose an extension $\tilde{f}\in\mathscr{Q}^{s,-\delta'}(\boldsymbol{D}^n)$ with sing supp $\tilde{f}\subset\bar{\partial}\Omega$. Then decompose first $\tilde{f}$ into $\tilde{f}_0+\tilde{f}_\infty$ such that

$$\text{sing supp}\,\tilde{f}_0\subset\overline{\overline{\partial\Omega}},\quad \text{sing supp}\,\tilde{f}_\infty\subset\bar{\partial}\Omega\cap\boldsymbol{S}_\infty^{n-1}.$$

The equation with the second member $\tilde{f}_0$ can be solved as in Theorem 2.1′ on account of our assumption. As for $\tilde{f}_\infty$, we can decompose it by means of the curvilinear Radon decomposition (1.10) to a finite sum

$$\tilde{f}_\infty=\sum\tilde{f}_\infty^\Delta,\qquad \text{S.S.}^{s,-\delta'}\tilde{f}_\infty^\Delta\subset(\bar{\partial}\Omega\cap\boldsymbol{S}_\infty^{n-1})\times\bar{\Delta}',$$

where $\bar{\Delta}'\subset\Delta$ is a compact subset. Therefore it only remains to solve the equation with the second member $\tilde{f}_\infty^\Delta$, and this follows from the following decomposition replacing (2.27):

LEMMA 2.9. *Assume that for each $\xi\in\Delta$ we have*

$$\bigcap_{j=1}^{N_\Delta}K_\xi^{\Delta,j}=\{0\}. \tag{2.28}$$

*Then there exists a decomposition of $\boldsymbol{S}_\infty^{n-1}\times\Delta$ into relatively closed subsets $Y^{\Delta,j}$, $j=1,\cdots,N_\Delta$, such that*

$$(a\infty,\xi)\in Y^{\Delta,j}\text{ implies }a\notin -K_\xi^{\Delta,j}. \tag{2.29}$$

PROOF. We first decompose $\boldsymbol{S}_\infty^{n-1}\times\bar{\Delta}'$ into closed subsets $Y^{\bar{\Delta}',j}$ as above. (This suffices for our present application.) Choose $a\infty\in\boldsymbol{S}_\infty^{n-1}$ and $\xi\in\bar{\Delta}'$ arbitrarily. By the assumption (2.28) there exists $j$ such that

$$a\notin -K_\xi^{\Delta,j}.$$

By the upper semi-continuity of $\xi\mapsto K_\xi^{\Delta,j}$, this relation remains to be valid in some (closed) neighborhood of $(a\infty,\xi)$ in $\boldsymbol{S}_\infty^{n-1}\times\bar{\Delta}'$. Since $\boldsymbol{S}_\infty^{n-1}\times\bar{\Delta}'$ is a compact set, it can be covered by a finite number of such neighborhoods. Thus it suffices to classify such closed neighborhoods according to the index $j$ and take their individual union as $Y^{\bar{\Delta}',j}$.

Now we decompose $\Delta$ by a locally finite family of compact subsets $\bar{\Delta}'\subset\Delta$ and construct $Y^{\bar{\Delta}',j}$ for each $\bar{\Delta}'$ as above. Since for each fixed $j$ $\{Y^{\bar{\Delta}',j}\}_{\bar{\Delta}'}$ again constitutes a locally finite family, their union $Y^{\Delta,j}$ is closed and satisfies (2.29). q.e.d.

To finish the proof of Theorem 2.8 it suffices to notice that for a locally hyperbolic operator the cones $K_\xi^{\Delta,j}$ are proper, hence $K_\xi^{\Delta,j}\cap -K_\xi^{\Delta,j}=\{0\}$ and the assumption (2.28) of the above lemma is obviously satisfied.

The same proof based on Corollary 2.7 gives the following

COROLLARY 2.10. *Let $P(D)$ be locally hyperbolic and assume that there exists a global upper semi-continuous correspondence $\xi\mapsto K_\xi$ of local propagation cones on $\boldsymbol{S}^{n-1}$. Let $\Omega\subset\boldsymbol{R}^n$ be a connected open set satisfying the following geometric condition: There exists a decomposition by closed subsets*

$$(2.30)\qquad \overline{\overline{\partial\Omega}}\times\boldsymbol{S}^{n-1}=X^+\cup X^-$$

*such that*

$$(2.31)\qquad (a,\xi)\in X^\pm \text{ implies } (\{a\}\pm\overline{\overline{K_\xi}})\cap\Omega=\varnothing \text{ respectively.}$$

*Then we have $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$.*

REMARK. For the present level of our theory we cannot weaken the geometric condition of the above theorem to the following one posed only on the finite boundary points $\partial\Omega$ of $\Omega\subset\boldsymbol{D}^n$: *There exists a decomposition by closed subsets*

$$(2.26)'\qquad \partial\Omega\times\Delta=\bigcup_{j=1}^{N_\Delta}X^{\Delta,j}$$

*such that*

$$(2.27)'\qquad (a,\xi)\in X^{\Delta,j} \text{ implies } (\{a\}+K_\xi^{\Delta,j})\cap\Omega=\varnothing$$

(and similarly for Corollary 2.10). In fact, consider e.g. the region in Figure 2a in relation to the operator $P(D)=D_1$. We have $\pm K_\xi=\{\pm x_1\geq 0,\ x_2=0\}$ for all $\xi\in N(P_m)$. Therefore we must choose the plus sign at every finite boundary point $a$ of $\Omega$ in order to have $(\{a\}+K_\xi)\cap\Omega=\varnothing$. Thus in order to obtain a decomposition by closed sets, we must choose the same sign even at the limit point $a\infty\in\overline{\overline{\partial\Omega}}$. We have, however, $\overline{\overline{\partial\Omega}}\cap\boldsymbol{S}_\infty^{n-1}=\bar{\bar{\partial}}\Omega\cap\boldsymbol{S}_\infty^{n-1}=\{a\infty\}$ with $a=(-1,0)$ in view of the definition of our compactification $\boldsymbol{D}^n$, and hence $(\{a\infty\}+\overline{\overline{K_\xi}})\cap\Omega\neq\varnothing$ by the definition of the vector sum given in Theorem 1.16. Summing up, our domain does not satisfy the geometric condition of Theorem 2.8 though it does the weakened one (2.26)′–(2.27)′. It may seem rather paradoxical that on the contrary we can apply our theorem to the domain in Figure 2b (for which we can employ $-K_\xi$ on a neighborhood of $a\infty$). Domains as in Figure 2a will be treated by our theory by

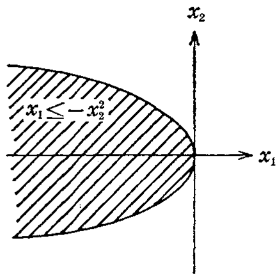


Figure 2a

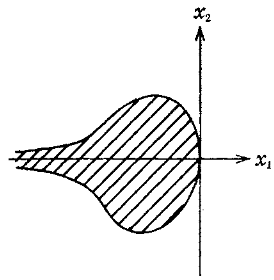


Figure 2b

the introduction of more refined compactification of $\boldsymbol{R}^n$ depending on the respective domains.

Related with this is the advantage of the abstract assumption of Theorem 2.1. It is known that if $P(D)$ possesses a set of micro-local fundamental solutions as in Theorem 2.1′, then $P(D)$ is necessarily locally hyperbolic. (For a detailed proof of this fact see Zampieri [22].) Therefore the abstract form of Theorem 2.1′ has a meaning if we can treat the case where $K_\xi$ are not necessarily proper. However, in view of the case 4) of Theorem 1.15 the compactification $\boldsymbol{D}^n$ of $\boldsymbol{R}^n$ is not adequate for this, and again various types of compactification related with the operators are required. We leave these subjects to a forthcoming paper. (See Zampieri [21] for an example of such study.)

On the contrary, in the case where there are no finite boundary points, that is $\Omega=\boldsymbol{R}^n$, we have the following

COROLLARY 2.11 (Andersson [2]). *Let $P(D)$ be locally hyperbolic. Then we have $P(D)\mathscr{A}(\boldsymbol{R}^n)=\mathscr{A}(\boldsymbol{R}^n)$.*

Andersson's proof is apparently similar to ours because he also employs the decomposition of the second member $f$ in relation to the good property of the fundamental solution on the propagation of S.S. However he does not employ the rapidly decreasing extension and hence he has to employ some approximation procedure instead of the direct calculus of convolution. Therefore our proof will be more instructive even in this simplest case of unbounded domain.

## § 3. Refinement of geometric condition.

Now we try to replace the geometric condition in Theorem 2.8 or in Corollary 2.10 by the one posed at each point:

(3.1) *For every point* $(a, \xi) \in \partial\Omega \times \Delta$ *there exists* $K_\xi^{\Delta, j}$ *such that*

$$(\{a\}+K_\xi^{\Delta, j}) \cap \Omega = \emptyset ;$$

or respectively

(3.2) *For every point* $(a, \xi) \in \partial\Omega \times S^{n-1}$ *we have either*

$$(\{a\}+K_\xi) \cap \Omega = \emptyset \quad \text{or} \quad (\{a\}-K_\xi) \cap \Omega = \emptyset .$$

In fact, in [20] Zampieri has shown that for a locally hyperbolic operator $P(D)$ the "pointwise" condition (3.2) suffices to prove $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$ for *convex* $\Omega$. Since his proof is based on Hörmander's result [5], the convexity of $\Omega$ is indispensable. Therefore it will be more advantageous to refine our argument to manage with the above "pointwise" conditions.

We would have to examine first if these "pointwise" conditions imply the "uniform" conditions (2.8) resp. (2.3) treated in § 2. The problem is to regroup $(a, \xi)$ with respect to the index $j$ resp. the sign $\pm$ in order to construct a decomposition by *closed* subsets satisfying (2.8) resp. (2.3). At the end of § 2 we have already seen that if $\Omega$ is unbounded there are some difficulties at infinite boundary points. The fact is that even in the case of bounded $\Omega$, this passage from "pointwise" to "uniform" is not so simple: Kawai in his article [12] gave several sufficient conditions which assure this in the case corresponding to Corollary 2.10. Among them he mentioned that if $\Omega$ has $C^1$-boundary we can deduce (2.2)–(2.3) from (3.2). It was pointed out by Zampieri [19], however, that if some of $K_\xi$ is larger than a finite set of half lines, then a bounded domain with $C^1$-boundary never satisfies the condition (3.2) itself. More recently, Zampieri constructed even an example where the pointwise condition (3.2) actually does not imply the uniform condition of Theorem 2.8:

*Example* 3.1. Consider

$$P(D)=D_1^2D_2^2-D_2^2D_3^2-D_3^4-D_4^4-D_3^2D_4^2. \tag{3.3}$$

This is locally hyperbolic. In fact, we have

$$\operatorname{grad}_\xi P(\xi)=(2\xi_1\xi_2^2,\ 2(\xi_1^2-\xi_3^2)\xi_2,\ -2\xi_3(2\xi_3^2+\xi_2^2+\xi_4^2),\ -2\xi_4(2\xi_4^2+\xi_3^2)) ,$$

hence $\operatorname{grad}_\xi P(\xi)\neq 0$ except for the two pairs of directions $\xi=(\pm 1, 0, 0, 0)$ or $\xi=(0, \pm 1, 0, 0)$. The localization at the former is equal to $\eta_2^2$. We see easily that $P(D)$ is locally hyperbolic at these points with the vector $v=(0, \pm 1, 0, 0)$. (Note however that $v$ is a characteristic direction of the original operator.)

