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Applications of the Malliavin calculus, Part II

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Introduction.

In the first part of this paper [6] we developed the basic machinery needed to study regularity properties of uniformly elliptic Itô processes by means of Malliavin's calculus (cf. Section 3 of [6]). In particular, we showed that the distribution of such a process at any fixed positive time admits a smooth density (densities will always be taken with respect to Lebesque measure unless we explicitly state that we are using a different reference at a particular place). Of course, in the case when the Itô process is Markovian and is therefore a diffusion, this result says nothing more (and, in fact, says somewhat less) than is known from the classicical theory of non-degenerate second order parabolic differential equations.

In the present article, we investigate what more can be said when we restrict our attention to the Markovian setting. As we saw in [6], the quantity which one must learn to control in order to obtain regularity via Malliavin's calculus is the *Malliavin covariance matrix* $A(T, x)$ (cf. (2.4)). The reason why one should expect to be able to do much better in the Markovian setting than one can with general Itô processes is that, when the process is Markovian, $A(T, x)$ admits a quite tractable expression (cf. (2.5) and (2.6)) to which one can apply a more refined analysis than seems to be possible in general. Indeed, Malliavin himself took advantage of this observation in his groundbreaking article Malliavin [7], [8] and indicated there how one might proceed. Since Malliavin's article appeared, there have been various attempts (cf. Ikeda-Watanabe [4], Bismut [1], and Stroock [12]) to put Malliavin's ideas on firmer mathematical footing. The approach which we have adopted here is basically the same as the one which we adopted in [12], although we hope that the present rendition makes it easier to follow what is being done. Theorem (2.17) contains the estimates about $A(T, x)$ on which the whole of our analysis rests. Unfortunately, we have not discovered a simple way to derive these estimates. In particular, the reader will find that Theorem (2.17) itself depends on the rather heavy stochastic analysis involved in the proof of Theorem (A. 6)

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in the appendix. Our justification for presenting this analysis in so much detail is that, so far as we know, our results in this direction are both the most general (we have been able to include Hörmander's "drift term") as well as the most quantitative which are known thus far.

Starting in Section 3), we apply the estimates obtained in Theorem (2.17) to the study of the fundamental solution to the equation (3.12) (cf. Theorem (3.17) and Corollary (3.25)). Although these initial applications are similar to results which others have obtained by purely analytic means, they are, in some ways, less refined than the recent estimates proved by S. Sanchez in [Sanchez] but are, in other ways, more general and more global. Section 4) is devoted to localization of the results in Section 3) (cf. Theorem (4.5) and Corollary (4.10)). Given the estimates in Section 3), the techniques used are rather standard applications of probability theory. In Section 5), we "microlocalize" our results. (From the probabilistic standpoint, this simply means that we apply our regularity theory to the marginal distribution of our diffusion on a submanifold.) In the strictly elliptic context, a preliminary version of such microlocalization was carried out already in [12]. More recently, S. Taniguchi [14] discussed a similar but somewhat less general procedure. Our principal result here is the one contained in Corollary (5.12).

In Section 6), we transfer our considerations from the fundamental solution to the resolvent kernel. Perhaps the most interesting aspect of this transfer is the way in which we have passed from Hörmander's "restricted condition" (i.e. the one which guarantees hypoellipticity of the parabolic operator) to Hörmander's general condition (i.e. the one for the elliptic operator). See Theorem (6.8) for our conclusions about the resolvent kernel.

The results in Section 7) seem to be new, although they have antecedents in the work of Oleinik and Radekevich [10] and were anticipated by Malliavin [7], [8]. What we provide are criteria (cf. Corollary (7.4)) which guarantee that the fundamental solution is regular even when Hörmander's condition fails to hold. Here again our analysis rests on Theorem (A.6). Finally, Section 8) contains an application of the preceding regularity theory to the study of hypoellipticity. The advantage which our approach (cf. Theorem (8.16)) to this topic has over more traditional ones is that we have circumvented the use of intermediate subelliptic estimates. This fact enables us to prove the hypoellipticity of operators which do not satisfy Hörmander's condition (cf. Corollary (8.18) and the remark (8.19)). We conclude this section with an example of a very special class of operators (cf. (8.20)) for which our analysis enables us to give necessary and sufficient conditions under which hypoellipticity obtains (cf.

Theorem (8.41)).

1. Some regularity properties of solutions to stochastic integral equations.

We continue with the notation used in [6]. In particular, $\Theta=\{\theta\in C([0,\infty), R^d):\theta(0)=0\}$, $\{\mathscr{B}_t : t\geqq 0\}$ is the standard filtration of the Borel field $\mathscr{B}$ over Θ, and W on $(\Theta, \mathscr{B})$ denotes standard Winer measure.

Throughout this paper, $\{V_0, \cdots, V_d\}$ will be a subset of $C_\uparrow^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ (cf. Section 1) of [6]) and $\sigma\in C_\uparrow^\infty(\boldsymbol{R}^N, \text{H.S.}(\boldsymbol{R}^d, \boldsymbol{R}^N))$ and $b\in C_\uparrow^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ will stand for the associated quantities described below:

$$\sigma(x)=(V_1(x), \cdots, V_d(x))=((V_k^i(x)))_{\substack{1\leq i\leq N\\ 1\leq k\leq d}} \tag{1.1}$$

and

$$b(x)=V_0(x)+\frac{1}{2}\sum_{k=1}^{d}\sum_{j=1}^{N}V_k^j(x)\frac{\partial V_k}{\partial x_j}(x)\,. \tag{1.2}$$

Unless the contrary is stated, we will be assuming that

$$\sup_{x\in\boldsymbol{R}^N}(\|\sigma^{(1)}(x)\|_{\text{H.S.}(\boldsymbol{R}^N;\,\text{H.S.}(\boldsymbol{R}^d,\boldsymbol{R}^N))}\vee\|b^{(1)}(x)\|_{\text{H.S.}(\boldsymbol{R}^N,\boldsymbol{R}^N)})<\infty\,. \tag{1.3}$$

(See Section 1) of [6] for the notation $F^{(n)}, n\geqq 1$, when $F\in C_\uparrow^\infty(E_1, E_2)$.) Given $x\in\boldsymbol{R}^N$, we use $X(\cdot, x)$ to denote the right continuous, $\mathscr{W}$-almost surely continuous, progressively measurable (progressively measurable is defined relative to $\{\mathscr{B}_t : t\geqq 0\}$ unless the contrary is stated) solution to

$$X(T, x)=x+\int_0^T\sigma(X(t, x))d\theta(t)+\int_0^T b(X(t, x))dt\,, \qquad T\geqq 0\,. \tag{1.4}$$

The existence of $X(\cdot, x)$ as well as its $\mathscr{W}$-almost sure uniqueness is guaranteed by (1.3). Moreover, a selection of $x\to X(\cdot, x)$ can be chosen so that for each $T>0$ the map $(t, x)\in[0, T]\times\boldsymbol{R}^N\to X(t, x)$ is an element of $C^{0,\infty}([0, T]\times\boldsymbol{R}^N, \boldsymbol{R}^N)$ $(=C([0, T], C^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)))$. (See, for example, Kunita [5].) In particular, using the notation

$$J(T, x)=X^{(1)}(T, x)=\left(\left(\frac{\partial X_i(T, x)}{\partial x_j}\right)\right)_{1\leq i,j\leq N} \tag{1.5}$$

to emphasize that we are thinking of $X^{(1)}(T, x)$ as a matrix, we see that $J(\cdot, x)$ is determined by:

$$J(T, x)=I+\sum_{k=1}^{d}\int_0^T V_k^{(1)}(X(t, x))J(t, x)d\theta_k(t) \tag{1.6}$$

$$+\int_0^T b^{(1)}(X(\cdot, x))J(t, x)dt\,, \qquad T\geqq 0\,.$$

From (1.6) it is an easy matter to see that $J(T, x)$ is $\mathcal{W}$-almost surely invertible and that its inverse $J^{-1}(T, x)$ is determined by:

$$J^{-1}(T, x)=I-\sum_{k=1}^{d}\int_0^T J^{-1}(t, x)V_k^{(1)}(X(t, x))d\theta_k(t) \tag{1.7}$$

$$-\int_0^T J^{-1}(t, x)\Big(b^{(1)}(X(t, x))-\sum_{k=1}^{d}(V_k^{(1)})^2(X(t, x))\Big)dt\,, \qquad T\geqq 0\,.$$

Given $m\geqq 1$, set $E^{(m)}=\mathrm{H.S.}((\boldsymbol{R}^N)^{\otimes m}, \boldsymbol{R}^N)$. Then, starting from (1.6), it is an easy inductive argument to show that for each $m\geqq 2$ there are universal (i.e., independent of $\{V_0, \cdots, V_d\}$) polynomials $P_{m,k}$, $0\leqq k\leqq d$, on $\left(\prod_{\mu=2}^{m}E^{(\mu)}\right)\times\left(\prod_{\mu=1}^{m-1}E^{(\mu)}\right)$ into $E^{(m)}$ such that:

$$\begin{aligned} X^{(m)}(T, x)=&\sum_{k=1}^{d}\int_0^T V_k^{(1)}(X(t, x))X^{(m)}(t, x)d\theta_k(t)\\ &+\int_0^T b^{(1)}(X(t, x))X^{(m)}(t, x)dt\\ &+\sum_{k=1}^{d}\int_0^T P_{m,k}(t)d\theta_k(t)\\ &+\int_0^T P_{m,0}(t)dt\,, \qquad T\geqq 0\,, \end{aligned} \tag{1.8}$$

where

$$P_{m,k}(t)=P_{m,k}(V_k^{(2)}(X(t, x)), \cdots, V_k^{(m)}(X(t, x)), X^{(1)}(t, x), \cdots, X^{(m-1)}(t, x))\,,$$

for $1\leqq k\leqq d$, and

$$P_{m,0}(t)=P_{n,0}(b^{(2)}(X(t, x)), \cdots, b^{(m)}(t, x), X^{(1)}(t, x), \cdots, X^{(m-1)}(t, x))\,.$$

Thus, by the method of variation of parameters, we conclude that

$$X^{(m)}(T, x)=J(T, x)\left[\sum_{k=1}^{d}\int_0^T J^{-1}(t, x)P_{m,k}(t)d\theta_k(t)+\int_0^T J^{-1}(t, x)Q_m(t)dt\right],$$

where

$$Q_m(t)=P_{m,0}(t)-\sum_{k=1}^{d}V_k^{(1)}(X(t, x))P_{m,k}(t)\,.$$

We are now ready to summarize the results of Sections 1) and 2) in [6] as they relate to the situation at hand. However, before doing so, we list, for the convenience of the reader, the place in [6] where some of the quantities and notation used below are explained.

a) The operator $\mathcal{L}$ and the associated bilinear operation $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{L}}$ are described at the beginning of Section 1).

b) The Hilbert space H and the associated space $\mathcal{H}(E)$ and operator D are defined in the discussion preceding Lemma (1.7). The relationship between these quantities and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{L}}$ is the content of Theorem (1.8).

c) The space $\mathcal{FG}(\mathcal{L}, E)$ and the associated semi-norms $\|\cdot\|_{q,T,E}^{(n)}$ and $\|\cdot\|_{q,T,E}^{(n)}$ are defined in the discussion preceding Theorem (2.19).

(1.9) THEOREM. *For each* $m \geqq 0$ *and* $x \in \boldsymbol{R}^N$, $X^{(m)}(\cdot, x) \in \mathcal{FG}(\mathcal{L}, E^{(m)})$ $(X^{(0)}(\cdot, x) \equiv X(\cdot, x)$ *and* $E^{(0)} \equiv \boldsymbol{R}^N)$. *More precisely, suppose* $\{C_m : m \geqq 0\} \subseteqq (0, \infty)$ *and* $\{\gamma_m : m \geqq 0\} \subseteqq [0, \infty)$ *are chosen so that*

$$(1.10) \qquad \|\sigma^{(m)}(x)\|_{\mathrm{H.S.}(\boldsymbol{R}^d, E^{(m)})} \vee \|b^{(m)}(x)\|_{E^{(m)}} \leqq C_m(1+\|x\|_{\boldsymbol{R}^N})^{\gamma_m}, \qquad x \in \boldsymbol{R}^N.$$

(By (1.3), we may and do assume that $\gamma_1 = 0$.*) Then for each* $q \in [2, \infty)$ *and* $T > 0$, *there exist* $A(q, T) \in (0, \infty)$ *and* $B(q, T) \in (0, \infty)$, *depending only on* C_0 *and* C_1, *respectively, such that*

$$(1.11) \qquad \|(X(\cdot, x) - x)\|_{q,T,\boldsymbol{R}^N}^{(0)} \leqq A(q, T)(1+\|x\|_{\boldsymbol{R}^N})^{\gamma_0}, \qquad x \in \boldsymbol{R}^N,$$

and

$$(1.12) \qquad \|X^{(1)}(\cdot, x)\|_{q,T,E^{(1)}}^{(0)} < B(q, T), \qquad x \in \boldsymbol{R}^N.$$

Moreover, for each $q \in [2, \infty)$, $T > 0$, *and* $n \geqq 2$, *there exist* $C_n(q, T) \in (0, \infty)$ *and* $\gamma_n(q, T) \in (0, \infty)$, *depending only on* $\{C_\nu : 0 \leqq \nu \leqq n\}$ *and* $\{\gamma_\nu : 0 \leqq \nu \leqq n\}$, *respectively, such that*

$$(1.13) \qquad |||DX(\cdot, x)|||_{q,t,\mathcal{H}(\boldsymbol{R}^N)}^{(n)} \vee |||\mathcal{L}X(\cdot, x)|||_{q,t,\boldsymbol{R}^N}^{(n-1)} \leqq C_{n+1}(q, T)(1+\|x\|_{\boldsymbol{R}^N})^{\gamma_{m+1}(q,T)} t^{1/2},$$

$$(1.14) \qquad |||DX^{(1)}(\cdot, x)|||_{q,t,\mathcal{H}(E^{(1)})}^{(n)} \vee |||X^{(1)}(\cdot, x)|||_{q,t,E^{(1)}}^{(n-1)} \leqq C_{n+2}(q, T)(1+\|x\|_{\boldsymbol{R}^N})^{\gamma_{n+1}(q,T)} t^{1/2},$$

and, for $m \geqq 2$:

$$(1.15) \qquad |||X^{(m)}(\cdot, x)|||_{q,t,E^{(m)}}^{(n)} \leqq C_{m+n}(q, T)(1+\|x\|_{\boldsymbol{R}^N})^{\gamma_{m+n}(T)} t^{1/2}$$

for all $t \in [0, T]$ *and* $x \in \boldsymbol{R}^N$. *Finally,* $\gamma_n(q, T) = 0$ *if* $\gamma_0 = \cdots = \gamma_n = 0$.

PROOF. The facts that $X(\cdot, x) \in \mathcal{FG}(\mathcal{L}, \boldsymbol{R}^N)$ and that (1.11) and (1.13) hold are the content of Theorem (2.19) in [6]. Furthermore, (1.12) is a simple consequence of (1.6) (recall that $X^{(1)}(\cdot, x) = J(\cdot, x)$ and that $\sigma^{(1)}$ and $b^{(1)}$

are bounded). To prove (1.14), consider the equations (1.4) and (1.6) simultaneously as a system defining $\begin{pmatrix} X(\cdot, x) \\ X^{(1)}(\cdot, x) \end{pmatrix}$. Then, Theorem (2.19) of [6] applies to this system and yields $X^{(1)}(\cdot, x) \in \mathscr{F}\mathscr{G}(\mathscr{L}, E^{(1)})$ as well as (1.14) as a consequence. To complete the proof, we can work by induction on $m \geqq 2$. Indeed, on the basis of what we already know about $X(\cdot, x)$ and $X^{(1)}(\cdot, x)$, Theorem (1.10) plus Lemma (2.2) of [6] applied to (1.8) show that $X^{(2)}(\cdot, x) \in \mathscr{F}\mathscr{G}(\mathscr{L}, E^{(2)})$ and that (1.15) holds when $m=2$. Next, if $m \geqq 3$ and we assume that $X^{(\mu)}(\cdot, x) \in \mathscr{F}\mathscr{G}(\mathscr{L}, E^{(\mu)})$ and that (1.15) holds for $2 \leqq \mu \leqq m-1$, then a repetition of the preceding argument completes the inductive step. Q. E. D.

2. The basic estimates on Malliavin's covariance matrix.

The notation in this section is the same as in the preceding one. In particular, $\{V_0, \cdots, V_d\} \subseteq C_{\uparrow}^{\infty}(\boldsymbol{R}^N, \boldsymbol{R}^N)$ satisfies (1.3) and $x \to X(\cdot, x)$ is a smooth selection of the solutions to (1.4).

For many purposes, it is more convenient to rewrite (1.4), (1.6), and (1.7) in their equivalent Stratonovich form. That is, $X(\cdot, x)$, $J(\cdot, x)$, and $J^{-1}(\cdot, x)$ are determined by the Stratonovich stochastic integral equations:

$$X(T, x) = x + \sum_{k=1}^{d} \int_0^T V_k(X(t, x)) \circ d\theta_k(t) + \int_0^T V_0(X(t, x))dt, \qquad T \geqq 0, \tag{2.1}$$

$$J(T, x) = I + \sum_{k=1}^{d} \int_0^T V_k^{(1)}(X(t, x))J(t, x) \circ d\theta_k(t) + \int_0^T V_0^{(1)}(X(t, x))J(t, x)dt, \qquad T \geqq 0, \tag{2.2}$$

and

$$J^{-1}(T, x) = I - \sum_{k=1}^{d} \int_0^T J^{-1}(t, x) V_k^{(1)}(X(t, x)) \circ d\theta_k(t) - \int_0^T J^{-1}(t, x) V_0^{(1)}(X(t, x))dt, \qquad T \geqq 0, \tag{2.3}$$

respectively.

Our aim (cf. Theorem (2.17) below) in this section is to provide lower bounds on the *Malliavin covariance matrix*

$$A(T, x) = \langle\!\langle X(T, x), X(T, x) \rangle\!\rangle_{\mathscr{L}}. \tag{2.4}$$

(See Section 1) of [6].) The first step is to find a tractable expression for $A(T, x)$. By Theorem (1.8) in [6],

$$A(T, x) = (DX(T, x), DX(T, x))_H .$$

At the same time, after converting (2.10) of [6] to its Stratonovich form, we have for each $h \in H$:

$$\begin{aligned} DX(T, x)(h) = & \sum_{k=1}^{d} \int_0^T V_k^{(1)}(X(t, x)) DX(t, x)(h) \circ d\theta_k(t) \\ & + \int_0^T V_0^{(1)}(X(t, x)) DX(t, x)(h) dt \\ & + \sum_{k=1}^{d} \int_0^T V_k(X(t, x)) h_k'(t) dt , \qquad T \geqq 0 . \end{aligned}$$

Hence, by the method of variation of parameters and the equations (2.2) and (2.3), we see that

$$J^{-1}(T, x) DX(T, x)(h) = \sum_{k=1}^{d} \int_0^T J^{-1}(t, x) V_k(X(t, x)) h_k'(t) dt .$$

Therefore, if we define

$$\tilde{A}(T, x) = \sum_{k=1}^{d} \int_0^T (J^{-1}(t, x) V_k(X(t, x)))^{\otimes 2} dt , \tag{2.5}$$

then

$$A(T, x) = J(T, x) \tilde{A}(T, x) J(T, x)^* . \tag{2.6}$$

(Given $v \in \boldsymbol{R}^N$, $v^{\otimes 2} = v \otimes v$ is the tensor product of v with itself; and in this context is thought of as the matrix $((v_i v_j))_{1 \leq i, j \leq N}$.

(2.7) REMARK. There are several pleasing features possessed by the preceding representation of $A(T, x)$. In the first place, (2.5) explicitly displays $\tilde{A}(T, x)$ as a non-negative definite element of $T_x^N(\boldsymbol{R}^N) \otimes T_x^N(\boldsymbol{R}^N)$. ($T_x(\boldsymbol{R}^N)$ denotes the tangent space to $\boldsymbol{R}^N$ at x when $\boldsymbol{R}^N$ is thought of as a differentiable manifold.) In particular, (2.5) shows that $\tilde{A}(T, x)$ is well-defined from a differential geometric (i.e., coordinate independent) point of view. Since $\tilde{A}(T, x)$ is well-defined as an element of $T_x(\boldsymbol{R}^N) \otimes T_x(\boldsymbol{R}^N)$, (2.6) tells us that $A(T, x)$ is well-defined as an element of $T_{X(T,x)}(\boldsymbol{R}^N) \otimes T_{X(T,x)}(\boldsymbol{R}^N)$. For the moment we will not be making any essential use of these remarks. Nonetheless, it should be comforting to find that $A(T, x)$ is indeed a kind of path by path covariance matrix, at least in the sense that it transforms in the way a covariance matrix should.

It is clear from (2.5) that $\tilde{A}(T, x)$ is non-degenerate for all $T > 0$ if $\operatorname{span}\{V_1(x), \cdots, V_d(x)\} = \boldsymbol{R}^N$. Since $\operatorname{span}\{V_1(x), \cdots, V_d(x)\} = \boldsymbol{R}^N$ is equivalent to the non-degeneracy of the matrix $a(x)$ given by

(2.8)
$$a(x)=\sum_{k=1}^{d}(V_k(x))^{\otimes 2}=\sigma\sigma^*(x)$$

we see that $\tilde{A}(T, x)$, and therefore $A(T, x)$, is strictly positive definite if the infinitesimal generator of $X(\cdot, x)$ is strictly elliptic at x. In order to obtain more delicate criteria which guarantee the non-degeneracy of $\tilde{A}(T, x)$ we are going to develop the quantities $J^{-1}(t, x)V_k(t, x)$, $1\leqq k\leqq d$, in a Taylor's expansion with respect to t. However, before doing so, it will be useful to have some new notation.

Given $V\in C^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$, we can associate with V the directional derivative

(2.9)
$$\sum_{i=1}^{N} V^i(x)\frac{\partial}{\partial x_i}\,.$$

As is common in the differential geometry literature, we will indulge in the notational convenience afforded by letting V stand for both the element of $C^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ as well as the directional derivative in (2.9). In this connection, given $V, W\in C^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$, we use $W(V)$ to denote the element of $C^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ with i^{th} coordinate given by

$$[W(V)]^i(x)=[W(V^i)](x)=\sum_{j=1}^{N} W^j(x)\frac{\partial V^i}{\partial x_j}(x)\,.$$

Related to the preceding is the *Lie bracket* $[V, W]$ of $V, W\in C^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ which is given by

$$[V, W]=V(W)-W(V)=\sum_{j=1}^{N}\left(V^j\frac{\partial W}{\partial x_j}-W^j\frac{\partial V}{\partial x_j}\right)\frac{\partial}{\partial x_j}\,.$$

Given $V\in C^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$, we now use Itô's formula in conjunction with equations (2.1) and (2.3) to obtain:

$$\begin{aligned} d(J^{-1}(t, x)V(X(t, x))=&\sum_{k=1}^{d}J^{-1}(t, x)(-V_k^{(1)}(X(t, x))V(X(t, x))\\ &+[V_k(V)](X(t, x)))\circ d\theta_k(t)\\ &+J^{-1}(t, x)(-V_0^{(1)}(X(t, x))V(X(t, x))\\ &+[V_0(V)](X(t, x)))dt\,. \end{aligned}$$

Noting that

$$-V_k^{(1)}(y)V(y)+[V_k(V)](y)=[V_k, V](y)\,,\qquad 0\leqq k\leqq d\,,$$

we conclude that

$$(2.10)\qquad J^{-1}(T,x)V(X(T,x))=V(x)+\sum_{k=1}^{d}\int_0^T J^{-1}(t,x)[V_k,V](X(t,x))dt$$
$$+\int_0^T J^{-1}(t,x)[V_0,V](X(t,x))dt\,,\qquad t\geqq 0\,.$$

Equation (2.10) is the basis for the Taylor's expansion which we have in mind. In order to describe the expansion, we define the set $\mathcal{A}=\{\emptyset\}\cup\bigcup_{l=1}^{\infty}(\{0,\cdots,d\})^l$, and the quanties $|\alpha|$, $\|\alpha\|$, α', and α_* for $\alpha\in\mathcal{A}$ as in the appendix (cf. (A.1), (A.2) and the discussion accompanying these). Given $V\in C_\uparrow^\infty(\boldsymbol{R}^N,\boldsymbol{R}^N)$, we define $V_{(\alpha)}$, $\alpha\in\mathcal{A}$, inductively on $|\alpha|$ by:

$$(2.11)\qquad V_{(\alpha)}=\begin{cases}V & \text{if}\quad \alpha=\emptyset\\ [V_{\alpha_*},V_{(\alpha')}] & \text{if}\quad \alpha\neq\emptyset\,.\end{cases}$$

Also, define $\theta^{(\alpha)}(\cdot)$, $\alpha\in\mathcal{A}$, as in (A.4).

(2.12) THEOREM. *For given $L\geqq 1$ and $0<\varepsilon\leqq 1$ there exist $C_{L,\varepsilon}<\infty$ and $\mu_{L,\infty}\in(0,\infty)$ such that for all $x\in\boldsymbol{R}^N$ and $V\in C_\uparrow^\infty(\boldsymbol{R}^N,\boldsymbol{R}^N)$:*

$$(2.13)\qquad J^{-1}(T,x)V(X(T,x))=\sum_{\|\alpha\|\leq L-1}\theta^{(\alpha)}(T)V_{(\alpha)}(x)+R_L(T,x,V)$$

where

$$(2.14)\qquad \sup_{0<T\leqq 1}\mathcal{W}\left(\frac{1}{T^L}\int_0^{T/K}|R_L(t,x,V)|^2dt\geqq 1/K^{L+1-\varepsilon}\right)$$
$$\leqq C_{L,\varepsilon}\exp(-K^{\mu_{L,\varepsilon}}/(1+M(x))^2)\,,\qquad K\in(0,\infty)\,,$$

with

$$M(x)=(\max\{\|V_k\|_{C_b^2(B(x,1),\boldsymbol{R}^N)}:0\leqq k\leqq d\})$$
$$\vee(\max\{\|V_{(\alpha)}\|_{C_b(B(x,1),\boldsymbol{R}^N)}:|\alpha|\leqq L+1\})\,.$$

PROOF. By repeated application of (2.10), we see that

$$J^{-1}(T,x)V(X(T,x))=\sum_{|\alpha|\leq L-1}\theta^{(\alpha)}(T)V_{(\alpha)}(x)+\sum_{|\alpha|=L}S^{(\alpha)}(T,Z_{(\alpha)})$$

where

$$Z_{(\alpha)}(T)=J^{-1}(T,x)V_{(\alpha)}(X(T,x))$$
$$=V_{(\alpha)}(x)+\sum_{k=1}^{d}\int_0^T J^{-1}(t,x)[V_k,V_{(\alpha)}](X(t,x))\circ d\theta_k(t)$$
$$+\int_0^T J^{-1}(t,x)[V_0,V_{(\alpha)}](X(t,x))dt$$

$$=V_{(\alpha)}(x)+\sum_{k=1}^{d}\int_0^T J^{-1}(t,x)[V_k, V_{(\alpha)}](X(t,x))d\theta_k(t)$$

$$+\int_0^T J^{-1}(t,x)\Big([V_0, V_{(\alpha)}]+\frac{1}{2}\sum_{k=1}^{d}[V_k,[V_k, V_{(\alpha)}]]\Big)(X(t,x))dt$$

and $S^{(\alpha)}(\cdot, Z_{(\alpha)})$ is defined as in the appendix. Thus, (2.13) holds when we define

$$R_L(T,x,V)=\sum_{|\alpha|=L}S^{(\alpha)}(T,Z_{(\alpha)})+\sum_{\substack{\|\alpha\|\geq L\\ |\alpha|\leqq L-1}}\theta^{(\alpha)}(T)V_{(\alpha)}(x)\,.$$

In order to show that the estimate (2.14) holds, it suffices to show that each term making up $R_L(\cdot, x, V)$ satisfies such an estimate.

Given an $f\in C([0,1])$, note that

$$\int_0^{T/K}|f(t)|^2dt\leqq\frac{2}{2L-\varepsilon}\Big(\sup_{0<t\leqq 1}|f(t)|/t^{L-\varepsilon/2}\Big)(T/K)^{L+1-\varepsilon/2}\,,\qquad T\in[0,1]\,.$$

Hence, it suffices for us to show that for $|\alpha|=L$ the quantity

$$\mathcal{W}\Big(\frac{2}{2L-\varepsilon}\sup_{0<t\leqq 1}|S^{(\alpha)}(t,Z_{(\alpha)})|/t^{L-\varepsilon/2}\geqq K^{\varepsilon/2}\Big)$$

and for $|\alpha|\leqq L-1$ with $\|\alpha\|\geqq L$ the quantity

$$\mathcal{W}\Big(\frac{2}{2L-\varepsilon}\sup_{0<t\leqq 1}|\theta^{(\alpha)}(t)V_{(\alpha)}(x)|/t^{L-\varepsilon/2}\geqq K^{\varepsilon/2}\Big)$$

can be estimated by expressions of the sort on the right hand side of (2.14). For quantities of the latter type, such an estimate is immediate from Theorem (A. 5). To handle quantities of the former type, set

$$\zeta=\inf\{t\geqq 0: \|J^{-1}(t,x)-I\|_{\mathrm{H.S.}(R^N,R^N)}\vee\|X(t,x)-x\|_{R^N}\geqq 1\}\,.$$

Then, by standard estimates, there exist $C<\infty$ and a universal $\lambda\in(0,\infty)$ such that

$$\mathcal{W}\Big(\zeta\leqq\frac{1}{K}\Big)\leqq C\exp(-\lambda K/(1+M(x))^2)\,,\qquad K\in(0,\infty)\,,$$

and

$$\mathcal{W}\Big(\sup_{0\leqq t\leqq 1\wedge\zeta}\|Z_{(\alpha)}(t)-Z_{(\alpha)}(0)\|_{R^N}\geqq K\Big)\leqq C\exp(-\lambda K^2/(1+M(x))^2)\,,$$
$$K\in(0,\infty)\,.$$

Combining these with Theorem (A. 50), the required estimate on terms of the former type follows easily. Q. E. D.

Given $L\geqq 1$, define

(2.15) $$\mathcal{V}_L(x,\eta)=\sum_{k=1}^{d}\sum_{\|\alpha\|\leq L-1}((V_k)_{(\alpha)}(x),\eta)_{N}^{2}, \qquad x\in \boldsymbol{R}^N \quad\text{and}\quad \eta\in\boldsymbol{R}^N.$$

Also, given a closed set $F\subseteqq S^{N-1}$, define

(2.16) $$\mathcal{V}_L(x,F)=\inf_{\eta\in F}(\mathcal{V}_L(x,\eta)\wedge 1).$$

The main result of this section is the following theorem which allows us to estimate the non-degeneracy of $A(T,x)$ in terms of the quantities just described.