The localization at the latter is $\eta_1^2-\eta_3^2$. Since $P(\xi)=0$ solved for $\xi_1$ gives real roots $\pm(\xi_3^2+(\xi_3^4+\xi_3^2\xi_4^2+\xi_4^4)/\xi_2^2)^{1/2}$ on a neighborhood of $(0, \pm1, 0, 0)$, $P(D)$ is locally hyperbolic there with $v=(\pm1, 0, 0, 0)$ and the propagation cones $\pm K_\xi$ of $\eta_1^2-\eta_3^2$, where

$$(3.4)\qquad K_\xi=\{x_1\geq|x_3|,\ x_2=x_4=0\},\qquad \xi=(0, \pm1, 0, 0)$$

serve as local propagation cones of $P(D)$. On the other hand, $P(\xi)=0$ solved for $\xi_3$ gives

$$\xi_3^2=-\frac{\xi_2^2+\xi_4^2}{2}\pm\sqrt{\frac{(\xi_2^2+\xi_4^2)^2}{4}+\xi_1^2\xi_2^2-\xi_4^4}\,.$$

Therefore on a neighborhood of $(0, \pm1, 0, 0)$ the four roots may be either "2 reals and 2 imaginaries" or "4 imaginaries" according to the sign of $\xi_1^2\xi_2^2-\xi_4^4$. Hence $P(D)$ is not locally hyperbolic with $v=(0, 0, \pm1, 0)$ though its localization is hyperbolic to these directions. Now consider any $\Omega$ which lies in $x_1+x_3<0$ and of which the boundary possesses a flat part $B$ in common with the hyperplane $x_1+x_3=0$. At any point of this part $B$ the pointwise geometric condition (3.1) is obviously satisfied. Note however that for $\xi=(0, \pm1, 0, 0)$ we have a unique choice for $K_\xi$, that is, (3.4). Therefore, if we wish to realize the uniform geometric condition (2.3), we must choose as $K_\xi$ for each simply characteristic direction $\xi\sim(0, \pm1, 0, 0)$ either of the half lines parallel to $\pm\mathrm{grad}_\xi P(\xi)$ which tends to (3.4) but not to the opposite one as $\xi\to(0, \pm1, 0, 0)$. But there exists a sequence of such half lines which always penetrate into $\Omega$. In fact, putting $\xi_2=1$, $\xi_4=0$ for simplicity, we have

$$\mathrm{grad}_\xi P(\xi)=(2\xi_1,\ 2(\xi_1^2-\xi_3^2),\ -2\xi_3(2\xi_3^2+1),\ 0)\,,$$

hence we must choose the plus sign for $\xi_1>0$, $\xi_3>0$, and then along this half line we have

$$x_1+x_3=2\xi_1-2\xi_3-4\xi_3^3=2\sqrt{\xi_3^2+\xi_3^4}-2\xi_3-4\xi_3^3<0\,.$$

Thus we have no way of dividing $B\times S^{n-1}$ by closed subsets in order to satisfy the uniform geometric condition (2.3).

The above example shows that our abstract theorem 2.1 (or 2.1′) does not suffice to prove Zampieri's result even for convex $\Omega$. Therefore we now try to improve the abstract theorem in order to compensate this shortage. For this purpose we consider a stratification on $\Delta\subset S^{n-1}$. This is by definition a disjoint decomposition of $\Delta$ by locally closed subsets $\Xi_k$:

$$(3.5)\qquad \Delta=\Xi_0\sqcup\Xi_1\sqcup\cdots\sqcup\Xi_{M_\Delta}$$

satisfying

$$\Xi_{k+1}\subset\overline{\Xi_k},\quad \Xi_k=\overline{\Xi_k}\setminus\overline{\Xi_{k+1}}$$

for each $k$. ($\Xi_{M_\Delta+1}=\emptyset$ by convention.)

THEOREM 3.2. *Assume that $P(D)$ admits a set of micro-local fundamental solutions $E^{\Delta,j}(x)$, $j=1,\cdots,N_\Delta$, as described in Theorem 2.1′. Assume that there exists a stratification* (3.5) *of each $\Delta$ such that for each $k$, $\bar{\bar{\partial}}\Omega\times\Xi_k$ can be decomposed into relatively closed subsets*

$$\bar{\bar{\partial}}\Omega\times\Xi_k=\bigcup_{j=1}^{N_\Delta}X^{\Delta,k,j} \tag{3.6}$$

*in such a way that*

$$(a,\xi)\in X^{\Delta,k,j} \text{ implies } (\{a\}+\overline{\overline{K_\xi^{\Delta,j}}})\cap\Omega=\emptyset\,. \tag{3.7}$$

*Then we have $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$.*

PROOF. Let $\tilde{f}\in\mathscr{B}(\boldsymbol{R}^n)$ be an extension of $f$ with support in $\bar{\Omega}$, and let $\tilde{f}=\sum_\Delta\tilde{f}^\Delta$ be a decomposition into finitely many $\tilde{f}^\Delta\in\mathscr{B}(\boldsymbol{R}^n)$ with S.S. $\tilde{f}^\Delta\subset\partial\Omega\times\bar{\Delta}'$, where $\Delta$ are some of the neighborhoods on which the micro-local fundamental solutions $E^{\Delta,j}$ and the stratification (3.5) are assumed to exist, and $\Delta'\Subset\Delta$. For each fixed $\Delta$ we shall find $u^\Delta\in\mathscr{B}(\boldsymbol{R}^n)$ such that $u^\Delta$ is real analytic in $\Omega$ and that $g^\Delta:=P(D)u^\Delta-\tilde{f}^\Delta$ is real analytic on $\boldsymbol{R}^n$. Then $u=\sum_\Delta u^\Delta$ will satisfy $P(D)u=\tilde{f}+\sum_\Delta g^\Delta$. We shall see later that for $g=\sum_\Delta g^\Delta\in\mathscr{A}(\boldsymbol{R}^n)$ we can find a solution $v\in\mathscr{A}(\boldsymbol{R}^n)$ of $P(D)v=g$ rather easily. Thus $(u-v)|_\Omega$ will be a required solution.

To construct $u^\Delta$ we proceed by the induction on $k$: Assume that we have obtained a micro-local solution $u^{\Delta,k}\in\mathscr{B}(\boldsymbol{R}^n)$ of $P(D)u^{\Delta,k}=\tilde{f}^\Delta$ on $\boldsymbol{R}^n\times(\Delta\setminus\overline{\Xi_{k+1}})$ in the following sense.

1) S.S.$u^{\Delta,k}\subset(\boldsymbol{R}^n\setminus\Omega)\times\bar{\Delta}'$.
2) S.S.$(P(D)u^{\Delta,k}-\tilde{f}^\Delta)\subset\partial\Omega\times(\bar{\Delta}'\cap\overline{\Xi_{k+1}})$.

We construct $u^{\Delta,k+1}$ from $u^{\Delta,k}$ employing the micro-local fundamental solutions as follows. (This argument also shows how to construct the first one $u^{\Delta,0}$.) Put

$$g^{\Delta,k+1}=P(D)u^{\Delta,k}-\tilde{f}^\Delta\,.$$

We have S.S. $g^{\Delta,k+1}\subset\partial\Omega\times(\bar{\Delta}'\cap\overline{\Xi_{k+1}})$. Modulo $\mathscr{A}(\boldsymbol{R}^n)$ we extend $g^{\Delta,k+1}$ to $h^{\Delta,k+1}\in\mathscr{Q}^{s,-\delta'}(\boldsymbol{D}^n)$ such that S.S.$^{s,-\delta'}h^{\Delta,k+1}\subset\bar{\bar{\partial}}\bar{\bar{\Omega}}\times(\bar{\Delta}'\cap\overline{\Xi_{k+1}})$. Then by the geometric condition we decompose $h^{\Delta,k+1}$ into the sum of $h^{\Delta,k+1,j}$ such that, putting $X^{\bar{\Delta}',k,j}=X^{\Delta,k,j}\cap \boldsymbol{D}^n\times\bar{\Delta}'$, we have

$$\mathrm{S.S.}^{s,-\delta'}h^{\Delta,k+1,j}\subset X^{\bar{\Delta}',k+1,j}\cup(\overline{\overline{\partial\Omega}}\times(\bar{\Delta}'\cap\overline{\Xi_{k+2}}))\,.$$

(As usual these procedures are possible in view of the flabbiness of the sheaf $\mathscr{R}^{s,-\delta'}$ of the corresponding class of Fourier microfunctions. For the treatment not relying on the flabbiness of $\mathscr{R}^{s,-\delta'}$ at infinity see the end of this proof.) Now put

$$v^{\Delta,k+1}=\sum_{j=1}^{N_\Delta}E^{\Delta,j}*h^{\Delta,k+1,j}\,.$$

By the geometric condition the S.S. with the direction components in $\Xi_{k+1}$ does not propagate into $\Omega$. Hence we have

$$\mathrm{S.S.}^{s,\delta''}v^{\Delta,k+1}\subset\overline{\overline{\boldsymbol{R}^n\setminus\Omega}}\times(\bar{\Delta}'\cap\overline{\Xi_{k+1}})\cup\overline{\overline{\Omega}}\times(\bar{\Delta}'\cap\overline{\Xi_{k+2}})\,,$$

where $\delta''$ is the one given by (1.13) via $\delta$ and $\delta'$. Hence

$$\mathrm{S.S.}(v^{\Delta,k+1}|_{\boldsymbol{R}^n})\subset(\boldsymbol{R}^n\setminus\Omega)\times(\bar{\Delta}'\cap\overline{\Xi_{k+1}})\cup\bar{\Omega}\times(\bar{\Delta}'\cap\overline{\Xi_{k+2}})\,.$$

Now choose $w^{\Delta,k+1}\in\mathscr{B}(\boldsymbol{R}^n)$ such that

$$\begin{aligned}&\mathrm{S.S.}v^{\Delta,k+1}|_{\boldsymbol{R}^n}-w^{\Delta,k+1}\subset\bar{\Omega}\times(\bar{\Delta}'\cap\overline{\Xi_{k+2}})\,,\\&\mathrm{S.S.}w^{\Delta,k+1}\subset(\boldsymbol{R}^n\setminus\Omega)\times(\bar{\Delta}'\cap\overline{\Xi_{k+1}})\,.\end{aligned}$$

(It suffices to take the component of a decomposition of $v^{\Delta,k+1}|_{\boldsymbol{R}^n}$ according to the above closed covering. Here is necessitated the flabbiness of the usual sheaf $\mathscr{C}$ hence only on $\boldsymbol{R}^n$ but in a very sharp form.) By the construction we have on $\boldsymbol{R}^n$

$$\begin{aligned}P(D)w^{\Delta,k+1}-g^{\Delta,k+1}&=(P(D)v^{\Delta,k+1}-h^{\Delta,k+1})+(h^{\Delta,k+1}-g^{\Delta,k+1})\\&\quad+P(D)(w^{\Delta,k+1}-v^{\Delta,k+1})\\&=P(D)(w^{\Delta,k+1}-v^{\Delta,k+1})\qquad\mathrm{mod}\ \mathscr{A}(\boldsymbol{R}^n)\,,\end{aligned}$$

hence

$$\begin{aligned}\mathrm{S.S.}(P(D)w^{\Delta,k+1}-g^{\Delta,k+1})&\subset(\boldsymbol{R}^n\setminus\Omega)\times(\bar{\Delta}'\cap\overline{\Xi_{k+1}})\cap\bar{\Omega}\times(\bar{\Delta}'\cap\overline{\Xi_{k+2}})\\&\subset\partial\Omega\times(\bar{\Delta}'\cap\overline{\Xi_{k+2}})\,.\end{aligned}$$

Hence $u^{\Delta,k+1}=u^{\Delta,k}-w^{\Delta,k+1}$ satisfies the inductive assumption 1)–2) with $k$ replaced by $k+1$. Thus after a finite step we finally obtain a desired microlocal solution $u^{\Delta}$.

Now we solve $P(D)u=f\in\mathscr{A}(\boldsymbol{R}^n)$ for $u\in\mathscr{A}(\boldsymbol{R}^n)$ under our geometric condition. Note that we cannot employ the above argument directly, because by once taking the convolution the Fourier hyperfunction loses the rapidly decreacing property and it can no more accept the convolution by the fundamental solutions. However, we can apply Remark after the proof of Theorem 2.1. That is, if $\xi \to K_\xi^{\Delta,j}$ is upper semi-continuous, the condition $(\{a\infty\}+\overline{\overline{K_\xi^{\Delta,j}}})\cap\Omega=\emptyset$ implies $(\{a'\infty\}+\overline{\overline{K_{\xi'}^{\Delta,j}}})\cap\Omega=\emptyset$ for $a'$ and $\xi'$ sufficiently close to $a$ and $\xi$. Note that we have the pointwise condition (3.1) for every point at infinity which is not necessarily in $\bar{\bar{\partial}}\Omega$. (This is also a consequence of our condition (3.6)–(3.7); see the proof of Proposition 3.3 below.) Therefore we can always decompose $S_\infty^{n-1}\times\Delta$ into relatively closed subsets $X_\infty^{\Delta,j}$ for which the uniform condition like (2.2)–(2.3) is valid. The fact that we can enlarge $X_\infty^{\Delta,j}$ a little allows us to manage with Proposition 1.21 instead of the flabbiness of $R^{s,-\delta'}$ at infinity. The same Remark applies also to choose $h^{\Delta,k+1}$ or $h^{\Delta,k+1,j}$ in the above proof. In that case we had better replace (3.6) by an equivalent condition

$$(3.6)' \qquad \partial\Omega\times\Xi_k\cup S_\infty^{n-1}\times\Delta=\bigcup_{j=1}^{N\Delta}X^{\Delta,k,j},$$

and likewise the definition of $X^{\bar{\Delta}',k,j}$ by $X^{\Delta,k,j}\cap(R^n\times\bar{\Delta}')\cup(S_\infty^{n-1}\times\bar{\Delta}')$, and we must allow to enlarge $\bar{\Delta}'$ in $\Delta$ for so many times as requires the induction procedure. q.e.d.