(2.17) THEOREM. *Given $L\geqq 1$, there exist $C(L)$, $\tilde{C}(L)\in(0,\infty)$ and μ_L, $\tilde{\mu}_L\in(0,1]$, all of which are independent of $\{V_0,\cdots,V_d\}$, such that for all $T\in(0,1]$, all closed $F\subseteqq S^{N-1}$, and all $K\in[1,\infty)$:*

(2.18) $$\mathcal{W}(\tilde{\lambda}(T/K^{1/(L+1)},x,F)/T^L\leqq 1/K)$$
$$<\tilde{C}(L)\exp(-(\mathcal{V}_L(x,F)^{L+2}K)^{\tilde{\mu}_L}/(1+M(x))^2)$$

and

(2.19) $$\mathcal{W}(\lambda(T/K^{1/(L+1)},x,F)/T^L\leqq 1/K)$$
$$\leqq C(L)\exp(-(\mathcal{V}_L(x,F)^{L+2}K)^{\mu_L}/(1+M(x))^2),$$

where

$$\tilde{\lambda}(t,x,F)=\inf_{\eta\in F}(\eta,\tilde{A}(t,x)\eta)_{R^N},$$

$$\lambda(t,x,F)=\inf_{\eta\in F}(\eta,A(t,x)\eta)_{R^N},$$

and

$$M(x)=\max\{\|(V_k)_{(\alpha)}\|_{C_b^2(B(x,1),R^N)}:0\leqq k\leqq d \text{ and } |\alpha|\leqq L+1\}.$$

PROOF. We first prove (2.18). Noting that:

$$(\eta,\tilde{A}(T,x)\eta)_{R^N}=\int_0^T\sum_{k=1}^{d}(J^{-1}(t,x)V_k(X(t,x)),\eta)_{R^N}^2dt$$

and applying (2.13), we see that:

$$(\eta,\tilde{A}(T/K,x)\eta)_{R^N}^{1/2}\geqq\left(\sum_{k=1}^{d}\int_0^{T/K}\left(\sum_{\|\alpha\|\leq L-1}((V_k)_{(\alpha)}(x),\eta)_{R^N}\theta^{(\alpha)}(t)\right)^2dt\right)^{1/2}$$
$$-\left(\sum_{k=1}^{d}\int_0^{T/K}|R_L(t,x,V_k)|^2dt\right)^{1/2}.$$

Hence,

$$\inf_{\eta\in F}\sum_{k=1}^{d}\int_0^{T/K}\Big(\sum_{\|\alpha\|<L-1}((V_k)_{(\alpha)}(x),\eta)_{R^N}\theta^{(\alpha)}(t)\Big)^2dt$$

$$\leqq 2\tilde{\lambda}(T/K,x,F)+2\sum_{k=1}^{d}\int_0^{T/K}|R_L(t,x,V_k)|^2dt\,.$$

Since, for any $\eta\in F$:

$$\sum_{k=1}^{d}\sum_{\|\alpha\|\leqq L-1}((V_k)_{(\alpha)}(x),\eta)^2\geqq \mathcal{V}_L(x,\eta)\,,$$

we see that

$$\inf_{\eta\in F}\sum_{k=1}^{d}\int_0^{T/K}\Big(\sum_{\|\alpha\|\leqq L-1}((V_k)_{(\alpha)}(x),\eta)_R\theta^{(\alpha)}(t)\Big)^2dt$$

$$\geqq(\mathcal{V}_L(x,F)/d)\inf\Big\{\int_0^{T/K}\Big(\sum_{\|\alpha\|\leqq L-1}b_\alpha\theta^{(\alpha)}(t)\Big)^2dt:\sum_{\|\alpha\|\leqq L-1}b_\alpha^2=1\Big\}.$$

Combining this with the preceding, we conclude that for any $0<\varepsilon<1$, $T\in(0,1]$, and $K\in[1,\infty)$:

$$\mathcal{W}\Big(\frac{1}{T^L}\tilde{\lambda}(T/K,x,F)\leqq 1/K^{L+1-\varepsilon}\Big)$$

$$\leqq\mathcal{W}\Big((K/T)^L\inf\Big\{\int_0^{T/K}\Big(\sum_{\|\alpha\|\leqq L-1}b_\alpha\theta^{(\alpha)}(t)\Big)^2dt:\sum_{\|\alpha\|\leqq L-1}b_\alpha^2=1\Big\}\leqq\frac{4d}{\mathcal{V}_L(x,F)}1/K_L^{1-\varepsilon}\Big)$$

$$+\mathcal{W}\Big(\sum_{k=1}^{d}\frac{1}{T^L}\int_0^{T/K}|R_L(f,x,V_k)|^2dt\geqq 1/K^{L+1-\varepsilon}\Big).$$

By Theorem (A. 6), the first term on the right is dominated by

$$C(L)\exp\Big(-\Big(\frac{\mathcal{V}_L(x,F)}{4d}K^{1-\varepsilon}\Big)^{\mu_L}\Big).$$

At the same time, by (2.14), the second term is dominated by

$$dC(L,\varepsilon)\exp(-(K/d)^{\mu_{L,\varepsilon}}/(1+M(x))^2).$$

Thus, after replacing K by $K^{1/(L+1-\varepsilon)}$ and taking $\varepsilon=1/(L+1)$, it is an easy matter to deduce (2.18).

The estimate (2.19) is really a consequence of (2.18). To see this, set $F'=\{\eta\in S^{N-1}:\mathcal{V}_L(x,\eta)\geqq(1/2)\mathcal{V}_L(x,F)\}$ and define $D=D_L=\mathrm{card}\{\alpha\in\mathcal{A}:\|\alpha\|\leqq L-1\}$. Then it is easy to see that $F'\supseteqq\{\eta'\in S^{D-1}:|\eta'-F|_{R^D}<\delta\}$, where $\delta^2=\mathcal{V}_L(x,F)/2dDM(x)^2$. Hence, if $\zeta=\inf\{t\geqq 0:\|J(t,x)-I\|_{\mathrm{H.S.}}\leqq\delta/4\}$, then, by (2.6), $\zeta \geqq t$ implies that

$$(\eta, \tilde{A}(t, x)\eta)_{R^N} = (J(t, x)^*\eta, \tilde{A}(t, x)J(t, x)^*\eta)_{R^N}$$
$$\geqq \frac{1}{4}\left(\frac{J(t, x)^*\eta}{\|J(t, x)^*\eta\|_{R^N}}, A(t, x)\frac{J(t, x)^*\eta}{\|J(t, x)^*\eta\|_{R^N}}\right)$$
$$\geqq \frac{1}{4}\tilde{\lambda}(t, x, F')$$

for all $\eta \in F$; and so:

$$\mathcal{W}\left(\frac{1}{T^L}\lambda(T/K^{1/(L+1)}, x, F) \leqq 1/K\right)$$
$$\leqq \mathcal{W}\left(\frac{1}{T^L}\tilde{\lambda}(T/K^{1/(L+1)}, x, F') \leqq 4/K\right) + \mathcal{W}(\zeta \leqq 1/K^{1/(L+1)}).$$

Since $\mathcal{V}_L(x, F') \geqq (1/2)\mathcal{V}_L(x, F)$, the first term on the right can be estimated by using (2.18). At the same time, standard estimates show that there exist universal $C < \infty$ and $\lambda \in (0, \infty)$ such that $\mathcal{W}(\zeta \leqq t) \leqq C\exp(-\lambda(\delta^2 \wedge 1)/(1+M(x))^2 t)$ for $t \in (0, 1]$. Thus (2.19) follows. Q. E. D.

3. Preliminary applications to the transition probability function.

Set $\Omega = C([0, \infty), \boldsymbol{R}^N)$ and for each $t \geqq 0$, define $\omega \to x(t, \omega) \in \boldsymbol{R}^N$ so that $x(t, \omega)$ is the position of ω at time t. Denote by $\mathcal{M}_t$, $t \geqq 0$, the σ-algebra over Ω which is generated by the maps $x(s)$, $0 \leqq x \leqq t$. Then $\bigcup_{t \geqq 0} \mathcal{M}_t$ generates the Borel field $\mathcal{M}$ over Ω.

Given $\{V_0, \cdots, V_d\} \subseteq C_\uparrow^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ satisfying (1.3), think of the V_k's as directional derivatives (cf. (2.9)) and define

$$\boldsymbol{L} = \frac{1}{2}\sum_{k=1}^{d} V_k^2 + V_0. \tag{3.1}$$

A second expression for $\boldsymbol{L}$ is

$$\boldsymbol{L} = \frac{1}{2}\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial}{\partial x_i}, \tag{3.1$'$}$$

where $a(x)$ is defined in (2.8) and $b(x)$ is defined in (1.2). Let $x \to X(\cdot, x)$ be a smooth selection of solutions to (2.1) (equivalently, to (1.4)); and note that, since $X(\cdot, x)$ is $\mathcal{W}$-almost surely continuous, $X(\cdot, x)$ determines a probability measure P_x on $(\Omega, \mathcal{M})$ via

$$P_x = \mathcal{W} \cdot (X(\cdot, x))^{-1}. \tag{3.2}$$

The following theorem simply summarizes well known facts about the distribution of solutions of stochastic integral equations with smooth coefficients. (See, for example, Chapters 5 and 6 of [13].)

(3.3) THEOREM. *For each $x \in \boldsymbol{R}^N$, P_x is the unique probability measure on $(\Omega, \mathcal{M})$ with the property that*

$$\left(\phi(x(t)) - \phi(x) - \int_0^t \boldsymbol{L}\phi(x(s))ds, \mathcal{M}_t, P_x\right)$$

is a mean-zero martingale for all $\phi \in C_0^\infty(\boldsymbol{R}^N)$ (functions in $C^\infty(\boldsymbol{R}^N)$ having compact support). Moreover, the family $\{P_x : x \in \boldsymbol{R}^N\}$ is Feller continuous (i.e., $x_n \to x$ in $\boldsymbol{R}^N$ implies P_{x_n} tends weakly to P_x). Finally, if $\tau : \Omega \to [0, \infty) \cup \{\infty\}$ is an $\{\mathcal{M}_t : t \geqq 0\}$-stopping time, then for all bounded $\mathcal{M}_\tau$-measurable $\Phi : \Omega \to \boldsymbol{R}^1$ and all bounded $\mathcal{M}$-measurable $\Psi : \Omega \to \boldsymbol{R}^1$:

$$E^{P_x}[\Phi \cdot (\Psi \cdot S_\tau), \tau < \infty] = \int_{\{\omega : \tau(\omega) < \infty\}} \Phi(\omega) E^{P_{x(\tau(\omega), \omega)}}[\Psi] P_x(d\omega) , \tag{3.4}$$

where $S_\tau : \{\tau < \infty\} \to \Omega$ is defined so that

$$x(t, S_\tau\omega) = x(t + \tau(\omega), \omega) , \qquad t \geqq 0 .$$

(In other words, $\{P_x : x \in \boldsymbol{R}^N\}$ is a *time-homogeneous strong Markov family.*)

Let $\mathcal{C}$ be the set of all $c \in C_\uparrow^\infty(\boldsymbol{R}^N)$ such that $c_+ \equiv c \vee 0$ is bounded. Given $c \in \mathcal{C}$, define

$$\begin{aligned} {}^cP(T, x, \Gamma) &= E^{P_x}\left[\exp\left(\int_0^T c(x(t))dt\right), x(T) \in \Gamma\right] \\ &= E^{\mathcal{W}}\left[\exp\left(\int_0^T c(X(t, x))dt\right), X(T, x) \in \Gamma\right] \end{aligned} \tag{3.5}$$

for $(T, x) \in (0, \infty) \times \boldsymbol{R}^N$ and $\Gamma \in \mathcal{B}_{\boldsymbol{R}^N}$. By (1.11):

$$\begin{aligned} &\sup\left(\int \|y - x\|_{\boldsymbol{R}^N}^q {}^cP(t, x, dy)\right)^{1/q} \\ &< A(q, T)(1 + \|x\|_{\boldsymbol{R}^N})^{\gamma_0} \exp\left(\frac{T}{q} \|c_+\|_{C_b(\boldsymbol{R}^N)}\right) \end{aligned} \tag{3.6}$$

for all $T > 0$ and $q \in [2, \infty)$. Combining (3.6) with the Feller continuity of $\{P_x : x \in \boldsymbol{R}^N\}$, we see that the operators ${}^cP_T$, $T > 0$, given by

$${}^cP_T\phi(x) = \int \phi(y) {}^cP(T, x, dy) , \qquad x \in \boldsymbol{R}^N , \tag{3.7}$$

map $C_\uparrow(\boldsymbol{R}^N)$ into itself. Moreover, the Markov property of $\{P_x : x \in \boldsymbol{R}^N\}$ plus (3.6) imply that $\{{}^cP_T : T > 0\}$ is a semigroup on $C_\uparrow(\boldsymbol{R}^N)$. At the same time, the continuity of the paths ω together with the Feller continuity of $\{P_x : x \in \boldsymbol{R}^N\}$ and the estimate (3.6) allows us to conclude that, for each $\phi \in C_\uparrow(\boldsymbol{R}^N)$, ${}^cP_T\phi \to \phi$ uniformly on compacts as $T \downarrow 0$. Finally, because of the martingale property characterizing P_x and (3.6):

$$\exp\left(\int_0^T c(x(t))dt\right)\phi(x(T)) - \phi(x) - \int_0^T \exp\left(\int_0^t c(x(s))ds\right)(L+c)\phi(x(t))dt$$

is a mean-zero P_x-martingale relative to $\{\mathcal{M}_T : T>0\}$ for each $\phi \in C_\uparrow^\infty(\boldsymbol{R}^N)$. Hence

$$\text{(3.8)} \qquad {}^cP_T\phi(x) - \phi(x) = \int_0^T {}^cP_t(L+c)\phi(x)dt\,, \qquad (T, x) \in (0, \infty) \times \boldsymbol{R}^N\,,$$

for all $\phi \in C_\uparrow^\infty(\boldsymbol{R}^N)$.

We next show that ${}^cP_T$ maps $C_\uparrow^\infty(\boldsymbol{R}^N)$ into itself. In order to simplify the notation, we set

$${}^cY(T, x) = \int_0^T c(X(t, x))dt$$

and

$${}^cZ(T, x) = \begin{pmatrix} X(T, x) \\ {}^cY(T, x) \end{pmatrix}.$$

(3.9) LEMMA. *If $\phi \in C_\uparrow^\infty(\boldsymbol{R}^N)$, then, for each $T>0$, ${}^cP_T\phi \in C_\uparrow^\infty(\boldsymbol{R}^N)$. In fact, for each $n \geqq 1$ there exist universal (i.e., independent of $\{V_0, \cdots, V_d\}$ and c) polynomials $\mathcal{P}_{n,m} : \Pi\, \mathrm{H.S.}((\boldsymbol{R}^N)^{\otimes \nu}, \boldsymbol{R}^{N+1}) \to \mathrm{H.S.}((\boldsymbol{R}^N)^{\otimes m}, (\boldsymbol{R}^N)^{\otimes n})$, $0 \leqq m \leqq n$, such that*

$$\text{(3.10)} \quad ({}^cP_T\phi)^{(n)}(x) = \sum_{m=0}^n E^{\mathcal{W}}[\exp({}^cY(T, x))\mathcal{P}_{n,m}({}^cZ^{(1)}(T, x), \cdots, {}^cZ^{(n)}(T, x))\phi^{(m)}(X(T, x))]$$

for $\phi \in C_\uparrow^\infty(\boldsymbol{R}^N)$. In particular, if $T>0$ and B is a bounded subset of $C_\uparrow^\infty(\boldsymbol{R}^N)$, then $\{{}^cP_t\phi : 0 \leqq t \leqq T \text{ and } \phi \in B\}$ is also a bounded subset of $C_\uparrow^\infty(\boldsymbol{R}^N)$.

PROOF. Note that, by Theorem (1.9), for each $n \geqq 0$, $T>0$, and $q \in [2, \infty)$ there exist $C_n(q, T) < \infty$ and $\gamma_n(q, T) \in [0, \infty)$ such that

$$E^{\mathcal{W}}\left[\sup_{0 \leqq t \leqq T} \left\| \left({}^cZ(t, x) - \begin{pmatrix} x \\ 1 \end{pmatrix}\right)^{(n)} \right\|^q_{\mathrm{H.S.}((\boldsymbol{R}^N)^{\otimes n}, \boldsymbol{R}^{N+1})}\right]^{1/q} \leqq C_n(q, T)(1 + \|x\|_{\boldsymbol{R}^N})^{\gamma_n(q,T)}\,, \qquad x \in \boldsymbol{R}^N\,.$$

Thus we can differentiate the expression

$$E^{W}[\exp({}^{c}Y(T, x))\phi(X(T, x))]$$

underneath the integral sign. Using this observation, we can easily derive the representation (3.10) by induction; and the other assertions made are quite easy consequences of (3.10) combined with the above estimate on the derivatives of ${}^{c}Z(T, x)$. Q. E. D.

(3.11) THEOREM. *For each* $c\in \mathcal{C}$, $\{{}^{c}P_{T}: T>0\}$ *is a semigroup on* $C_{\uparrow}^{\infty}(\boldsymbol{R}^{N})$. *Moreover, given* $\phi\in C_{\uparrow}^{\infty}(\boldsymbol{R}^{N})$, $(t, x)\in(0, \infty)\times\boldsymbol{R}^{N}\rightarrow P_{t}\phi(x)$ *is the unique function* $u\in C^{\infty}([0, \infty), C_{\uparrow}^{\infty}(\boldsymbol{R}^{N}))$ *such that*

$$\begin{aligned}\frac{\partial u}{\partial t}&=(\boldsymbol{L}+c)u\,, \qquad t>0\,,\\ u(0, \cdot)&=\phi\,.\end{aligned} \tag{3.12}$$

PROOF. To prove that $(t, x)\rightarrow{}^{c}P_{t}\phi(x)$ solves (3.12), it suffices to show that

$$\lim_{h\downarrow 0}\frac{{}^{c}P_{T+h}\phi(x)-{}^{c}P_{T}\phi(x)}{h}=(\boldsymbol{L}+c)^{c}P_{T}\phi(x)$$

for $(T, x)\in[0, \infty)\times\boldsymbol{R}^{N}$. But, by Lemma (3.9), ${}^{c}P_{T}\phi\in C_{\uparrow}^{\infty}(\boldsymbol{R}^{N})$; and so, by (3.8):

$${}^{c}P_{T+h}\phi(x)-{}^{c}P_{T}\phi(x)=({}^{c}P_{h}-I)\cdot{}^{c}P_{T}\phi(x)=\int_{0}^{h}{}^{c}P_{t}(\boldsymbol{L}+c)^{c}P_{T}\phi(x)dt\,;$$

Since $\{{}^{c}P_{t}(\boldsymbol{L}+c)^{c}P_{T}\phi: 0\leqq t\leqq h\}$ is bounded in $C_{\uparrow}^{\infty}(\boldsymbol{R}^{N})$ and ${}^{c}P_{t}(\boldsymbol{L}+c)^{c}P_{T}\phi\rightarrow(\boldsymbol{L}+c)^{c}P_{T}\phi$ pointwise, we have now proved that

$$\frac{d}{dt}{}^{c}P_{t}\phi(x)=(\boldsymbol{L}+c)^{c}P_{t}\phi(x)\,, \qquad (t, x)\in(0, \infty)\times\boldsymbol{R}^{N}\,. \tag{3.13}$$

From (3.13) and the estimates on ${}^{c}P_{t}\phi$ provided by Lemma (3.9), we conclude that $(t, x)\rightarrow{}^{c}P_{t}\phi(x)$ is an element of $C^{\infty}([0, \infty), C_{\uparrow}^{\infty}(\boldsymbol{R}^{N}))$ which solves (3.12). To prove that it is the only such element of $C^{\infty}([0, \infty), C_{\uparrow}^{\infty}(\boldsymbol{R}^{N}))$, let u be a second one. Given $T>0$, set $w(t, x)=[{}^{c}P_{t}u(T-t, \cdot)](x)$, $t\in[0, T]$. Then, $(d/dt)w(t, x)=0$ for $t\in[0, T]$ and so $u(T, x)=w(0, x)=w(T, x)={}^{c}P_{T}\phi(x)$. Q. E. D.

Related to the preceding is the following statement that we can make when all the coefficients have bounded derivatives of all orders.

(3.14) THEOREM. *Suppose that* $\{V_{0}, \cdots, V_{d}\}\subseteq C_{b}^{\infty}(\boldsymbol{R}^{N}, \boldsymbol{R}^{N})$ *and that* $c\in C_{b}^{\infty}(\boldsymbol{R}^{N})$. *If* $T>0$ *and* B *is a bounded subset of* $C_{b}^{n}(\boldsymbol{R}^{N})(\mathcal{S}(\boldsymbol{R}^{N}))$, *then* $\{{}^{c}P_{t}\phi: 0\leqq t\leqq T$: *and* $\phi\in B\}$ *is also a bounded subset of* $C_{b}^{n}(\boldsymbol{R}^{N})(\mathcal{S}(\boldsymbol{R}^{N}))$. *Furthermore, if*

$$\tilde{V}_0 \equiv -V_0 + \frac{1}{2}\sum_{k=1}^{d} \operatorname{div}(V_k) V_k$$

and

$$\tilde{c} \equiv c - \operatorname{div}(V_0) + \frac{1}{2}\sum_{k=1}^{d} V_k(\operatorname{div}(V_k)) + \frac{1}{2}\sum_{k=1}^{d}(\operatorname{div} V_k)^2,$$

and if $\{{}^c\tilde{P}_T : T>0\}$ *is the semigroup associated with* $\{\tilde{V}_0, V_1, \cdots, V_d\}$ *and* $\tilde{c}$, *then* $\{{}^c\tilde{P}_T : T>0\}$ *on* $C_b^n(\boldsymbol{R}^N)$ $(C_\uparrow^\infty(\boldsymbol{R}^N)$ *or* $\mathcal{S}(\boldsymbol{R}^N))$ *has the same regularity properties as* $\{{}^cP_T : T>0\}$. *Finally, if* $\phi \in \mathcal{S}(\boldsymbol{R}^N)$ *and* $\psi \in C_\uparrow^\infty(\boldsymbol{R}^N)$ *or* $\phi \in C_\uparrow^\infty(\boldsymbol{R}^N)$ *and* $\psi \in \mathcal{S}(\boldsymbol{R}^N)$, *then*

$$\int \phi(y)\,{}^c\tilde{P}_T\psi(y)dy = \int \psi(x)\,{}^cP_T\phi(x)dx\,, \qquad T>0\,. \tag{3.15}$$

(That is, ${}^c\tilde{P}_T$ *is the formal adjoint of* ${}^cP_T$.*)*

PROOF. Let $\mathcal{P}_{n,m}$ be the polynomials introduced in Lemma (3.9) and set

$$\mathcal{P}_{n,m}(t, x) = \mathcal{P}_{n,m}({}^cZ^{(1)}(t, x), \cdots, {}^cZ^{(n)}(t, x))\,.$$

By Theorem (1.9) with $\gamma_m = 0$, $m \geqq 0$, we know that for all $n \geqq 1$, $T>0$, and $q \in [1, \infty)$:

$$\sup_{0\leq t\leq T}\sup_{x\in R^N} E^{\mathcal{W}}\left[\left(\sum_{m=0}^{n} \|\mathcal{P}_{n,m}(t, x)\|^2_{E^{(m,n)}}\right)^{q/2}\right]^{1/q} = K_n(q, T) < \infty\,,$$

where $E^{(m,n)} \equiv \mathrm{H.S.}((\boldsymbol{R}^N)^{\otimes m}, (\boldsymbol{R}^N)^{\otimes n})$. Using this together with (3.10), we see that $\{{}^cP_t\phi : 0 \leqq t \leqq T$ and $\phi \in B\}$ is bounded in $C_b^n(\boldsymbol{R}^N)$ whenever B is.

Next, let B be a bounded subset of $\mathcal{S}(\boldsymbol{R}^N)$ and define

$$A = \max_{0\leq m\leq n}\sup_{x\in R}\sup_{\phi\in B}(1+\|x\|^2_{R^N})^{n/2}\|\phi^{(m)}(x)\|_{E^{(m)}}\,,$$

where $E^{(m)} \equiv \mathrm{H.S.}((\boldsymbol{R}^N)^{\otimes m}, \boldsymbol{R}^1)$. Given $0 \leqq m \leqq n$, $t \in [0, T]$, and $\phi \in \mathcal{S}(\boldsymbol{R}^N)$, (3.10) shows us that

$$\begin{aligned}
&(1+\|x\|^2_{R^N})^{n/2}\|({}^cP_t\phi)^{(m)}(x)\|_{E^{(m)}} \\
&\leqq (1+\|x\|^2_{R^N})^{n/2}\exp(T\|c_+\|_{C_b(R^N)}) \\
&\quad\times E^{\mathcal{W}}\left[\left(\sum_{\mu=0}^{m}\|\mathcal{P}_{m,\mu}(t, x)\|_{E^{(m,\mu)}}\right)^{1/2}(\sum \phi^{(\mu)}(X(t, x))\|^2_{E^{(\mu)}})^{1/2}\right] \\
&\leqq K_n(2, T)\exp(T\|c_+\|_{C_b(R^N)})(1+\|x\|^2_{R^N})^{n/2} \\
&\quad\times E^{\mathcal{W}}\left[\sum_{\mu=0}^{m}\|\phi^{(\mu)}(X(t, x))\|^2_{(\mu)}\right]^{1/2}.
\end{aligned}$$

At the same time, if $\phi \in B$, then:

$$(1+\|x\|_{R^N}^2)^{n/2}\left(E^{\mathcal{W}}\left[\sum_{\mu=0}^{m}\|\phi^{(\mu)}(X(t,x))\|_{E(\mu)}^2\right]\right)^{1/2}$$

$$\leqq n^{1/2}A_n(2^n+[(1+\|x\|_{R^N}^2)^n\mathcal{W}(\|X(t,x)\|_{R^N}\leqq\|x\|_{R^N}/2)]^{1/2}).$$

Finally, by standard estimates, there exist $\lambda\in(0,\infty)$ and $C_T<\infty$ such that

$$\mathcal{W}\left(\sup_{0\leq t\leq T}\|X(t,x)-x\|_{R^N}\geqq R\right)\leqq C_T\exp(-\lambda R^2/T), \qquad R>0.$$

Combining this with the preceding, we conclude that $\{{}^cP_t\phi: 0\leqq t\leqq T$ and $\phi\in B\}$ is bounded in $\mathcal{S}(\boldsymbol{R}^N)$.

Turning to $\{{}^c\tilde{P}_T: T>0\}$, we first remark that $\{\tilde{V}_0, V_1,\cdots,V_d\}$ and $\tilde{c}$ have the regularity properties as $\{V_0,\cdots,V_d\}$ and c. Hence the regularity properties of $\{{}^c\tilde{P}_T: T>0\}$ on $C_b^\infty(\boldsymbol{R}^N)(C_\uparrow^\infty(\boldsymbol{R}^N)$ and $\mathcal{S}(\boldsymbol{R}^N))$ are the same as those of $\{{}^cP_T: T>0\}$. Moreover, if

$$\tilde{\boldsymbol{L}}=\frac{1}{2}\sum_1^d V_k^2+\tilde{V},$$

then $\tilde{\boldsymbol{L}}+\tilde{c}$ is the formal adjoint of $\boldsymbol{L}+c$ and

$$\frac{d}{dt}{}^c\tilde{P}_t\phi={}^c\tilde{P}_t(\tilde{\boldsymbol{L}}+\tilde{c})\phi=(\tilde{\boldsymbol{L}}+\tilde{c}){}^c\tilde{P}_t\phi$$

for all $\phi\in C_\uparrow^\infty(\boldsymbol{R}^N)$. Thus, if $\phi\in\mathcal{S}(\boldsymbol{R}^N)$ and $\psi\in C_\uparrow^\infty(\boldsymbol{R}^N)$ or $\phi\in C_\uparrow^\infty(\boldsymbol{R}^N)$ and $\psi\in\mathcal{S}(\boldsymbol{R}^N)$, then we can use the estimates just mentioned to show that

$$w(t)\equiv\int{}^cP_t\phi(x)\,{}^c\tilde{P}_{T-t}\psi(x)dx, \qquad t\in[0,T],$$

can be differentiated underneath the integral sign and has derivatives identically equal to zero. Clearly (3.15) results from $w(0)=w(T)$. Q.E.D.

Thus far none of our results depends on anything except the regularity properties of the V_k's and c. In particular, these results are equally true when $V_1,\cdots,V_d$ vanish identically and therefore do not deserve to be called elliptic regularity estimates. In order to obtain regularity properties which depend on non-degeneracy properties of $\{V_1,\cdots,V_d\}$, we must take into account the considerations contained in Section 2).

Recall the definition of Malliavin's covariance matrix $A(T,x)$ (cf. (2.4) and (2.5)) and set

(3.16) $$\Delta(T,x)=\det(A(T,x)).$$

(3.17) THEOREM. *Let U be an open subset of $\boldsymbol{R}^N$. Suppose that ρ: $(0,1]\to(0,\infty)$ and $p\in[1,\infty)\to M_p\in(0,\infty)$ are non-decreasing functions such that*

$$(3.18) \qquad \sup_{x\in U} \|1/\Delta(t, x)\|_{L^p(\mathcal{W})} \leqq M_p/\rho(t), \qquad t\in(0, 1].$$

Then, for each $c\in\mathcal{C}$, there is a ${}^c p\in C^\infty((0, \infty)\times U\times \boldsymbol{R}^N)$ such that ${}^c P(T, x, dy) = {}^c p(T, x, y)dy$ for each $(T, x)\in(0, \infty)\times \boldsymbol{R}^N$. In fact, suppose that $\{C_m : m\geqq 0\}$ and $\{\gamma_m : m\geqq 0\}$ are the numbers appearing in (1.10) *(recall that $\gamma_1=0$ and see* (1.1) *and* (1.2) *for an explanation of the notation in* (1.10)) *and assume in addition that*

$$\|c^{(m)}(x)\|_{\mathrm{H.S.}((\boldsymbol{R}^N)^{\otimes m}, \boldsymbol{R}^1)} \leqq C_{m+1}(1+\|x\|_{\boldsymbol{R}^N})^{\gamma_{m+1}}, \quad m\geqq 1 \text{ and } x\in \boldsymbol{R}^N.$$

Then, for each $n\geqq 0$, there exists an $m_n\geqq 2$, a $p_n\in[2, \infty)$, a $\nu_n\in(0, \infty)$, and a $\lambda_n\in(0, \infty)$, all of which are independent of $\{V_0, \cdots, V_d\}$ and c; a $\mu_n\in [0, \infty)$ which depends only on $\{\gamma_m : 0\leqq m\leqq m_n\}$ and is 0 if $\gamma_0=\cdots=\gamma_{m_n}=0$; and a non-decreasing map $T\in[1, \infty)\to K_n(T)\in(0, \infty)$ which depends only on $\{C_m : 0\leqq m\leqq m_n\}$ such that

$$(3.19) \quad (1+\|y-x\|_{\boldsymbol{R}^N}^2)^{n/2} |D_t^m D_x^\alpha D_y^\beta \, {}^c p(t, x, y)| \leqq M_{p_n}^{\nu_n} K_n(T)(1+\|x\|_{\boldsymbol{R}^N})^{\mu_n}$$
$$\times \exp(t\|c_+\|_{B(\boldsymbol{R}^N)} - \lambda_n(\|y-x\|_{\boldsymbol{R}^N}\wedge 1)^2/C_0^2(1+\|x\|_{\boldsymbol{R}^N}^2)^{\gamma_0} t)/(\rho(t\wedge 1)\wedge t)^{\nu_n}$$

for all $T\geqq 1$ and $(t, x, y)\in(0, T]\times U\times \boldsymbol{R}^N$ and all $m\geqq 0$, $\alpha\in\mathfrak{N}^N$, and $\beta\in\mathfrak{N}^N$ satisfying $m+|\alpha+\beta|\leqq n$. Moreover, if $\gamma_0=0$, then (3.19) *can be replaced by*

$$(3.20) \qquad |D_t^m D_x^\alpha D_y^\beta \, {}^c p(t, x, y)| \leqq M_{p_n}^{\nu_n} K_n(T)(1+\|x\|_{\boldsymbol{R}^N})^{\mu_n}$$
$$\times \exp(t\|c_+\|_{B(\boldsymbol{R}^N)} - \lambda_n \|y-x\|_{\boldsymbol{R}^N}^2/(1+C_0^2)t)/(\rho(t\wedge 1)\wedge t)^{\nu_n}.$$

Finally, if $\gamma_m=0$ for $0\leqq m\leqq m_n$, then the μ_n in (3.20) *can be taken to be* 0.