In the geometric condition (3.6) we may replace $\bar{\bar{\partial}}\Omega$ by $\overline{\overline{\partial\Omega}}$ if we assume in addition the pointwise geometric condition at points at infinity of $\bar{\bar{\partial}}\Omega$, because we have employed the condition on the part $\bar{\bar{\partial}}\Omega\setminus\overline{\overline{\partial\Omega}}$ only to deduce the latter. It is not at all obvious if we can do without this additional condition in the abstract argument (though all are obvious for locally hyperbolic operators).

Now we give a convenient sufficient condition for bounded $\Omega$. In view of possible later development we discuss a little generalized situation.

PROPOSITION 3.3. *Assume that there exists a stratification* (3.5) *of $\Delta$ such that* (*at each connected component of $\Xi_k$*), *the correspondence $\xi\to K_\xi^{\Delta,j}$ is continuous, that is, we have not only the upper semi-continuity*, (*cf.* (2.6)) *but also the following lower semi-continuity*:

(3.8) *For each fixed $\xi^0$ given any $\varepsilon>0$ there exists $\delta>0$ such that for $|\xi-\xi^0|<\delta$ and for any $x\in K_{\xi^0}^{\Delta,j}\cap\{|x|=1\}$ we have* $\mathrm{dis}(x, K_\xi^{\Delta,j})<\varepsilon$.

(Note that we do not require that $K_\xi$ remains homeomorphic as $\xi$ varies in each connected component of $\Xi_k$.) *Then for* bounded $\Omega$ *under the pointwise* *geometric condition* (3.1) *we can find a decomposition of* $(\boldsymbol{D}^n\setminus\Omega)\times\Xi_k$, *hence a fortiori of* $\partial\Omega\times\Xi_k$, *by relatively closed subsets as* (3.6)–(3.7).

PROOF. Let $R>0$ be a large constant and let $B_R=\{x\in\boldsymbol{R}^n;\ |x|\leq R\}$. Put

$$X_R^{\Delta,k,j}=\{(a,\xi)\in(B_R\setminus\Omega)\times\Xi_k;\ (\{a\}+\overline{\overline{K_\xi^{\Delta,j}}})\cap\Omega=\emptyset\}\,. \tag{3.9}$$

Then this set automatically becomes relatively closed. In fact, assume that $(a^{(l)},\xi^{(l)})\in X_R^{\Delta,k,j}$ and $(a^{(l)},\xi^{(l)})\rightarrow(a^{(0)},\xi^{(0)})\in(B_R\setminus\Omega)\times\Xi_k$. Assume $x\in(a^{(0)}+K_{\xi^{(0)}}^{\Delta,j})\cap\Omega\neq\emptyset$. Then by the lower semi-continuity (3.8) the set $a^{(l)}+K_{\xi^{(l)}}^{\Delta,j}$ for sufficiently large $l$ would contain a point in the $\varepsilon$-neighborhood of $x$ which should be contained in $\Omega$ for $\varepsilon$ small. This is a contradiction.

Next, for a compact subset $\bar{\Delta}'\subset\Delta$ we try to decompose $(\boldsymbol{D}^n\setminus\mathrm{Int}(B_R))\times\bar{\Delta}'$ by closed subsets. (Note that the set which we obtain from (3.9) by replacing $B_R$ by $\boldsymbol{D}^n$ is not necessarily closed: For example, for $a\in K_\xi^{\Delta,j}\setminus\mathrm{Int}(K_\xi^{\Delta,j})$, the set $ta-\overline{\overline{K_\xi^{\Delta,j}}}$ may be disjoint of $\Omega$ for each $t>0$ though its limit $a\infty-\overline{\overline{K_\xi^{\Delta,j}}}$ agrees with the whole $\boldsymbol{D}^n$.) Suppose $(a^{(0)},\xi^{(0)})\in\boldsymbol{S}^{n-1}\times\bar{\Delta}'$. Then in view of the pointwise geometric condition (3.1) we can find $j$ such that $a^{(0)}\notin-K_{\xi^{(0)}}^{\Delta,j}$. In fact, choose $a\in\partial\Omega$ and $b\in\Omega$ such that $a^{(0)}/\!/a-b$. (This is possible if $\Omega$ is bounded, or more generally if $a^{(0)}\infty\Subset\bar{\partial}\Omega$.) We must have $(\{a\}+K_{\xi^{(0)}}^{\Delta,j})\cap\Omega=\emptyset$ for some $j$. Hence $b\Subset\{a\}+K_{\xi^{(0)}}^{\Delta,j}$, that is, $a^{(0)}\Subset-K_{\xi^{(0)}}^{\Delta,j}$. By the upper semi-continuity of $\xi\rightarrow K_\xi^{\Delta,j}$, we can find a closed neighborhood $\overline{\Delta_{\xi^{(0)}}}$ of $\xi^{(0)}$ and $\overline{\Gamma_{a^{(0)}}}$ of $a^{(0)}$ such that

$$a\Subset-K_\xi^{\Delta,j}\quad\text{for}\quad a\in\overline{\Gamma_{a^{(0)}}},\quad \xi\in\overline{\Delta_{\xi^{(0)}}}\,.$$

Note that if $R$ is sufficiently large, then we have

$$a\Subset-K_\xi^{\Delta,j}\ \text{implies}\ (ta+\overline{\overline{K_\xi^{\Delta,j}}})\cap\Omega=\emptyset\quad\text{for}\ R\leq t\leq+\infty\,. \tag{3.10}$$

Again by the upper semi-continuity of $\xi\mapsto K_\xi^{\Delta,j}$ we can assume that (3.10) holds with the same $R$ for $a\in\overline{\Gamma_{a^{(0)}}}$, $\xi\in\overline{\Delta_{\xi^{(0)}}}$. Now we cover $\boldsymbol{S}^{n-1}\times\bar{\Delta}'$ by a finite number of closed neighborhoods $\overline{\Gamma_{a^{(0)}}}\times\overline{\Delta_{\xi^{(0)}}}$ like this. Let $R=R(\bar{\Delta}')$ be the maximum of $R$'s attached to them. Just as in the proof of Lemma 2.9 we can find closed subsets $Y^{\bar{\Delta}',j}$, $j=1,\cdots,N_\Delta$, of $(\boldsymbol{D}^n\setminus\mathrm{Int}(B_R))\times\bar{\Delta}'$ such that

$$(a,\xi)\in Y^{\bar{\Delta}',j}\ \text{implies}\ (a+\overline{\overline{K_\xi^{\Delta,j}}})\cap\Omega=\emptyset\,.$$

Now let $\bar{\Delta}'$ run in a locally finite closed covering of $\Delta$, and put

$$X^{\Delta,k,j}=\bigcup_{\bar{\Delta}'}(X_{R(\bar{\Delta}')}^{\Delta,k,j}\cap(B_{R(\bar{\Delta}')}\times\bar{\Delta}'))\cup(Y^{\bar{\Delta}',j}\cap(\boldsymbol{D}^n\times\Xi_k))\,.$$

This becomes a closed subset of $(\boldsymbol{D}^n\setminus\Omega)\times\Delta$ because the union in the right hand side is locally finite. Thus it gives the required decomposition. q.e.d.

From these arguments we can deduce in particular the global existence theorem for locally hyperbolic operators on *bounded* domain under the pointwise condition (3.1). Recall that on $S^{n-1}$ there always exists a stratification as follows: Put

$$\Xi_k=\{\xi\in S^{n-1};\ \deg(P_m)_\xi(\eta)=k\},\quad k=0,1,\cdots,m\,, \tag{3.11}$$

where $\deg(P_m)_\xi(\eta)$ denotes the degree of the localization of $P_m(\xi)$ at $\xi$. $\Xi_k$ is a locally closed subset of $S^{n-1}$. In fact, in view of the expansion

$$P_m(\xi+t\eta)=P_m(\xi)+\eta\nabla_\xi P_m(\xi)t+\cdots+(\eta\nabla_\xi)^m P_m(\xi)t^m\,,$$

we have

$$\Xi_k=\{\xi\in S^{n-1};\ (\eta\nabla_\xi)^j P_m(\xi)=0\ \text{ for }\ j\le k-1,\ \text{and}\ (\eta\nabla_\xi)^k P_m(\xi)\not\equiv 0\}\,.$$

Especially $\Xi_0$ is nothing but the set of non-characteristic directions of $P(D)$ and $\Xi_1$ that of simply characteristic ones. From $\Xi_2$ on there may be some void $\Xi_k$'s. Then we omit them and re-index the rest. Then we have obviously (3.5) for any $\Delta\subset S^{n-1}$ by the strata $\Delta\cap\Xi_k$ with a fixed $M\le m$.

Lemma 3.4. *Along the strata* (3.11) *of the above stratification the correspondence* $\xi\mapsto K_\xi$ *is lower semi-continuous.*

Proof. Let us denote the localization $(\eta\nabla_\xi)^k P_m(\xi)$ by $Q(\eta;\xi)$. It suffices to consider $\xi$ in a small neighborhood $\Delta$ for which $Q(\eta;\xi)$ is hyperbolic to a fixed vector $v$, and hence $K_\xi=\Gamma(Q(\eta;\xi),v)^\circ$, where $\Gamma(Q(\eta;\xi),v)$ denotes the connected component of $R^n_\eta\setminus N(Q(\eta;\xi))$ containing $v$ (which is well known to be a convex proper cone). Thus the problem is reduced to the following: Let $Q(\eta;\xi)$ be a family of homogeneous operators of a fixed degree depending continuously on the parameter $\xi$. Assume that $Q(\eta;\xi)$ are hyperbolic to the direction $v=(1,0,\cdots,0)$. Then the propagation cone $K_\xi$ of $Q(\eta;\xi)$ satisfies the condition (3.8) of lower semi-continuity.

Put $\Gamma_\xi=\Gamma(Q(\eta;\xi),v)$. Passing to the dual cones we can rewrite the condition (3.8) as follows:

(3.12) For each unit vector $x$ satisfying $-\{x\}^\circ\cap\Gamma_{\xi^0}=\emptyset$, given any $\varepsilon>0$ we can find $\delta>0$ such that for $|\xi-\xi^0|<\delta$ we have $-\Delta_\varepsilon(x)^\circ\cap\overline{\Gamma_\xi}=\emptyset$.

Here $\{x\}^\circ$ resp. $\Delta_\varepsilon(x)^\circ$ denotes the dual cone of the half line $\{tx;t\ge 0\}$ resp. the cone generated by the $\varepsilon$-neighborhood of $x$ in the unit sphere. For $\eta'\in R^{n-1}$ let $\tau(\eta';\xi)$ be the maximal root of the algebraic equation $Q(\tau,\eta';\xi)=0$ for $\tau$. Then $\partial\Gamma_\xi$ is generated by the vectors $(\tau(\eta';\xi),\eta')$, $|\eta'|=1$. The assertion $-\{x\}^\circ\cap\Gamma_{\xi^0}=\emptyset$ implies that

$$\langle x, (\tau(\eta'; \xi^0), \eta')\rangle \geq 0 \qquad \text{for } |\eta'|=1.$$

Since $Q(1, 0, \cdots, 0; \xi) \neq 0$ by assumption, the root $\tau(\eta'; \xi)$ is continuous in $(\eta'; \xi)$, hence given $\varepsilon > 0$ we can find $\delta > 0$ such that for $|\xi - \xi^0| < \delta$ we have

$$\langle x, (\tau(\eta'; \xi), \eta')\rangle > -\varepsilon \qquad \text{for } |\eta'|=1,$$

that is, the cone $(\eta; \langle x, \eta\rangle \leq -\varepsilon)$ does not intersect $\overline{\Gamma_\xi}$. q.e.d.

Summing up, for a bounded domain $\Omega$ we can conclude this article by the following rather agreeable result:

THEOREM 3.5. *Let $P(D)$ be locally hyperbolic and let $\Omega$ be bounded. Assume that for a set of micro-local fundamental solutions $\{E^{\Delta,j}\}$ (resp. a pair of global "good fundamental solutions" $E^{\pm}$) the pointwise geometric condition* (3.1) *(resp.* (3.2)*) is satisfied in relation to the local propagation cones $K_\xi^{\Delta,j}$ (resp. $\pm K_\xi$). Then we have $P(D)\mathscr{A}(\Omega) = \mathscr{A}(\Omega)$.*

FINAL REMARKS. 1) For convex $\Omega$ Zampieri [20] proves in fact a stronger assertion by assuming (3.1) not for the local propagation cone of $P_m$ itself but for that of each (local) irreducible component of $P_m$ (and by this refined form he has succeeded in proving also the necessity of the geometric condition when the localizations are at most of order 2). His result cannot be entirely covered by the mere enlargement of the possibility of choice of $K_\xi$ since it can treat an operator which is locally a product of locally hyperbolic factors but not in itself locally hyperbolic. Here we will only mention that a suitable decomposition of $P(D)$ would allow us to prove his result from our viewpoint. In fact, we could employ an argument similar to the proof of Theorem 3.2 in order to solve the equation successively by its factors. This process is in fact practicable for a homogeneous operator $P(D) = P_m(D)$. (In that case the condition (3.6) should be replaced by

$$(3.6)'' \qquad (D^n \setminus \Omega) \times \Xi_k = \bigcup_{j=1}^{N_\Delta} X^{\Delta,k,j}.$$

The generalized form of Proposition 3.3 assumes this application.)[3] We do not know yet, however, how to decompose $P(D)$ itself when it has lower

3) This was a misunderstanding of the author. In fact, solving e.g. $P_1(D)P_2(D)u = f$ successively, we first obtain $P_2(D)u = \tilde{f} * E_1$ by the fundamental solution $E_1$ of $P_1$, and the right-hand side has S.S. extended over $D^n \setminus \Omega$ in general. However, we may here replace the left-hand side by an extension of $\tilde{f} * E_1|_\Omega$ with minimal support. Hence a condition like (3.6) for each factor will be sufficient. But we here leave Proposition 3.3 as it is because its generalized form may still have some interest. The author is grateful to the referee for pointing out this.

order parts. (And we should notice that for $\Omega$ not necessarily convex we do not know yet a priori whether $P(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$ is equivalent to $P_m(D)\mathscr{A}(\Omega)=\mathscr{A}(\Omega)$ or not.)

2) The discussion of the necessary condition for the global existence from our (i.e. micro-local or non-topological) point of view is a chalenging problem. As far as we know, there is yet no theory applicable to the converse implication of the arrow (0.4)$\Rightarrow$(0.3) given in the introduction.

## Appendix: On the foundation of Fourier hyperfunctions of type $[s, \delta]$

Here we give briefly the proofs to the theorems on the foundation of our new class of Fourier hyperfunctions listed up in § 1. In the general line we follow the argument of Kawai [11], but we adopt simplifications introduced afterwards mainly by T. Oshima. Note that we can always assume $\delta=0$. In fact, the multiplication by $\exp(-\delta(z^2+1)^{s/2})$ induces a sheaf isomorphism between $\tilde{\mathcal{O}}^{s,\delta}$ and $\tilde{\mathcal{O}}^{s,0}$ on the complex neighborhood $C_s(C_1, C_2)$ of $\boldsymbol{D}^n$, where

(1.2) bis
$$C_s(C_1, C_2)=\{z\in \boldsymbol{C}^n; |\mathrm{Im}\, z|\leq C_1|\mathrm{Re}\, z|+C_2\}$$
$$(0<C_1<\min\{1, \tan^{-1}(\pi/4s)\},\ C_2>0)\,.$$

Nevertheless we shall retain the suffix $\delta$ (which is assumed to be equal to zero in the sequel) in order to keep the identity of the symbol. On the contrary we cannot reduce the parameter $s$ to 1. (A real analytic coordinate transformation of the type $x^s\mapsto x$ changes the topological structure of the product $\boldsymbol{D}^n\times\boldsymbol{R}^n\simeq\boldsymbol{D}^n+i\boldsymbol{R}^n$ when it is continued to the complex neighborhood.)

First we let $\tilde{L}^{s,\delta}_{2,loc}$ denote the sheaf of germs of local $L_2$-functions on $\boldsymbol{D}^n+i\boldsymbol{R}^n$ obeying the same growth (or decay) condition as (1.1) for $\tilde{\mathcal{O}}^{s,\delta}$. That is, for $U\subset\boldsymbol{D}^n+i\boldsymbol{R}^n$, $f(z)\in\tilde{L}^{s,\delta}_{2,loc}(U)$ implies that for any $K\Subset U$ and for any $\varepsilon>0$

$$f(z)e^{-(\delta+\varepsilon)|\mathrm{Re}\, z|^s}|_{K\cap\boldsymbol{C}^n}\in L_2(K\cap\boldsymbol{C}^n)\,.$$

(As is well known, for $f(z)\in\mathcal{O}(U\cap\boldsymbol{C}^n)$ the estimates by seminorms of $L_2$- and of maximum type are equivalent on account of the arbitrariness of $K\Subset U$ and $\varepsilon>0$.) Then we let $\tilde{L}^{s,\delta(0,p)}_{2,loc}$ denote the sheaf of germs of $(0,p)$-forms with coefficients in $\tilde{L}^{s,\delta}_{2,loc}$. Finally we let $\tilde{\mathscr{L}}^{s,\delta(0,p)}_{2,loc}$ denote the domain of the $\bar{\partial}$-operator $\tilde{L}^{s,\delta(0,p)}_{2,loc}\to\tilde{L}^{s,\delta(0,p+1)}_{2,loc}$, that is, the subsheaf consisting of those germs $f\in\tilde{L}^{s,\delta(0,p)}_{2,loc}$ such that $\bar{\partial}f$, calculated in the sense of distributions, belong to $\tilde{L}^{s,\delta(0,p+1)}_{2,loc}$. Since the multiplication by bounded $C^1$-functions with bounded first order derivatives conserves the domain of $\bar\partial$, $\tilde{\mathscr{L}}^{s,\delta(0,p)}_{2,loc}$ is a fine sheaf on $\boldsymbol{D}^n+i\boldsymbol{R}^n$ as well as $\tilde{L}^{s,\delta(0,p)}_{2,loc}$ or a fortiori $\tilde{L}^{s,\delta}_{2,loc}$.

PROPOSITION A.1. *Let $U\subset \boldsymbol{D}^n+i\boldsymbol{R}^n$ be a pseudo-convex (in the sense of Definition 1.2) open set such that $U\cap \boldsymbol{C}^n$ is contained in a set $C_s(C_1, C_2)$ of the form (1.2) bis. Then we have the following exact sequence of section modules:*

$$\text{(A.1)}\quad 0\longrightarrow\tilde{\mathscr{O}}^{s,\delta}(U)\longrightarrow\tilde{\mathscr{L}}^{s,\delta}_{2,loc}(U)\xrightarrow{\bar\partial}\tilde{\mathscr{L}}^{s,\delta(0,1)}_{2,loc}(U)\xrightarrow{\bar\partial}\cdots\longrightarrow\tilde{\mathscr{L}}^{s,\delta(0,n)}_{2,loc}(U)\longrightarrow 0\,.$$

We prove this by means of Hörmander's $L_2$-theory of the $\bar\partial$-complex. Write $C_s=C_s(C_1, C_2)$ for simplicity. First we prepare

LEMMA A.2. *Let $f(z)\in\tilde{L}^{s,\delta}_{2,loc}(U)$ with $U\subset \boldsymbol{D}^n+i\boldsymbol{R}^n$ satisfying $U\cap\boldsymbol{C}^n\subset C_s$. Then we can find $Q(z)\in\tilde{\mathscr{O}}^{s,\delta}(C_s)$ such that it is invertible in $C_s$ and that for every $K\Subset U$*

$$f(z)/Q(z)|_{K\cap \boldsymbol{C}^n}\in L_2(K\cap\boldsymbol{C}^n)\,.$$

PROOF. This is an $L_2$-variant of the calculus done in Kaneko [6]. Therefore we try to reduce it to the latter. There exists an absorbing sequence $\{K_j\}$ of compact subsets of $U$. Put

$$g_j(r)=\left(\int_{K_j\cap\{r-1\le|\mathrm{Re}\,z|\le r\}}|f(z)|^2d\mu\right)^{1/2},$$

where $d\mu$ is the Lebesgue measure of $\boldsymbol{C}^n$. Then the growth condition for $f\in\tilde{L}^{s,\delta}_{2,loc}(U)$ is equivalent to the following one on $g_j(r)$:

$$\text{(A.2)}\qquad \text{For } \forall\varepsilon>0 \text{ we have } g_j(r)=O(e^{\varepsilon r^s})\,.$$

(Recall that we are assuming $\delta=0$.) In fact, for $f\in\tilde{L}^{s,\delta}_{2,loc}(U)$ we have

$$\begin{aligned}&\left(\int_{K_j\cap\{r-1\le|\mathrm{Re}\,z|\le r\}}|f(z)|^2d\mu\right)^{1/2}\\ &\le\left(\int_{K_j\cap\{r-1\le|\mathrm{Re}\,z|\le r\}}|f(z)|^2e^{-2\varepsilon|\mathrm{Re}\,z|^s}d\mu\right)^{1/2}\cdot\max_{r-1\le|\mathrm{Re}\,z|\le r}e^{\varepsilon|\mathrm{Re}\,z|^s}\\ &\le\|f(z)e^{-\varepsilon|\mathrm{Re}\,z|^s}\|_{L_2(K_j\cap\boldsymbol{C}^n)}\cdot e^{\varepsilon r^s}\,,\end{aligned}$$

and conversely if $g_j(r)$ for $f$ satisfy the above condition then

$$\begin{aligned}&\|f(z)e^{-2\varepsilon|\mathrm{Re}\,z|^s}\|_{L_2(K_j\cap\boldsymbol{C}^n)}\\ &=\sum_{r=1}^{\infty}\|f(z)e^{-2\varepsilon|\mathrm{Re}\,z|^s}\|_{L_2(K_j\cap\{r-1\le|\mathrm{Re}\,z|\le r\})}\\ &\le\sum_{r=1}^{\infty}g_j(r)e^{-2\varepsilon(r-1)^s}\le\sum_{r=1}^{r_\varepsilon}g_j(r)+\sum_{r=r_\varepsilon}^{\infty}(g_j(r)e^{-\varepsilon r^s})e^{-(\varepsilon/2)r^s}<\infty\,,\end{aligned}$$

where $r_s \geq 1/(1-(3/4)^{1/s})$. Now put $h_j(r)=g_j(r^{1/s})$. Then $h_j(r)$ are of infra-exponential growth. By the proof of Lemma in Kaneko [6] we can find another continuous function $h(r)$ of infra-exponential growth (which can be assumed monotone increasing) such that $r^{2/s}h_j(r)=O(h(r))$ for all $j$. Then by Lemma 1.2 in Kaneko [6bis] we can construct an entire infra-exponential function $J(w)$ such that it is invertible in

$$\text{(A.3)} \qquad |\mathrm{Im}\, w| \leq c_1 |\mathrm{Re}\, w| + c_2$$

and that

$$|J(w)| \geq h(c_3|w|+c_4)$$

there. Here $c_1$ to $c_4$ are positive constants which can be assigned arbitrarily except the condition $c_1<1$. (In the cited article they are so chosen that $c_1=1/\sqrt{3}$, $c_2=1$. This generalization is obvious.) Put finally

$$Q(z)=J((1+z^2)^{s/2}).$$

Then with an appropriate choice of the constants $c_j$ this function satisfies all the requirements. In fact, for a sufficiently large choice of $c_1(<1)$, $c_2$ and $1/a$, the domain $C_s(C_1, C_2)$ will be mapped by the transformation $w=(a+z^2)^{s/2}$ into a subdomain of (A.3). There we have, for a suitable choice of $c_3$, $c_4$, (assuming $a=1$ for simplicity),

$$\begin{aligned}|J((1+z^2)^{s/2})| &\geq h(c_3|(1+z^2)^{s/2}|+c_4) \geq h((|\mathrm{Re}\, z|+1)^s) \\ &\geq \varepsilon_j(|\mathrm{Re}\, z|+1)^2 g_j(|\mathrm{Re}\, z|+1)\end{aligned}$$

with some constant $\varepsilon_j>0$. Thus

$$\begin{aligned}\|f(z)/Q(z)\|_{L_2(K_j\cap C^n)} &\leq \sum_{r=1}^{\infty} \|f(z)/J((1+z^2)^{s/2})\|_{L_2(K_j\cap\{r-1\leq|\mathrm{Re}\, z|\leq r\})} \\ &\leq \sum_{r=1}^{\infty} g_j(r)\frac{1}{\varepsilon_j g_j(r) r^2} < \infty .\end{aligned}$$

It is obvious that $Q(z) \in \tilde{\mathcal{O}}^{s,\delta}(C_s)$. q.e.d.