PROOF. We first note that for $T\geqq 1$:

$$\begin{aligned} A(T, x) &= J(T, x)\tilde{A}(T, x)J(T, x)^* \\ &\geqq J(T, x)\tilde{A}(1, x)J(T, x)^* \\ &= J_1(T, x)A(1, x)J_1(T, x)^* \end{aligned}$$

where $J_1(T, x)=J(T, x)J^{-1}(1, x)$. Starting from (2.2) and (2.3), we see that:

$$J_1^{-1}(T, x) = I - \sum_{k=1}^d \int_1^T J_1^{-1}(t, x) V_k^{(1)}(X(t, x))\circ d\theta_k(t)$$
$$- \int_0^T J_1^{-1}(t, x) V_0^{(1)}(X(t, x))dt, \qquad T\geqq 1.$$

After converting this equation to its equivalent Itô form, one can easily check that for each $p\in[1, \infty)$ there exist $A_p\in(0, \infty)$ and $B_p\in[0, \infty)$, depending on C_1 alone, such that

$$\| \|J_1^{-1}(T, x)\|^2_{\text{H.S.}(\boldsymbol{R}^N, \boldsymbol{R}^N)}\|_{L^p(\mathcal{W})} \leqq A_p \exp(B_p(T-1)) , \qquad T \geqq 1 .$$

Combining this with the above, we conclude that

$$\|1/\Delta(T, x)\|_{L^p(\mathcal{W})} \leqq A_{2p} \exp(B_{2p}(T-1)) \|1/\Delta(1, x)\|_{L^{2p}(\mathcal{W})} , \qquad T \geqq 1 , \tag{3.21}$$

for all $x \in \boldsymbol{R}^N$. In particular, by Theorem (1.10) in [6] and (3.18) above, $1/\Delta(T, x) \in \mathcal{G}(\mathcal{L})$ for all $(T, x) \in (0, \infty) \times U$.

Next, let $\alpha \in \mathfrak{N}^N$ be given and set

$$^c\Xi_{\alpha,\alpha'}(T, x) = \exp(^cY(T, x))(\mathcal{P}_{|\alpha|,|\alpha'|}(^cZ^{(1)}(T, x), \cdots , {}^cZ^{(n)}(T, x)))_{\alpha,\alpha'} ,$$

where we have used the notation introduced in Lemma (3.9) and the paragraph preceding that lemma. Then

$$D^\alpha\, {}^cP_t\phi(x) = \sum_{|\alpha'| \leqq |\alpha|} E^{\mathcal{W}}[^c\Xi_{\alpha,\alpha'}(T, x)(D^{\alpha'}\phi)(X(T, x))]$$

for all $\phi \in C_\uparrow^\infty(\boldsymbol{R}^N)$. Hence, if $\mathcal{R}_\gamma(T, x)$, $\gamma \in \mathfrak{N}^N$, is the operator on $\mathcal{G}(\mathcal{L})$ associated with $X(T, x)$ as in Theorem (1.20) of [6], then for $(T, x) \in (0, \infty) \times U$:

$$D^\alpha\, {}^cP_T\phi(x) = E^{\mathcal{W}}[^c\Psi_\alpha(T, x)\phi(X(T, x))] , \qquad \phi \in C_\uparrow^\infty(\boldsymbol{R}^N)$$

where

$$^c\Psi_\alpha(T, x) = \sum_{|\alpha'| \leqq |\alpha|} (-1)^{|\alpha'|}\mathcal{R}_{\alpha'}(T, x)(^c\Xi_{\alpha,\alpha'}(T, x)/(\Delta(T, x))^{(2|\alpha'|-1)0})$$

is again an element of $\mathcal{G}(\mathcal{L})$. In other words, if

$$^cQ_\alpha(T, x, \cdot) = (^c\Psi_\alpha(T, x)\mathcal{W})(X(T, x)-x)^{-1} , \qquad (T, x) \in (0, \infty) \times U ,$$

then

$$D^\alpha\, {}^cP_T\phi(x) = \int \phi(x+y)\, {}^cQ_\alpha(T, x, dy) , \qquad \phi \in C_\uparrow^\infty(\boldsymbol{R}^N) . \tag{3.22}$$

Next, choose $\eta \in C^\infty(\boldsymbol{R}^N)$ so that $0 \leqq \eta \leqq 1$, $\eta = 0$ on $B(0, 1/2)$, and $\eta = 1$ off $B(0, 3/4)$. For $n \geqq 0$, define

$$\eta_{n,\varepsilon}(y) = \begin{cases} (1+\|y\|^2_{\boldsymbol{R}^N})^{n/2} & \text{if } \varepsilon = 0 \\ (1+\|y\|^2_{\boldsymbol{R}^N})^{n/2}\eta(y/\varepsilon) & \text{if } \varepsilon > 0 . \end{cases}$$

Then, for $\beta \in \mathfrak{N}^N$, $\phi \in C_\uparrow^\infty(\boldsymbol{R}^N)$, and $(T, x) \in (0, \infty) \times U$:

$$\begin{aligned} &\int (D^\beta\phi)(y)\eta_{n,\varepsilon}(y)\, {}^cQ_\alpha(T, x, dy) \\ &= E^{\mathcal{W}}[^c\Psi_\alpha(T, x)\eta_{n,\varepsilon}(X(T, x)-x)(D^\beta\phi)(X(T, x)-x)] \\ &= (-1)^{|\beta|}E^{\mathcal{W}}[^c\Psi_{\alpha,\beta,n,\varepsilon}(T, x)\phi(X(T, x)-x)] \end{aligned}$$

where

$$^c\Psi_{\alpha,\beta,n,\varepsilon}(T,x)=\mathcal{R}_\beta(T,x)(\Psi_\alpha(T,x)\eta_{n,\varepsilon}(X(T,x)-x)/(\Delta(T,x))^{(2|\beta|-1)\vee 0})$$

is again an element of $\mathcal{G}(\mathcal{L})$. We can therefore apply Theorem (1.31) of [6] to conclude that for each $(T,x)\in(0,\infty)\times U$ there is a $q_\alpha(T,x,\cdot)\in C_b^\infty(\boldsymbol{R}^N)$ such that

$$\eta_{n,\varepsilon}(y)^cQ_\alpha(T,x,dy)=\eta_{n,\varepsilon}(y)q_\alpha(T,x,y)dy\,.$$

Moreover, that same theorem tells us that there is a universal $C\in(0,\infty)$ such that

$$\begin{aligned}&\|\eta_{n,\varepsilon}(\cdot)^cq_\alpha(t,x,\cdot)\|_{C_b^m(\boldsymbol{R}^N)}\\ &\leqq C\Big(\sum_{|\gamma|\leqq m+1}\|\mathcal{R}_\gamma(t,x)[\Psi_\alpha(t,x)\eta_{n,\varepsilon}(X(t,x)-x)/(\Delta(t,x))^{(2|\alpha|-1)\vee 0}]\|_{L^{N+1}(\mathcal{W})}\Big)\\ &\quad\times\Big(\sum_{|\gamma|=1}\|\mathcal{R}_\gamma(t,x)(1/\Delta(t,x))\|_{L^{N+1}(\mathcal{W})}^{N2/(N+1)}\Big)\,.\end{aligned}$$

By combining the estimates in (1.22) and Lemma (1.14) of [6] with those in (3.18), (3.21), and Theorem (1.9) of the present article, we see that for given $n\geqq 0$ there exist $p_n\in[1,\infty)$, $q_n\in[1,\infty)$, $m_n\geqq 2$, and $\nu_n\in(p,\infty)$, which are independent of $\{V_0,\cdots,V_d\}$ and c, such that, for all $T\geqq 1$ and $\alpha,\beta\in\mathcal{N}^N$ satisfying $|\alpha+\beta|<n$, the right hand side of the preceding is dominated by an expression of the form

$$\begin{aligned}&M_{p_n}^{\nu_n}K_n(T)\exp(t\|c_+\|_{B_0(\boldsymbol{R}^N)})(1+\|x\|_{\boldsymbol{R}^N}^2)^{\mu_n}\|\eta(\cdot/\varepsilon)\|_{C_b^{n+1}(\boldsymbol{R}^N)}\\ &\times\mathcal{W}\Big(\|X(t,x)-x\|_{\boldsymbol{R}^N}\geqq\frac{\varepsilon}{2}\Big)^{1/q_n}\Big/\rho(t\wedge 1)^{\nu_n}\,,\qquad (t,x)\in(0,T]\times U\,,\end{aligned}$$

where $T\in[1,\infty)\to K_n\in(0,\infty)$ depends only on $\{C_m:0\leqq m\leqq m_n\}$ and $\mu_n\in[0,\infty)$ depends only on $\{\gamma_m:0\leqq m\leqq m_n\}$. In particular, if $\gamma_0=\cdots=\gamma_{m_n}=0$, then μ_n can be taken to be 0. Given $y\in\boldsymbol{R}^N$, we now apply this estimate with $\varepsilon=0$ and $\varepsilon=\|y\|_{\boldsymbol{R}^N}$. After making an obvious modification of $K_n(T)$, we then arrive at:

$$\begin{aligned}\text{(3.23)}\qquad &\max_{|\alpha+\beta|\leqq n}|D_y^\beta(1+\|y\|_{\boldsymbol{R}^N}^2)^{n/2}\,{}^cq_\alpha(t,x,y)|\\ &\leqq M_{P_n}^{\nu_n}K_n(T)\exp(t\|c_+\|_{B(\boldsymbol{R}^N)})(1+\|x\|_{\boldsymbol{R}^N}^2)^{\mu_n}/\rho(t\wedge 1)^{\nu_n}\\ &\quad\times[(\mathcal{W}(\|X(T,x)-x\|_{\boldsymbol{R}^N}\geqq\|y\|_{\boldsymbol{R}^N}/2)^{1/q_n}/\|y\|_{\boldsymbol{R}^N}^{n+1})\wedge 1]\end{aligned}$$

for all $(t,x,y)\in(0,T]\times U\times\boldsymbol{R}^N$.

To complete the proof of (3.19) and (3.20) when $m=0$, we must provide an appropriate estimate for $\mathcal{W}(\|X(t,x)-x\|_{\boldsymbol{R}^N}\geqq R)$. To this end, set

$$M(x, R)=\sup_{y\in B(x,R)}\|\sigma(y)\|_{\mathrm{H.S.}(\boldsymbol{R}^N,\boldsymbol{R}^N)}\vee\|b(y)\|_{\boldsymbol{R}^N}\,,$$

where $\sigma(\cdot)$ and $b(\cdot)$ are given by (1.1) and (1.2). Then, by standard estimates, so long as $t\leqq R/2M(x, R)$:

$$\mathcal{W}\Big(\sup_{0\leqq s\leqq t}\|X(s, x)-x\|_{\boldsymbol{R}^N}\geqq R\Big)\leqq 2N\exp(-R^2/8NM(x, R)^2t)\,.$$

Thus, for all $t>0$ and $R>0$:

$$\text{(3.24)}\qquad \mathcal{W}\Big(\sup_{0\leqq s\leqq t}\|X(x, s)-x\|\geqq R\Big)$$
$$\leqq 2N\exp(R/4N)\exp(-R^2/8N(1+M(x, R)^2)t)\,.$$

Using (3.24) in conjunction with (3.22) and (3.23), one can easily complete the derivative of (3.19) and (3.20) when $m=0$. (Recall that $\gamma_1=0$ and therefore that we may always assume that $\gamma_0\in[0, 1]$.)

To obtain (3.19) and (3.20) for general $m\geqq 0$, simply observe that, by (3.13) and the regularity results already proved about $x\in U\rightarrow {}^cp(t, x, y)$:

$$\frac{\partial^m}{\partial t^m}{}^cp(t, x, y)=(\boldsymbol{L}_x+c(x))^m\,{}^cp(t, x, y)$$

on $(0, \infty)\times U\times\boldsymbol{R}^N$. Q. E. D.

(3.25) COROLLARY. *Define* $\mathcal{V}_L(x, \eta)$ *for* $L\geqq 1$, $x\in\boldsymbol{R}^N$, *and* $\eta\in S^{N-1}$ *as in* (2.15) *and set* $\mathcal{V}_L(x)=(\inf_{\eta\in S^{n-1}}\mathcal{V}_L(x, \eta))\wedge 1$. *Given* $L\geqq 1$, *define* $U_L=\{x\in\boldsymbol{R}^N:\mathcal{V}_L(x)>0\}$. *Then there exists an* $m(L)\geqq 2$ *and a non-decreasing map* $p\in[1, \infty)\rightarrow M_p(L)\in(0, \infty)$, *both of which are independent of* $\{V_0, \cdots, V_d\}$ *and* c; *a* $\mu(L)\in(0, \infty)$ *which depends only on* $\{\gamma_m: 0\leqq m\leqq m(L)\}$ *and is* 0 *if* $\gamma_0=\cdots=\gamma_{m(L)}=0$; *and a* $K(L)\in(0, \infty)$ *which depends only on* $\{C_m: 0\leqq m\leqq m(L)\}$ *such that*:

$$\text{(3.26)}\qquad \|1/\Delta(T, x)\|_{L^p(\mathcal{W})}\leqq M_p(L)K(L)(1+\|x\|^2_{\boldsymbol{R}^N})^{\mu(L)}/(\mathcal{V}_L(x)^{1+2/L}T)^{NL}\,,$$
$$(T, x)\in(0, 1]\times U_L\,.$$

Next, set $U=\bigcup_{L=1}^{\infty}U_L$. *Then there is a* ${}^cp\in C^\infty((0, \infty)\times U\times\boldsymbol{R}^N)$ *such that* ${}^cP(T, x, dy)={}^cp(T, x, y)dy$ *for each* $(T, x)\in(0, \infty)\times U$. *In fact, for given* $n\geqq 0$ *and* $L\geqq 1$, *there exists an* $m_n(L)\geqq 2$, $\nu_n(L)\in(0, \infty)$, *and a* $\lambda_n\in(0, \infty)$, *all of which are independent of* $\{V_0, \cdots, V_d\}$ *and* c; *a* $\mu_n(L)\in(0, \infty)$ *which depends only on* $\{\gamma_m: 0\leqq m\leqq m_n(L)\}$ *and is* 0 *if* $\gamma_0=\cdots=\gamma_{m_n(L)}=0$; *and a non-decreasing function* $T\in[1, \infty)\rightarrow K_n(T, L)\in(0, \infty)$ *which depends only on* $\{C_m: 0\leqq m\leqq m_n(L)\}$ *such that*

(3.27) $$(1+\|y-x\|_{\boldsymbol{R}^N}^2)^{n/2}|D_t^m D_x^\alpha D_y^\beta {}^c p(t, x, y)| \leqq K_n(T, L)(1+\|x\|_{\boldsymbol{R}^N})^{\mu_n(L)}$$
$$\times \exp(t\|c_+\|_{B(\boldsymbol{R}^N)} - \lambda_n(\|y-x\|_{\boldsymbol{R}^N} \wedge 1)^2 / C_0^2(1+\|x\|_{\boldsymbol{R}^N})^{\gamma_0} t) / (\mathcal{V}_L(x)^{1+2/L} t)^{\nu_n(L)}$$

for all $T \geqq 1$, *all* m, α, *and* β *satisfying* $2m+|\alpha|+|\beta| \leqq n$ *and all* $(t, x, y) \in (0, T] \times U_L \times \boldsymbol{R}^N$. *Moreover, if* $\gamma_0=0$, *then* (3.27) *can be replaced by*

(3.28) $$|D_t^m D_x^\alpha D_y^\beta {}^c p(t, x, y)| \leqq K_n(T, L)(1+\|x\|_{\boldsymbol{R}^N})^{\mu_n(L)}$$
$$\times \exp(t\|c_+\|_{B(\boldsymbol{R}^N)} - \lambda_n \|y-x\|_{\boldsymbol{R}^N}^2 / (1+C_0^2) t) / (\mathcal{V}_L(x)^{1+2/L} t)^{\nu_n(L)} .$$

Finally, if $\gamma_0 = \cdots = \gamma_{m_n(L)} = 0$, *then* (3.28) *holds with* $\mu_n(L)=0$.

PROOF. In view of Theorem (3.17), we need only check (3.27). Referring to the notation used in Theorem (2.17), set $\tilde{\lambda}(T, x) = \tilde{\lambda}(T, x, S^{N-1})$. *Then*

$$1/\Delta(T, x) = (\det J^{-1}(T, x))^2 / \det \tilde{A}(T, x)$$
$$\leqq (\det J^{-1}(T, x))^2 / \tilde{\lambda}(T, x)^N .$$

By the same argument as the one which led to (3.21), there exists a non-decreasing map $p \in [1, \infty) \to M'_p \in (0, \infty)$, depending on C_1 alone, such that

$$\sup_{0 \leq T \leq 1} \sup_{x \in \boldsymbol{R}^N} \|(\det J^{-1}(T, x))^2\|_{L^p(\mathcal{W})} \leqq M'_p .$$

At the same time, since $t \to \tilde{\lambda}(t, x)$ is non-decreasing, (2.18) tells us that for $(T, x) \in (0, 1] \times U_L$:

$$\mathcal{W}(\tilde{\lambda}(T, x)/T^L \leqq 1/K) \leqq \mathcal{W}(\tilde{\lambda}(T/K, x)/T^L \leqq 1/K)$$
$$\leqq \tilde{C}(L) \exp(-(\mathcal{V}_L(x)^{L+2} K)^{\mu_L} / (1+M(x))^2) , \qquad K \geqq 1 ,$$

where $\tilde{C}(L)$ and μ_L are universal and $M(x)$ is described in Lemma (2.17). From this and the preceding, the deduction of (3.27) is easy. Q. E. D.

4. Localization.

Thus far we have been working under rather rigid global regularity assumptions on our coefficients. The purpose of this section is to show that so long as we are interested only in local conclusions we need only impose local regularity assumptions (cf. Theorem (4.5) below). Since the procedures which we have in mind are quite general and do not rely particularly on the detailed structure of the diffusion under consideration, we will begin with a somewhat abstract formulation of our localization procedures.

Throughout this section, our notation is standardized as follows:

i) $\{Q_x : x\in \boldsymbol{R}^N\}$ is a strong Markov, Feller continuous family of probability measures on $(\Omega, \mathscr{M})$ and $c\in C(\boldsymbol{R}^N)$ satisfies $\|c_+\|_{B(\boldsymbol{R}^N)}<\infty$. For $(T, x) \in(0, \infty)\times \boldsymbol{R}^N$ and $\Gamma\in\mathscr{B}_{\boldsymbol{R}^N}$:

$$^cQ(T, x, \Gamma)=E^{Q_x}\left[\exp\left(\int_0^T c(x(t))dt\right), x(T)\in\Gamma\right]$$

and $\{^cQ_T : T>0\}$ is the associated semigroup on $B(\boldsymbol{R}^N)$: $^cQ_Tf(x)=\int f(y)^cQ(T, x, dy)$.

ii) W is a non-empty open subset of $\boldsymbol{R}^N$ and $\hat{\boldsymbol{R}}^N$ is a Polish space containing $\boldsymbol{R}^N$ as an open subset, $\hat{\Omega}\equiv C([0, \infty), \hat{\boldsymbol{R}}^N)$ and $\hat{\mathscr{M}}$ and $\{\hat{\mathscr{M}}_t : t\geqq 0\}$ are the naturally associated σ-algebra over $\hat{\Omega}$.

iii) $\{P_x : x\in\hat{\boldsymbol{R}}^N\}$ is a strong Markov, Feller continuous family of probability measures on $(\hat{\Omega}, \hat{\mathscr{M}})$, $c\in C(\hat{\boldsymbol{R}}^N)$ satisfies $\|\hat{c}_+\|_{B(\hat{\boldsymbol{R}}^N)}<\infty$, and $^{\hat{c}}P(T, x, \cdot)$ and $\{^{\hat{c}}P_T : T>0\}$ are defined by analogy with $^cQ(T, x, \cdot)$ and $\{^cQ_T : T>0\}$.

iv) $\zeta^W=\inf\{t\geqq 0 : x(t)\notin W\}$; and, for each $x\in W$, $P_x|_{\mathscr{M}_{\zeta^W}}=Q_x|_{\mathscr{M}_{\zeta^W}}$ and $\hat{c}(x)=c(x)$.

(4.1) LEMMA. *Let U and V be non-empty open subsets of W satisfying $\bar{U}\subseteqq V$ and $\bar{V}\subseteqq W$, and let $(\Sigma, \|\cdot\|_\Sigma)$ be a Banach space which is contained in the space of finite measures on U. Suppose that $(T, x)\in(0, 1]\times\bar{W}\to{}^cQ_U(T, x, dy)\equiv\chi_U(y)^cQ(T, x, dy)$ is a measurable mapping into Σ, and assume that*

$$K(t)\equiv\sup_{0<s\leqq t}\ \sup_{\xi\in\bar{W}\setminus U}\|^cQ_U(s, \xi, \cdot)\|_\Sigma<\infty\,, \qquad t\in(0, 1]\,.$$

In addition, suppose that

$$\sup_{x\in\bar{V}} Q_x(\zeta^W\leqq\mu)\leqq\frac{1}{e}\,,$$

where μ is some element of $(0, 1]$. Then

$$(T, x)\in(0, 1]\times\hat{\boldsymbol{R}}^N \longrightarrow {}^{\hat{c}}P_U(T, x, dy)\equiv\chi_U(y)^{\hat{c}}P(T, x, dy)$$

is a measurable mapping into Σ. In fact,

$$\|^{\hat{c}}P_U(T, x, \cdot)\|_\Sigma\leqq\|^cQ(T, x, \cdot)\|_\Sigma+5K(T)\exp(T\|\hat{c}_+\|_{B(\hat{\boldsymbol{R}}^N)})Q_x(\zeta^W\leqq T)/\mu^2\,,$$

if $(T, x)\in(0, 1]\times\bar{W}$, and

$$\|^{\hat{c}}P_U(T, x, \cdot)\|_\Sigma\leqq 5K(T)\exp(T\|\hat{c}_+\|_{B(\hat{\boldsymbol{R}}^N)})/\mu^2$$

if $(T, x)\in(0, 1]\times(\hat{\boldsymbol{R}}^N\setminus\bar{V})$. Finally, if $\sup_{\xi\in\bar{W}}\|^cQ_U(1, \xi, \cdot)\|_\Sigma<\infty$, then $(T, x)\in(1, \infty)\times\hat{\boldsymbol{R}}^N\to{}^{\hat{c}}P_U(T, x, \cdot)$ is a measurable mapping into Σ and

$$\|{}^{\hat{c}}P_U(T, x, \cdot)\|_{\Sigma} \leqq \left(\sup_{\xi\in\overline{W}} \|{}^{c}Q_U(1, \xi, \cdot)\|_{\Sigma}\right) \exp((T-1)\|\hat{c}_+\|_{B(\hat{R}^N)})$$
$$+5K(1)\exp(T\|\hat{c}_+\|_{B(\hat{R}^N)}/\mu^2)$$

if $(T, x)\in(1, \infty)\times\hat{\boldsymbol{R}}^N$.

PROOF. First, define

$${}^{c}Q^W(T, x, \Gamma) = E^{Q_x}\left[\exp\left(\int_0^T c(x(t))dt\right), x(T)\in\Gamma \text{ and } \zeta^W > T\right].$$

Then, by the strong Markov property;

$${}^{c}Q^W(T, x, \Gamma)$$
$$= {}^{c}Q(T, x, \Gamma) - E^{Q_x}\left[\exp\left(\int_0^{\zeta^W} c(x(t))dt\right){}^{c}Q(T-\zeta^W, x(\zeta^W), \Gamma), \zeta^W < T\right]$$

for $(T, x)\in(0, \infty)\times W$ and $\Gamma\in\mathcal{B}_U$. It is obvious from this and our hypotheses that $(T, x)\in(0, 1]\times W\to{}^{c}Q_U^W(T, x, dy)\equiv\chi_U(y){}^{c}Q^W(T, x, dy)$ is measurable into Σ and that

$$\|{}^{c}Q_U^W(T, x, \cdot)\|_{\Sigma} \leqq \|{}^{c}Q(T, x, \cdot)\|_{\Sigma} + K(T)\exp(T\|\hat{c}\|_{B(\hat{R}^N)})Q_x(\zeta^W \leqq T). \tag{4.2}$$

Next, set $\tau_0\equiv 0$ and define $\{\sigma_n\}_1^\infty$ and $\{\tau_n\}_1^\infty$ inductively by:

$$\sigma_n = \inf\{t\geqq\tau_{n-1} : x(t)\in\overline{V}\}$$
$$\tau_n = \inf\{t\geqq\sigma_n : x(t)\notin W\}$$

for $n\geqq 1$. Then, for $(T, x)\in(0, 1]\times\hat{\boldsymbol{R}}^N$ and $\Gamma\in\mathcal{B}_U$:

$${}^{\hat{c}}P(T, x, \Gamma) = \sum_{n=1}^{\infty} E^{P_x}\left[\exp\left(\int_0^T \hat{c}(x(t))dt\right), x(T)\in\Gamma \text{ and } \sigma_n < T < \tau_n\right]$$
$$= \sum_{n=1}^{\infty} E^{P_x}\left[\exp\left(\int_0^{\sigma_n} c(x(t))dt\right){}^{c}Q^W(T-\sigma_n, x(\sigma_n), \Gamma), \sigma_n < T\right].$$

Now, suppose that we knew that

$$\sup_{x\in R^N}\sum_1^\infty P_x(\sigma_n < 1) \leqq 5/\mu^2. \tag{4.3}$$

Then, it would be clear from the preceding that $(T, x)\in(0, 1]\times\hat{\boldsymbol{R}}^N\to{}^{c}P_U(T, x, \cdot)\in\Sigma$ is measurable and that the desired estimates hold so long as $T\in(0, 1]$. Moreover, since

$${}^{\hat{c}}P_U(T, x, \cdot) = \int{}^{\hat{c}}P(1, \xi, \cdot){}^{\hat{c}}P_U(T-1, x, d\xi)$$

for $T>1$, the final assertion of the lemma follows immediately from the earlier ones. Hence, all that remains is to provide the proof of (4.3).

Clearly

$$\sum_1^\infty P_x(\sigma_n \leqq 1) \leqq 1 + \sum_1^\infty P_x(\tau_n \leqq 1).$$

Moreover, if $n \geqq 1$, then

$$P_x(\tau_n \leqq \mu) = \int_{\{\omega:\sigma_n(\omega)\leqq\mu\}} Q_{x(\sigma_n(\omega),\omega)}(\zeta^W \leqq \mu - \sigma_n(\omega))P_x(d\omega) \leqq \frac{1}{e} P_x(\sigma_n \leqq \mu)$$

$$\leqq \frac{1}{e} P_x(\sigma_n \leqq \mu).$$

Hence, $P_x(\tau_n \leqq \mu) \leqq e^{-n}$, $\mu \geqq 1$. Also, for any $l \geqq 1$ and $m \geqq 1$:

$$P_x(\tau_{(l+1)m} \leqq (l+1)\mu) \leqq P_x(\tau_{lm} \leqq l\mu) + P_x(\tau_{(l+1)m} - \tau_{lm} \leqq \mu, \tau_{lm} < \infty).$$

Hence, $P_x(\tau_{lm} \leqq l\mu) \leqq l/e^m$. From here it is clear that:

$$P_x(\tau_n \leqq 1) \leqq [(1+\mu)/\mu]e^{-n\mu/(1+\mu)}$$

and therefore that

$$\sum_{n=1}^\infty P_x(\tau_n \leqq 1) \leqq (1+\mu)/\mu(e^{\mu/(1+\mu)} - 1)$$

$$\leqq 4/\mu^2;$$

Clearly this leads to (4.3). Q.E.D.

Lemma (4.1) provides us with a way to localize results about the "forward variables". We now want to develop an analogous procedure for the "backward variable".

(4.4) LEMMA. *Let U and V be open subsets of W satisfying $\bar{U} \subseteqq V$ and $\bar{V} \subseteqq W$, and let $(B, \|\cdot\|_B)$ be a Banach space contained in $B(\hat{\boldsymbol{R}}^N) \cap C(U)$ with the property that if $\{f_n\}_1^\infty$ is a bounded subset of B and $f_n(x) \to f(x)$ for each $x \in U$, then $f \in C_b(U)$. Assume that ${}^cQ_t : B(\hat{\boldsymbol{R}}^N) \to B$ for all $t \in (0, 1]$ and that there exist non-decreasing functions $\alpha : (0, 1] \to (0, \infty)$, $\beta : (0, 1] \to (0, \infty)$, and $\gamma : (0, 1] \to (0, \infty)$ satisfying*

$$K(t) \equiv \sum_{m=1}^\infty (\alpha(t/2^m) + \beta(t/2^m)/\gamma(t/2^m)) < \infty, \qquad t \in (0, 1],$$

and such that

$$\|{}^cQ_t f\|_B \leqq \exp(t\|c_+\|_{B(\boldsymbol{R}^N)})\|f\|_{B(\boldsymbol{R}^N)}/\gamma(t), \qquad f \in B(\boldsymbol{R}^N),$$

$$\|{}^cQ_t \chi_{V^c} f\|_B \leqq \exp(t\|c_+\|_{B(\boldsymbol{R}^N)})\alpha(t)\|f\|_{B(\boldsymbol{R}^N)}, \quad f \in B(\boldsymbol{R}^N),$$

and

$$\sup_{x\in\bar{V}} Q_x(\zeta^W \leqq t) \leqq \beta(t)$$

for each $t\in(0,1]$. *Then* ${}^{\hat{c}}P_t : B(\hat{\boldsymbol{R}}^N)\to B$ *for all* $t\in(0,\infty)$. *In fact, if* $C=\|c_+\|_{B(\boldsymbol{R}^N)} \vee \|\hat{c}_+\|_{B(\hat{\boldsymbol{R}}^N)}$, *then for all* $f\in B(\hat{\boldsymbol{R}}^N)$:

$$\|{}^{\hat{c}}P_t f\|_B \leqq \|{}^c Q_t f\|_B + 2K(t)\exp(tC)\|f\|_{B(\hat{\boldsymbol{R}}^N)}$$

when $t\in(0,1]$ *and*

$$\|{}^{\hat{c}}P_t f\|_B \leqq (1/\gamma(1)+2K(1))\exp(tC)\|f\|_{B(\hat{\boldsymbol{R}}^N)}$$

when $t\in(1,\infty)$.

PROOF. Since, for $t>1$, ${}^{\hat{c}}P_t = {}^{\hat{c}}P_1 \circ {}^{\hat{c}}P_{t-1}$, we may and will restrict our attention to $t\in(0,1]$. Also, observe that it suffices to prove the desired conclusions about ${}^{\hat{c}}P_t f$, $t\in(0,1]$, when $f\in C_b(\hat{\boldsymbol{R}}^N)$.