Proof of Proposition A.1. Take $f \in \mathscr{L}^{s,\delta(0,p)}_{2,loc}(U)$ such that $\bar{\partial}f=0$. By the above lemma we can find $Q(z) \in \tilde{\mathcal{O}}^{s,\delta}(U)$ such that $f/Q|_{K\cap C^n} \in L_2^{(0,p)}(K \cap C^n)$ for any compact subset $K \subset U$. We have obviously $\bar{\partial}(f/Q)=0$. Since $U \cap C^n$ is pseudo-convex by assumption, we can find a continuous plurisubharmonic function $\varphi(z)$ on $U \cap C^n$ with the properties 1)–2) in Definition 1.2. Moreover, for any convex continuous function $\chi(t)$ of one variable, the composed function $\chi(\varphi(z))$ also satisfies them. Now put

$$K_j=\overline{\{z\in U\cap C^n;\ \varphi(z)<j\}},\quad j=1,2,\cdots.$$

These $K_j$ constitute an absorbing sequence of compact subsets of $U$. We can obviously choose $\chi$ so that

$$\|f/Q\|_{L_2((K_j\setminus K_{j-1})\cap C^n)}\leq e^{\chi(j)}=\inf_{z\in(K_j\setminus K_{j-1})\cap C^n}|e^{\chi(\varphi(z))}|.$$

Thus we have found a plurisubharmonic function $\psi(z)=\chi(\varphi(z))$ such that $(f/Q)e^{-\psi(z)}\in L_2(U\cap C^n)$. Then Theorem 2.2.1′ of Hörmander [4] gives a solution $g$ of $\bar\partial g=f/Q$ such that $g(1+|z|^2)^{-2}e^{-\psi(z)}\in L_2(U\cap C^n)$. $Qg$ is our final solution. q.e.d.

The proof of the above proposition shows especially that

$$(A.4)\qquad 0\longrightarrow\tilde{\mathcal{O}}^{s,\delta}\longrightarrow\tilde{\mathscr{L}}^{s,\delta}_{2,loc}\xrightarrow{\bar\partial}\tilde{\mathscr{L}}^{s,\delta(0,1)}_{2,loc}\xrightarrow{\bar\partial}\cdots\longrightarrow\tilde{\mathscr{L}}^{s,\delta(0,n)}_{2,loc}\longrightarrow 0$$

is a fine resolution of the sheaf $\tilde{\mathcal{O}}^{s,\delta}$. Hence Theorem 1.3 follows at once from the same proposition.

Next, to prove Theorem 1.4 we also consider the dual resolution

$$0\longrightarrow\underset{\sim}{\mathcal{O}}^{s,\delta}\longrightarrow\underset{\sim}{\mathscr{L}}^{s,\delta}_{2,loc}\xrightarrow{\bar\partial}\underset{\sim}{\mathscr{L}}^{s,\delta(0,1)}_{2,loc}\xrightarrow{\bar\partial}\cdots\longrightarrow\underset{\sim}{\mathscr{L}}^{s,\delta(0,n)}_{2,loc}\longrightarrow 0.$$

Here $\underset{\sim}{L}^{s,\delta}_{2,loc}(U)$ is defined by the same growth condition as $\underset{\sim}{\mathcal{O}}^{s,\delta}(U)$ in (1.6) (with the supremum norm replaced by the $L_2$-norm) and the definitions of the other symbols are deduced from this one just as for the case of upper $\sim$. By a similar argument we can show $H^p(K,\underset{\sim}{\mathcal{O}}^{s,\delta})=0$ for $p\geq 1$ for a compact set $K\subset \boldsymbol{D}^n+i\boldsymbol{R}^n$ which admits a fundamental system of neighborhoods of pseudo-convex open sets as in Theorem 1.3. Moreover, for $U\subset\boldsymbol{D}^n+i\boldsymbol{R}^n$ we have the following dual complexes:

$$0\longrightarrow\tilde{\mathscr{L}}^{(0,0)}_{2,loc}(U)\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\tilde{\mathscr{L}}^{(0,n)}_{2,loc}(U)\longrightarrow 0$$
$$0\longleftarrow\underset{\sim}{\mathscr{L}}^{(0,n)}_{2,c}(U)\xleftarrow{-\bar\partial}\cdots\xleftarrow{-\bar\partial}\underset{\sim}{\mathscr{L}}^{(0,n)}_{2,c}(U)\longleftarrow 0,$$

where the subscript $c$ denotes the sections with compact supports. These complexes consist of FS*- resp. DFS*-spaces in the sense of Komatsu [24] and have the same cohomology groups as the complexes

$$0\longrightarrow\tilde{L}^{(0,0)}_{2,loc}(U)\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\tilde{L}^{(0,n)}_{2,loc}(U)\longrightarrow 0$$
$$0\longleftarrow\underset{\sim}{L}^{(0,n)}_{2,c}(U)\xleftarrow{-\bar\partial}\cdots\xleftarrow{-\bar\partial}\underset{\sim}{L}^{(0,n)}_{2,c}(U)\longleftarrow 0,$$

which consist of the same type of spaces but with unbounded operators. Thus we can apply the Serre-Komatsu duality ([24], Theorem 19) for each pair of cohomology groups if one of the complexes consists of operators with closed range. (See also addendum 2 below.) From these facts we can deduce Theorem 1.4 just as in the case of ordinary hyperfunctions following the argument of Martineau-Harvey (see Kawai [11], pp. 480–482). The outline is as follows:

Sketch of Proof of Theorem 1.4. Let $U \supset K$ be a pseudo-convex neighborhood. From the long exact sequence for the pair $U \setminus K \subset U$ we obtain, in view of Theorem 1.3,

$$\begin{aligned} &H^1_K(U, \tilde{\mathcal{O}}^{s,\delta}) = \tilde{\mathcal{O}}^{s,\delta}(U \setminus K)/\tilde{\mathcal{O}}^{s,\delta}(U) \\ &H^p_K(U, \tilde{\mathcal{O}}^{s,\delta}) = H^{p-1}(U \setminus K, \tilde{\mathcal{O}}^{s,\delta}) \qquad \text{for } p \geq 2 . \end{aligned} \tag{A.5}$$

On the other hand, from the long exact sequence with compact support for the pair $K \subset U$ we obtain, taking into account the principle of analytic continuation and the fact $H^p(K, \underset{\sim}{\mathcal{O}}^{s,\delta}) = 0$ for $p \geq 1$,

$$\begin{aligned} &H^1_c(U \setminus K, \underset{\sim}{\mathcal{O}}^{s,\delta})/\underset{\sim}{\mathcal{O}}^{s,\delta}(K) = H^1_c(U, \underset{\sim}{\mathcal{O}}^{s,\delta}) \\ &H^p_c(U \setminus K, \underset{\sim}{\mathcal{O}}^{s,\delta}) = H^p_c(U, \underset{\sim}{\mathcal{O}}^{s,\delta}) \qquad \text{for } p \geq 2 . \end{aligned}$$

By the above mentioned duality we obtain

$$\begin{aligned} &H^p_c(U, \underset{\sim}{\mathcal{O}}^{s,\delta}) = H^{n-p}(U, \tilde{\mathcal{O}}^{s,\delta})' = 0 \qquad \text{for } 0 \leq p \leq n-1 , \\ &H^n_c(U, \underset{\sim}{\mathcal{O}}^{s,\delta}) = \tilde{\mathcal{O}}^{s,\delta}(U)' , \end{aligned}$$

hence

$$\begin{aligned} &H^1_c(U \setminus K, \underset{\sim}{\mathcal{O}}^{s,\delta}) = \underset{\sim}{\mathcal{O}}^{s,\delta}(K) , \\ &H^p_c(U \setminus K, \underset{\sim}{\mathcal{O}}^{s,\delta}) = 0 \quad (2 \leq p \leq n-1) , \\ &H^n_c(U \setminus K, \underset{\sim}{\mathcal{O}}^{s,\delta}) = \tilde{\mathcal{O}}^{s,\delta}(U)' . \end{aligned} \tag{A.6}$$

We can show that the first and the third isomorphisms of (A.6) are even topological. (This can be shown e.g. by proving the closed range property of the concerning mappings an in the proof of Theorem 1.5 below.) Thus we can again apply the duality theorem with $U \setminus K$ in place of $U$ to obtain

$$\begin{aligned} &H^n(U \setminus K, \tilde{\mathcal{O}}^{s,\delta}) = H^0_c(U \setminus K, \underset{\sim}{\mathcal{O}}^{s,\delta})' = 0 , \\ &H^{n-1}(U \setminus K, \tilde{\mathcal{O}}^{s,\delta}) = \underset{\sim}{\mathcal{O}}^{s,\delta}(K)' , \\ &H^p(U \setminus K, \tilde{\mathcal{O}}^{s,\delta}) = 0 \quad (1 \leq p \leq n-2) , \\ &\tilde{\mathcal{O}}^{s,\delta}(U \setminus K) = H^n_c(U \setminus K, \underset{\sim}{\mathcal{O}}^{s,\delta})' = \tilde{\mathcal{O}}^{s,\delta}(U) . \end{aligned}$$

Combined with the isomorphisms (A.5) at the beginning these establish Theorem 1.4. (On account of the excision theorem, $U$ may finally be arbitrary.)

PROOF OF THEOREM 1.5. This is to show the exactness of $\tilde{\mathscr{L}}^{s,\delta(0,n-1)}_{2,loc}(U)\xrightarrow{\bar\partial}\tilde{\mathscr{L}}^{s,\delta(0,n)}_{2,loc}(U)\to 0$ or equivalently of $\underset{\sim}{L}^{s,\delta(0,n-1)}_{2,loc}(U)\xrightarrow{\bar\partial}\underset{\sim}{L}^{s,\delta(0,n)}_{2,loc}(U)\to 0$. In view of the duality theorem it suffices to show that equivalently its dual

$$\text{(A.7)}\qquad \underset{\sim}{L}^{s,\delta(0,1)}_{2,c}(U)\xleftarrow{-\bar\partial}\underset{\sim}{L}^{s,\delta(0,0)}_{2,c}(U)\longleftarrow 0$$

has closed range[4]. This is surely true if $U$ is replaced by a larger pseudo-convex open set $V$ because then $H^p(V,\tilde{\mathcal{O}}^{s,\delta})=0$ for $p\geq 1$. Thus the problem reduces to the estimation of support of the solution of the system $\bar\partial u=f$ employing the principle of analytic continuation: Note that in a DFS*-space we can test the closedness by converging *sequences*. So assume that $u_k\in\underset{\sim}{L}^{s,\delta}_{2,c}(U)$, and $-\bar\partial u_k\to f$ in $\underset{\sim}{L}^{s,\delta(0,1)}_{2,c}(U)$. Then the convergence takes place in $\underset{\sim}{L}^{s,\delta(0,1)}_{2,c}(V)$, hence by the closed range property of

$$\text{(A.7)}'\qquad \underset{\sim}{L}^{s,\delta(0,1)}_{2,c}(V)\xleftarrow{-\bar\partial}\underset{\sim}{L}^{s,\delta(0,0)}_{2,c}(V)\longleftarrow 0$$

we can find $u\in\underset{\sim}{L}^{s,\delta}_{2,c}(V)$ such that $-\bar\partial u=f$. Since $u$ is holomorphic outside supp $f$, in view of the uniqueness of analytic continuation we see that supp $u\subset K$, where $K$ is a compact subset of $V$ which may contain some connected components of $(\boldsymbol{D}^n+i\boldsymbol{R}^n)\setminus U$ contained in $V$ but on which $u$ is holomorphic. It remains to show that $u$ is in fact zero on these components. Again by the closedness of (A.7)′ and the stability of DFS* property for closed subspaces, we can apply the open mapping theorem and find some sequence $v_k$, $v\in\underset{\sim}{L}^{s,\delta}_{2,c}(V)$ such that $-\bar\partial v_k=-\bar\partial u_k$, $-\bar\partial v=-\bar\partial u$ and that $v_k\to v$ in $\underset{\sim}{L}^{s,\delta}_{2,c}(V)$. But by the condition of support we must have $v_k=u_k$, $v=u$. Hence $u_k\to u$ in $\underset{\sim}{L}^{s,\delta}_{2,c}(V)$ and especially in $\mathscr{D}'(V\cap\boldsymbol{C}^n)$. By the local character of the topology of the latter space, we thus conclude that supp $u$ must be contained in $U$. This shows that (A.7) has closed range. q.e.d.