Given $t\in(0,1]$ and $f\in C_b(\hat{\boldsymbol{R}}^N)$, note that

$${}^{\hat{c}}P_t f(x) - {}^c Q_t f(x) = \sum_{m=0}^{\infty} {}^c Q_{t/2^{m+1}} g_m(x)\,, \qquad x\in U\,,$$

where

$$g_m = ({}^{\hat{c}}P_{t/2^{m+1}} - Q_{t/2^{m+1}})\circ {}^{\hat{c}}P_{t(1-1/2^{m+1})} f\,.$$

Set $g'_m = \chi_V g_m$ and $g''_m = \chi_{V^c} g_m$. Then

$$\|g'_m\|_{B(\boldsymbol{R}^N)} \leqq 2\exp(t(1-1/2^{m+1})C)\beta(t/2^{m+1})\|f\|_{B(\hat{\boldsymbol{R}}^N)}$$

and so

$$\|{}^c Q_{t/2^{m+1}} g'_m\|_B \leqq 2\exp(tC)\beta(t/2^{m+1})\|f\|_{B(\hat{\boldsymbol{R}}^N)}/\gamma(t/2^{m+1})\,.$$

At the same time:

$$\|{}^c Q_{t/2^{m+1}} g''_m\|_B \leqq \exp(tC)\alpha(t/2^{m+1})\|f\|_{B(\hat{\boldsymbol{R}}^N)}\,.$$

Combining these, we conclude that ${}^{\hat{c}}P_t f\in B$ and that the desired estimates hold. Q. E. D.

We now specialize our choice of $\{Q_x : x\in \boldsymbol{R}^N\}$ so that we can apply these localization procedures to the situation treated in Section 3). Let $V_0, \cdots, V_d \in C_b^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ and $c\in C_b^\infty(\boldsymbol{R}^N)$ be given. Set

$$\boldsymbol{L} = \frac{1}{2}\sum_1^d V_k^2 + V_0$$

and let $\{Q_x : x\in\boldsymbol{R}^N\}$ be the diffusion generated by $\boldsymbol{L}$. We point out that the condition $P_x|_{\mathcal{M}_{\zeta^W}} = Q_x|_{\mathcal{M}_{\zeta^W}}$, $x\in W$, is, in this case, equivalent to the statement that

$$\left(\phi(x(t\wedge\zeta^W))-\phi(x)-\int_0^{t\wedge\zeta^W}\boldsymbol{L}\phi(x(s))ds,\ \hat{\mathcal{M}}_t,\ P_x\right)$$

is a mean-zero martingale for all $x\in W$ and $\phi\in C_0^\infty(\boldsymbol{R}^N)$. Finally, let $\{C_m: m\geqq 0\}\subseteq[0,\infty)$ be chosen so that (1.10) holds with $\gamma_m=0$ for each $m\geqq 0$, $\|\hat{c}_+\|_{B(\hat{R}^N)}\leqq C_0$, and

$$\sup_{x\in R^N}\|c^{(m)}(x)\|_{\mathrm{H.S.}((R^N)^{\otimes m},R^1)}\leqq C_m\,,\qquad m\geqq 0\,.$$

(4.5) THEOREM. *Referring to the situation just described, define* $\Delta(T,x)$, $(T,x)\in(0,\infty)\times\boldsymbol{R}^N$, *as in* (3.16). *Let* $\rho:(0,1]\to(0,\infty)$ *be a non-decreasing function with the property that* $\lim_{t\downarrow 0}t\log\rho(t)=0$, *and suppose that there exist* $\{M_p: p\in[1,\infty)\}\subseteq(0,\infty)$ *such that*

$$\sup_{x\in R^N}\|1/\Delta(t,x)\|_{L^p(\mathcal{W})}\leqq M_p/\rho(t)\,,\qquad t\in(0,1]\,,\tag{4.6}$$

for each $p\in[1,\infty)$. *Then there exists a measurable map* $x\in\hat{\boldsymbol{R}}^N\to{}^{\hat{c}}p(\cdot,x,*)$ $\in C^\infty((0,\infty)\times W)$ *such that*

$$\chi_W(y)\,{}^{\hat{c}}P(t,x,dy)={}^{\hat{c}}p(t,x,y)dy\,,\qquad (t,x)\in(0,\infty)\times\hat{\boldsymbol{R}}^N$$

and

$$\frac{\partial}{\partial t}{}^{\hat{c}}p(t,x,y)=(\boldsymbol{L}_y^*+c(y))\,{}^{\hat{c}}p(t,x,y)\,,\qquad (t,x,y)\in(0,\infty)\times\hat{\boldsymbol{R}}^N\times W\,,$$

where $\boldsymbol{L}^*$ *denotes the formal adjoint of* $\boldsymbol{L}$. *In fact, given* $n\geqq 0$, *there exist an* $m_n\geqq 2$, *a* $p_n\in[2,\infty)$, *a* $\nu_n\in(0,\infty)$, *and a* $\lambda_n\in(0,\infty)$, *all of which are independent of* $\{V_0,\cdots,V_d\}$, c *and* $\hat{c}$, *and a* $K_n\in(0,\infty)$ *depending only on* $\{C_m: 0\leqq m\leqq m_n\}$, *such that for each* $0<\delta<(\mathrm{rad}\,W)\wedge 1$ $(\mathrm{rad}\,W\equiv\sup\{\delta>0: (\exists x\in W)B(x,\delta)\subseteq W\})$:

$$\begin{aligned}\max_{|\gamma|\leqq n}|D_y^\gamma\,{}^{\hat{c}}p(t,x,y)|&\\ \leqq M_{p_n}^{\nu_n}B_n(\delta)K_n\exp[t\|\hat{c}_+\|_{B(\hat{R}^N)}&-\lambda_n(\|y-x\|_{R^N}\wedge\delta)^2/(1+C_0^2)t]/(\rho(t\wedge 1)t)^{\nu_n}\end{aligned}\tag{4.7}$$

for $(t,x,y)\in(0,\infty)\times\hat{\boldsymbol{R}}^N\times W_{(\delta)}$,

$$\begin{aligned}\max_{|\gamma|\leqq n}|D_x^\gamma\,{}^{\hat{c}}P_tf(x)|\leqq{}&M_{p_n}^{\nu_n}K_n\exp(t\|c_+\|_{B(\hat{R}^N)})\\ &\times\left[\left(\int_{\mathrm{supp}(f)}\exp[-\lambda_n\|\xi-x\|_{R^N}^2/(1+C_0^2)t]d\xi\right)\wedge 1/(\rho((t/2)\wedge 1)\wedge t)^{\nu_n}\right.\\ &\left.+B_n(\delta)\exp[-\lambda_n\delta^2/(1+C_0^2)t]\right]\|f\|_{B(\hat{R}^N)}\end{aligned}\tag{4.8}$$

for all $(t, x) \in (0, \infty) \times W_{(\delta)}$ *and all* $f \in B(\hat{\boldsymbol{R}}^N)$, *and*

(4.9)
$$\max_{|\alpha| \vee |\beta| \leq n} |D_x^\alpha D_y^\beta \,{}^{\hat{c}}p(t, x, y)| \leq M_{p_n}^{2\nu_n} B_n(\delta)^2 K_n^2 \exp[t \| c_+ \|_{B(\hat{R}^N)} - \lambda_n (\| y - x \|_{R^N} \wedge \delta)^2 / (1 + C_0^2) t] / (\rho((t/2) \wedge 1) \wedge t)^{2\nu_n}$$

for $(t, x, y) \in (0, \infty) \times W_{(\delta)} \times W_{(\delta)}$; *where*

$$W_{(\delta)} = \{x \in W : \mathrm{dist}(x, W^c) > \delta\}$$

and

$$B_n(\delta) = \left[\sup_{0<t<1} \exp[-\lambda_n \delta^2 / (1 + C_0^2) t] / (\rho(t \wedge 1) \wedge t)^{\nu_n}\right] \vee \delta^{-4}.$$

PROOF. Under the stated hypotheses, Theorem (3.17) applies to $\{Q_x : x \in \boldsymbol{R}^N\}$ and says that ${}^cQ(T, x, dy) = {}^cq(t, x, y)dy$ where ${}^cq \in C^\infty((0, \infty) \times \boldsymbol{R}^N \times \boldsymbol{R}^N)$ satisfies (3.20), with $\mu_n = 0$, for each $n > 0$. Moreover, one sees from standard estimates (e.g., (3.24)) that there is a universal $\varepsilon > 0$ such that

$$\sup_{x \in \boldsymbol{R}^N} Q_x\left(\sup_{0 \leq t \leq \mu} \| x(t) - x \|_{R^N} \geqq \delta\right) \leqq 1/e$$

so long as $\mu \leqq \varepsilon(\delta / C_0)^2$. Now let $n \geqq 1$ be given. Using the preceding, taking $U = W_{(\delta)}$, $V = W_{(\delta/2)}$, and $\Sigma = B = C_0^n(U)$; and applying Lemmas (4.1) and (4.4) ; we conclude that there is a measurable mapping

$$(T, x) \in (0, \infty) \times \hat{\boldsymbol{R}}^N \longrightarrow {}^{\hat{c}}p(T, x, \cdot) \in C^\infty(W)$$

such that $\chi_W(y)\,{}^{\hat{c}}P(T, x, dy) = {}^{\hat{c}}p(T, x, y)dy$. Moreover, for each $0 < \delta < (\mathrm{rad}\, W) \wedge 1$, the same argument shows that (4.7) and (4.8) hold.

To prove that ${}^{\hat{c}}p(\cdot, x, *) \in C^\infty((0, \infty) \times W)$ and than $\frac{\partial}{\partial t}\,{}^{\hat{c}}p(t, x, y) = (L_y^* + c(y))\,{}^{\hat{c}}p(t, x, y)$ on $(0, \infty) \times \hat{\boldsymbol{R}}^N \times W$, it suffices to check that ${}^{\hat{c}}P_t\phi - \phi = \int_0^t {}^{\hat{c}}P_s(L + c)\phi ds$, $t \geqq 0$, for each $\phi \in C_0^\infty(W)$. Given $\phi \in C_0^\infty(W)$, choose an open V so that $\mathrm{supp}\, \phi \Subset V$ and $\bar{V} \Subset W$, and define $\{\tau_m\}_0^\infty$ and $\{\sigma_m\}_1^\infty$ accordingly as in the proof of Lemma (4.1). Setting $\Xi(t) = \exp\left(\int_0^t c(x(s))ds\right)$ and $\hat{\Xi}(t) = \exp\left(\int_0^t \hat{c}(x(s))ds\right)$, we then have:

$$\begin{aligned} {}^{\hat{c}}P_t\phi(x) - \phi &= \sum_{m=1}^\infty E^{P_x}[\hat{\Xi}(t)\phi(x(t)), \sigma_m < t < \tau_m] - \phi(x) \\ &= \sum_{m=1}^\infty \int_{\{\omega : \sigma_m(\omega) < t\}} \hat{\Xi}(\sigma_m(\omega), \omega) \\ &\quad \times E^{P_{x(\sigma_m(\omega), \omega)}}[\hat{\Xi}(t - \sigma_m(\omega))\phi(x((t - \sigma_m(\omega)) \wedge \zeta^W)]P_x(d\omega) - \phi(x) \end{aligned}$$

$$=\sum_{m=1}^{\infty} E^{P_x}\left[\int_0^t \chi_{(\sigma_m,\tau_m)}(s)\hat{\Xi}(s)(L+c)\phi(x(s))ds\right]$$

$$=\int_0^t {}^cP_s(L+c)\phi(x)ds\,.$$

In order to prove (4.9), we combine (4.7) and (4.8) as follows. Let $0<\delta<(\mathrm{rad}\,W)\wedge 1$ be given, choose $x, y\in W_{(\delta)}$, and set $\varepsilon=\|y-x\|_{R^N}\wedge\delta$. Given $\alpha, \beta\in\mathfrak{N}^N$, we have

$$D_x^\alpha D_y^\beta\, {}^cp(t, x, y)=D_x^\alpha\, {}^cP_{t/2}f(x)\,,$$

where $f=D_y^\beta\, {}^cp(t/2, \cdot, y)$. If $t\in[1, \infty)$, (4.9) follows directly from this, (4.7), and (4.8). If $t\in(0, 1]$, set $f'=\chi_{B(y,\varepsilon/2)}f$ and $f''=f-f'$. Then

$$|D_x^\alpha D_y^\beta\, {}^cp(t, x, y)|\leqq|D_x^\alpha\, {}^cP_{t/2}f'(x)|+|D_x^\alpha\, {}^cP_{t/2}f''(x)|\,.$$

Assuming that $|\alpha|\vee|\beta|\leqq n$, we have, from (4.7):

$$\|f'\|_{B(\hat{R}^N)}\leqq(2M_{p_n})^{\nu n}B_n(\delta)K_n\exp(t\|c_+\|_{B(\hat{R}^N)}/2)/(\rho(t/2)\wedge t)^{\nu n}\,.$$

Hence, by (4.8):

$$|D_x^\alpha\, {}^cP_{t/2}f'(x)|\leqq 2^{\nu n}M_{p_n}^{\nu n}K_n^2(\omega_N(\varepsilon/2)^N+B_n(\delta))B_n(\delta)$$
$$\times\exp[t\|c_+\|_{B(\hat{R}^N)}-\lambda_n\varepsilon^2/2(1+C_0^2)t]/(\rho(t/2)\wedge t)^{2\nu n}$$

where ω_N denotes the volume of the unit ball in $\boldsymbol{R}^N$. At the same time, by (4.7):

$$\|f''\|_{B(\hat{R}^N)}\leqq(2M_{p_n})^{\nu n}B_n(\delta)K_n\exp[t\|c_+\|_{B(\hat{R}^N)}/2-\lambda_n\varepsilon^2/2(1+C_0^2)t]/(\rho(t/2)\wedge t)^{\nu n}$$

and so, by (4.8):

$$|D_x^\alpha\, {}^cP_{t/2}f''(x)|\leqq 2^{\nu n}M_{p_n}^{2\nu n}K_n((\pi(1+C_0^2)t/\lambda_n)^{N/2}+B_n(\delta))$$
$$\times\exp[t\|c_+\|_{B(\hat{R}^N)}-\lambda_n\varepsilon^2/2(1+C_0^2)t]/(\rho(t/2)\wedge t)^{2\nu n}\,.$$

Clearly (4.9) follows from these, once minor adjustments to K_n and λ_n have been made. Q. E. D.

(4.10) COROLLARY. *Again referring to the situation in Theorem* (4.5), *define* $\mathcal{V}_L(x)$, $L\leqq 1$ *and* $x\in\boldsymbol{R}^N$, *as in Corollary* (3.25) *and assume that, for some* $L\geqq 1$ *and* $0<\varepsilon\leqq 1$, $\inf_{x\in R^N}\mathcal{V}_L(x)\geqq\varepsilon$. *Then there is a measurable map* $x\in\hat{\boldsymbol{R}}^N\to{}^cp(\cdot, x, *)\in C^\infty((0, \infty)\times W)$ *such that*

$$\chi_W(y)\,{}^cP(t, x, dy)={}^cp(t, x, y)dy$$

and

$$\frac{\partial}{\partial t}{}^{\hat{c}}p(t, x, y)=(\boldsymbol{L}_y^*+c(y))^{\hat{c}}p(t, x, y)$$

for all $(t, x, y)\in(0, \infty)\times\hat{\boldsymbol{R}}^N\times W$. *In fact, given* $n\geqq 0$, *let* $m_n(L)\geqq 2$ *and* $\nu_n(L)\in(0, \infty)$ *be as in Corollary* (3.25) *and let* $\lambda_n\in(0, \infty)$ *be as in Theorem* (4.5). *Given* $0<\delta<(\mathrm{rad}\, W)\wedge 1$, *define* $B_n(\delta)$ *as in Theorem* (4.5). *Then there is a* $K_n(L)\in(0, \infty)$, *depending only on* $\{C_m : 0\leqq m\leqq m_n(L)\}$ *such that*

$$\text{(4.11)}\qquad \max_{|\gamma|\leq n}|D_y^\gamma{}^{\hat{c}}p(t, x, y)|\leqq K_n(L)B_n(\delta)$$
$$\times\exp[t\|\hat{c}_+\|_{B(\hat{R}^N)}-\lambda_n(\|y-x\|_{R^N}\wedge\delta)^2/(1+C_0^2)t]/(\varepsilon^{1+2/L}(t\wedge 1))^{\nu_n(L)}$$

for $(t, x, y)\in(0, \infty)\times\hat{\boldsymbol{R}}^N\times W_{(\delta)}$,

$$\text{(4.12)}\qquad \max_{|\gamma|\leq n}|D_x^\gamma{}^{\hat{c}}P_t f(x)|\leqq K_n(L)B_n(\delta)\exp(t\|\hat{c}_+\|_{B(\hat{R}^N)})$$
$$\times\Big[\Big(\int_{\mathrm{supp}(f)}\exp[-\lambda_n\|\xi-x\|_{R^N}^2/(1+C_0^2)t]d\xi\Big)\wedge 1/t^{\nu_n(L)}$$
$$+B_n(\delta)\exp[-\lambda_n\delta^2/(1+C_0^2)t]\Big]\|f\|_{B(\hat{R}^N)}$$

for all $(t, x)\in(0, \infty)\times W_{(\delta)}$ *and* $f\in B(\hat{\boldsymbol{R}}^N)$, *and*

$$\text{(4.13)}\qquad \max_{|\alpha|\vee|\beta|\leq n}|D_x^\alpha D_y^\beta{}^{\hat{c}}p(t, x, y)|\leqq K_n(L)^2B_n(\delta)^2$$
$$\times\exp[t\|\hat{c}_+\|_{B(\hat{R}^N)}-\lambda_n(\|y-x\|_{R^N}\wedge\delta)^2/(1+C_0^2)t]/(\varepsilon^{1+3/L}(t\wedge 1))^{2\nu_n(L)}$$

for $(t, x, y)\in(0, \infty)\times W_{(\delta)}\times W_{(\delta)}$.

PROOF. This result is an immediate consequence of Theorem (4.5) and the estimate (3.26). Q.E.D.

5. Microlocalization.

Let $\{V_0, \cdots, V_d\}\subseteq C_\uparrow^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ satisfy (1.3), define $\boldsymbol{L}$ accordingly as in (3.1), let $\{P_x : x\in\boldsymbol{R}^N\}$ be the diffusion generated by $\boldsymbol{L}$ (cf. Theorem (3.3)), and set $P(t, x, \Gamma)=P_x(x(T)\in\Gamma)$. In the preceding section we learned how to localize (in space) regularity results about $P(T, x, \cdot)$. In this section, we want to *microlocalize* such results. Rather than give a formal definition of what that means, we will instead give an example of the kind of result toward which we will be working. Namely, let $1\leqq N_1\leqq N$, write $\boldsymbol{R}^N=\boldsymbol{R}^{N_1}\times\boldsymbol{R}^{N_2}$, and denote by $\Pi : \boldsymbol{R}^N\to\boldsymbol{R}^{N_1}$ the natural projection map. Recalling the definition of $\mathcal{V}_L(x, \eta)$ ($L\geqq 1$, $x\in\boldsymbol{R}^N$, and $\eta\in S^{N-1}$) given in (2.15), suppose that

$$\inf_{x\in R^N} \mathcal{V}_L(x,\eta)\geqq\varepsilon\|\Pi\eta\|_{R^N}^2, \qquad \eta\in S^{N-1},$$

for some $L\geqq 1$ and $\varepsilon>0$. When $N_1=N$, Corollary (3.25) says that $P(T,x,\cdot)$ possesses a great deal of regularity. When $N_1<N$, it is reasonable to suppose that the marginal distribution $P_\pi(T,x,\cdot)\equiv P(T,x,\cdot)\circ\Pi^{-1}$ of $P(T,x,\cdot)$ on $\boldsymbol{R}^{N_1}$ should enjoy similar properties. The purpose of the present section is to prove such results (cf. Theorem (5.9) and its corollaries).

(5.1) LEMMA. *Given a non-empty closed set $F\subseteqq S^{N-1}$, define $\mathcal{V}_L(x,F)$, $L\geqq 1$ and $x\in\boldsymbol{R}^N$, as in (2.16) and $\lambda(t,x,F)$, $(t,x)\in[0,\infty)\times\boldsymbol{R}^N$, as in Theorem (2.17). Assume that $\inf_{x\in R^N}\mathcal{V}_L(x,F)\geqq\varepsilon$ for some $0<\varepsilon\leqq 1$ and that $M\equiv\max\{\|(V_k)_{\langle\alpha\rangle}\|_{C_b^2(R^N,R^N)}: 0\leqq k\leqq d$ and $|\alpha|\leqq L+1\}$ is finite. Then for all $(T,x)\in(0,\infty)\times\boldsymbol{R}^N$ and $K\in[1,\infty)$:*

$$\mathcal{W}(\lambda(T,x,F)/\eta(T,K)^L\leqq 1/K)\leqq C(L)\exp(-(\varepsilon^{L+2}K)^{\mu_L}/(1+M)^2), \tag{5.2}$$

where

$$\eta(T,K)=\begin{cases}1 & \text{if } T\geqq 1/K^L\\ T & \text{if } T<1/K^L\end{cases}$$

and the numbers $C(L)$ and μ_L are those described in Theorem (2.17).

PROOF. Given $0\leqq s\leqq T$, define

$$J_s(T,x)=J(T,x)J^{-1}(s,x),$$

$$A_s(T,x)=\int_s^T J_t(T,x)a(X(t,x))J_t(T,x)^*dt,$$

and

$$\lambda_s(T,x,F)=\inf_{\eta\in F}(\eta,A_s(T,x)\eta)_{R^N},$$

where $a(\cdot)$ is the matrix in (1.1). Next, let $\theta\to\mathcal{W}_\theta^s$ be the regular conditional distribution of $\mathcal{W}$ given $\mathcal{B}_s$. Then, for $\mathcal{W}$-almost every $\theta\in\Theta$, the distribution of

$$\begin{pmatrix}X(T,x)\\ A_s(T,x)\end{pmatrix}$$

under $\mathcal{W}_\theta^s$ coincides with the distribution of

$$\begin{pmatrix}X(T-s,X(s,x,\theta))\\ A(T-s,X(s,x,\theta))\end{pmatrix}$$

under $\mathcal{W}$.

Now take $s=T-\eta(T,K)/K^L$. Since $A_x(T,x)\leqq A(T,x)$, we have from the above and (2.19):

$$\begin{aligned}&\mathcal{W}(\lambda(T,x,F)/\eta(T,K)^L\leqq 1/K)\leqq\mathcal{W}(\lambda_s(T,x,F)/\eta(T,K)^L\leqq 1/K)\\&=\int\mathcal{W}(\lambda(\eta(T,K)/K^L,X(s,x,\theta),F)/\eta(T,K)^L\leqq 1/K)\mathcal{W}(d\theta)\\&\leqq C(L)\exp(-(\varepsilon^{L+2}K)^{\mu_L}/(1+M)^2)\,.\end{aligned}$$

Q.E.D.

(5.3) LEMMA. *Let W be a non-empty open set in $\boldsymbol{R}^N$, F a non-empty closed subset of S^{N-1}, $L\geqq 1$, and $0<\varepsilon\leqq 1$. Assume that*

$$\mathcal{V}_L(x,F)\geqq\varepsilon\,,\qquad x\in W\,,$$

and that

$$M=\max\{\|(V_k)_{(\alpha)}\|_{C_b^2(W)}:0\leqq k\leqq d\ \text{and}\ |\alpha|<L+1\}<\infty\,.$$

Then there exist $C(L)\in(0,\infty)$, $\lambda_L\in(0,\infty)$, $\gamma_L\in(0,\infty)$, and $\mu_L\in(0,\infty)$ (all of which are independent of $\{V_0,\cdots,V_d\}$, W, and ε) such that for each $0<\delta<(\mathrm{rad}\,W)\wedge 1$, all $K\in[1,\infty)$, and all $(T,x)\in(0,\infty)\times\boldsymbol{R}^N$:

$$\begin{aligned}(5.4)\qquad&\mathcal{W}(\lambda(T,x,F)/(T\wedge 1)^L\leqq 1/K,\ X(T,x)\in W_{(\delta)})\\&\leqq C(L)(1+M^2)^2\exp(-\lambda_L\delta^{\gamma_L}(\varepsilon^{L+2}K)^{\mu_L}/(1+M)^2)/\delta^4\,,\end{aligned}$$

where $W_{(\delta)}=\{y\in W:\mathrm{dist}(y,W^c)>\delta\}$.

PROOF. We note first that it suffices to prove (5.4) when $T\in(0,1]$. Indeed, given $T>1$, set $s=T-1$ and observe (cf. the proof of Lemma (5.1)) that

$$\begin{aligned}&\mathcal{W}(\lambda(T,x,F)\leqq 1/K,\ X(T,x)\in W_{(\delta)})\leqq\mathcal{W}(\lambda_s(T,x,F)\leqq 1/K,\ X(T,x)\in W_{(\delta)})\\&\quad=\int\mathcal{W}(\lambda(1,X(s,x,\theta),F)\leqq 1/K,\ X(1,X(s,x,\theta))\in W_{(\delta)})\mathcal{W}(d\theta)\,.\end{aligned}$$

Before turning to the proof of (5.4) for $T\in(0,1]$, we need to make a simple construction. Namely, given $0<\delta<(\mathrm{rad}\,W)\wedge 1$, choose a $\phi\in C_0^\infty(W)$ and a $\psi\in C_0^\infty(W_{(\delta/4)})$ so that $0\leqq\phi$, $\psi\leqq 1$, $\phi=1$ on $W_{(\delta/4)}$, and $\psi=1$ on $W_{(\delta/2)}$. Note that the choice of ϕ and ψ can be made so that $\|\phi\|_{C_b^n(R^N)}\vee\|\psi\|_{C_b^n(R^N)}\leqq B_n/\delta^n$, $n\geqq 0$, where $B_n\in(0,\infty)$ is independent of W. Next, set

$$\hat{V}_k=\phi V_k\,,\qquad 0\leqq k\leqq d\,,$$

and

$$\hat{V}_k=(1-\psi)\frac{\partial}{\partial x_{k-d}}\,,\qquad d+1\leqq k\leqq\hat{d}\equiv d+N\,.$$

We will use "^" on top of quantities associated with $\{\hat{V}_0,\cdots,\hat{V}_{\hat{d}}\}$ in order to distinguish them from the analogous quantities associated with $\{V_0, \cdots, V_d\}$.

Note that

$$\hat{\mathcal{V}}_L(x, F) \geqq \varepsilon, \qquad x \in \boldsymbol{R}^N,$$

and that

$$\hat{M} \equiv \max\{\|(\hat{V}_k)_{(\alpha)}\|_{C_b^0(\boldsymbol{R}^N)} : 0 \leqq k \leqq \hat{d} \text{ and } |\alpha| \leqq L+1\}$$

is dominated by $(B_{L+3})^{L+1} M/\delta^{\gamma_L/2}$, where $\gamma_L = 2(L+1)(L+3)$.

Now let $T \in (0, 1]$ be given, set $\tau_0(x) \equiv 0$, and define

$$\sigma_m(x) = \inf\{t \geqq \tau_{m-1}(x) : X(t, x) \in W_{(3\delta/4)}\} \wedge T$$

$$\tau_m(x) = \inf\{t \geqq \sigma_m(x) : X(t, x) \notin W_{(\delta/2)}\} \wedge T$$

for $m \geqq 1$. Also, set $T_m(x, \theta) = T - \sigma_m(x, \theta)$, $Y_m(x, \theta) = X(\sigma_m(x, \theta), x, \theta)$, $\hat{Y}_m(x, \theta) = \hat{X}(T_m(x, \theta), Y_m(x, \theta))$, and $\zeta(y) = \inf\{t \geqq 0 : X(t, y) \notin W_{(\delta/2)}\}$ for $y \in \boldsymbol{R}^N$. Then

$$\begin{aligned}
&\mathcal{W}(\lambda(T, x, F)/T^L \leqq 1/K, X(T, x) \in W_{(\delta)}) \\
&\leqq \sum_{m=1}^{\infty} \mathcal{W}(\lambda_{\sigma_m(x)}(T, x, F)/T^L \leqq 1/K, X(T, x) \in W_{(\delta)}, \sigma_m(x) < T < \tau_m(x)) \\
&= \sum_{m=1}^{\infty} \int_{\{\theta : \sigma_m(x, \theta) < T\}} \mathcal{W}(\lambda(T_m(x, \theta), Y_m(x, \theta), F)/T^L \leqq 1/K, \\
&\qquad X(T_m(x, \theta), Y_m(x, \theta)) \in W_{(\delta)}, \ \zeta(Y_m(x, \theta)) > T_m(x, \theta))\mathcal{W}(d\theta) \\
&\leqq \sum_{m=1}^{\infty} I_m(T, x),
\end{aligned}$$

where

$$\begin{aligned}
I_m(T, x) = \int_{\{\theta : \sigma_m(x, \theta) < T\}} &\hat{\mathcal{W}}(\hat{\lambda}(T_m(x, \theta), Y_m(x, \theta), F)/T^L \leqq 1/K, \\
&\hat{Y}_m(x, \theta) \in W_{(\delta)})\mathcal{W}(d\theta).
\end{aligned}$$

In the preceding, we have used two facts. The first of these is the strong Markov property in the form which says that, for each $m \geqq 1$ and $\mathcal{W}$-almost all $\theta \in \{\sigma_m(x) < T\}$, the distribution of

$$\begin{pmatrix} X(T, x) \\ A_{\sigma_m(x)}(T, x) \end{pmatrix}$$

under the regular conditional probability distribution $\mathcal{W}_\theta^{\sigma_m(x)}$ of $\mathcal{W}$ given $\mathcal{B}_{\sigma_m(x)}$ coincides with the distribution of

$$\begin{pmatrix} X(T_m(x, \theta), Y_m(x, \theta)) \\ A(T_m(x, \theta), Y_m(x, \theta)) \end{pmatrix}$$

under $\mathcal{W}$. The second fact is a consequence of uniqueness and states that, for each $y \in \boldsymbol{R}^N$, the distribution of

$$\begin{gathered} \zeta(y) \\ X(\cdot \wedge \zeta(y), y) \\ A(\cdot \wedge \zeta(y), y) \end{gathered}$$

under $\mathcal{W}$ coincides with that of

$$\begin{gathered} \hat{\zeta}(y) \\ \hat{X}(\cdot \wedge \hat{\zeta}(y), y) \\ \hat{A}(\cdot \wedge \hat{\zeta}(y), y) \end{gathered}$$

under $\hat{\mathcal{W}}$.

In order to complete the proof, we will now show that there exist $C(L) \in (0, \infty)$ and $\lambda_L \in (0, \infty)$ (which are independent of $\{V_0, \cdots, V_d\}$ and W) such that, for all $m \geqq 1$ and $K \in [1, \infty)$:

$$I_m(T, x) \leqq C(L) \exp(-\lambda_L \delta^{\gamma_L} (\varepsilon^{L+2} K)^{\mu_L} / (1+M)^2) \mathcal{W}(\sigma_m(x) < T)\,, \tag{5.5}$$

where γ_L is the same as above and μ_L is the same as in Lemma (5.1). Since (cf. the proof of Theorem (4.5)) there is a universal $B \in (0, \infty)$ such that

$$\sum_{m=1}^{\infty} \mathcal{W}(\sigma_m(x) < 1) < B(1+M^2)^2/\delta^4\,,$$

(5.4) will follow immediately from (5.5).

In the derivation of (5.5), we consider two cases. First, suppose that $x \in W_{(3\delta/4)}$. Then $\sigma_1(x) \equiv 0$ and

$$I_1(T, x) \leqq \hat{\mathcal{W}}(\lambda(T, x, F)/T^L \leqq 1/K)\,.$$

Thus, this case is covered by (5.2). Next, suppose $x \notin W_{(3\delta/4)}$ and $m \geqq 1$ or that $x \in W_{(3\delta/4)}$ and $m \geqq 2$. Then $Y_m(x, \theta) \in \partial W_{(3\delta/4)}$ for all θ with $\sigma_m(x, \theta) < T$. Hence, in this case, (5.5) will follow from

$$\begin{aligned} &\hat{\mathcal{W}}(\lambda(t, y, F) \leqq 1/K, \hat{X}(t, y) \in W_{(\delta)}) \\ &\quad \leqq C(L) \exp(-\lambda_L \delta^{\gamma_L} (\varepsilon^{L+2} K)^{\mu_L} / (1+M)^2) \end{aligned} \tag{5.6}$$

for all $(t, y) \in (0, 1] \times \partial W_{(3\delta/4)}$. To see (5.6), first suppose that $0 < t \leqq 1/K^L$. Then, by standard estimates:

$$\sup_{y \in \partial W_{(3\delta/4)}} \hat{\mathcal{W}}(\hat{X}(t, y) \in W_{(\delta)}) \leqq 2N \exp(-\lambda \delta^2 K^L / (1+M)^2)$$

where $\lambda \in (0, \infty)$ is universal. Second, suppose that $1/K^L < t \leqq 1$ and apply (5.2) to $\hat{\mathcal{W}}(\hat{\lambda}(t, y, F) \leqq 1/K)$. Q.E.D

(5.7) LEMMA. *Let* $1 \leqq D \leqq N$, $L \geqq 1$, $0 < \varepsilon \leqq 1$, *and a non-empty open set* W *in* $\boldsymbol{R}^N$ *be given; and define*

$$F^W(L, \varepsilon) = \{\eta \in S^{N-1} : \mathcal{V}_L(x, \eta) \geqq \varepsilon \quad \text{for all } x \in W\}.$$

Assume that $F^W(L, \varepsilon) \neq \emptyset$ *and that*

$$M \equiv \max\{\|(V_k)_{(\alpha)}\|_{C_b^0(W)} : 0 \leqq k \leqq d \quad \text{and} \quad |\alpha| \leqq L+1\} < \infty.$$

Next, let $\pi \in C_\uparrow^\infty(\boldsymbol{R}^N, \boldsymbol{R}^D)$ *and define*

$$\hat{X}(\cdot, x) = \pi \circ X(\cdot, x),$$

$$\hat{A}(\cdot, x) = \langle\!\langle \hat{X}(\cdot, x), \hat{X}(\cdot, x) \rangle\!\rangle_{\mathcal{L}},$$

and

$$\hat{\lambda}(\cdot, x) = \inf_{\hat{\eta} \in S^{D-1}} (\hat{\eta}, \hat{A}(\cdot, x)\hat{\eta})_{R^D}.$$

If for all $y \in W$ *and* $\hat{\eta} \in S^{D-1}$:

$$\|(\pi^*)_y \hat{\eta}\| \geqq \varepsilon$$

and

$$(\pi^*)_y \hat{\eta} \in \operatorname{span} F^W(L, \varepsilon)$$

then for each $0 < \delta < (\operatorname{rad} W) \wedge 1$:

$$\mathcal{W}(\hat{\lambda}(T, x)/(T \wedge 1)^L \leqq \varepsilon^2/K, X(T, x) \in W_{(\delta)}) \tag{5.8}$$
$$\leqq C(L)(1+M^2)^2 \exp(-\lambda_L \delta^{\gamma L}(\varepsilon^{L+2}K)^{\mu_L}(1+M)^2)/\delta^4$$

for all $(T, x) \in (0, \infty) \times \boldsymbol{R}^N$ *and all* $K \in [1, \infty)$. *(The quantities* $C(L)$, λ_L, *and* μ_L *in* (5.8) *are the same as those in* (5.4).)

PROOF. Because $\langle \phi \circ \Phi, \Psi \rangle = \phi' \circ \Phi \langle \Phi, \Psi \rangle$ for all $\Phi, \Psi \in \mathcal{G}(\mathcal{L})$ and $\phi \in C_\uparrow^\infty(\boldsymbol{R}^1)$, we have:

$$(\hat{\eta}, \hat{A}(T, x)\hat{\eta})_{R^D} = ((\pi^*)_{X(T,x)}\hat{\eta}, A(T, x)(\pi^*)_{X(T,x)}\hat{\eta})_{R^N}$$
$$\geqq \varepsilon^2 \lambda(T, x, F^W(L, \varepsilon))$$

for all $\hat{\eta} \in S^{D-1}$ whenever $X(T, x) \in W$. Hence, (5.8) follows immediately from (5.4). Q.E.D.

(5.9) THEOREM. *With the notation and hypotheses the same as they are in Lemma* (5.7), *define*:

$$P_\phi^\pi(T, x, \cdot)=(\phi(X(T, x))\mathcal{W})\circ(\hat{X}(T, x))^{-1}$$

for $(T, x)\in(0, \infty)\times \boldsymbol{R}^N$ *and* $\phi\in C_b^\infty(\boldsymbol{R}^N)$. *Assuming that* $\operatorname{supp}(\phi)\subseteq W_{(\delta)}$ *for some* $0<\delta<(\operatorname{rad} W)\wedge 1$, *there exists a* $p_\phi^\pi\in C^\infty((0, \infty)\times \boldsymbol{R}^N\times \boldsymbol{R}^D)$ *such that*

$$P_\phi^\pi(T, x, d\hat{y})=p_\phi^\pi(T, x, \hat{y})d\hat{y}$$

for all $(T, x)\in(0, \infty)\times \boldsymbol{R}^N$. *In fact, if* $d_\pi^W(x, \hat{y})=\operatorname{dist}(x, \pi^{-1}(\hat{y})\cap W)$ ($\equiv\infty$ *if* $\hat{y}\in \boldsymbol{R}^D\backslash\pi(W)$), *then, for each* $n\geqq 0$, p_ϕ^π *satisfies an estimate of the form*

$$(5.10)\qquad (1+d_\pi^W(x, \hat{y})^2)^{n/2}|D_t^m D_x^\alpha D_{\hat{y}}^\beta p_\phi^\pi(t, x, \hat{y})|\leqq K_n(T)(1+\|x\|_{R^N})^{\mu_n}\|\phi\|_{C_b^n(R^N)}$$
$$\times\exp(-\lambda_n(d_\pi^W(x, \hat{y})\wedge 1)^2/C_0(1+\|x\|_{R^N}^2)^{\gamma_0}t)/(\varepsilon\delta(t\wedge 1))^{\nu_n}$$

for all $(t, x, \hat{y})\in(0, T]\times \boldsymbol{R}^N\times \boldsymbol{R}^D$ *and all* $(m, \alpha, \beta)\in \boldsymbol{N}\times \boldsymbol{N}^N\times \boldsymbol{N}^D$ *satisfying* $m+|\alpha|+|\beta|\leqq n$. *The quantities* $K_n(T)$, μ_n, λ_n, *and* ν_n *in* (5.10) *are elements of* $(0, \infty)$ *which depend on* $\{V_0, \cdots, V_d\}$, n, *and* L *in the same way as the analogous quantities in* (3.27). *The* C_0 *and* γ_0 *are those in* (1.10). *Finally, under the growth conditions stated in the last part of Corollary* (3.25) (*cf.* (3.28) *and the statement following* (3.28)), *the analogous modifications of* (5.10) *hold.*

PROOF. Given the estimate (5.8), the proof of this theorem differs in no essential way from that of Theorem 3.27). Q. E. D.

(5.11) COROLLARY. *Let* $1\leqq D\leqq N$, $\pi\in C_\uparrow^\infty(\boldsymbol{R}^N, \boldsymbol{R}^D)$, *and a non-empty open set* $W\subseteq \boldsymbol{R}^N$ *be given; and, for* $\phi\in C_0^\infty(W)$, *define* $P_\phi^\pi(T, x, \cdot)$, $(T, x)\in(0, \infty)\times \boldsymbol{R}^N$, *on* $\boldsymbol{R}^D$ *as in Theorem* (5.9). *If*

$$\operatorname{span}\{(\pi_*)_x((V_k)_{(\alpha)}) : 0\leqq k\leqq d \text{ and } \alpha\in\mathcal{A}\}=\boldsymbol{R}^D$$

for each $x\in W$, *then there is a* $p_\phi^\pi\in C^\infty((0, \infty)\times \boldsymbol{R}^N\times \boldsymbol{R}^D)$ *such that* $P_\phi^\pi(T, x, d\hat{y})=p_\phi^\pi(T, x, \hat{y})d\hat{y}$, $(T, x)\in(0, \infty)\times \boldsymbol{R}^N$. *Moreover, for each* $n\geqq 0$ *and* $T\in[1, \infty)$, $(t, x, \hat{y})\in(0, T]\times \boldsymbol{R}^N\times \boldsymbol{R}^D\to p_\phi^\pi(t, x, \hat{y})$ *satisfies an estimate of the form in* (5.10).

PROOF. Note that for each $x\in W$ there is an $L\geqq 1$ such that

$$\sum_{k=0}^d\sum_{\|\alpha\|\leqq L-1}((V_k)_{(\alpha)}(x), (\pi^*)_x\hat{\eta})^2>0\,,\qquad \hat{\eta}\in S^{D-1}\,.$$

Indeed, simply choose L so that

$$\boldsymbol{R}^D=\operatorname{span}\{(V_k)_{(\alpha)}(x) : 0\leqq k\leqq d \text{ and } \|\alpha\|\leqq L-1\}\,.$$

Thus, for each $x\in W$, we can find an open set $W(x)\ni x$, a $0<\varepsilon\leqq 1$, and an $L\geqq 1$ such that

$$\|(\pi^*)_y\hat{\eta}\|_{R^N}\geqq\varepsilon$$

and

$$(\pi^*)\hat{\eta}\in\operatorname{span}F^{W(x)}(L,\varepsilon)$$

for all $(y,\hat{\eta})\in W(x)\times S^{D-1}$ (cf. the statement of Theorem (5.9) for the notation $F^W(L,\varepsilon)$). Our result now follows from Theorem (5.9) and an easy partition of unity argument. Q. E. D.

We conclude this discussion with a result obtained by combining the preceding with Lemma (4.1). In describing this result, we use the notation $\{P_x : x\in\hat{\boldsymbol{R}}^N\}$ explained in (iii) of Section 4) (cf. the second paragraph of that section). Also, we use $\boldsymbol{L}$ to denote the operator

$$\frac{1}{2}\sum_{k=1}^{d}V_k^2+V_0$$

and ζ^W to denote the escape time

$$\inf\{t\geqq 0 : x(t)\notin W\}$$

from W. Finally, we assume that

$$\left(\phi(x(t\wedge\zeta^W))-\phi(x)-\int_0^{t\wedge\zeta^W}\boldsymbol{L}\phi(x(s))ds,\ \hat{\mathcal{M}}_s,\ P_x\right)$$

is a mean-zero martingale for all $x\in W$ and $\phi\in C_0^\infty(W)$.

(5.12) COROLLARY. *Referring to the preceding, let* $1\leqq D\leqq N$, $\pi\in C_\uparrow^\infty(\boldsymbol{R}^N,\boldsymbol{R}^D)$, *and* $\phi\in C_0^\infty(W)$ *be given and set*

$$P_\phi^\pi(T,x,\cdot)=(\phi((T))P_x)\circ(\pi\circ x(T))^{-1},\qquad (T,x)\in(0,\infty)\times\hat{\boldsymbol{R}}^N.$$

Assume that there exists an open set U *such that*: $\operatorname{supp}(\phi)\Subset U$, $\bar{U}\Subset W$, $d^W(x,\pi(y))\geqq\delta>0$ *for all* $x\in W\backslash U$ *and* $y\in\operatorname{supp}(\phi)$ *(cf. Theorem* (5.9) *for the definition of* d_π^W), *and* $\operatorname{span}\{(\pi_*)_x(V_k)_{(\alpha)} : 0\leqq k\leqq d$ *and* $\alpha\in\mathcal{A}\}=\boldsymbol{R}^D$ *for all* $x\in U$. *Then there exists a measurable mapping* $(T,x)\in(0,\infty)\times\hat{\boldsymbol{R}}^N\to p_\phi^\pi(T,x,\cdot)\in C_0^\infty(\boldsymbol{R}^D)$ *such that* $P_\phi^\pi(T,x,d\hat{y})=p_\phi^\pi(T,x,\hat{y})d\hat{y}$. *Moreover, for each* $n\geqq 0$, p_ϕ^π *satisfies an estimate of the form*

$$|D_{\hat{y}}^r p_\phi^\pi(T,x,\hat{y})|\leqq K_n\exp(-\lambda_n(d_\pi^W(x,\hat{y})\wedge 1)^2/T\wedge 1)/(t\wedge 1)^{\nu_n}, \tag{5.13}$$

where K_n, λ_n, *and* ν_n *are all elements of* $(0,\infty)$.

PROOF. In Lemma (4.1), take Σ to be the space of Radon measures μ on $\boldsymbol{R}^N$ such that $(\phi\mu)\circ\pi^{-1}$ has a density in $C_0^n(\boldsymbol{R}^D)$. Now use Corollary (5.11) and apply Lemma (4.1). Q. E. D.

6. Regularity of the resolvent operator.

Let $V_0, \cdots, V_d \in C_b^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ and define $\mathrm{Lie}(V_0, \cdots, V_d)$ and $\mathcal{J}_{V_0}(V_1, \cdots, V_d)$ to be, respectively, the smallest Lie algebra of vector fields containing $\{V_0, \cdots, V_d\}$ and the smallest Lie algebra of vector fields containing $\{V_1, \cdots, V_d\}$ and closed under Lie multiplication by V_0. Then, it is easy to check that

$$\mathrm{Lie}\,(V_0, \cdots, V_d)(x) = \mathrm{span}\{(V_k)_{\langle\alpha\rangle}(x) : 0 \leqq k \leqq d \text{ and } \alpha \in \mathcal{A}\} \tag{6.1}$$

and

$$\mathcal{J}_{V_0}(V_1, \cdots, V_d)(x) = \mathrm{span}\{(V_k)_{\langle\alpha\rangle}(x) : 1 \leqq k \leqq d \text{ and } \alpha \in \mathcal{A}\}\,. \tag{6.2}$$

In particular,

$$\mathrm{Lie}(V_0, \cdots, V_d)(x) = \mathrm{span}(\mathcal{J}_{V_0}(V_1, \cdots, V_d)(x) \cup \{V_0(x)\})\,. \tag{6.3}$$

As we have seen, the regularity of $P_t = e^{tL}$ when $\boldsymbol{L} = \frac{1}{2}\sum_1^d V_k^2 + V_0$ is related to the rank of $\mathcal{J}_{V_0}(V_1, \cdots, V_d)$. We will show in this section that the regularity of $R_\lambda = (\lambda I - \boldsymbol{L})^{-1}$ is related in the same way to the rank of $\mathrm{Lie}\,(V_0, \cdots, V_d)$.

Throughout this section, we will be working with the following situation:

i) $\{V_0, \cdots, V_d\} \subseteq C_b^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$, $c \in C_b^\infty(\boldsymbol{R}^N)$, and $\{C_m : m \geqq 0\} \subseteq (0, \infty)$ are quantities for which (1.10) holds with $\gamma_m = 0$ and $\sup_{x \in \boldsymbol{R}^N} \|c^{(m)}(x)\|_{\mathrm{H.S.}((\boldsymbol{R}^N)^{\otimes m}, \boldsymbol{R}^1)} \leqq C_m$ for each $m \geqq 0$. Given $L \geqq 1$ and $x \in \boldsymbol{R}^N$, $\mathcal{V}_L(x)$ is defined as in Corollary (3.25). Finally, set $\boldsymbol{L} = \frac{1}{2}\sum_1^d V_k^2 + V_0$.

ii) W is a non-empty open subset of $\boldsymbol{R}^N$; $\hat{\boldsymbol{R}}^N$ is a Polish space in which $\boldsymbol{R}^N$ is an open subset; $\hat{\Omega}$, $\hat{\mathcal{M}}$ and $\{\hat{\mathcal{M}}_t : t > 0\}$ are defined relative to $\hat{\boldsymbol{R}}^N$ (cf. ii) in the second paragraph of Section 2)); and $\zeta^W(\hat{\omega}) = \inf\{t > 0 : x(t, \hat{\omega}) \notin W\}$, $\hat{\omega} \in \hat{\Omega}$.

iii) $\{P_x : x \in \hat{\boldsymbol{R}}^N\}$ is a strong Markov, Feller continuous family of probability measures on $(\hat{\Omega}, \hat{\mathcal{M}})$ such that, for each $x \in \hat{\boldsymbol{R}}^N$ and $\phi \in C_0^\infty(W)$,

$$\left(\phi(x(t \wedge \zeta^W)) - \phi(x) - \int_0^{t \wedge \zeta^W} \boldsymbol{L}\phi(x(s))ds,\ \hat{\mathcal{M}}_t,\ P_x\right)$$

is a mean-zero martingale; and $c \in C_b(\hat{\boldsymbol{R}}^N)$ equals c on W.

iv) $\rho(\xi) = 2 + \sin\xi$ for $\xi \in \boldsymbol{R}^1$; $\{\tilde{V}_k : 0 \leqq k \leqq d+1\} \subseteq C_b^\infty(\boldsymbol{R}^{N+1})$ are defined by:

$$\tilde{V}_0(z) = \rho(\xi)\sum_{i=1}^N V_0^i(x)\frac{\partial}{\partial x_i}\,, \qquad z = (x, \xi) \in \boldsymbol{R}^{N+1}\,,$$

$$\tilde{V}_k(z)=\rho(\xi)^{1/2}\sum_{i=1}^{N} V_k^i(x)\frac{\partial}{\partial x_i}, \quad z=(x,\xi)\in \boldsymbol{R}^{N+1} \text{ and } 1\leqq k\leqq d,$$

$$\tilde{V}_{d+1}(z)=\frac{\partial}{\partial \xi}, \qquad z=(x,\xi)\in \boldsymbol{R}^{N+1},$$

and $\tilde{L}=\frac{1}{2}\sum_{1}^{d+1}\tilde{V}_k^2+\tilde{V}_0$.

v) $\Theta=\{\theta\in C([0,\infty),\boldsymbol{R}^1):\theta(0)=0\}$, and $\mathcal{B}$ and $\{\mathcal{B}_t:t\geqq 0\}$ are defined accordingly; $\mathcal{W}$ is Wiener measure on $(\Theta,\mathcal{B})$; $\tilde{\Omega}=C([0,\infty),\hat{\boldsymbol{R}}^N\times\boldsymbol{R}^1)$, and $\tilde{\mathcal{M}}$ and $\{\tilde{\mathcal{M}}_t:t>0\}$ are defined accordingly; and, for $z=(x,\xi)\in\hat{\boldsymbol{R}}^N\times\boldsymbol{R}^1$, $\hat{P}_z$ on $(\tilde{\Omega},\tilde{\mathcal{M}})$ is the distribution of

$$(\tilde{\omega},\theta)\in\tilde{\Omega}\times\Theta\longrightarrow\left(x\left(\int_0^{\cdot}\rho(\xi+\theta(s))ds,\tilde{\omega}\right),\theta(\cdot)\right)$$

under $P_x\times\mathcal{W}$.

vi) For $z=(x,\xi)\in\boldsymbol{R}^N\times\boldsymbol{R}^1$, $\tilde{c}(z)\equiv\rho(\xi)c(x)$. Given $x\in\hat{\boldsymbol{R}}^N$ and $z\in\hat{\boldsymbol{R}}^N\times\boldsymbol{R}^1$, ${}^{\hat{c}}R(x,\cdot)$ and ${}^{c}\tilde{R}(z,\cdot)$ are defined on $\mathcal{B}_{\hat{R}^N}$ by:

$$\text{(6.4)}\qquad {}^{\hat{c}}R(x,\Gamma)E^{P_x}\left[\int_0^\infty \exp\left(\int_0^t \hat{c}(x(s))ds\right)\chi_\Gamma(x(t))dt\right]$$

and

$${}^{\tilde{c}}\tilde{R}(z,\Gamma)=E^{\tilde{P}_z}\left[\int_0^\infty \exp\left(\int_0^t \tilde{c}(z(s))ds\right)\rho(\xi(t))\chi_\Gamma(x(t))dt\right],$$

respectively.

The following lemma is a consequence of straight-forward computations.

(6.5) LEMMA. *The family is* $\{\tilde{P}_z:z\in\hat{\boldsymbol{R}}^N\times\boldsymbol{R}^1\}$ *is strongly Markov and Feller continuous. Moreover, if* $\zeta^W(\tilde{\omega})=\inf\{t\geqq 0:z(t,\tilde{\omega})\notin W\times\boldsymbol{R}^1\}$, $\tilde{\omega}\in\tilde{\Omega}$, *then for all* $z\in\boldsymbol{R}^N\times\boldsymbol{R}^1$ *and* $\phi\in C_0^\infty(W\times\boldsymbol{R}^1)$:

$$\left(\phi(z(t\wedge\zeta^W))-\phi(z)-\int_0^{t\wedge\zeta^W}\tilde{\boldsymbol{L}}\phi(z(s))ds,\tilde{\mathcal{M}}_t,\tilde{P}_z\right)$$

is a mean-zero martingale. Finally, if $z=(x,\xi)\in\hat{\boldsymbol{R}}^N\times\boldsymbol{R}^1$, *then* ${}^{c}\hat{\boldsymbol{R}}(x,\cdot)={}^{\tilde{c}}\tilde{R}(z,\cdot)$.

(6.6) LEMMA. *For each* $L\geqq 4$:

$$\text{(6.7)}\qquad \mathcal{C}\tilde{V}_L(z)\geqq\frac{1}{4}\inf\{(V_0(x),\eta)_{R^N}^2+\mathcal{C}V_L(x,\eta):\eta\in S^{N-1}\},$$

where $\mathcal{C}\tilde{V}_L(z)$ *is defined in terms of* $\{\tilde{V}_0,\cdots,\tilde{V}_{d+1}\}$ *in the same way as* $\mathcal{C}V_L(x)$ *was in terms of* $\{V_0,\cdots,V_d\}$.

PROOF. First note that if $1 \leqq k \leqq d$ and $\alpha \in \mathcal{A}$, then $(\tilde{V}_k)_{\langle\alpha\rangle}(z) = \rho(\xi)^{\|\alpha\|/2}(V_k)_{\langle\alpha\rangle}(x)$ for $z=(x, \xi) \in \boldsymbol{R}^N \times \boldsymbol{R}^1$. Next, observe that $[\tilde{V}_0, \tilde{V}_{d+1}](z) = (-\cos\xi)V_0(x)$ and $[\tilde{V}_{d+1}, [V_0, V_{d+1}]](z) = (\sin\xi)V_0(x)$ for $z=(x, \xi) \in \boldsymbol{R}^N \times \boldsymbol{R}^1$. Combining these remarks, we arrive at (6.9) immediately. Q. E. D.

(6.8) THEOREM. *Assume that there exist an* $0 < \varepsilon \leqq 1$ *such that*

$$\sum_{k=0}^{d} \sum_{\|\alpha\| \leqq L-1} ((V_k)_{\langle\alpha\rangle}(x),)_{R^N}^2 \geqq \varepsilon\,, \qquad x \in \boldsymbol{R}^N \quad \textit{and} \quad \eta \in S^{N-1}\,, \tag{6.9}$$

for some $L \geqq 4$. *Also assume that* $c(x) \leqq -\lambda$, $x \in \boldsymbol{R}^N$, *and* $\hat{c}(x) \leqq -\lambda$, $x \in \hat{\boldsymbol{R}}^N$, *for some* $\lambda > 0$. *Then there is a measurable map* $x \in \hat{\boldsymbol{R}}^N \to {}^{\hat{c}}r(x, \cdot) \in C^\infty(W \backslash \{x\})$ *such that* ${}^{\hat{c}}r$ *is* C^∞ *on* $(W^2)^* \equiv \{(x, y) \in W^2 : x \neq y\}$ *and* $\chi_W(y)^{\hat{c}}R(x, dy) = {}^{\hat{c}}r(x, y)dy$ *for all* $x \in \hat{\boldsymbol{R}}^N$. *In fact, given* $n \geqq 0$ *and* $0 < \delta < (\mathrm{rad}\, W) \wedge 1$, *there exist* $m_n(L) \geqq 2$ *and* $\nu_n(L) \in (0, \infty)$, *with the same dependence as those in Corollary* (3.25), *and a* $K_n(L) \in (0, \infty)$, *depending only on* $\{C_m : 0 \leqq m \leqq m_n(L)\}$, *such that*:

$$\max_{|\beta| \leqq n} |D_y^\beta\, {}^{\hat{c}}r(x, y)| \leqq K_n(L)B_n(\delta)/\lambda(\varepsilon(\|x-y\|_{\hat{R}^N} \wedge \delta))^{\nu_n(L)}\,, \qquad x \in \hat{\boldsymbol{R}}^N \quad \textit{and} \quad y \in W\{x\}\,, \tag{6.10}$$

and

$$\max_{|\alpha| \vee |\beta| \leqq n} |D_x^\alpha D_y^\beta\, {}^{\hat{c}}r(x, y)| \leqq (K_n(L)B_n(\delta))^2/\lambda(\varepsilon(\|x-y\|_{R^N} \wedge \delta))_{(\delta)}^{2\nu_n(L)}\,, \qquad (x, y) \in (W_{(\delta)}^2)^*\,, \tag{6.11}$$

where $W_{(\delta)} \equiv \{x \in W : \mathrm{dist}(x, W) > \delta\}$ *and* $B_n(\delta) \in (0, \infty)$ *is as in Theorem* (4.5). *Finally, for each* $x \in \hat{\boldsymbol{R}}^N$,

$$(\hat{c}(y) + \boldsymbol{L}_y^*)^{\hat{c}}r(x, y) = -\delta_x(y)\,, \qquad y \in W\,, \tag{6.12}$$

where $\boldsymbol{L}$ *denotes the formal adjoint of* $\boldsymbol{L}^*$.

PROOF. Set $\tilde{\Theta} = \{\theta \in C([0, \infty), \boldsymbol{R}^{d+1}) : \theta(0) = 0\}$ and let $\tilde{\mathcal{W}}$ on $(\tilde{\Theta}, \mathcal{B}_{\tilde{\Theta}})$ denote Wiener measure. Let $z \in \boldsymbol{R}^N \to Z(\cdot, z) = (X(\cdot, z), \ \Xi(\cdot, z)) \in C([0, \infty), \boldsymbol{R}^N \times \boldsymbol{R}^1)$ be a smooth selection of solutions to

$$Z(T, z) = z + \sum_{k=1}^{d+1} \int_0^T \tilde{V}_k(Z(t, z)) \circ d\tilde{\theta}_k(t) + \int_0 \tilde{V}_0(Z(t, z))dt\,, \qquad T \geqq 0\,;$$

and define

$$\tilde{Q}(T, z, \Gamma) = E^{\tilde{\mathcal{W}}}\Big[\exp\Big(\int_0^T \rho(\Xi(t, z))c(X(t, x))dt\Big)\rho(\Xi(T, z)),\ X(T, z) \in \Gamma\Big]$$

for $(T, z) \in [0, \infty) \times \boldsymbol{R}^{N+1}$ and $\Gamma \in \mathcal{B}_{R^N}$. Since, by Lemma (6.6), $\mathcal{V}_L(z) \geqq \varepsilon/4$, $z \in \boldsymbol{R}^{N+1}$, we can repeat the argument which led us to Corollary (3.25) and thereby derive the existence of a $\tilde{q}\in C^{\infty}((0,\infty)\times\boldsymbol{R}^{N+1}\times\boldsymbol{R}^{N})$ such that $Q(T,z,dy)=\tilde{q}(T,z,y)dy$, $(T,z)\in(0,\infty)\times\boldsymbol{R}^{N+1}$. In fact, that line of reasoning allows us to assert that for each $n\geqq 0$, there exist $m_n(L)\geqq 2$, $\nu_n(L)\in(0,\infty)$, and $K_n(L)$ (with the desired dependence properties) and a universal $\lambda_n\in(0,\infty)$ such that:

$$\max_{|\alpha|\vee|\beta|\leq n}|D_z^{\alpha}D_y^{\beta}\tilde{q}(t,z,y)|$$

$$\leqq K_n(L)\exp[-\lambda t-\lambda_n\|y-x\|_{\boldsymbol{R}^N}^2/(1+C_0^2)t]/(\varepsilon^{1+2/L}t)^{\nu_n(L)}$$

for all $t\in(0,1]$, $z=(x,\xi)\in\boldsymbol{R}^N\times\boldsymbol{R}^1$, and $y\in\boldsymbol{R}^N$.

Next, define

$${}^{\tilde{c}}\tilde{P}(T,z,\Gamma)=E^{\tilde{P}_z}\left[\exp\left(\int_0^T\tilde{c}(z(t))dt\right)\rho(\xi(T)),\,x(T)\in\Gamma\right]$$

for $(T,z)\in(0,\infty)\times(\hat{\boldsymbol{R}}^N\times\boldsymbol{R}^1)$ and $\Gamma\in\mathscr{B}_{\boldsymbol{R}^N}$. Noting that the distribution of $Z(\cdot,z)$ under $\mathcal{W}$ and of $z(\cdot)$ under $\tilde{P}_z$ are equal on $\tilde{\mathscr{M}}_{\zeta W}$, we use the localization procedures introduced in Section 4) to show that there is a measurable map $(T,z)\in(0,\infty)\times\hat{\boldsymbol{R}}^N\to{}^{\tilde{c}}\tilde{p}(T,z,\cdot)\in C^{\infty}(W)$ such that:

$${}^{\tilde{c}}\tilde{p}(T,\cdot,*)|_{(W\times R^1)\times W}\in C^{\infty}((W\times\boldsymbol{R}^1)\times W)\,,\qquad T>0\,,$$

and

$$\chi_W(y){}^{\tilde{c}}\tilde{P}(T,z,dy)={}^{\tilde{c}}p(T,z,y)dy\,,\qquad (T,z)\in(0,\infty)\times(\hat{\boldsymbol{R}}^N\times\boldsymbol{R}^1)\,.$$

Moreover, given $n\geqq 0$ and $0<\delta<(\mathrm{rad}\,W)\wedge 1$, we can find $K_n(L)\in(0,\infty)$, with the desired dependence properties, so that

$$\max_{|\beta|<n}|D_y^{\beta}\,{}^{\tilde{c}}\tilde{p}(t,z,y)|$$

$$\leqq K_n(L)B_n(\delta)\exp[-\lambda t-\lambda_n(\|y-x\|_{\boldsymbol{R}^N}\wedge\delta)^2/(1+C_0^2)t]/(\varepsilon^{1+2/L}(t\wedge 1))^{\nu n(L)}$$

for all $t>0$, $z=(x,\xi)\in\hat{\boldsymbol{R}}^N\times\boldsymbol{R}^1$, and $y\in W_{(\delta)}$; and

$$\max_{|\alpha|\vee|\beta|\leq n}|D_z^{\alpha}D_y^{\beta}\,{}^{\tilde{c}}\tilde{p}(t,z,y)|$$

$$\leqq(K_n(L)B_n(L))^2\exp[-\lambda t-\lambda_n(\|y-x\|_{\boldsymbol{R}^N}\wedge\delta)^2/(1+C_0^2)t]/(\varepsilon^{1+2/L}(t\wedge 1))^{2\nu n(L)}$$

for all $t>0$, $z=(x,\xi)\in W_{(\delta)}\times\boldsymbol{R}^1$, and $y\in W_{(\delta)}$. Since ${}^{\hat{c}}R(x,\cdot)={}^{\tilde{c}}\tilde{R}((x,0),\cdot)=\int_0^{\infty}{}^{\tilde{c}}\tilde{P}(t,(x,0),\cdot)dt$, the existence and regularity properties of ${}^{\tilde{c}}r$ are now proved.