Sketch of Proof of Theorem 1.6. Let $\Omega\subset\boldsymbol{D}^n$ be open and let $U\supset\bar\Omega$ be a pseudo-convex neighborhood. Applying Theorems 1.4 and 1.5 to the fundamental long exact sequence for the pair $\partial\Omega\subset\bar\Omega$:

$$\cdots H^p_{\partial\Omega}(U,\tilde{\mathcal{O}}^{s,\delta})\longrightarrow H^p_{\bar\Omega}(U,\tilde{\mathcal{O}}^{s,\delta})\longrightarrow H^p_{\Omega}(U,\tilde{\mathcal{O}}^{s,\delta})\longrightarrow\cdots,$$

we see at once

$$H^p_{\Omega}(U,\tilde{\mathcal{O}}^{s,\delta})=0\qquad\text{for } p\neq n-1,\ n\,.$$

To prove $H^{n-1}_{\Omega}(U,\tilde{\mathcal{O}}^{s,\delta})=0$ we further need the injectivity of $H^n_{\partial\Omega}(U,\tilde{\mathcal{O}}^{s,\delta})\to H^n_{\bar\Omega}(U,\tilde{\mathcal{O}}^{s,\delta})$ or equivalently the denseness of range of the dual map $\underset{\sim}{Q}^{s,\delta}(\bar\Omega)\to\underset{\sim}{Q}^{s,\delta}(\partial\Omega)$. Since the proof of Theorem 2.2.1 in Kawai [11] is rather long

4) The algebraic exactness of (A.7) is obvious from the principle of analytic continuation.

though it provides a more general approximation theorem, we give here a short proof for our need. Let $\{\Gamma_j\}_{j=1}^N$ denote a set of open convex polyhedral cones whose dual $\{\Gamma_j^\circ\}_{j=1}^N$ defines a decomposition of $S^{n-1}$, and let $W_s(z, \Gamma_j^\circ)$ be the holomorphic function defined by (1.11) with $\Gamma_j^\circ \cap S^{n-1}$ in place of $\Delta^\circ$. Now let $\varphi(z) \in \mathcal{O}^{s,\delta}(\partial\Omega)$ and assume that $\varphi(z) \in \mathcal{O}^{s,\delta}(K)$ for a compact complex neighborhood $K \Supset \partial\Omega$. Put

$$F_j(z)=\int_{K \cap R^n} W_s(z-\zeta, \Gamma_j^\circ)\varphi(\zeta)d\zeta, \qquad j=1, \cdots, N.$$

Then as a special case of curved Radon decomposition we can show that $F_j(z)$ becomes an element of $\mathcal{O}^{s,\delta}((D^n+i\Gamma_j 0) \cup \mathrm{Int}(K))$ and that

$$\varphi(z)=\sum_{j=1}^N F_j(z) \qquad \text{on } \mathrm{Int}(K).$$

Therefore it suffices to approximate each $F_j(z)$ by elements of $\mathcal{O}^{s,\delta}(\bar{\Omega})$ or rather of $\mathcal{O}^{s,\delta}(D^n)$. Put

$$F_j^\varepsilon(z)=F_j(z+i\varepsilon y_j), \qquad j=1, \cdots, N,$$

where $y_j \in \Gamma_j$ is a fixed element. Then it is obvious that $F_j^\varepsilon(z) \in \mathcal{O}^{s,\delta}(D^n)$ for sufficiently small $\varepsilon>0$ and $F_j^\varepsilon(z) \to F_j(z)$ in $\mathcal{O}^{s,\delta}(\partial\Omega)$ when $\varepsilon \to 0$. q.e.d.

Now we turn to the foundation of micro-analyticity of Fourier hyperfunctions. We follow the argument in Kaneko [9bis] for the case of ordinary hyperfunctions which unites the ideas of Komatsu, Kashiwara and Kataoka. The following lemma is fundamental:

LEMMA A.3. *Let $\Omega \subset D^n$ be open, $(x^0, \xi^0) \in \Omega$ and let $f(x) \in \mathcal{Q}^{s,\delta}(\Omega)$. Let $D \Subset \Omega$ be a neighborhood of $(x^0, \xi^0)$ and let $f(x)=\sum_{j=1}^N F_j(x+i\Gamma_j 0)$ be a boundary vlaue expression on a neighborhood of $\bar{D}$. Then we have $(x^0, \xi^0) \notin$ S.S.$^{s,\delta}f$ if and only if for one (equivalently any) choice of sufficiently small $y^{(j)} \in \Gamma_j$ the function*

$$(f * W_s)_{D,\{F_j\},\{y^{(j)}\}}(z, \zeta)=\sum_{j=1}^N \int_{D+iy^{(j)}} F_j(x) W_s(z-x, \zeta)\, dx \tag{A.8}$$

*which is in general holomorphic in $(z, \zeta)$ on the infinitesimal half-space $\mathrm{Im}\{z\zeta+i(z^2-(z\zeta)^2)\}>0$ on a neighborhood of $(x^0, \xi^0)$, extends as a section of $\tilde{\mathcal{O}}_z^{s,\delta}\hat{\otimes}\mathcal{O}_\zeta$ to a full neighborhood of $(x^0, \xi^0)$.*

(Here we denote by $\tilde{\mathcal{O}}_z^{s,\delta}\hat{\otimes}\mathcal{O}_\zeta$ the sheaf on $(D^n+iR^n)\times C^n$ of functions holomorphic in $(z, \zeta)$ and satisfying the same growth condition as $\tilde{\mathcal{O}}^{s,\delta}$ with respect to $z$ locally uniformly in the parameter $\zeta$. The symbol of the tensor product may be considered only formal.)

PROOF. By the Cauchy-Poincaré theorem we can deform the path of integral $D+iy^{(j)}$ as close to the real axis as we desire in the part $\mathrm{Int}(D)$. Hence (A.8) is always holomorphic on the infinitesimal half-space $\mathrm{Im}\{z\zeta+i(z^2-(z\zeta)^2)\}>0$ on a neighborhood of $(x^0, \xi^0)$. The change of the domain of integral $D$ or of the imaginary levels $y^{(j)}\in\Gamma_j$ (if the latter are sufficiently small compared to the size of $D$) affect the result at $(x^0, \xi^0)$ only by a section of $\tilde{\mathcal{O}}_z^{s,\delta}\hat{\otimes}\mathcal{O}_\zeta$ on a full neighborhood of $(x^0, \xi^0)$. Further, the change of the boundary value expression is realized by the repetition of the calculus such as $F_j(x+i\Gamma_j 0)+F_k(x+i\Gamma_k 0)=(F_j+F_k)(x+i(\Gamma_j\cap\Gamma_k)0)$ which obviously commutes with the integral if the imaginary level is chosen to a common value in $\Gamma_j\cap\Gamma_k$. (See Remark after the proof of Theorem 1.15 below.) Thus (A.8) neither depends on the local boundary value expression modulo a section of $\tilde{\mathcal{O}}_z^{s,\delta}\hat{\otimes}\mathcal{O}_\zeta$ at $(x^0, \xi^0)$.

Now the $j$-th individual term in (A.8) is originally holomorphic in the open set

$$\mathrm{Im}\{(z-iy^{(j)})\zeta+i((z-iy^{(j)})^2-((z-iy^{(j)})\zeta)^2)\}>0$$

which will contain $(\boldsymbol{R}^n+i\{|y|<\varepsilon\})\times\{|\xi-\xi^0|<\varepsilon\}$ for $\varepsilon$ small if $\xi\in\Gamma_j^\circ$ and $y^{(j)}\in\Gamma_j$ is so chosen that $\langle y^{(j)}, \xi^0\rangle<0$ with $|y^{(j)}|$ sufficiently small. The growth condition is obviously verified. Thus we see that if $(x^0, \xi^0)\notin \mathrm{S.S.}^{s,\delta}f$, then under a suitable choice of boundary value expression (which comes from the definition of S.S.) the function (A.8) extends to a section of $\tilde{\mathcal{O}}_z^{s,\delta}\hat{\otimes}\mathcal{O}_\zeta$ on a neighborhood of $(x^0, \xi^0)$.

Conversely put $G(z, \zeta)=(f*W_s)_{D,\{F_j\},\{y^{(j)}\}}(z, \zeta)$ and assume that $G(z, \zeta)$ defines a section of $\tilde{\mathcal{O}}_z^{s,\delta}\hat{\otimes}\mathcal{O}_\zeta$ on a real neighborhood $\bar{D}\times\bar{\Delta}\subset\boldsymbol{D}^n\times\boldsymbol{S}^{n-1}$ of $(x^0, \xi^0)$. Let $\{\Delta_k\}_{k=1}^M$ denote a covering by pyramids such that $\boldsymbol{S}^{n-1}\setminus\mathrm{Int}(\Delta)\subset\bigcup_{k=1}^M\bar{\Delta}_k\subset\boldsymbol{S}^{n-1}\setminus\{\xi^0\}$. Then in view of the Radon decomposition formula (1.10) we have

$$f(x)=\sum_{k=1}^M G_k(x+i\Delta_k 0) \qquad \mathrm{mod}\ \tilde{\mathcal{O}}^{s,\delta}(D)$$

on $D$, where

$$G_k(z)=\int_{\Delta_k} G(z, \xi)d\xi\,.$$

This shows that $(x^0, \xi^0)\notin \mathrm{S.S.}^{s,\delta}f$ by definition. q.e.d.

A precise proof of Theorem 1.13 follows immediately from this lemma. The same argument also gives the following

COROLLARY A.4. *Let $\Omega\subset\boldsymbol{D}^n$ be open, $\Gamma$ be an open convex cone and* *assume that* $f \in \mathscr{Q}^{s,\delta}(\boldsymbol{D})$ *satisfy* $\mathrm{S.S.}^{s,\delta} f \subset \Omega \times i\Gamma^\circ dx\infty$. *Then we can find* $F(z) \in \tilde{\mathscr{O}}^{s,\delta}(\Omega + i\Gamma 0)$ *such that* $f(x) = F(x + i\Gamma 0)$. *Especially, if* $\mathrm{S.S.}^{s,\delta} f = 0$, *then* $f(x) \in \mathscr{P}^{s,\delta}(\Omega)$.

In fact, local defining functions with such a property are given by the Radon decomposition (1.10) as in the above proof (first with a cone $\Delta \Subset \Gamma$). Since we have obviously the local uniqueness for such defining functions from a single wedge (which comes at once from the corresponding assertion for $\mathscr{B}$), the local ones can be glued together to a global one (and finally with the initial $\Gamma$).

PROOF OF THEOREM 1.15 is performed by induction on $N$. For $N=2$, putting $f(x) = F_1(x + i\Gamma_1 0) = -F_2(x + i\Gamma_2 0)$, we know that $\mathrm{S.S.}^{s,\delta} f \subset \boldsymbol{D}^n \times i(\Gamma_1^\circ \cap \Gamma_2^\circ) dx\infty = \boldsymbol{D}^n \times i(\Gamma_1 + \Gamma_2)^\circ dx\infty$. Hence by the above corollary we can find $H_{12}(z) \in \tilde{\mathscr{O}}^{s,\delta}(\boldsymbol{D}^n + i(\Gamma_1 + \Gamma_2)0)$ such that $f(x) = H_{12}(x + i(\Gamma_1 + \Gamma_2)0)$. Assume that the assertion is true for $N$, and let $\sum_{j=1}^{N+1} F_j(x + i\Gamma_j 0) = 0$. Choose $\Delta'_j$ such that $\Delta_j \Subset \Delta'_j \Subset \Gamma_j$. Put $f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0) = -F_{N+1}(x + i\Gamma_{N+1} 0)$. Then we have

$$\mathrm{S.S.}^{s,\delta} f \subset \boldsymbol{D}^n \times \bigcup_{j=1}^{N} i(\Gamma_j^\circ \cap \Gamma_{N+1}^\circ) dx\infty \subset \boldsymbol{D}^n \times i \,\mathrm{Int}\Big(\bigcup_{j=1}^{N} (\Delta'_j + \Delta'_{N+1})^\circ\Big) dx\infty \,.$$

By Theorem 1.13 we can find $H_{N+1,j}(z) \in \tilde{\mathscr{O}}^{s,\delta}(\boldsymbol{D}^n + i(\Delta'_j + \Delta'_{N+1})0)$ such that

$$F_{N+1}(x + i\Gamma_{N+1} 0) = \sum_{j=1}^{N} H_{N+1,j}(x + i(\Delta'_j + \Delta'_{N+1})0) \,.$$

In view of the uniqueness of the defining function from one wedge this implies

$$F_{N+1}(z) = \sum_{j=1}^{N} H_{N+1,j}(z) \qquad \text{in} \quad \boldsymbol{D}^n + i\Delta'_{N+1} 0 \,.$$

Now apply the induction hypothesis to the relation $\sum_{j=1}^{N} (F_j + H_{N+1,j})(x + i\Delta'_j 0) = 0$ to obtain $H_{jk}(z) \in \tilde{\mathscr{O}}^{s,\delta}(D^n + i(\Delta_j + \Delta_k)0)$ satisfying

$$F_j(z) + H_{N+1,j}(z) = \sum_{k=1}^{N} H_{jk}(z) \qquad \text{in} \quad \boldsymbol{D}^n + i\Delta_j 0 \,.$$

Putting $H_{j,N+1}(z) = -H_{N+1,j}(z)$ we thus obtain a set of required functions.

q.e.d.