Finally, to prove (6.14), we need only show that

$$\int(\hat{c}\phi+L\phi)(y){}^{\hat{c}}R(x,dy)=-\phi(x)\,,\qquad x\in\hat{\boldsymbol{R}}^N\,,\tag{6.13}$$

for all $\phi\in C_0^\infty(W)$. But (cf. the proof of Theorem (4.5)), for any $\phi\in C_0^\infty(W)$, $x\in \boldsymbol{R}^N$, and $T>0$:

$$E^{P_x}\left[\exp\left(\int_0^T \hat{c}(x(s))ds\right)\phi(x(T))\right]-\phi(x)$$

$$=E^{P_x}\left[\int_0^T \exp\left(\int_0^t \hat{c}(x(s))ds\right)(\hat{c}\phi+\boldsymbol{L}\phi)(x(t))dt\right].$$

Since $c(\cdot)\leqq-\lambda$, (6.13) results from this after one lets $T\uparrow\infty$. Q. E. D.

7. Regularity in the presence of degeneracy on thin sets.

Let $V_0,\cdots,V_d\in C_\uparrow^\infty(\boldsymbol{R}^N,\boldsymbol{R}^N)$ satisfy (1.3), set $\mathcal{J}_{V_0}(V_1,\cdots,V_d)=\operatorname{span}\{(V_k)_{(\alpha)}: 1\leqq k\leqq d \text{ and } \alpha\in\mathcal{A}\}$, and define the processes $X(\cdot,x)$, $A(\cdot,x)$, and $\Delta(\cdot,x)$ on $(\Theta,\mathcal{B},\mathcal{W})$ accordingly. The basic result of Section 2) is that $\{1/\Delta(T,x): 0<T\leqq 1\}\subseteqq \bigcap_{p\in[1,\infty)} L^p(\mathcal{W})$ whenever $\mathcal{J}_{V_0}(V_1,\cdots,V_d)(x)=\boldsymbol{R}^N$. Before turning to the topic of the present section, it may be helpful to know that when the V_k's are real analytic near x the condition $\mathcal{J}_{V_0}(V_1,\cdots,V_d)(x)=\boldsymbol{R}^N$ is not only sufficient but is also necessary for $\{1/\Delta(T,x): 0<T\leqq 1\}\subseteqq \bigcap_{p\in[1,\infty)} L^p(\mathcal{W})$ to be true. To see the necessity, suppose that $\mathcal{J}_{V_0}(V_1,\cdots,V_d)(x)\neq\boldsymbol{R}^N$. Then, thinking of the V_k's as vector fields on $\boldsymbol{R}^{N+1}=\boldsymbol{R}^1\times\boldsymbol{R}^N$, we see that $\operatorname{Lie}(\partial/\partial t+V_0, V_1,\cdots,V_d)((0,x))\neq\boldsymbol{R}^{N+1}$. Therefore, by the Nagano Theorem (cf. [9]), there exists an open real-analytic submanifold M of $\boldsymbol{R}^{N+1}$ passing through $(0,x)$ and integrating $\operatorname{Lie}(\partial/\partial t+V_0, V_1,\cdots,V_d)$. Hence, there is an open neighborhood W of $(0,x)$ in $\boldsymbol{R}^{N+1}$ such that $\mathcal{W}((t,X(t,x))\notin M$ for some $t\in[0,\zeta^W(x)))=0$, where $\zeta^W(x)=\inf\{t\geqq 0:(t,X(t,x))\notin W\}$. In particular, the measure $\mu=\chi_{(0,1)}(t)dt\times P(t,x,dy)$ assigns M positive mass. Since M has dimension strictly less than $(N+1)$, this means that μ has a non-zero $\boldsymbol{R}^{N+1}$-Lebesgue singular part. On the other hand, by Theorem (3.17), if $\{1/\Delta(T,x): 0<T\leqq 1\}\subseteqq \bigcap_{p\in[0,\infty)} L^p(\mathcal{W})$, then μ would be $\boldsymbol{R}^{N+1}$-Lebesgue absolutely continuous. Thus, in the real-analytic category, $\mathcal{J}_{V_0}(V_1,\cdots,V_d)(x)\neq\boldsymbol{R}^N$ implies that $\{1/\Delta(T,x): 0<T\leqq 1\}\nsubseteqq \bigcap_{p\in[1,\infty)} L^p(\mathcal{W})$ (in fact, a closer look reveals that $\mathcal{W}(\Delta(t,x)=0)>0$ for all $t\in(0,1]$ if $\mathcal{J}_{V_0}(V_1,\cdots,V_d)(x)\neq\boldsymbol{R}^N$).

In spite of the preceding remarks, one can still ask whether for C^∞ (as opposed to analytic) vector fields V_k it is possible that $\{1/\Delta(T,x): 0<T\leqq 1\}\subseteqq \bigcap_{p\in[1,\infty)} L^p(\mathcal{W})$ can hold in situations where $\mathcal{J}_{V_0}(V_1,\cdots,V_d)(x)=\boldsymbol{R}^N$ fails. That such situations might exist was already indicated by Oleinik and Radekevich in their extensive study of hypoellipticity [10]. The probabilistic intuition underlying our analysis of this question is very simple.

Namely, in the C^∞-category, the analogue of Nagano's theorem is false; thus, the diffusion can "escape instantaneously" from $\{x\in \boldsymbol{R}^N : \mathcal{I}_{V_0}(V_1,\cdots, V_d)(x) \neq \boldsymbol{R}^N\}$. By making this intuition quantitative, we will now produce criteria which guarantee that $\{1/\Delta(T, x): 0<T\leqq 1\}\subseteqq \bigcap_{p\in[1,\infty)} L^p(\mathcal{W})$ even when $\mathcal{I}_{V_0}(V_1,\cdots, V_d)(x)\neq \boldsymbol{R}^N$.

In the discussion below, we will be using the following notation. Let $\mathcal{A}$ be the index set described in the appendix; and, for $\alpha\in\mathcal{A}$, recall the definitions of $|\alpha|$ and $\|\alpha\|$ and, when $\alpha\neq\emptyset$, of α' and α_*. Given $\phi\in C^\infty(\boldsymbol{R}^N)$, define $\phi_{(\alpha)}$, $\alpha\in\mathcal{A}$, inductively by: $\phi_{(\emptyset)}=\phi$ and $\phi_{(\alpha)}=V_{\alpha_*}\phi_{(\alpha')}$, $\alpha\neq\emptyset$. Finally, given $L\geqq 1$, set

$$\sigma^2_{(L)}(\phi)(x)=\sum_{\|\alpha\|\leqq L-1}(\phi_{(\alpha)}(x))^2\,. \tag{7.1}$$

(7.2) LEMMA. *For each $L\geqq 1$ there exist universal (i.e., independent of $\{V_0,\cdots, V_d\}$ and ϕ) constants $C(L)\in(0,\infty)$ and $\mu_L\in(0,1]$ such that*

$$\mathcal{W}\Big(\sup_{0\leqq t\leqq TK^{-2/(L+1)}}\phi(X(t, x))^2/T^{L-1}\leqq 1/K\Big)$$

$$\leqq C(L)\exp[-(\sigma^2_{(L)}(\phi)(x)^{L+2}K^2)^{\mu_L}/(1+M(x))^2]\,,\qquad K\in[1,\infty)$$

where

$$M(x)=\max\{\|\phi_{(\alpha)}\|_{C^2_b(B(x,1))}: |\alpha|\leqq L+1\}\,.$$

PROOF. We relate this estimate to one which is similar to the estimate proved in Theorem (2.17). Indeed, since

$$\frac{K}{T^L}\int_0^{T/K}\phi(X(t, x))^2dt\leqq \sup_{0\leqq t\leqq T/K}\phi(X(t, x))^2/T^{L-1}\,,$$

it suffices for us to estimate

$$\mathcal{W}\Big(\int_0^{TK^{-2/(L+1)}}\phi(X(t, x))^2dt/T^L\leqq 1/K^2\Big)\,.$$

To this end, we proceed as in Theorem (2.12) and develop $\phi(X(t, x))$ in a Taylor's expansion:

$$\phi(X(t, x))=\sum_{\|\alpha\|\leqq L-1}\phi_{(\alpha)}(x)\theta^{(\alpha)}(t)+R_L(t, x, \phi)$$

where

$$R_L(t, x, \phi)=\sum_{|\alpha|=L}S^{(\alpha)}(t, \phi_{(\alpha)}(X(\cdot, x)))+\sum_{\substack{|\alpha|\leqq L-1\\ \|\alpha\|\geqq L}}\phi_{(\alpha)}(x)\theta^{(\alpha)}(t)$$

and the notation $\theta^{(\alpha)}(\cdot)$ and $S^{(\alpha)}$ is explained in the appendix. Starting from here, the argument differs in no essential way from the one used to prove Theorem (2.17). Q. E. D.

(7.3) THEOREM. *Let $h:(0,\infty)\to(0,1)$ be a non-decreasing function satisfying: $h(0)\equiv\lim_{\lambda\downarrow 0}h(\lambda)=0$ and $\lim_{\lambda\downarrow 0}\lambda^{\rho}\log(1/h(\lambda))=0$ for each $\rho>0$. For given $L\geqq 1$ and non-empty closed $F\subseteq S^{N-1}$, define $\mathcal{V}_L(*,F)$ and $\tilde{\lambda}(\cdot,*,F)$ as in Section 2) (cf. (2.15) and Theorem (2.17)); and assume that $\mathcal{V}_L(*,\overline{F^{(\delta)}})\geqq h\circ\phi(*)$ for some $0<\delta<1$ and $\phi\in C^{\infty}(\boldsymbol{R}^N)$, where $F^{(\delta)}=\{\eta\in S^{N-1}:\|\eta-F\|_{R^N}<\delta\}$. If $x\in\boldsymbol{R}^N$ satisfies $\sigma^2_{(L)}(\phi)(x)\geqq 1$, then for each $\rho\in(0,1]$:*

$$\mathcal{W}(\tilde{\lambda}(T/(\log K)^{1/\rho},x,F)/T^L\leqq 1/4K)\leqq C(L)\exp[-\lambda\delta^2(\log K)^2/(1+M(x))^2]$$

for all $T\in(0,T_\rho(L)]$ and $K\geqq\exp[(4/T)^\rho]$, where: $C(L)\in(0,\infty)$ depends only on L; $T_\rho(L)\in(0,1]$ depends only on h, ρ and L; λ is universal; and

$$M(x)=(\max\{\|\phi_{(\alpha)}\|_{C_b^0(B(x,1))}:|\alpha|\leqq L+1\})$$
$$\vee(\max\{\|(V_k)_{(\alpha)}\|_{C_b^0(B(x,2);R^N)}:0\leqq k\leqq d \text{ and } |\alpha|\leqq L+1\}).$$

PROOF. Let $\mu\in(0,1]$ be the smaller of the constants $\tilde{\mu}_L$ and μ_L appearing in Theorem (2.17) and Lemma (7.2) and let C be the larger of the constants $C(L)$ appearing in those results. Set $\gamma=(L+1)/(\rho\wedge\mu)$ and choose $K_\rho(L)\in[1,\infty)$ so that all the inequalities:

$$(K(h((\log K)^{-\gamma})^{L+2}))^{\mu}\geqq(\log K)^2$$
$$3K\geqq(\log K)^{2/\rho}$$
$$(\log K)^{1/\rho}\geqq 4$$

hold whenever $K\geqq K_\rho(L)$. Define $T_\rho(L)=4(\log K_\rho(L))^{-1/\rho}$.

Given $T\in(0,T_\rho(L)]$ and $K\geqq\exp((4/T)^\rho)$, we have:

$$\mathcal{W}(\tilde{\lambda}(T/(\log K)^{1/\rho},x,F)/T^L\leqq 1/4K)$$
$$\leqq\mathcal{W}(\tau(x)\geqq 1/(\log K)^{2/\rho})$$
$$+\mathcal{W}(\zeta(x)\leqq 1/(\log K)^{(2/\rho)}$$
$$+\mathcal{W}(\tilde{\lambda}(T(\log K)^{1/\rho},x,F)/T^L\leqq 1/4K,\ \tau(x)<1/(\log K)^{2/\rho}<\zeta(x))$$
$$\equiv I_1(K)+I_2(K)+I_3(K)\,,$$

where

$$\tau(x)\equiv\inf\{t\geqq 0:\phi(X(t,x))^2\geqq 1/(\log K)^{\gamma}\}$$

and

$$\zeta(x)\equiv\inf\{t\geqq 0:\|X(t,x)-x\|_{R^N}\geqq 1 \text{ or } \|J^{-1}(t,x)-I\|_{\mathrm{H.S.}(R^N,R^N)}\geqq\delta/(2+\delta)\}\,.$$

By Lemma (7.2):

$$I_1(K)=\mathcal{W}\Big(\sup_{0\le t\le (\log K)^{2T/L+1}}\psi(X(t,x))^2\leqq 1/(\log K)^{\gamma}\Big)$$

$$\leqq C\exp(-(\log K)^{\mu\gamma}/(1+M(x))^2)$$

$$\leqq C\exp(-(\log K)^2/(1+M(x))^2).$$

By standard estimates on the growth of solutions to stochastic integral equations, there exist universal $C\in(0,\infty)$ and $\lambda\in(0,\infty)$ such that

$$I_2(K)\leqq C\exp[-\lambda\delta^2(\log K)^{2/\rho}/(1+M(x))^2]$$

$$\leqq C\exp[-\lambda\delta^2(\log K)^2/(1+M(x))^2].$$

To estimate $I_3(K)$, first note that if $0\leqq s<\zeta(x)\wedge t$, then

$$\tilde{\lambda}_s(t,x,\overline{F^{(\delta)}})\leqq 4\tilde{\lambda}(t,x,F),$$

where

$$\tilde{\lambda}_s(t,x,\overline{F^{(\delta)}})=\inf\{(\eta,\tilde{A}_s(t,x)\eta)_{R^N}:\eta\in\overline{F^{(\delta)}}\}$$

and

$$\tilde{A}_s(t,x)=\sum_{k=1}^{d}J(s,x)\Big(\int_s^t(J^{-1}(u,x)V_k(X(u,x)))^{\otimes 2}du\Big)J(s,x)^*.$$

Next, note that

$$T/(\log K)^{1/\rho}\geqq 4/(\log K)^{2/\rho},$$

and so, after applying the strong Markov property for

$$\begin{pmatrix}X(\cdot,x)\\ \tilde{A}(\cdot,x)\end{pmatrix}$$

we obtain:

$$I_3(K)\leqq\int_{\{\theta:\tau(x,\theta)<(1/\log K)^{2/\rho}\wedge\zeta(x,\theta)\}}\mathcal{W}(\tilde{\lambda}(\sigma(x,\theta),Y(x,\theta),\overline{F^{(\delta)}})\leqq 1/K)\mathcal{W}(d\theta)$$

where

$$\sigma(x,\theta)=T/(\log K)^{1/\rho}-\tau(x,\theta)>3/(\log K)^{2/\rho}$$

and

$$Y(x,\theta)=X(\tau(x,\theta),x,\theta).$$

Since $t\to\tilde{\lambda}(t,Y(x,\theta),\overline{F^{(\delta)}})$ is non-decreasing and

$$T/K\leqq 1/K\leqq 3/(\log K)^{2/\rho},$$

we conclude that

$$I_3(K)\leqq\int_{\{\theta:\tau(x,\theta)<(1/\log K)^{2/\rho}\wedge\zeta(x,\theta)\}}\mathcal{W}(\tilde{\lambda}(T/K,Y(x,\theta),\overline{F^{(\delta)}})\leqq 1/K)\mathcal{W}(d\theta).$$

Hence, by Theorem (2.17):

$$I_3(K) \leqq C \exp[-(K(h((\log K)^{-\gamma}))^{L+2})^{\mu}/(1+M(x))^2]$$
$$\leqq C \exp[-(\log K)^2/(1+M(x))^2]. \qquad \text{Q. E. D.}$$

(7.4) COROLLARY. *Let* $h:(0,\infty)\to(0,1]$ *be as in the preceding theorem and assume that* $\mathcal{V}_L(\cdot) \geqq h\circ\phi(\cdot)$ *for some* $L\geqq 1$ *and* $\phi \in C^\infty(\boldsymbol{R}^N)$. *If* $x\in \boldsymbol{R}^N$ *and* $\sigma^2_{(L)}(\phi)(x)\geqq 1$, *then, for each* $p\in[1,\infty)$ *and* $\rho\in(0,1]$, *there exist an* $M_p(\rho)\in(0,\infty)$, *depending only on* $h(\cdot)$, L, p, ρ *and* $\{\|\phi_{(\alpha)}\|_{C_b^0(B(x,1))} : |\alpha|\leqq L+1\} \cup \{\|(V_k)_{(\alpha)}\|_{C_b^0(B(x,2);\boldsymbol{R}^N)} : 1\leqq k\leqq d$ *and* $|\alpha|\leqq L+1\} \cup \{\|V_k^{(1)}\|_{C_b(\boldsymbol{R}^N;\mathrm{H.S.}(\boldsymbol{R}^N,\boldsymbol{R}^N))} : 0\leqq k \leqq d\}$, *such that*:

$$\|1/\Delta(T,x)\|_{L^p(\mathcal{W})} \leqq M_p(\rho)\exp[1/T^\rho], \qquad T\in(0,1]. \tag{7.5}$$

In particular, Theorems (3.17) *and* (4.5) *apply.*

PROOF. Since $\Delta(T,x) \geqq (\det J(T,x))^2(\tilde{\lambda}(T,x))^N$ and $\|\|J^{-1}(T,x)\|_{\mathrm{H.S.}(\boldsymbol{R}^N,\boldsymbol{R}^N)}\|_{L^p(\mathcal{W})} \leqq A_p\exp[B_pT]$, $p\in[1,\infty)$ and $T\geqq 0$, where $A_p\in(0,\infty)$ and $B_p\in(0,\infty)$ depend only on p and $\{\|V_k^{(1)}\|_{C_b(\boldsymbol{R}^N;\mathrm{H.S.}(\boldsymbol{R}^N,\boldsymbol{R}^N))} : 0\leqq k\leqq d\}$, (7.5) is an easy consequence of the estimate obtained in Theorem (7.3). Q. E. D.

8. Hypoellipticity.

Our aim in this section is to show that our results about the fundamental solution to $\partial u/\partial t = \boldsymbol{L}u$ provide us with enough information to prove hypoellipticity properties for $\boldsymbol{L}$. Indeed, it is clear that $\int_0^1 p(t,x,y)dt$ is, in some sense, a "parametrix" for $\boldsymbol{L}$ and therefore that hypoelliplicity cannot be far off. However, we do not know whether $\int_0^1 p(t,x,y)dt$ is in some algebra of pseudodifferential operators. Hence, it is not immediately obvious how to use $\int_0^1 p(t,x,y)dt$ to prove hypoelliplicity. For this reason, we need to begin with a few preliminaries.

Let $\boldsymbol{L}$ be a linear differential operator on $C^\infty(\boldsymbol{R}^N)\to C^\infty(\boldsymbol{R}^N)$. Given an open set $W\subseteq \boldsymbol{R}^N$, we say that $\boldsymbol{L}$ is *hypoelliptic on* W if, for every $u\in \mathcal{D}'(\boldsymbol{R}^N)$, sing supp$(u|_W)\subseteq$sing supp$(\boldsymbol{L}u|_W)$. We say that $\boldsymbol{L}$ is *parabolic hypoelliptic on* W if $\partial/\partial t+\boldsymbol{L}$ is hypoelliptic on $\boldsymbol{R}^1\times W$.

Lemma (8.5) and Theorem (8.6) refer to the following situation. There is a $q\in C^\infty((0,2)\times \boldsymbol{R}^N\times\boldsymbol{R}^N)$ such that

$$\frac{\partial q}{\partial t}(t,x,y) = (\boldsymbol{L}_y^* q)(t,x,y), \quad (t,x,y)\in(0,2)\times\boldsymbol{R}^N\times\boldsymbol{R}^N, \tag{8.1}$$

where L^* denotes the formal adjoint of L and the subscript "y" indicates the variable on which the operator is acting. For $\phi \in C_0^\infty(\boldsymbol{R}^N)$, $Q_t\phi(x) \equiv \int \phi(y)q(t, x, y)dy$ and $Q_t^*\phi(y) = \int \phi(x)q(t, x, y)dx$. We assume that for each $\phi \in C_0^\infty(\boldsymbol{R}^N)$:

$$\lim_{t \downarrow 0} Q_t\phi(x) = \phi(x), \qquad x \in \boldsymbol{R}^N, \tag{8.2}$$

and that for each $n \geqq 0$ there is a $C_n \in (0, \infty)$ such that

$$\sup_{0<t\leqq 1} \|Q_t\phi\|_{C_b^n(\boldsymbol{R}^N)} \leqq C_n \|\phi\|_{C_b^n(\boldsymbol{R}^N)}, \tag{8.3}$$

and

$$\sup_{0<t\leqq 1} \|Q_t^*\phi\|_{C_b^n(\boldsymbol{R}^N)} \leqq C_n \|\phi\|_{C_b^n(\boldsymbol{R}^N)}. \tag{8.4}$$

Finally, $\rho \in C_0^\infty((1, 2))$ satisfies $\int_{R^1} \rho(t)dt = 1$, and $\rho_\tau(t) \equiv (1/\tau)\rho(t/\tau)$ for $\tau \in (0, 1/2]$ and $t \in \boldsymbol{R}^1$.

(8.5) LEMMA. *Given $\phi \in C_0^\infty(\boldsymbol{R}^N)$, $Q_t^*\phi \in \mathcal{E}(\boldsymbol{R}^N)$ ($= C^\infty(\boldsymbol{R}^N)$, the test function space for distributions with compact support) for each $t \in (0, 1]$ and $Q_t^*\phi \to \phi$ in $\mathcal{E}(\boldsymbol{R}^N)$ as $t \downarrow 0$. Also, if $\phi \in C_0^\infty(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$ and*

$$\phi_\tau^*(t, y) \equiv \int_{R^1 \times R^N} \rho_\tau(t-s)\phi(s, x)q(t-s, x, y)dsdx$$

for $\tau \in (0, 1/2)$ and $(t, y) \in \boldsymbol{R}^1 \times \boldsymbol{R}^N$, then $\phi_\tau^* \in \mathcal{E}(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$ for each $\tau \in (0, 1/2)$ and $\phi_\tau^* \to \phi$ in $\mathcal{E}(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$ as $\tau \downarrow 0$.

PROOF. Since the proof of the first part is essentially the same as that of the second (only easier), we will only prove the second part. To this end, note that

$$\phi_\tau^*(t, y) = \int_{R^1 \times R^D} \phi(t-s, x)\rho_\tau(s)q(s, x, y)dsdx.$$

Thus, by (8.4):

$$\sup_{0<\tau\leqq 1/2} \|\phi_\tau^*\|_{C_b^n(\boldsymbol{R}^N)} < \infty, \qquad n \geqq 0.$$

In particular, $\{\phi_\tau^* : 0 < \tau \leqq 1/2\}$ is relatively compact in $\mathcal{E}(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$. Thus, the proof will be complete once we show that $\phi_\tau^* \to \phi$ in $\mathcal{D}'(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$. But, given $\psi \in C_0^\infty(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$:

$$\int_{R^1 \times R^N} \phi_\tau^*(t, y)\psi(t, y)dtdy = \int_{R^1 \times R^N} \phi(s, x)\psi_\tau(s, x)dsdx,$$

where

$$\phi_\tau(s, x) \equiv \int \rho_\tau(t)\phi(s+\tau t, y)q(t, x, y)dydt\,.$$

Since, by (8.2) and (8.3), $\phi_\tau \to \phi$ boundedly and pointwise, we now see that $\phi_\tau^* \to \phi$ in $\mathcal{D}'(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$. Q. E. D.

(8.6) THEOREM. *If for each* $n \geqq 0$ *and* $\varepsilon > 0$

$$\max_{|\alpha|+|\beta|\leqq n}\ \sup_{0<t\leqq 1}\ \sup_{\|y-x\|_{\boldsymbol{R}^N}\geqq \varepsilon} |D_x^\alpha D_y^\beta q(t, x, y)| < \infty\,, \tag{8.7}$$

then $\boldsymbol{L}$ *is hypoelliptic on* $\boldsymbol{R}^N$. *If for each* $n \geqq 0$, $\varepsilon > 0$, *and* $\nu > 0$:

$$\max_{|m|+|\alpha|+|\beta|\leqq n}\ \sup_{0<t\leqq 1}\ \sup_{\|y-x\|_{\boldsymbol{R}^N}\geqq \varepsilon} \frac{1}{t^\nu} |D_t^m D_x^\alpha D_y^\beta q(t, x, y)| < \infty\,, \tag{8.8}$$

then $\boldsymbol{L}$ *is parabolic hypoelliptic.*

PROOF. Assume that (8.7) holds. Let $u \in \mathcal{D}'(\boldsymbol{R}^N)$ and let W be an open set for which $\boldsymbol{L}u|_W \in C^\infty(W)$. Given $x^0 \in W$, choose $\eta \in C_0^\infty(W)$ so that $\eta = 1$ in a neighborhood of x^0. Set $\tilde{u} = \eta u$, $\tilde{f} = \eta \boldsymbol{L}u$, and $\tilde{v} = \boldsymbol{L}\tilde{u} - \tilde{f}$. Then: $\tilde{u} \in \mathcal{E}'(\boldsymbol{R}^N)$, $\tilde{f} \in C_0^\infty(W)$, and there is an $\varepsilon > 0$ such that $\overline{B(x^0, 2\varepsilon)} \Subset W$, $\eta = 1$ on $B(x^0, 2\varepsilon)$, and $\mathrm{supp}(\tilde{v}) \Subset W \backslash \overline{B(x^0, 2\varepsilon)}$. Given $t \in (0, 1]$, set $\tilde{u}_t(x) = \tilde{u}(q(t, x, \cdot))$. Clearly $\tilde{u}_t \in C_b^\infty(\boldsymbol{R}^N)$, $t \in (0, 1]$, and, by the first part of Lemma (8.5), $\tilde{u}_t(\phi) = \tilde{u}(Q_t^*\phi) \to \tilde{u}(\phi)$ as $t \downarrow 0$ for each $\phi \in C_0^\infty(\boldsymbol{R}^N)$. Hence, $\tilde{u}_t \to \tilde{u}$ in $\mathcal{D}'(\boldsymbol{R}^N)$. Thus, we will be done once we show that $\sup_{0<t\leqq 1} \|\tilde{u}_t\|_{C_b^n(B(x^0,\varepsilon))} < \infty$, $n \geqq 0$. But, by (8.1), $(\partial/\partial t)\tilde{u}_t(x) = (\boldsymbol{L}\tilde{u})(q(t, x, \cdot)) = Q_t\tilde{f}(x) + \tilde{v}_t(x)$, where $\tilde{v}_t(x) \equiv \tilde{v}(q(t, x, \cdot))$. Thus, for $\tau \in (0, 1]$:

$$\tilde{u}_\tau = \tilde{u}_1 - \int_\tau^1 Q_t\tilde{f}\,dt - \int_\tau^1 \tilde{v}_t dt\,.$$

By (8.3), $\sup_{0<t\leqq 1} \|Q_t\tilde{f}\|_{C_b^n(\boldsymbol{R}^N)} < \infty$, $n \geqq 0$; and, by (8.7) plus $\mathrm{supp}(\tilde{v}) \Subset W \backslash \overline{B(x^0, 2\varepsilon)}$, $\sup_{0<t\leqq 1} \|\tilde{v}_t\|_{C_b^n(B(x^0,\varepsilon))} < \infty$, $n \geqq 0$.