REMARK. In the basis of the proof of Lemma A.3 to establish the well-definedness (modulo $\tilde{\mathscr{O}}_z^{s,\delta} \hat{\otimes} \mathscr{O}_\zeta$) of the definite integral $f * \underset{\sim}{W}_s$ we have employed an assertion which is a consequence of a local, weakest form of the edge of the wedge theorem of Martineau's type: That is, two different local boundary value expressions can be deformed from one to the other by the repetition of the calculus of the form $F_j(x+i\Gamma_j 0)+F_k(x+i\Gamma_k 0)=(F_j+F_k)(x+i(\Gamma_j\cap\Gamma_k)0)$. For our standpoint this assertion follows from Lemma 1.8 rather by the definition. Since Lemma A.3 is so fundamental we give in the Addendum another proof not employing Lemma 1.8 for the reader who cannot avail [9bis].

PROOF OF PROPOSITION 1.21. First we decompose $f|_{\boldsymbol{R}^n}$ into the sum $f_1+\cdots+f_N$ such that $\mathrm{S.S.}f_j\subset X^j\cap(\boldsymbol{R}^n\times\boldsymbol{S}^{n-1})$ employing the flabbiness of the usual sheaf $\mathscr{C}$ of microfunctions. In order to modify this at infinity to obtain the required decomposition employing the flabbiness of $\mathscr{Q}^{s,\delta}/\mathscr{P}^{s,\delta}$ and the Radon decomposition (1.10), we must enter some technical details as follows:

For each $j$ choose a finite number of closed sets $\bar{\Delta}^{j,k}\subset\boldsymbol{S}^{n-1}$ (without common interior points) and correspondingly closed cones $K^{j,k}$ (not necessarily connected nor convex) satisfying

$$X^j\cap(\boldsymbol{S}^{n-1}_\infty\times\boldsymbol{S}^{n-1})\Subset\bigcup_k K^{j,k}_\infty\times\bar{\Delta}^{j,k}\Subset W^j\,,$$

where $K^{j,k}_\infty=\overline{\overline{K^{j,k}}}\cap\boldsymbol{S}^{n-1}_\infty$. Decompose $X^j$ correspondingly to the union of closed subsets $\bigcup_k X^{j,k}$ in such a way that

$$X^{j,k}\cap(\boldsymbol{S}^{n-1}_\infty\times\boldsymbol{S}^{n-1})\Subset K^{j,k}_\infty\times\bar{\Delta}^{j,k},\qquad\text{in}\quad\boldsymbol{S}^{n-1}_\infty\times\boldsymbol{S}^{n-1}\,.$$

Now decompose $f_j$ to the sum $\sum_k f_{j,k}$ such that

$$\mathrm{S.S.}f_{j,k}\subset X^{j,k}\cap(\boldsymbol{R}^n\times\boldsymbol{S}^{n-1})\,.$$

By virtue of Theorem 1.10 we can find $\tilde{f}_{j,k}\in\mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$ such that

$$\tilde{f}_{j,k}|_{\boldsymbol{R}^n}-f_{j,k}\in\mathscr{A}(\boldsymbol{R}^n),\qquad\operatorname{sing\,supp}\tilde{f}_{j,k}\subset\overline{\overline{\operatorname{sing\,supp}f_{j,k}}}\,,$$

hence

$$\operatorname{sing\,supp}\tilde{f}_{j,k}\cap\boldsymbol{S}^{n-1}_\infty\subset K^{j,k}_\infty\,.\tag{A.9}$$

Now let $\bar{\Delta}'^{j,k}\supset\bar{\Delta}^{j,k}$ be such that

$$K^{j,k}_\infty\times\bar{\Delta}^{j,k}\subset K^{j,k}_\infty\times\bar{\Delta}'^{j,k}\Subset W^j\,.$$

Without loss of generality we can assume that

$$\mathrm{S.S.}^{s,\delta}\tilde{f}_{j,k}\cap\boldsymbol{S}^{n-1}_\infty\times\boldsymbol{S}^{n-1}\subset K^{j,k}_\infty\times\bar{\Delta}'^{j,k}\,.\tag{A.10}$$

In fact, we have

$$\tilde{f}_{j,k}=\tilde{f}_{j,k}*\underset{\sim}{W}_s(x, \boldsymbol{S}^{n-1}\setminus\bar{\Delta}'^{j,k})+\tilde{f}_{j,k}*\underset{\sim}{W}_s(x, \bar{\Delta}'^{j,k}).$$

The second term obviously satisfies (A.10). In view of (A.9) the singular support of the first term splits into the two disjoint parts, the one which is a compact subset of $\boldsymbol{R}^n$ and the other in $\boldsymbol{S}_\infty^{n-1}$. Hence in view of the flabbiness of $\mathscr{Q}^{s,\delta}/\mathscr{P}^{s,\delta}$ we can pick up the part $\tilde{f}'_{j,k}\in\mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$ which conveys the singularity of $\tilde{f}_{j,k}$ in $\boldsymbol{R}^n$ and adopt it as the new $\tilde{f}_{j,k}$. Thus we have obtained $\tilde{f}_j=\sum_k\tilde{f}_{j,k}$ such that

$$\text{S.S.}^{s,\delta}\tilde{f}_j\subset(X^j\cap(\boldsymbol{R}^n\times\boldsymbol{S}^{n-1}))\sqcup W^j, \qquad (f-\textstyle\sum\tilde{f}_j)|_{\boldsymbol{R}^n}\in\mathscr{A}(\boldsymbol{R}^n).$$

Now it remains to give a decomposition of $g=f-\sum\tilde{f}_j$ satisfying $\text{S.S.}^{s,\delta}g\subset\bigcup_{j=1}^N W^j$ to the sum $\sum g_j$ satisfying $\text{S.S.}^{s,\delta}g_j\subset W^j$. We can do this just in the same way as above by choosing an intermediate covering of the form $\bigcup_k K_\infty^{j,k}\times\bar{\Delta}^{j,k}$ and employing the convolution by $\underset{\sim}{W}_s(x, \bar{\Delta}^{j,k})$. The final ambiguity is in $\mathscr{P}^{s,\delta}(\boldsymbol{D}^n)$ and may be added to any of thus obtained solutions $\tilde{f}_j+g_j$. q.e.d.

## Addendum 1. A direct proof for well-definedness of $f*\underset{\sim}{W}_s$.

Since the edge of the wedge theorem of Martineau's type is employed by some authors to show the well definedness of the fundamental operations such as the definite integral by an elementary approach, the argument employed here may seem a kind of vicious circle to the reader who cannot avail the reference [9bis]. Therefore we give here another way of showing the well definedness of the integral $f*\underset{\sim}{W}_s$ (mod $\tilde{\mathscr{O}}_z^{s,\delta}\hat{\otimes}\mathscr{O}_\zeta$).

First note that for $f\in\mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$ the definite integral for $f*\underset{\sim}{W}_s$ can be well defined by the inner product $\langle f(x), \underset{\sim}{W}_s(z-x,\zeta)\rangle_x$ between $\mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$ and $\mathscr{P}^{s,\delta}(\boldsymbol{D}^n)$. For $f(x)=F(x+i\Gamma 0)\in\mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)$ this agrees with

$$\int_{\boldsymbol{R}^n+iy}F(x)\underset{\sim}{W}_s(z-x,\zeta)\,dx$$

for any fixed $y\in\Gamma$. In fact, by a linear coordinate transformation we can assume without loss of generality that $\Gamma$ properly contains the first orthant and hence $F(x+i\Gamma 0)$ corresponds to the cohomology class defined by $F(z)$, $0,\cdots,0$ in the Čech cohomology representation

$$\tilde{\mathscr{O}}^{s,\delta}(U\#\boldsymbol{D}^n)/\sum_{j=1}^n\tilde{\mathscr{O}}^{s,\delta}(U\#_j\boldsymbol{D}^n)$$

of $\mathscr{Q}^{s,\delta}(\boldsymbol{D}^n)=H^n_{D^n}(U,\tilde{\mathscr{O}}^{s,\delta})$, where $U\supset \boldsymbol{D}^n$ is a pseudo-convex neighborhood satisfying the condition of Theorem 1.3 and

$$U_j=\{z\in U;\ \operatorname{Im} z_j\neq 0\},\quad j=1,\cdots,n\,,$$
$$U\sharp \boldsymbol{D}^n=\bigcap_{j=1}^n U_j,\qquad U\sharp_j \boldsymbol{D}^n=\bigcap_{k\neq j} U_k\,.$$

The analysis of the Martineau-Harvey duality by this representation gives in general

$$\langle [G(z)],\varphi(z)\rangle=\sum_\sigma \operatorname{sgn}\sigma\int_{R^n+iy^{(\sigma)}} G(z)\varphi(z)dz$$

for $G(z)\in\tilde{\mathscr{O}}^{s,\delta}(U\sharp \boldsymbol{D}^n)$ and $\varphi(z)\in\mathscr{Q}^{s,\delta}(U)$, where $\sigma=(\sigma_1,\cdots,\sigma_n)$, $\sigma_j=\pm 1$, $\operatorname{sgn}\sigma=\sigma_1\cdots\sigma_n$ and $y^{(\sigma)}$ is fixed in the $\sigma$-th orthant $\{\operatorname{Im}\sigma_j z_j>0, j=1,\cdots,n\}$. The above formula for $\langle F(x+i\Gamma 0),\ \underset{\sim}{W}_s(z-x,\zeta)\rangle_x$ is a particular case of this one.

Next consider the case where $\operatorname{supp} f$ is contained in a compact subset $K$ of $\boldsymbol{D}^n$. Then the above defined result of $f*\underset{\sim}{W}_s$ is in $\tilde{\mathscr{O}}^{s,\delta}_z\hat{\otimes}\mathscr{O}_\zeta$ on $(\boldsymbol{D}^n\setminus K)\times \boldsymbol{S}^{n-1}$. This is because the inner product can then be extended to the one between $\mathscr{Q}^{s,\delta}(K)$ and $\mathscr{P}^{s,\delta}(K)(=\mathscr{Q}^{s,\delta}(K))$ and $\underset{\sim}{W}_s(z-x,\zeta)$ is a $\mathscr{P}^{s,\delta}(K)$-valued holomorphic function on a neighborhood of $(\boldsymbol{D}^n\setminus K)\times \boldsymbol{S}^{n-1}$.

Consider finally $f(x)=F(x+i\Gamma 0)\in\mathscr{Q}^{s,\delta}(\Omega)$. Choose an extension $\tilde{f}(x)$ with support in $\overline{\overline{\Omega}}$. We again assume that $\Gamma$ properly contains the first orthant and employ the above Čech cohomology representation $[G(z)]$ of $\tilde{f}(x)$. By what is just remarked above, the inner product

$$\langle \tilde{f}(x),\ \underset{\sim}{W}_s(z-x,\zeta)\rangle=\sum_\sigma \operatorname{sgn}\sigma\int_{R^n+iy^{(\sigma)}} G(x)\underset{\sim}{W}_s(z\quad x,\zeta)dx$$

is independent of the choice of $\tilde{f}$ modulo $\tilde{\mathscr{O}}^{s,\delta}_z\hat{\otimes}\mathscr{O}_\zeta$ on a neighborhood of a fixed point $(x^0,\xi^0)\in\Omega\times \boldsymbol{S}^{n-1}$. As such we can even confine the integral region to a neighborhood $D\subset\Omega$ of $x^0$:

$$\sum_\sigma \operatorname{sgn}\sigma\int_{D+iy^{(\sigma)}} G(x)\underset{\sim}{W}_s(z-x,\zeta)dx\,,$$

the result being independent of the choice of small $y^{(\sigma)}$ or of $D$ just by the same argument as used in the proof of Lemma A.3. Recall that $[G(z)]=[F(z),0,\cdots,0]$ on $D$. Via similar Čech cohomology representation with $U$ replaced by a pseudo-convex neighborhood $V$ of $D$, this implies that the difference is in $\sum_{j=1}^n\tilde{\mathscr{O}}^{s,\delta}(V\sharp_j \boldsymbol{D}^n)$. Hence modulo $\tilde{\mathscr{O}}^{s,\delta}_z\hat{\otimes}\mathscr{O}_\zeta$ at $(x^0,\xi^0)$ we can deform the last integral to

$$\int_{D+iy} F(x)\underset{\sim}{W}_s(z-x,\zeta)dx$$

with $y \in \Gamma$. Since the definition is obviously linear in $f$, we have established that (A.8) modulo $\tilde{\mathcal{O}}_z^{s,\delta}\hat{\otimes}\mathcal{O}_\zeta$ depends only on the cohomology class of $f$. q.e.d.

Note that in our proof of the edge of the wedge theorem the study of $\mathscr{H}^n_{D^n+i\Gamma}(\tilde{\mathcal{O}}^{s,\delta})$ is replaced by the use of the curved Radon decomposition.

## Addendum 2: Remarks on the duality between cohomology groups of complexes of closed operators.

In the foundation of the theory of hyperfunctions or Fourier hyperfunctions we often encounter with "resolutions" of such types as

$$0 \longrightarrow \mathcal{O} \longrightarrow L^{(0,0)}_{2,loc} \xrightarrow{\bar{\partial}^{(0)}} L^{(0,1)}_{2,loc} \xrightarrow{\bar{\partial}^{(1)}} \cdots \longrightarrow L^{(0,n)}_{2,loc} \longrightarrow 0\,. \tag{B.1}$$

Here $L^{(0,p)}_{2,loc}$ denotes the sheaf of germs of $(0,p)$-forms with coefficients in locally square integrable functions. Therefore precisely speaking (B.1) is not a resolution of $\mathcal{O}$ in the usual sense of sheaf theory. To avoid this difficulty we usually escape to resolutions by $C^\infty$-coefficients at the cost of preparing the regularity theorem for the solutions of the Cauchy-Riemann system. (See for example Hörmander [23] and the cited references.) It has been pointed out, however, by T. Oshima (unpublished) that if we simply replace each term $L^{(0,p)}_{2,loc}$ in (B.1) by another sheaf $\mathscr{L}^{(0,p)}_{2,loc}$ consisting of the germs of $L^{(0,p)}_{2,loc}$ which are in the domain of $\bar{\partial}^{(p)}$, then we immediately obtain the following *soft* resolution of $\mathcal{O}$ in the usual sense:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{L}^{(0,0)}_{2,loc} \xrightarrow{\bar{\partial}^{(0)}} \mathscr{L}^{(0,1)}_{2,loc} \xrightarrow{\bar{\partial}^{(1)}} \cdots \longrightarrow \mathscr{L}^{(0,n)}_{2,loc} \longrightarrow 0\,. \tag{B.2}$$

Thus we can calculate the cohomology groups with coefficients in $\mathcal{O}$ employing only the solutions of $\bar{\partial}^{(p)}$ in the $L_2$-theory.

This idea of Oshima may not seem to apply to deduce the Martineau-Harvey duality theorem, because the dual space of the section module of such a sheaf as $\mathscr{L}^{(0,p)}_{2,loc}$ (naturally endowed with the usual graph topology) is not directly connected with the corresponding object for the formal dual operator of $\bar{\partial}^{(p)}$. We will show here, however, that in the level of cohomology groups there exists a very clear correspondence. In fact we have the two following theorems.

THEOREM B.1. *Let*

$$0 \longrightarrow X^{(0)} \xrightarrow{A^{(0)}} X^{(1)} \xrightarrow{A^{(1)}} \cdots \xrightarrow{A^{(n-1)}} X^{(n)} \longrightarrow 0 \tag{B.3}$$

*be a complex of FS*- (resp. DFS*-)spaces with densely defined closed oper-*

*ators, that is,* $\mathrm{Domain}(A^{(p)})\supset\mathrm{Image}(A^{(p-1)})$ *and* $A^{(p)}\circ A^{(p-1)}=0$ *for* $p=1,\cdots,n-1$. *Assume that each* $A^{(p)}$ *has closed range and let*

$$H^{(p)}=\mathrm{Ker}(A^{(p)})/\mathrm{Image}(A^{(p-1)}) \tag{B.4}$$

*be the cohomology groups of* (B.3). *Let* $\Gamma(A^{(p)})$ *denote the graph of* $A^{(p)}$ *endowed with the usual graph topology. Put*

$$\begin{array}{ccc} \tilde{A}^{(p)}: \Gamma(A^{(p)}) & \longrightarrow & \Gamma(A^{(p+1)}), \\ \cup & & \cup \\ (x, A^{(p)}x) & \longmapsto & (A^{(p)}x, 0) \end{array} \qquad p=0,\cdots,n-2. \tag{B.5}$$

*Then we have the following complex of continuous operators which has the same cohomology groups as* (B.3):

$$0\longrightarrow\Gamma(A^{(0)})\xrightarrow{\tilde{A}^{(0)}}\Gamma(A^{(1)})\xrightarrow{\tilde{A}^{(1)}}\cdots\xrightarrow{\tilde{A}^{(n-2)}}\Gamma(A^{(n-1)})\longrightarrow X^{(n)}\longrightarrow 0 \tag{B.6}$$

*(the last mapping being the natural projection to* $X^{(n)}$*).*

THEOREM B.2. *Let*

$$0\longleftarrow X'_{(0)}\xleftarrow{A'_{(0)}}X'_{(1)}\xleftarrow{A'_{(1)}}\cdots\xleftarrow{A'_{(n-1)}}X'_{(n)}\longleftarrow 0 \tag{B.7}$$

*be the dual complex of* (B.3) *consisting of the dual spaces* $X'_{(p)}=(X^{(p)})'$ *and the dual operators* $A'_{(p)}=(A^{(p)})'$. *Then* $A'_{(p)}$ *have closed range and the homology group*

$$H'_{(p)}=\mathrm{Ker}(A'_{(p-1)})/\mathrm{Image}(A'_{(p)}) \tag{B.8}$$

*becomes the dual space of* $H^{(p)}$ *in* (B.4) *by a natural correspondence (which will be given in the proof).*

An FS*-space is the projective limit of a sequence of Banach spaces by weakly compact mappings with dense range. (See Komatsu [24].) The space $L_{2,loc}(\Omega)$ of locally square integrable functions is a typical example. Here the weak compactness of the projective mapping $L_{2,loc}(\Omega_{j+1})\to L_{2,loc}(\Omega_j)$ for $\Omega_j\Subset\Omega_{j+1}\Subset\Omega$ is obvious because these spaces are Hilbert. Theorem B.1 thus assures that we can calculate the cohomology groups $H^p(\Omega,\mathcal{O})$ employing the complex of closed operators

$$0\longrightarrow L^{(0,0)}_{2,loc}(\Omega)\xrightarrow{\bar\partial^{(0)}}L^{(0,1)}_{2,loc}(\Omega)\xrightarrow{\bar\partial^{(1)}}\cdots\longrightarrow L^{(0,n)}_{2,loc}(\Omega)\longrightarrow 0\,. \tag{B.9}$$

Usually this fact is explained in a more complicated way with use of the Sobolev spaces $H^s(\Omega)$ and $C^\infty(\Omega)$.

The DFS*-space is the dual notion of the FS*-space: It is defined as the inductive limit of a sequence of Banach spaces by weakly compact *injective* mappings. (See again Komatsu [24].) The duality between the cohomology groups of (B.3) and those of (B.7) was shown directly in Komatsu [24], Theorem 19 even for closed unbounded operators $A^{(p)}$. Our present interest concerns therefore the relation between the dual sequence of (B.6) and the sequence formally corresponding to it:

$$\text{(B.10)}\quad 0\longleftarrow X'_{(0)}\longleftarrow \Gamma(A'_{(0)})\xleftarrow{\tilde{A}'_{(1)}}\Gamma(A'_{(1)})\xleftarrow{\tilde{A}'_{(2)}}\cdots\xleftarrow{\tilde{A}'_{(n-1)}}\Gamma(A'_{(n-1)})\longleftarrow 0\,.$$

As is seen below, it is explained by a kind of mapping cone.

We first prove Theorem B.1. Recall the definition (B.5) of the mapping $\tilde{A}^{(p)}$. It is clear that $\tilde{A}^{(p)}$ is continuous. We have obviously

$$\begin{aligned}&\mathrm{Ker}(\tilde{A}^{(p)})=\{(x, 0);\, A^{(p)}x=0\}\cong\mathrm{Ker}(A^{(p)})\,,\\&\mathrm{Image}(\tilde{A}^{(p)})=\{(A^{(p-1)}(x), 0)\}\cong\mathrm{Image}(A^{(p-1)})\end{aligned}$$

also including the topology. Thus the cohomology groups of the complex (B.6) agree with (B.4).

Next we prove Theorem B.2. It is well known that $A'_{(p)}$ becomes a closed operator, $\mathrm{Image}(A'_{(p+1)})\subset\mathrm{Domain}(A'_{(p)})$ and hence $A'_{(p)}\circ A'_{(p+1)}=(A^{(p+1)}\circ A^{(p)})'=0$. Also, $A'_{(p)}$ has closed range if $A^{(p)}$ does (see e.g. Komatsu [24], Theorem 19). Now we calculate the dual complex of (B.6).

LEMMA B.3. *We have the following canonical isomorphism*

$$\text{(B.11)}\qquad \Gamma(A^{(p)})'=X'_{(p)}\times X'_{(p+1)}/\Gamma(-A'_{(p)})\,.$$

*The correspondence is given by the canonical scalar product between* $X^{(p)}\times X^{(p+1)}$ *and* $X'_{(p)}\times X'_{(p+1)}$: *For* $(x, A^{(p)}x)\in\Gamma(A^{(p)})$, $(f, g)\in X'_{(p)}\times X'_{(p+1)}$ *we have*

$$\text{(B.12)}\qquad \langle (x, A^{(p)}x), (f, g)\rangle=\langle x, f\rangle+\langle A^{(p)}x, g\rangle\,.$$

In fact, by the routine polar analysis for the subspace $\Gamma(A^{(p)})\subset X^{(p)}\times X^{(p+1)}$ we have the canonical isomorphism

$$\Gamma(A^{(p)})'=X'_{(p)}\times X'_{(p+1)}/\Gamma(A^{(p)})^{\perp}\,.$$

Therefore it suffices to show the identity

$$\Gamma(A^{(p)})^{\perp}=\Gamma(-A'_{(p)})\,.$$

The inclusion $\Gamma(A^{(p)})^{\perp}\supset\Gamma(-A'_{(p)})$ is obvious from (B.12). The opposite inclusion also follows from (B.12) by the definition of the dual to an unbounded operator.

Note also that the dual operator $(\tilde{A}^{(p)})': \Gamma(A'_{(p+1)})' \to \Gamma(A'_{(p)})'$ of $\tilde{A}^{(p)}$ agrees with the continuous operator induced via the above isomorphism from the operator $\tilde{A}'_{(p)}: \Gamma(A'_{(p+1)}) \to \Gamma(A'_{(p)})$ which in turn is defined as in (B.5) employing $A'_{(p)}: X'_{(p+1)} \to X'_{(p)}$.

Now consider the following commutative diagram by the well understood continuous mappings:

$$
\text{(B.13)} \quad
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
 & 0 & \leftarrow \Gamma(A^{(0)})' & & \leftarrow \Gamma(A^{(1)})' & & \leftarrow \cdots \leftarrow & & \Gamma(A^{(n-1)})' & & \leftarrow X'_{(n)} \leftarrow 0 \\
 & \uparrow & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
0 \leftarrow & X'_{(0)} & \leftarrow X'_{(0)} \times X'_{(1)} & & \leftarrow X'_{(1)} \times X'_{(2)} & & \leftarrow \cdots \leftarrow & & X'_{(n-1)} \times X'_{(n)} & & \leftarrow X'_{(n)} \leftarrow 0\,. \\
 & \uparrow & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
0 \leftarrow & X'_{(0)} & \leftarrow \Gamma(-A'_{(0)}) & & \leftarrow \Gamma(-A'_{(1)}) & & \leftarrow \cdots \leftarrow & & \Gamma(-A'_{(n-1)}) & & \leftarrow 0 \\
 & \uparrow & \uparrow & & \uparrow & & & & \uparrow & & \\
 & 0 & 0 & & 0 & & & & 0 & &
\end{array}
$$

Each column is exact by (B.11). The middle row consists of the mappings

$$
\begin{array}{ccc}
X'_{(p-1)} \times X'_{(p)} & \longleftarrow & X'_{(p)} \times X'_{(p+1)}\,, \\
\cup & & \cup \\
(0, f) & \longleftarrow & (f, g)
\end{array}
$$

hence is trivially exact. In view of Theorem 1, the complex $\Gamma(-A'_{(\cdot)})$ in the third row gives (B.8) as the homology groups:

$$
\text{(B.14)} \qquad H_p(\Gamma(-A'_{(\cdot)})) \cong H'_{(p)}\,.
$$

(Here in general $H_p(C_{(\cdot)})$ denotes the $p$-th homology group of the chain complex $C_{(\cdot)}$.) The first row is the dual complex of (B.6), hence if we assume the duality between the cohomology and homology groups in the case of continuous operators $A^{(p)}$, it has the following as the homology groups:

$$
\text{(B.15)} \qquad H_p((\Gamma(A^{(\cdot)})') \cong (H^p(\Gamma(A^{(\cdot)}))' \cong (H^{(p)})'\,.
$$

These two are connected by the well known snake lemma for the diagram (B.13):

$$
\text{(B.16)} \qquad H_p(\Gamma(-A'_{(\cdot)})) \cong H_{p+1}(\Gamma(A^{(\cdot-1)}))\,.
$$

Combining these we thus obtain $H'_{(p)} = (H^{(p)})'$. (Note that the degree shift in the highest row is just compensated by that of the snake lemma (B.16).)

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