The proof of the second part is similar but somewhat more involved. Let $u \in \mathcal{D}'(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$, and suppose that W is an open set in $\boldsymbol{R}^1 \times \boldsymbol{R}^N$ for which $(\partial/\partial t + \boldsymbol{L})u|_W \in C^\infty(W)$. Given $(s^0, x^0) \in W$: choose $0 < \varepsilon < 1/10$ so that $[s^0 - 5\varepsilon, s^0 + 5\varepsilon] \times \overline{B(x^0, 5\varepsilon)} \Subset W$ and choose $\eta \in C_0^\infty((s^0 - 4\varepsilon, s^0 + 4\varepsilon) \times B(x^0, 4\varepsilon))$ so that $\eta \equiv 1$ on a neighborhood of $[s^0 - 3\varepsilon, s^0 + 3\varepsilon] \times \overline{B(x^0, 3\varepsilon)}$. Define $\tilde{u} = \eta u$, $\tilde{f} = \eta(\partial/\partial t + L)u$, and $\tilde{v} = (\partial/\partial t + L)\tilde{u} - f$. Clearly $\mathrm{supp}(\tilde{u}) \Subset (s^0 - 4\varepsilon, s^0 + 4\varepsilon) \times B(x^0, 4\varepsilon)$, $\tilde{f} \in C_0^\infty((s^0 - 4\varepsilon, s^0 + 4\varepsilon) \times B(x^0, 4\varepsilon))$, and $\mathrm{supp}(\tilde{v}) \Subset (s^0 - 4\varepsilon, s^0 + 4\varepsilon) \times B(x^0, 4\varepsilon) \backslash [s^0 - 3\varepsilon, s^0 + 3\varepsilon] \times \overline{B(x^0, 3\varepsilon)}$. Given $\tau \in (0, 1/2)$, define $\tilde{u}_\tau(s, x) = \tilde{u}(\rho_\tau(\cdot - s)q(\cdot - s, x, *))$. Then $\{\tilde{u}_\tau : \tau \in (0, 1/2)\} \subseteq \mathcal{E}(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$ and, by the second part of Lemma (8.5), for each $\phi \in C_0^\infty(\boldsymbol{R}^1 \times \boldsymbol{R}^N)$:

$$\int_{R^1\times R^N}\tilde{u}_\tau(s,x)\phi(s,x)dsdx=\tilde{u}(\phi_\tau^*)\longrightarrow u(\phi)$$

as $\tau\downarrow 0$. That is, $\tilde{u}_\tau\to\tilde{u}$ in $\mathscr{D}'(\boldsymbol{R}^1\times\boldsymbol{R}^N)$. Hence, we will be done if we show that

$$\sup_{0<\tau\leqq 5\varepsilon}\|\tilde{u}_\tau\|_{C_b^n((s^0-\varepsilon,s^0+\varepsilon)\times B(x^0,\varepsilon))}<\infty\,,\qquad n\geqq 0\,. \tag{8.9}$$

To this end, first note that $\tilde{u}_{5\varepsilon}(s,x)=0$ for all $|s-s^0|<\varepsilon$ and $x\in\boldsymbol{R}^N$. Hence

$$\tilde{u}_\tau(s,x)=-\int_\tau^{5\varepsilon}\frac{\partial\tilde{u}_\sigma}{\partial\sigma}(s,x)d\sigma\,,\qquad \tau\in(0,5\varepsilon]\quad\text{and}\quad|s-s^0|<\varepsilon\,. \tag{8.10}$$

We next compute $\partial\tilde{u}_\sigma/\partial\sigma$. First:

$$\frac{\partial\rho_\sigma(\xi)}{\partial\sigma}=-\frac{1}{\sigma}\rho_\sigma(\xi)-\frac{\xi}{\sigma}\left(\frac{\partial}{\partial\xi}\rho_\sigma\right)(\xi)=-\frac{\partial}{\partial\xi}\bar{\rho}_\sigma(\xi)$$

where $\bar{\rho}_\sigma(\xi)\equiv(\xi/\sigma^2)\rho(\xi/\sigma)$. Thus, by (8.1):

$$\frac{\partial}{\partial\sigma}(\rho_\sigma(t-s)q(t-s,x,y))$$

$$=-\frac{\partial}{\partial t}(\bar{\rho}_\sigma(t-s)q(t-s,x,y))+\boldsymbol{L}_y^*(\bar{\rho}(t-s)q(t-s,x,y))\,;$$

and so

$$\frac{\partial\tilde{u}_\sigma}{\partial\sigma}(s,x)=\left[\left(\frac{\partial}{\partial s}+\boldsymbol{L}\right)\tilde{u}\right](\bar{\rho}_\sigma(\cdot-s)q(\cdot-s,x,*)]$$

$$=\tilde{f}_\sigma(s,x)+\tilde{v}_\sigma(s,x)\,,$$

where

$$\tilde{f}_\sigma(s,x)=\int_{R^1\times R^N}\bar{\rho}_\sigma(t-s)\tilde{f}(t,y)q(t-s,x,y)dtdt$$

$$=\int_{R^1\times R^N}\bar{\rho}_\sigma(t)\tilde{f}(s+t,y)q(t,x,y)dtdy\,,$$

and

$$\tilde{v}_\sigma(s,x)=\tilde{v}(\bar{\rho}_\sigma(\cdot-s)q(\cdot-s,x,*))\,.$$

By (8.3), $\sup_{0<\sigma\leqq 5\varepsilon}\|\tilde{f}_\sigma\|_{C_b^n(R^1\times R^N)}<\infty$, $n\geqq 0$. Thus, in view of (8.10), (8.9) will be proved once we show that

$$\sup_{0<\sigma\leqq 5\varepsilon}\|\tilde{v}_\sigma\|_{C_b^n((s^0-\varepsilon,s^0+\varepsilon)\times B(x^0,\varepsilon))}<\infty\,,\quad n\geqq 0\,. \tag{8.11}$$

To prove (8.11), choose $\psi\in C_b^\infty(\boldsymbol{R}^1\times\boldsymbol{R}^N)$ so that $0\leqq\psi\leqq 1$, $\psi=0$ in a neighborhood of $[s^0-2\varepsilon,s^0+2\varepsilon]\times\overline{B(x^0,2\varepsilon)}$, and $\psi=1$ off of $(s^0-5\varepsilon/2,s^0+5\varepsilon/2)\times B(x^0,5\varepsilon/2)$. Then, because $\mathrm{supp}(\tilde{v})\cap[(s^0-3\varepsilon,s^0+3\varepsilon)\times B(x^0,3\varepsilon)]=\emptyset$, $\tilde{v}_\sigma(s,x)=$

$\tilde{v}(\Psi_\sigma(s, x, \cdot, *))$, where

$$\Psi_\sigma(s, x; t, y) = \bar{\rho}_\sigma(t-s)\phi(t, y)q(t-s, x, y) .$$

Thus, (8.11) will follow from:

$$\sup_{0<\sigma\leqq 5\varepsilon} \|\Psi_\sigma\|_{C_b^n([(s^0-\varepsilon, s^0+\varepsilon)\times B(x^0,\varepsilon)]\times[R^1\times R^N])} < \infty, \quad n \geqq 0 . \tag{8.12}$$

Because of (8.8),

$$\sup_{\varepsilon/2\leqq\sigma\leqq 5\varepsilon} \|\Psi_\sigma\|_{C_b^n((R^1\times R^N)^2)} < \infty , \quad n \geqq 0 .$$

Now suppose that $0<\sigma<\varepsilon/2$ and that $|s-s^0| \vee |x-x^0|_{R^N} < \varepsilon$. Note that $\Psi_\sigma(s, x; t, y)=0$ if either $t-s \notin (\sigma, 2\sigma)$ or $\|y-x^0\|_{R^N} \leqq 2\varepsilon$. Indeed: $\bar{\rho}_\sigma(t-s)=0$ if $(t-s) \notin (\sigma, 2\sigma)$, $\phi(t, y)=0$ if $|t-s^0| \vee \|y-x^0\|_{R^N} \leqq 2\varepsilon$, and $|t-s^0|>2\varepsilon$ implies $t-s>\varepsilon>2\sigma$. On the other hand, if $t-s\in(\sigma, 2\sigma)$ and $\|y-x^0\|_{R^N} \geqq 2\varepsilon$, then $\|y-x\|_{R^N} \geqq \varepsilon$ and so, by (8.8), for each $n\geqq 0$ and $\nu>0$ there is a $C_n \in (0, \infty)$ such that

$$|D_t^m D_x^\alpha D_y^\beta q(t-s, x, y)| \leqq C_n |t-s|^{m+1} \leqq 2^{m+1} C_n \sigma^{m+1} .$$

Since $\|D_t^m \bar{\rho}_\sigma\|_{C_b^m(R^1)} \leqq \|\bar{\rho}\|_{C_b^m(R^1)} \sigma^{-m-2}$, (8.12) is now proved. Q. E. D.

(8.13) THEOREM. *Let $\{V_0, \cdots, V_d\} \subseteqq C_b^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ and define $\Delta(\cdot, x)$, $x \in \boldsymbol{R}^N$, accordingly as in* (3.16). *Assume that there is a non-decreasing $\rho: (0, 1] \to (0, \infty)$ such that $\|1/\Delta(t, x)\|_{L^p(\mathscr{W})} \leqq M_p/\rho(t)$, $(t, x) \in (0, 1]\times \boldsymbol{R}^N$ for each $p\in[1, \infty)$ and some $M_p \in (0, \infty)$. If $t\log(1/\rho(t)) \to 0$ as $t \downarrow 0$, then for each $c \in C_b^\infty(\boldsymbol{R}^N)$ the operator ${}^c\boldsymbol{L} = 1/2 \sum_{k=1}^d V_k^2 + V_0 + c$ is parabolic hypoelliptic on $\boldsymbol{R}^N$.*

PROOF. Define ${}^cP(T, x, \cdot)$ as in (3.5), $\{{}^cP_T : T>0\}$ as in (3.7), and $\{{}^c\tilde{P}_T : T>0\}$ as in Theorem (3.14). By Theorem (3.14), for each $n\geqq 0$ and any bounded $B \subseteqq C_b^n(\boldsymbol{R}^N)$, both $\{{}^cP_t\phi : 0<t\leqq 1$ and $\varnothing \in B\}$ and $\{{}^cP_t\phi : 0<t\leqq 1$ and $\varnothing\in B\}$ are bounded subsets of $C_b^n(\boldsymbol{R}^N)$. Moreover, by (3.8)

$$P_T\phi - \phi = \int_0^T {}^cP_t {}^c\boldsymbol{L}\phi dt , \quad T>0 \quad \text{and} \quad \phi \in C_\uparrow^\infty(\boldsymbol{R}^N) , \tag{8.14}$$

and by (3.15)

$$\int_{R^N} \phi\, {}^cP_T \psi dx = \int \psi\, {}^c\tilde{P}_T \phi dx , \qquad T>0 , \tag{8.15}$$

for all $\phi, \psi \in \mathcal{S}(\boldsymbol{R}^N)$. Finally, by Theorem (3.17), there is a ${}^cp \in C^\infty((0, \infty) \times \boldsymbol{R}^N \times \boldsymbol{R}^N)^+$ such that: ${}^cP(T, x, dy) = {}^cp(T, x, y)dy$, $(T, x) \in (0, \infty)\times \boldsymbol{R}^N$; and, for each $n\geqq 0$, there exist C_n, λ_n, and ν_n from $(0, \infty)$ for which

$$(8.16)\qquad \max_{m+|\alpha|+|\beta|\leqq n} |D_t^m D_x^\alpha D_y^\beta {}^c p(t, x, y)| \leqq C_n \exp(-\lambda_n \|y-x\|_{R^N}^2/t)/(\rho(t)\wedge t)^{\nu n}$$

so long as $(t, x, y)\in(0, 2]\times \boldsymbol{R}^N\times\boldsymbol{R}^N$ (cf. (3.20)).

Clearly ${}^cP_t\phi(x)=\int\phi(y){}^cp(t,x,y)dy$ and, by (8.15), ${}^c\tilde{P}_t\phi(y)=\int\phi(x){}^cp(t,x,y)dx$. Moreover, from (8.14), we have that $(\partial^c p/\partial t)(t, x, y)=({}^c\boldsymbol{L}_y^{*c}p)(t, x, y)$ in $(0, \infty)\times\boldsymbol{R}^N\times\boldsymbol{R}^N$. Finally, from (8.16) and $\lim_{t\downarrow 0} t\log(1/\rho(t))=0$, we see that, for each $n\geqq 0$, $\varepsilon>0$, and $\nu>0$, (8.8) holds with ${}^cp$ in place of q. Thus, by Theorem (8.6), ${}^c\boldsymbol{L}$ is parabolic hypoelliptic on $\boldsymbol{R}^N$. Q. E. D.

(8.17) COROLLARY. *Let* $\{V_0, \cdots, V_d\}\subseteq C^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ *and define* $\mathcal{V}_L(x)$, $L\geqq 1$ *and* $x\in\boldsymbol{R}^N$, *as in Corollary* (3.25). *Assume that there exists a non-decreasing* $h:(0,\infty)\to(0,1]$, *satisfying* $h(0)=\lim_{\lambda\downarrow 0}h(\lambda)=0$ *and* $\lim_{\lambda\downarrow 0}\lambda^\rho\log(1/h(\lambda))=0$ *for each* $\rho\in(0,1]$, *and a* $\phi\in C^\infty(\boldsymbol{R}^N)$ *such that* $\mathcal{V}_L(\cdot)\geqq h\circ\phi$ *for some* $L\geqq 1$. *If* $\sigma^2_{(L)}(\phi)$ *is defined as in* (7.1) *and* W *is an open set in* $\boldsymbol{R}^N$ *such that* $\sigma^2_{(L)}(\phi)\geqq 1$ *on* W, *then for each* $c\in C^\infty(\boldsymbol{R}^N)$ *the operator* ${}^c\boldsymbol{L}=\frac{1}{2}\sum_1^d V_k^2+V_0+c$ *is parabolic hypoelliptic on* W.

PROOF. Suppose for each $x^0\in W$ we can find an $\varepsilon>0$ and an operator $\hat{\boldsymbol{L}}$ such that $\hat{\boldsymbol{L}}$ equals ${}^c\boldsymbol{L}$ on $C_0^\infty(B(x^0,\varepsilon))$ and $\hat{\boldsymbol{L}}$ is parabolic hypoelliptic on $\boldsymbol{R}^N$. Then, clearly, ${}^c\boldsymbol{L}$ is parabolic hypoelliptic on W.

Given $x^0\in W$, choose $\varepsilon>0$ so that $\overline{B(x^0,4\varepsilon)}\subseteq W$. Choose $\eta_1\in C_0^\infty(B(x^0,2\varepsilon))$ so that $\eta_1=1$ on a neighborhood of $\overline{B(x^0,\varepsilon)}$ and $\eta_2\in C^\infty(B(x^0,3\varepsilon))$ so that $\eta_2=1$ on a neighborhood of $\overline{B(x^0,2\varepsilon)}$. Set

$$\hat{V}_k=\eta_2 V_k\,,\qquad 0\leqq k\leqq d\,,$$

$$\hat{V}_{k+i}=(1-\eta_1)\frac{\partial}{\partial x_i}\,,\qquad 1\leqq i<N\,,$$

and

$$\hat{c}=\eta_1 c\,;$$

and define $\mathcal{V}_L$ and $\sigma^2_{(L)}(\phi)$ relative to $\{\hat{V}_0,\cdots,\hat{V}_{d+N}\}$. Clearly, $\hat{\mathcal{V}}_L\geqq\mathcal{V}_L$ and $\hat{\sigma}^2_{(L)}(\phi)\geqq\sigma^2_{(L)}(\phi)$ on $B(x^0,2\varepsilon)$. Moreover, $\hat{\mathcal{V}}_L\geqq\hat{\mathcal{V}}_1\geqq 1$ off of $B(x^0,2\varepsilon)$. Thus, $\hat{\mathcal{V}}_L\geqq h\circ\phi$ everywhere, $\hat{\sigma}^2_{(L)}(\phi)\geqq 1$ on $B(x,2\varepsilon)$, and $\hat{\mathcal{V}}_L\geqq 1$ off of $B(x^0,2\varepsilon)$. Combining these with Corollaries (7.4) and (3.25), we conclude that, for each $p\in[1,\infty)$, there is an $M_p\in(0,\infty)$ such that

$$\|1/\hat{\Delta}(T,x)\|_{L^p(\mathcal{W})}\leqq M_p\exp(1/T^{1/2})\,,\qquad (T,x)\in(0,1]\times\boldsymbol{R}^N\,,$$

where $\hat{\Delta}(\cdot, x)$ is defined relative to $\{\hat{V}_0, \cdots, \hat{V}_{d+N}\}$. Hence, by Theorem (8.6), ${}^{\hat{c}}\hat{\boldsymbol{L}}=\frac{1}{2}\sum_{k=1}^{d+N}\hat{V}_k^2+\hat{V}_0+\hat{c}$ is parabolic hypoelliptic on $\boldsymbol{R}^N$. Since ${}^c\boldsymbol{L}={}^{\hat{c}}\hat{\boldsymbol{L}}$ on $C_0^\infty(B(x^0, \varepsilon))$, the proof is now complete. Q. E. D.

(8.18) COROLLARY. *Let $\{V_0, \cdots, V_d\}\subseteqq C^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ and define $\overline{\mathcal{V}}_L(x)=\inf_{\eta\in S^{N-1}}(\sum_{k=0}^{d}\sum_{\|\alpha\|\leqq L-1}(V_k)_{\langle\alpha\rangle}(x), \eta)_{\boldsymbol{R}^N}^2)\wedge 1$ for $L\geqq 1$ and $x\in\boldsymbol{R}^N$. Assume that there exists a non-decreasing $h:(0,\infty)\to(0,1]$, satisfying $h(0)\equiv\lim_{\lambda\downarrow 0}h(\lambda)=0$ and $\lim_{\lambda\downarrow 0}\lambda^\rho\log(1/h(\lambda))=0$ for each $\rho\in(0,1]$, and a $\phi\in C^\infty(\boldsymbol{R}^N)$ such that $\overline{\mathcal{V}}_L\geqq h\circ\phi$. If $\sigma_{(L)}^2(\phi)$ is defined as in (7.1) and W is an open set in $\boldsymbol{R}^N$ such that $\sigma_{(L)}^2(\phi)\geqq 1$ on W, then for each $c\in C_b^\infty(\boldsymbol{R}^N)$ the operator ${}^c\boldsymbol{L}=\frac{1}{2}\sum_1^d V_k^2+V_0+c$ is hypoelliptic on W. In particular, if* Lie$(\{V_0, \cdots, V_d\})$ *has dimension N at each $x\in W$, then ${}^c\boldsymbol{L}$ is hypoelliptic on W.*

PROOF. The last assertion is obviously a consequence of the first, since Lie$(\{V_0, \cdots, V_d\})(x)=\boldsymbol{R}^N$ implies that $\overline{\mathcal{V}}_L(x)>0$ for some $L\geqq 1$.

To prove the first assertion, we use the same device as we used in Section 6). Namely, set $\rho(\xi)=2+\sin\xi$, $\xi\in\boldsymbol{R}^1$, and define $\{\tilde{V}_0, \cdots, \tilde{V}_{d+1}\}\subseteqq C^\infty(\boldsymbol{R}^{N+1}, \boldsymbol{R}^{N+1})$ by:

$$\tilde{V}_0(z)=\rho(\xi)\sum_{i=1}^N V_0^i(x)\frac{\partial}{\partial x_i},$$

$$\tilde{V}_k(z)=\rho(\xi)^{1/2}\sum_{i=1}^N V_k^i(x)\frac{\partial}{\partial x_i}, \qquad 1\leqq k\leqq d,$$

and

$$\tilde{V}_{d+1}(z)=\frac{\partial}{\partial\xi}$$

for $z=(x,\xi)\in\boldsymbol{R}^N\times\boldsymbol{R}^1$. Also, set $\tilde{c}((x,\xi))=\rho(\xi)c(x)$ for $(x,\xi)\in\boldsymbol{R}^N\times\boldsymbol{R}^1$. Clearly, if ${}^{\tilde{c}}\tilde{\boldsymbol{L}}=\frac{1}{2}\sum_1^{d+1}\tilde{V}_k^2+\tilde{V}_0+\tilde{c}$ is hypoelliptic on $W\times\boldsymbol{R}^1$, then ${}^c\boldsymbol{L}$ is hypoelliptic on W. At the same time, if $\overline{\tilde{\mathcal{V}}}_L(z)$ is defined as in Corollary (3.25) relative to $\{\tilde{V}_0, \cdots, \tilde{V}_{d+1}\}$, then it is easy to find an $\varepsilon>0$ such that $\overline{\tilde{\mathcal{V}}}_L((x,\xi))\geqq\varepsilon\overline{\mathcal{V}}_L(x)$, $(x,\xi)\in\boldsymbol{R}^N\times\boldsymbol{R}^1$. Finally, if $\tilde{\phi}(x,\xi)=\phi(x)$, $(x,\xi)\in\boldsymbol{R}^N\times\boldsymbol{R}^1$, and $\tilde{\sigma}_{(L)}^2(\phi)$ is defined as in (7.1) relative to $\{\tilde{V}_0, \cdots, \tilde{V}_{d+1}\}$, then $\tilde{\sigma}_{(L)}^2(\tilde{\phi})(x,\xi)\geqq\sigma_{(L)}^2(\phi)(x)$, $(x,\xi)\in\boldsymbol{R}^N\times\boldsymbol{R}^1$. Thus, Corollary (8.17) applies to ${}^{\tilde{c}}\tilde{\boldsymbol{L}}$ on $W\times R^1$. Q. E. D.

(8.19) REMARK. The last assertion in Corollary (8.18) is the criterion for hypoellipticity discovered by L. Hörmander [3]. In the case when the V_k's are real analytic, the reasoning given at the beginning of Section 7) shows that Hörmander's criterion is necessary as well as sufficient for hypoellipticity of ${}^c\boldsymbol{L}$ on W. When the V_k's are only C^∞, the criterion given in Corollary (8.18) clearly shows that Hörmander's criterion is no longer necessary for hypoellipticity. In fact, in general Hörmander's criterion guarantees that ${}^c\boldsymbol{L}$ is not only hypoelliptic but also *subelliptic* (indeed, subellipticity is the route taken by Hörmander to prove hypoellipticity); and, at least when ${}^c\boldsymbol{L}$ is formally self-adjoint, results of C. Fefferman and D. Phong [2] show that Hörmander's condition is necessary for subellipticity. From the point of view taken in this section, the distinction between subellipticity and hypoellipticity lies in the way in which the fundamental solution to $\partial u/\partial t=\boldsymbol{L}u$ explodes as $t\downarrow 0$: if it explodes polynomially, then $\boldsymbol{L}$ is subelliptic; if the explosion is faster than polynomial, then $\boldsymbol{L}$ may be hypoelliptic but the Fefferman-Phong result indicates that it will not be subelliptic.

We conclude this section with the examination of a special situation in which we can give a necessary and sufficient condition for the hypoellipticity of an operator having C^∞-coefficients. To be precise, let $\sigma\in C_b^\infty(\boldsymbol{R}^1)$ be a function with the properties that: $\sigma(\xi)=0$ if and only if $\xi=0$, $\sigma(\cdot)^2$ is non-decreasing on $[0,\infty)$, and $\sigma(\cdot)^2$ is even on $\boldsymbol{R}^1$. Given σ, define

$$\boldsymbol{L}_\sigma=1/2\left(\frac{\partial^2}{\partial x_1^2}+\left(\sigma(x_1)\frac{\partial}{\partial x_2}\right)^2+\frac{\partial^2}{\partial x_3^2}\right). \tag{8.20}$$

Our goal is to show that $\boldsymbol{L}_\sigma$ hypoelliptic on $\boldsymbol{R}^3$ if and only if:

$$\lim_{\xi\downarrow 0}\xi\log(1/|\sigma(\xi)|)=0. \tag{8.21}$$

The reason why we can get such a precise result in this situation is due to the simplicity of the diffusion generated by $\boldsymbol{L}_\sigma$. Indeed, the stochastic integral representation of this diffusion is:

$$X_1(T,x)=x_1+\theta_1(T)$$

$$X_2(T,x)=x_2+\int_0^T\sigma(x_1+\theta_1(t))d\theta_2(t)$$

$$X_3(T,x)=x_3+\theta_3(t).$$

Using (2.5), one easily sees that

$$\Delta(T,x)\geqq T^2\sum(T,x_1) \tag{8.22}$$

where

$$\sum(T,x_1)\equiv\int_0^T\sigma(x_1+\theta_1(t))^2dt$$

(8.23) LEMMA. *The condition* (8.21) *is equivalent to*

$$\lim_{T\downarrow 0} T\log\Big(\sup_{x_1\in\boldsymbol{R}^1}\|1/\Sigma(T,x_1)\|_{L^p(\mathcal{W})}\Big)=0\,,\qquad p\in[1,\infty)\,. \tag{8.24}$$

PROOF. We begin by recording some facts about Brownian motion. First, for any $\lambda>0$, $x_1\in(-\lambda,\lambda)$, $T>0$, and $\Gamma\in\mathscr{B}_{(-\lambda,\lambda)}$:

$$\mathcal{W}\Big(\{X_1(T,x)\in\Gamma\}\cap\Big\{\sup_{0\le t\le T}|X_1(t,x)|<\lambda\Big\}\Big)=\int_\Gamma \hat{r}(T,x_1,y_1)dy\,, \tag{8.25}$$

where

$$\hat{r}(T,x_1,y_1)=\frac{1}{\lambda}\sum_{n=1}^{\infty}\exp(-n^2\pi^2T/8\lambda^2)\cos(n\pi x_1/2\lambda)\cos(n\pi y_1/2\lambda)\,. \tag{8.26}$$

In particular, there exists a $K\in(0,\infty)$ such that

$$\mathcal{W}\Big(\sup_{0<t\le T}|\theta_1(t)|<\lambda\Big)\le K\exp(-\varepsilon T/\lambda^2),\quad \lambda>0\quad\text{and}\quad T>0\,, \tag{8.27}$$

where $\varepsilon=\pi^2/8$. Also, an easy comparison argument using $(t,x_1)\to e^{\pi^2t/8\lambda^2}\cos(\pi x_1/2\lambda)$ yields

$$\mathcal{W}\Big(\sup_{0\le t\le T}|\theta_1(t)|<\lambda\Big)\ge\exp(-\varepsilon T/\lambda^2),\quad \lambda>0\quad\text{and}\quad T>0\,. \tag{8.28}$$

Finally, we recall the familiar estimate

$$\mathcal{W}\Big(\sup_{0\le t\le T}|\theta_1(t)|>\lambda\Big)\le 2\exp(-\lambda^2/2T),\quad \lambda>0\quad\text{and}\quad T>0\,. \tag{8.29}$$

Now, assume that (8.24) holds. By (8.28):

$$\begin{aligned}E^{\mathcal{W}}[(1/\Sigma(T,0))^p]&\ge T^{-p}\sigma(\lambda)^{-2p}\mathcal{W}\Big(\sup_{0\le t\le T}|\theta_1(t)|<\lambda\Big)\\&\ge(T\sigma(\lambda)^2)^{-p}\exp(-\varepsilon T/\lambda^2)\,.\end{aligned}$$

Taking $\lambda=T$, we see that (8.24) implies that:

$$\varlimsup_{T\downarrow 0} T\log(1/|\sigma(T)|)\le\varepsilon/2p\,,\qquad p\in[1,\infty)\,.$$

Hence, (8.24) implies (8.21).

We now turn to the proof that (8.21) implies (8.24). Given $\lambda>0$, set $\tau_\lambda(x_1)=\inf\{t\ge 0:|x_1+\theta_1(t)|\ge 2\lambda\}$, $x_1\in\boldsymbol{R}^1$, and $\zeta_\lambda=\inf\{t\ge 0:|\theta_1(t)|\ge\lambda\}$. Observe that:

$$\mathcal{W}(\tau_\lambda(x_1)>T/2)\le\mathcal{W}\Big(\sup_{0\le t\le T/2}|\theta_1(t)|<4\lambda\Big)\le K\exp(-\varepsilon T/32\lambda^2)$$

for all $x_1\in\boldsymbol{R}^1$, $\lambda\in(0,1]$, and $T>0$. Thus, for $R>0$, $\lambda\in(0,1]$, and $T>0$:

$$\mathcal{W}(\Sigma(T, x_1)\leqq 1/R)\leqq \mathcal{W}\left(\int_{\tau_\lambda(x_1)}^{\tau_\lambda(x_1)+T/2}\sigma(x_1+\theta_1(t))^2dt\leqq 1/R,\ \tau_\lambda(x_1)\leqq T/2\right)$$
$$+K\exp(-\varepsilon T/32\lambda^2)\,.$$

By the strong Markov property:

$$\mathcal{W}\left(\int_{\tau_\lambda(x_1)}^{\tau_\lambda(x_2)+T/2}\sigma(x_1+\theta_1(t))^2dt\leqq 1/R,\ \tau_\lambda(x_1)\leqq T/2\right)$$
$$\leqq \sup_{|\xi|\geqq 2\lambda}\mathcal{W}(\Sigma(T/2, \xi)\leqq 1/R)$$
$$\leqq \mathcal{W}((T/2)\wedge\zeta_\lambda\leqq 1/R\sigma(\lambda)^2)\,.$$

Hence, we conclude from (8.29) and the above, that:

$$\text{(8.30)}\qquad \mathcal{W}(\Sigma(T, x_1)\leqq 1/R)\leqq 2\exp(-\lambda^2\sigma(\lambda)^2R/2)+K\exp(-\varepsilon T/32\lambda^2)$$

for all $\lambda\in(0,1]$, $T>0$, and $R\geqq 2/T\sigma(\lambda)^2$. Next, let $L\geqq 1$ be given. By (8.21), we can choose $\delta_L>0$ so $\lambda^2\sigma(\lambda)^2\geqq\delta_L\exp(-1/2L\lambda)$ for $\lambda\in(0,1]$. Thus, if $R\geqq e$ and $\lambda=1/L\log R$, then

$$R\lambda^2\sigma(\lambda)^2\geqq\delta_L R\exp(-\log R/2)=\delta_L R^{1/2}\,.$$

Hence, by (8.30):

$$\text{(8.31)}\quad \mathcal{W}(\Sigma(T, x_1)\leqq 1/R)\leqq 2\exp(-\delta_L R^{1/2}/2)+K\exp(-\varepsilon T(L\log R)^2/32)$$

for $T>0$ and $R\geqq(2/\delta_L T)^2\vee e$. In particular:

$$E^{\mathcal{W}}[(1/\Sigma(T, x_1))^{-p}]=p\int_0^\infty R^{p-1}\mathcal{W}(\Sigma(T, x_1)\leqq 1/R)dR$$
$$\leqq((2/\delta_L T)^2\vee e)^p+2p\int_0^\infty R^{p-1}\exp(-\delta_L R^{1/2}/2)dR$$
$$+pK\int_1^\infty R^{p-1}\exp(-\varepsilon TL^2(\log R)^2/32)dR$$
$$\leqq((2/\delta_L T)^2\vee e)^p+K_1p/\delta_L^{2p}+(K_2p/T^{1/2}L)\exp(\alpha p^2/TL^2)$$

for appropriately chosen positive numbers K_1, K_2 and α. From this we conclude that

$$\varlimsup_{T\downarrow 0}T\log\left(\sup_{x_1\in R^1}\|1/\Sigma(T, x_1)\|_{L^p(\mathcal{W})}\right)\leqq\alpha p/L^2\,.$$

Since L was an arbitrary element of $[1,\infty)$, we have now derived (8.24) from (8.21). Q.E.D.

In view of (8.22), Lemma (8.23), and Theorem (8.13), it is now clear that $\boldsymbol{L}_\sigma$ is hypoelliptic on $\boldsymbol{R}^3$ whenever (8.21) holds. Also, we now know that the converse statement will be proved once we show how to deduce (8.24) from the hypoellipticity of L_σ.

Given $x=(x_1, x_2)\in\boldsymbol{R}^2$, set $\bar{X}(\cdot, \bar{x})=(X_1(\cdot, (\bar{x}, 0)), X_2(\cdot, (\bar{x}, 0)))$ and $\bar{P}(t, \bar{x}, \cdot)=\mathcal{W}\circ(\bar{X}(t, \bar{x}))^{-1}$, $t>0$. Let $\eta\in C_0^\infty(\boldsymbol{R}^1)$ be chosen so that $0\leqq\eta\leqq 1$ and $\eta=1$ on $[-3, 3]$, and define $\bar{\mu}_0(d\bar{x})=(\eta(x_1)dx_1)\times\delta_0(dx_2)$ where δ_0 denotes the unit mass at 0. For $t>0$, define

$$\bar{\mu}(t, \cdot)=\int\bar{P}\,(t, \bar{x}, \cdot)\mu_0(d\bar{x})\,.$$

Finally, let $\bar{L}_\sigma=1/2((\partial^2/\partial x_1^2)+(\sigma(x_1)(\partial/\partial x_2))^2)$.

(8.32) LEMMA. *The map $t\in(0, \infty)\to\bar{\mu}_t(t, \cdot)\in\mathcal{S}'(\boldsymbol{R}^2)$ is differentiable and satisfies:*

$$\lim_{t\downarrow 0}\bar{\mu}(t, \cdot)=\bar{\mu}_0 \tag{8.33}$$

$$\frac{\partial\bar{\mu}}{\partial t}(t, \cdot)=L_\sigma\bar{\mu}(t, \cdot)\,, \qquad t>0\,,$$

in $\mathcal{S}'(\boldsymbol{R}^2)$. Furthermore, $\mu\in C^\infty((0, \infty)\times\boldsymbol{R}_*^2)$, where $\boldsymbol{R}_*^2=\{(x_1, x_2)\in\boldsymbol{R}^2: x_1\neq 0\}$. Finally, for each $p\in[1, \infty)$ and $\lambda>0$ there is a $K_p(\lambda)\in(0, \infty)$ such that:

$$\sup_{|x_1|\geqq\lambda}\|1/\Sigma(t, x_1)\|_{L^p(\mathcal{W})}\leqq K_p(\lambda)/(t\wedge 1), \quad t>0: \tag{8.34}$$

and for each $n\geqq 0$:

$$\frac{\partial^{2n}\bar{\mu}}{\partial x_2^{2n}}(t, x_1, 0)=\beta_n E^{\mathcal{W}}[(\Sigma(t, x_1))^{-(2n+1)/2}\eta(x_1+\theta_1(t))], \tag{8.35}$$
$$(t, x_1)\in(0, \infty)\times\boldsymbol{R}^1\backslash\{0\}$$

where $\beta_n=(-1)^n\dfrac{(2n)!}{n!}$.

PROOF. First, note that by (3.8):

$$\int\phi(\bar{y})\bar{P}(t, \bar{x}, d\bar{y})-\phi(x)=\int_0^t\int\bar{\boldsymbol{L}}_\sigma\phi(\bar{y})\bar{P}(s, \bar{x}, dy)ds$$

for all $\phi\in C_\uparrow^\infty(\boldsymbol{R}^2)$. Thus (8.33) follows from the symmetry of $\bar{\boldsymbol{L}}_\sigma$ on $\mathcal{S}(\boldsymbol{R}^2)$.

Next, observe that the symmetry of $\bar{\boldsymbol{L}}_\sigma$ implies that the semigroups $\{\bar{P}_t: t>0\}$ and $\{\tilde{\bar{P}}_t: t>0\}$, associated with $\bar{X}(\cdot, \bar{x})$ in Theorem (3.14), coincide. Thus the measure $\bar{P}_t(d\bar{x}\times d\bar{y})=d\bar{x}\;\bar{P}(t, x, d\bar{y})$ is symmetric on $\boldsymbol{R}^2\times\boldsymbol{R}^2$. At the same time, by Corollary (3.25), there is a $\bar{p}\in C^\infty((0, \infty)\times\boldsymbol{R}_*^2\times\boldsymbol{R}^2)$ such that $\bar{P}(t, \bar{x}, d\bar{y})=\bar{p}(t, \bar{x}, \bar{y})dy$ for $(t, \bar{x})\in(0, \infty)\times\boldsymbol{R}_*^2$. Clearly the symmetry of $\bar{P}_t(d\bar{x}\times d\bar{y})$ implies that $\bar{p}(t, \bar{x}, \bar{y})=\bar{p}(t, \bar{y}, \bar{x})$ for $(t, \bar{x}, \bar{y})\in(0, \infty)\times\boldsymbol{R}_*^2\times\boldsymbol{R}_*^2$. Hence, if $(t, \bar{y})\in(0, \infty)\times\boldsymbol{R}_*^2$, then

$$\int_{R^1} \eta(x_1)\bar{p}(t,(x_1,0)\bar{y})dx_1 = \int \eta(x_1)\bar{p}(t,\bar{y},(x_1,0))dx_1\,,$$

and so

$$\bar{\mu}(t,\bar{x}) = \int \eta(y_1)\bar{p}(t,\bar{x},(y_1,0))dy_1, \quad (t,\bar{x}) \in (0,\infty)\times \boldsymbol{R}^2\,. \tag{8.36}$$

In particular, since $\bar{p} \in C^\infty((0,\infty)\times \boldsymbol{R}_*^2 \times \boldsymbol{R}^2)$, we now see that $\bar{\mu} \in C^\infty((0,\infty)\times \boldsymbol{R}_*^2)$.

To prove (8.34), simply note that if $\lambda > 0$ and $|x_1| \geqq 2\lambda$, then

$$E^{\mathcal{W}}[(1/\Sigma(t,x_1))^{-p}] \leqq \sigma(\lambda)^{-2p} E^{\mathcal{W}}[(t\wedge \zeta_\lambda)^{-p}]$$

where $\zeta_\lambda = \inf\{t \geqq 0 : |\theta_1(t)| \geqq \lambda\}$. Hence, (8.34) is a consequence of (8.29).

The proof of (8.35) is now quite simple. Indeed, the conditional distribution under $\mathcal{W}$ of $X_1(t,(\bar{x},0))$ given $\theta_1(\cdot)$ is $g(\Sigma(t,x_1,\theta),y_2-x_2)dy_2$, where $g(\tau,\xi) = (2\pi\tau)^{-1/2}\exp(-\xi^2/2\tau)$. Thus, if $\rho \in C_0^\infty(\boldsymbol{R}^1)$ satisfies $\int_{\boldsymbol{R}^1}\rho(\xi)d\xi = 1$ and $\rho_\varepsilon(\xi) = (1/\varepsilon)\rho(\xi/\varepsilon)$, $\varepsilon > 0$ and $\xi \in R^1$, then for $(t,\bar{y}) \in (0,\infty) \in \boldsymbol{R}^2$:

$$\begin{aligned}
&\int \eta(y_1)\bar{p}(t,\bar{x},(y_1,0))dy_1 = \lim_{\varepsilon \downarrow 0}\int \eta(y_1)\rho_\varepsilon(y_2)\bar{p}(t,\bar{x},\bar{y})d\bar{y} \\
&= \lim_{\varepsilon\downarrow 0} E^{\mathcal{W}}\left[\left(\int g(\Sigma(t,x_1),y_2-x_2)\rho_\varepsilon(y_2)dy_2\right)\eta(x_1+\theta_1(t))\right] \\
&= E^{\mathcal{W}}[g(\Sigma(t,x_1),x_2)\eta(x_1+\theta_1(t))]\,,
\end{aligned}$$

where we have used (8.34) in the last step. Combining this with (8.36) and using (8.34) to justify differentiating under the integral sign, we can now easily deduce (8.35). Q. E. D.

Define

$$\gamma(t,x_3) = \mathcal{W}\left(\{x_3+\theta_3(t) \in [2,3)\} \cap \left\{\sup_{0\leqq s\leqq t}|x_3+\theta_3(s)| < 3\right\}\right)$$

on $(0,\infty)\times(-3,3)$. By (8.25),

$$\gamma(t,x_3) = \int_2^3 \hat{\gamma}(t,x_3,y_3)dy_3$$

where $\hat{\gamma}$ is given by (8.26). In particular,

$$\frac{\partial\gamma}{\partial t} = \frac{1}{2}\frac{\partial^2\gamma}{\partial^2 x_3^2} \qquad \text{on } (0,\infty)\times(-3,3) \tag{8.37}$$

and

$$|\gamma(t,x_3)| \leqq K\exp(-\varepsilon t/9), \quad (t,x_3) \in (0,\infty)\times(-3,3)\,. \tag{8.38}$$

Also,

$$(8.39)\qquad 0\leqq\frac{1}{(2\pi t)^{1/2}}\int_2^3 e^{-(\xi-x_3)^2/2t}d\xi-\gamma(t,x_3)\leqq\mathcal{W}\Big(\sup_{0\leqq s\leqq t}|x_3+\theta_3(s)|\geqq 3\Big).$$

Let $\psi\in C_0^\infty((-2,2))$ be chosen so that $\psi=1$ in a neighborhood of $[-1,1]$. Then, in view of the preceding and (8.33), one can easily check that

$$\phi\in\mathcal{S}(\boldsymbol{R}^3)\longrightarrow\int_0^\infty\mu\Big(t,\int_{-2}^2\psi(x_3)\phi(\cdot,x_3)\gamma(t,x_3)dx_3\Big)dt$$

determines an element ν of $\mathcal{S}'(\boldsymbol{R}^3)$ and that $\boldsymbol{L}_\sigma\nu=0$ in $\boldsymbol{R}^2\times(-1,1)$. In particular, if $\boldsymbol{L}_\sigma$ is hypoelliptic in $\boldsymbol{R}^3$, then $\nu\in C^\infty(\boldsymbol{R}^2\times(-1,1))$. Moreover, after combining (8.34), (8.36), (8.38), and (8.39), we see that

$$\frac{\partial^{2n}\nu}{\partial x_2^{2n}}(x_1,0,0)=\beta_n\int_0^\infty E^{\mathcal{W}}[(\Sigma(t,x_1))^{-(2n+1)/2}\eta(x_1+\theta_1(t))]\gamma(t,0)dt$$

for all $n\geqq 0$ and $x_1\in(-1,1)\backslash\{0\}$. Thus, for each $p\in[1,\infty)$:

$$(8.40)\qquad M_p\equiv\sup_{|x_1|\leqq 1}\int_0^\infty E^{\mathcal{W}}[(\Sigma(t,x_1))^{-p}\eta(x_1+\theta_1(t))]\gamma(t,0)dt<\infty\,.$$

In order to pass from (8.40) to (8.24), note that if $|x_1|\leqq 1$ and $\tau(x_1)=\inf\{t\geqq 0:|x_1+\theta(t)|\geqq 2\}$, then

$$\begin{aligned}
&E^{\mathcal{W}}[(\Sigma(t,x_1))^{-p}(1-\eta(x_1+\theta_1(t)))]\\
&\leqq E^{\mathcal{W}}[(\Sigma(t,x_1))^{-p},\ |x_1+\theta_1(t)|\geqq 3]\\
&\leqq E^{\mathcal{W}}\Big[\Big(\int_{\tau(x_1)}^t\sigma(x_1+\theta_1(t))^2dt\Big)^{-p},\ \{\tau(x_1)<\infty\}\cap\{|x_1+\theta_1(t)|>3\}\Big]\\
&\leqq E^{\mathcal{W}}[(\Sigma(t/2,2))^{-p}]+\mathcal{W}(\{\tau(x_1)\geqq t/2\}\cap\{|x_1+\theta_1(t)|>3\}]\\
&\leqq 2^pK_p(2)t^{-p}+\mathcal{W}\Big(\sup_{0\leqq s\leqq t/2}|\theta_1(s)|\geqq 1\Big)\\
&\leqq 2^p\kappa_p(2)t^{-p}+2e^{-1/t}\,,
\end{aligned}$$

where we have used (8.34) and (8.29). Thus, for each $p\in[1,\infty)$ there is a $K_p\in(0,\infty)$ such that

$$\begin{aligned}
&E^{\mathcal{W}}[(\Sigma(t,x_1))^{-p}]\\
&\leqq E^{\mathcal{W}}[(\Sigma(t,x_1))^{-p}\eta(x_1+\theta_1(t))]+K_pt^{-p},\ (t,x_1)\in(0,1]\times(1,1)\,.
\end{aligned}$$

At the same time, by (8.39) and (8.29), there is a $\delta>0$ such that

$$\gamma(t,0)\geqq\delta\,e^{-2/t}\,,\qquad t\in(0,1]\,.$$

Combining these, we arrive at:

$$
\begin{aligned}
E^{\mathscr{W}}[(\Sigma(t, x_1))^{-p}] &\leqq \frac{1}{t}\int_t^{2t} E^{\mathscr{W}}[(\Sigma(s, x_1))^{-p}]ds \\
&\leqq \frac{e^{2/t}}{\delta t}\int_t^{2t} E^{\mathscr{W}}[(\Sigma(s, x_1))^{-p}\eta(x_1+\theta_1(s))]\gamma(s, 0)ds + K_p\int_t^{2t} s^{-p}ds \\
&\leqq M_p e^{2/t}/\delta t + (K_p/(p-1))t^{-(p-1)}
\end{aligned}
$$

for $p\in(1, \infty)$ and $(t, x_1)\in(0, 1/2)\times(-1, 1)$. In conjunction with (8.34), this leads to

$$\varlimsup_{t\downarrow 0} t \log\Big(\sup_{x_1\in \boldsymbol{R}^1} E^{\mathscr{W}}[(\Sigma(t, x_1))^{-p}]\Big) \leqq 2$$

for all $p\in[1, \infty)$. Hence, for any $p\in[1, \infty)$ and $n\geqq 1$:

$$
\begin{aligned}
&\varlimsup_{t\downarrow 0} t \log\Big(\sup_{x_1\in \boldsymbol{R}^1} E^{\mathscr{W}}[(\Sigma(t, x_1))^{-p}]\Big) \\
&\leqq \frac{1}{n}\varlimsup_{t\downarrow 0} t \log\Big(\sup_{x_1\in \boldsymbol{R}^1} E^{\mathscr{W}}[(\Sigma(t, x_1))^{-np}]\Big) \leqq \frac{2}{n},
\end{aligned}
$$

and so (8.24) follows upon letting $n\to\infty$.

We state our conclusions as a theorem.

(8.41) THEOREM. *Let $\sigma\in C_b^\infty(\boldsymbol{R}^1)$ be a function which satisfies: $\sigma(\xi)=0$ if and only if $\xi=0$, $\sigma(\xi)^2$ is non-decreasing in $\xi\in[0, \infty)$, and $\sigma(-\xi)^2=\sigma(\xi)^2$ for all $\xi\in\boldsymbol{R}^1$. Define $\boldsymbol{L}_\sigma=1/2(\partial^2/\partial x_1^2+(\sigma(x_1)(\partial/\partial x_2))^2+\partial^2/\partial x_3^2)$ on $C^\infty(\boldsymbol{R}^3)$. Then $\boldsymbol{L}_\sigma$ is hypoelliptic on $\boldsymbol{R}^3$ if and only if $\lim_{\xi\downarrow 0}\xi\log(1/|\sigma(\xi)|)=0$.*

Appendix.

Our whole program rests on certain facts about Brownian stochastic integrals. The purpose of this appendix is to collect these facts together in one place so that the development of the main program need not be interrupted each time we need one of them. We begin by stating the results toward which our efforts will be directed.

Let $\mathscr{A}=\{\emptyset\}\cup\bigcup_{l=1}^{\infty}(\{0, \cdots, d\})^l$. Given $\alpha\in\mathscr{A}$, set

(A.1) $$|\alpha|=\begin{cases} 0 & \text{if } \alpha=\emptyset \\ l & \text{if } \alpha\in(\{0, \cdots, d\})^l \end{cases}$$

and

(A.2) $$\|\alpha\|=\begin{cases} 0 & \text{if } \alpha=\emptyset \\ |\alpha|+\operatorname{card}\{j: \alpha_j=0\} & \text{if } |\alpha|\geqq 1. \end{cases}$$

Also, if $\alpha=(\alpha_1, \cdots, \alpha_l)\in\mathscr{A}\backslash\{\emptyset\}$, define

$$\alpha_* = \alpha_l$$

and

$$\alpha' = \begin{cases} \emptyset & \text{if } l=1 \\ (\alpha_1, \cdots, \alpha_{l-1}) & \text{if } l \geqq 2 . \end{cases}$$

For notational convenience, we will often use $\theta_0(\cdot)$ to denote the function

$$\theta_0(t) = t , \qquad t \geqq 0 .$$

With this convention in mind, suppose that $Z:[0, \infty) \times \Theta \to \boldsymbol{R}^1$ has the form

$$Z(T) = Z_0 + \sum_{k=0}^{d} \int_0^T Y_k(t) d\theta_k(t) , \qquad T \geqq 0 , \tag{A.3}$$

where $Z_0 \in \boldsymbol{R}^1$ and $Y:[0, \infty) \times \Theta \to \boldsymbol{R}^{d+1}$ is a continuous $\{\mathcal{B}_t : t \geqq 0\}$-progressively measurable function. We then define $S^{(\alpha)}(\cdot, Z)$, $\alpha \in \mathcal{A}$, inductively on $|\alpha|$ by:

$$S^{(\alpha)}(T, Z) = \begin{cases} Z(T) & \text{if } \alpha = \emptyset \\ \int_0^T S_{\alpha'}(t, Z) \circ d\theta_{\alpha_*}(t) & \text{if } |\alpha| \geqq 1 . \end{cases}$$

In particular, we set

$$\theta^{(\alpha)}(T) = S^{(\alpha)}(T, 1) . \tag{A.4}$$

Our goal is to prove the following theorems.

(A.5) THEOREM. *Given $L \geqq 1$ and $\varepsilon > 0$, there exist $C(L, \varepsilon) < \infty$ and $\lambda(L, \varepsilon) > 0$ such that for all $Z(\cdot)$ of the form given in* (A.3) *and all $\alpha \in \mathcal{A}$ with $\|\alpha\| = L$:*

$$\mathcal{W}\left(\sup_{0<t\leqq 1} |S^{(\alpha)}(t, Z)| / t^{L/2-\varepsilon} \geqq K^{2L}, \sup_{0<t\leqq 1} |Z(t)| \leqq K, \left\{ \sum_{k=1}^{d} \int_0^1 |Y_k(t)|^2 dt \right\}^{1/2} \leqq K \right)$$

$$\leqq C(L, \varepsilon) \exp(-\lambda(L, \varepsilon) K) , \quad K > 0 .$$

(A.6) THEOREM. *Given $L \geqq 1$, there exists $C(L) < \infty$ and $\mu_L \in (0, \infty)$ such that for all $T \in (0, 1)$:*

$$\mathcal{W}\left(\inf_{\sum_{\|\alpha\| \leqq L-1} b_\alpha^2 = 1} \frac{1}{T^L} \int_0^T \left(\sum_{\|\alpha\| \leqq L-1} b_\alpha \theta^{(\alpha)}(t) \right)^2 dt \leqq 1/K \right)$$

$$\leqq C(L) \exp(-K^{\mu_L}) , \qquad K > 0 .$$

Theorem (A.5) is a quite easy application of familiar facts about stochastic integrals. It depends on the following two lemmas.

(A.7) LEMMA. *Let* $\beta:[0,\infty)\times\Theta\to\boldsymbol{R}^d$ *be a continuous progressively measurable function. Set* $\xi(T)=\sum_{k=1}^{d}\int_0^T\beta_k(t)d\theta_k(t)$ *and* $V(T)=\int_0^T|\beta(t)|^2dt$ *for* $T\geqq 0$. *Then, for any* $r\in[0,1/2)$ *there is a* $C_r<\infty$ *and a* $\lambda_r>0$ *such that*

$$\mathcal{W}\left(\sup_{0\leqq s<t\leqq T}\frac{|\xi(t)-\xi(s)|}{|V(t)-V(s)|^r}\geqq K_1,\ V(T)\leqq K_2\right)\leqq C_r\exp(-\lambda_r K_1^2/K_2^{1-2r})$$

for all positive K_1 *and* K_2.

PROOF. Without loss in generality, we assume that $|\beta(t)|\geqq\varepsilon$ for some $\varepsilon>0$ and all $t\geqq 0$. It is then easy to check that $B(\cdot)=\xi\circ V^{-1}(\cdot)$ is a 1-dimensional Brownian motion under $\mathcal{W}$. Moreover, it is clear that

$$\left\{\sup_{0\leqq s<t\leqq T}\frac{|\xi(t)-\xi(s)|}{|V(t)-V(s)|^r}\geqq K_1,\ V(T)\leqq K_2\right\}$$

$$\subseteqq\left\{\sup_{0\leqq s<t\leqq K_2}\frac{|B(t)-B(s)|}{|t-s|^r}\geqq K_1\right\}.$$

Thus, all that we need to know is that

$$\mathcal{W}\left(\sup_{0\leqq s<t\leqq K_2}\frac{|B(t)-B(s)|}{|t-s|^r}\geqq K_1\right)\leqq C_r\exp(-\lambda_r K_1^2/K_2^{1-2r}) \tag{A.8}$$

for some $C_r<\infty$ and $\lambda_r>0$, and for all $K_1>0$ and $K_2>0$.

There are various ways of deriving (A.8). To begin with, observe that, by Brownian scaling invariance, it suffices to treat the case when $K_2=1$. To see (A.8) when $K_2=1$, one need only note that $B(\cdot)$ is Gaussian and that for each $r\in(0,1/2)$ the distribution of $B(\cdot)|_{[0,1]}$ is supported on the Banach space $C^r([0,1])=\{\phi\in C([0,1]):\phi(0)=0$ and $\|\phi\|_r\equiv\sup_{0\leqq s<t\leqq 1}\frac{|\phi(t)-\phi(s)|}{|t-s|^r}<\infty\}$. Thus, by Fernique's Theorem, there exists a $\lambda_r>0$ such that

$$E^{\mathcal{W}}[\exp(\lambda_r\|B(\cdot)\|_r^2)]=C_r<\infty .$$

(See Lemma (8.27) in [12] for a more self-contained derivation of (A.8).)

Q. E. D.

(A.9) LEMMA. *Given* $a\in[0,\infty)$ *and* $r\in(0,1/2)$, *there exist* $C(a,r)<\infty$ *and* $\lambda(a,r)\in(0,\infty)$ *such that*

$$\mathcal{W}\left(\sup_{0<T\leqq 1}|\xi(T)|/T^{a+r}\geqq K^2/2,\ \sup_{0<t\leqq 1}|\beta(t)|/t^a\leqq K\right)$$

$$\leqq C(a,r)\exp(-\lambda(a,r)K),\qquad K\in(0,\infty),$$

whenever $\beta(\cdot)$ *and* $\xi(\cdot)$ *are given as in Lemma* (A.7).

PROOF. Define $V(T)=\int_0^T|\beta(t)|^2dt,\ T\geqq 0$. Then

$$V(T)\leqq \sup_{0<t\leqq 1}(|\beta(t)|/t^a)^2\frac{T^{2a+1}}{2a+1}\leqq \sup_{0<t\leqq 1}(|\beta(t)|/t^a)^2T^{2a+1}\quad \text{for } 0<T\leqq 1\,.$$

Thus, if $\rho=(a+r)/(2a+1)$, then

$$V(T)^\rho\leqq \sup_{0<t\leqq 1}(|\beta(t)|/t^a)^{2\rho}T^{a+r}\,,\qquad 0<T\leqq 1\,;$$

and therefore, by Lemma (A.7):

$$\begin{aligned}&\mathcal{W}\Big(\sup_{0<T\leqq 1}|\xi(T)|/T^{a+r}\geqq K^2/2,\ \sup_{0<t\leqq 1}|\beta(t)|/t^a\leqq K\Big)\\ &\leqq \mathcal{W}\Big(\sup_{0<T\leqq 1}|\xi(T)|/(V(T))^\rho\geqq K^{2-2\rho}/2,\ V(1)\leqq K^2\Big)\\ &\leqq C_\rho\exp(-\lambda_\rho K)\,,\end{aligned}$$

where C_ρ and λ_ρ are as in Lemma (A.7). Q. E. D.

(A.10) PROOF OF (A.5). Clearly it suffices to prove our inequality for $0<\varepsilon<1/2$ and $K\in(1,\infty)$. Thus we always choose such ε and K. Given K, set

$$E(K)=\Big\{\sup_{0<t\leqq 1}|Z(t)|\leqq K\quad\text{and}\quad\Big\{\sum_{k=1}^d\int_0^1|Y_k(t)|^2dt\Big\}^{1/2}\leqq K\Big\}\,.$$

We work by induction on $L\geqq 1$.

First, suppose that $\|\alpha\|=L=1$ and therefore that $\alpha=k_0\in\{1,\cdots,d\}$. Then

$$S^{(\alpha)}(T,Z)=\int_0^T Z(t)d\theta_{k_0}(t)+1/2\int_0^T Y_{k_0}(t)dt$$

and so

$$\begin{aligned}&\mathcal{W}\Big(\sup_{0<T\leqq 1}|S^{(\alpha)}(T,Z)|/T^{1/2-\varepsilon}\geqq K^2,\ E(K)\Big)\\ &\leqq \mathcal{W}\Big(\sup_{0<T\leqq 1}\Big|\int_0^T Z(t)d\theta_{k_0}(t)\Big|/T^r\geqq K^2/2,\ \sup_{0<t\leqq 1}|Z(t)|\leqq K\Big)\end{aligned}$$

where $r=1/2-\varepsilon$. Hence the desired inequality is an immediate consequence of Lemma (A.9) with $a=0$.

Next, let $L\geqq 2$ and assume the result for all $\alpha\in\mathcal{A}$ with $\|\alpha\|\leqq L-1$. Given $\alpha=(\alpha_1,\cdots,\alpha_l)$ with $\|\alpha\|=L$, suppose first that $\alpha_l=0$. Then

$$|S^{(\alpha)}(T,Z)|\leqq T\sup_{0<t\leqq 1}|S^{(\alpha')}(t,Z)|$$

for $0<T\leqq 1$. If $\alpha'=\emptyset$, there is nothing more to be said. If $\alpha'\neq\emptyset$, then

$$\begin{aligned}&\mathcal{W}\Big(\sup_{0<T\leq 1}|S^{(\alpha)}(T,Z)|/T^{L/2-\varepsilon}\geqq K^{2^L},\ E(K)\Big)\\ &\leqq \mathcal{W}\Big(\sup_{0<T\leq 1}|S^{(\alpha')}(T,Z)|/T^{L/2-1-\varepsilon}\geqq K^{2^L},\ E(K)\Big).\end{aligned}$$

Since $L/2-1=\|\alpha'\|/2$ and $2L\geqq 2\|\alpha'\|$, the desired result now follows from the induction hypothesis. Next, suppose that $\alpha_l=k_0\in\{1,\cdots,d\}$ and that $\alpha_{l-1}\neq k_0$. Then

$$S^{(\alpha)}(T,Z)=\int_0^T S^{(\alpha')}(t,Z)d\theta_k(t)$$

and so

$$\begin{aligned}&\mathcal{W}\Big(\sup_{0<T\leq 1}|S^{(\alpha)}(T,Z)|/T^{L/2-\varepsilon}\geqq K^{2^L},\ E(K)\Big)\\ &\leqq \mathcal{W}\Big(\sup_{0<T<1}\Big|\int_0^T S^{(\alpha')}(t,Z)d\theta_{k_0}(t)\Big|/T^{L/2-\varepsilon}\geqq K^{2^L},\\ &\qquad \sup_{0<t\leq 1}|S^{(\alpha')}(t,Z)|/t^{(L-1)/2-\varepsilon/2}\leqq K^{2^{(L-1)}}\Big)\\ &\quad+\mathcal{W}\Big(\sup_{0<T\leq 1}|S^{(\alpha')}(T,Z)|/T^{(L-1)/2-\varepsilon/2}\geqq K^{2^{(L-1)}},\ E(K)\Big).\end{aligned}$$

Thus, the desired result follows in this case from Lemma (A.9) with $a=(L-1)/2-\varepsilon/2$ and $r=\varepsilon/2$ and from the induction hypothesis. Finally, suppose that $\alpha_l=k_0\in\{1,\cdots,d\}$ and that $\alpha_{l-1}=k_0$. Then

$$S^{(\alpha)}(T,Z)=\int_0^T S^{(\alpha')}(t,Z)d\theta_{k_0}(t)+1/2\int_0^T S^{(\alpha'')}(t,Z)dt,$$

where $\alpha''=(\alpha_1,\cdots,\alpha_{l-2})$ if $l\geqq 3$ and $\alpha''=\emptyset$ if $l=2$. Thus:

$$\begin{aligned}&\mathcal{W}\Big(\sup_{0<T\leq 1}|S^{(\alpha)}(T,Z)|/T^{L/2-\varepsilon}\geqq K^{2^L},\ E(K)\Big)\\ &\leqq \mathcal{W}\Big(\sup_{0<T\leq 1}\Big|\int_0^T S^{(\alpha')}(t,Z)d\theta_{k_0}(t)\Big|/T^{L/2-\varepsilon}\geqq K^{2^L}/2,\\ &\qquad \sup_{0<t\leq 1}|S^{(\alpha')}(t,Z)|/t^{(L-1)/2-\varepsilon/2}\leqq K^{2^{(L-1)}}\Big)\\ &\quad+\mathcal{W}\Big(\sup_{0<T\leq 1}|S^{(\alpha')}(T,Z)|/T^{(L-1)/2-\varepsilon/2}\geqq K^{2^{(L-1)}},\ E(K)\Big)\\ &\quad+\mathcal{W}\Big(\sup_{0<T\leq 1}|S^{(\alpha'')}(T,Z)|/T^{L/2-1-\varepsilon}\geqq K^{2^L},\ E(K)\Big).\end{aligned}$$

The first of these terms is handled by Lemma (A.9); the second and third are covered by the induction hypothesis. Q.E.D.

We must now turn to the proof of Theorem (A.6). Our first step is to convert the problem from one involving Stratonovich integrals to the corresponding one for Itô integrals. To be precise, given a continuous progressively measurable function $Z:[0,\infty)\times\Theta\to R^1$, define $I^{(\alpha)}(\cdot,Z)$, $\alpha\in\mathcal{A}$, inductively by:

$$I^{(\alpha)}(T,Z)=\begin{cases} Z(T) & \text{if } \alpha=\emptyset \\ \int_0^T I^{(\alpha')}(t,Z)d\theta_{\alpha_*}(t) & \text{if } \alpha\neq\emptyset\,. \end{cases}$$

In particular, we set

$$\theta^{(\alpha)}(T)=I^{(\alpha)}(T,1)\,. \tag{A.11}$$

(A.12) LEMMA. *For each $L\geqq 0$ there is an invertible matrix $\{a^{(L)}_{\alpha,\beta}:\|\alpha\|=\|\beta\|=L\}$ such that*

$$\theta^{(\alpha)}(\cdot)=\sum_{\|\beta\|=L} a^{(L)}_{\alpha,\beta}\theta^{(\beta)}(\cdot)\,,\qquad \|\alpha\|=L\,.$$

PROOF. We work by induction on $L\geqq 0$. When $L\in\{0,1\}$ there is nothing to prove. Suppose that $L_0\geqq 2$ and that the results holds for $L\leqq L_0-1$. Given $\alpha\in\mathcal{A}$ with $\|\alpha\|=L_0$, suppose that $\alpha_*=0$. Then:

$$\theta^{(\alpha)}(T)=\int_0^T\theta^{(\alpha')}(t)dt=\sum_{\|\beta'\|=L_0-2} a^{(L_0-2)}_{\alpha',\beta'}\int_0^T\theta^{(\beta')}(t)dt\,.$$

Thus, we take

$$a^{(L_0)}_{\alpha,\beta}=\begin{cases} a^{(L_0-2)}_{\alpha',\beta'} & \text{if } \beta_*=0 \\ 0 & \text{if } \beta_*\neq 0 \end{cases},\quad \text{for } \alpha_*=0\,.$$

Next, suppose that $\alpha_*=k_0\in\{1,\cdots,d\}$ and that $(\alpha')_*\neq k_0$. Then

$$\theta^{(\alpha)}(T)=\int_0^T\theta^{(\alpha')}(t)d\theta_{k_0}(t)=\sum_{\|\beta'\|=L_0-1} a^{(L_0-1)}_{\alpha',\beta'}\int_0^T\theta^{(\beta')}(t)d\theta_{k_0}(t)\,.$$

Thus we take

$$a^{(L_0)}_{\alpha,\beta}=\begin{cases} a^{(L_0-1)}_{\alpha',\beta'} & \text{if } \beta_*=\alpha_* \\ 0 & \text{if } \beta_*\neq\alpha_* \end{cases},\qquad \text{for } (\alpha')_*\neq\alpha_*\in\{1,\cdots,d\}\,.$$

Finally, suppose that $(\alpha')_*=\alpha_*=k_0\in\{1,\cdots,d\}$. Then

$$\begin{aligned}\theta^{(\alpha)}(T)&=\int_0^T\theta^{(\alpha')}(t)d\theta_{k_0}(t)+1/2\int_0^T\theta^{(\alpha'')}(t)dt\\ &=\sum_{\|\beta'\|=L_0-1} a^{(L_0-1)}_{\alpha',\beta'}\int_0^T\theta^{(\beta')}(t)d\theta_{k_0}(t)\\ &\quad+\sum_{\|\beta'\|=L_0-2} 1/2a^{(L_0-2)}_{\alpha',\beta'}\int_0^T\theta^{(\beta')}(t)dt\,,\end{aligned}$$

where $\alpha''=(\alpha')'$. Thus, we take:

$$a^{(L_0)}_{\alpha,\beta}=\begin{cases} a^{(L_0-1)}_{\alpha',\beta'} & \text{if } \beta_*=\alpha_* \\ 1/2a^{(L_0-2)}_{\alpha'',\beta'} & \text{if } \beta_*=0\,, \\ 0 & \text{if } \beta_*\notin\{0,\alpha_*\} \end{cases} \qquad \text{for } (\alpha')_*=\alpha_*\in\{1,\cdots,d\}\,.$$

It is an easy matter to check from the induction hypothesis that $\{a^{(L_0)}_{\alpha,\beta}: \|\alpha\|=\|\beta\|=L_0\}$ is an invertible matrix. Q.E.D.

As an immediate consequence of Lemma (A.12), we see that Theorem (A.6) is equivalent to proving that for each $L\geqq 1$ there is a $C_L<\infty$ and $\mu_L\in(0,\infty)$ such that for all $T\in(0,1]$:

$$\text{(A.13)}\quad \mathcal{W}\Bigg(\inf_{\sum_{\|\alpha\|\leqq L-1} b^2_\alpha=1} \frac{1}{T^L}\int_0^T\Bigg(\sum_{\|\alpha\|\leqq L-1} b_\alpha\theta^{(\alpha)}(t)\Bigg)^2 dt<1/K\Bigg)\leqq C_L\exp(-K^{\mu_L})\,, \qquad K>0\,.$$

Given a bounded interval $I\subseteqq R^1$ and an $f\in C(R^1)$, define the mean f_I of f by

$$f_I=\frac{1}{|I|}\int_I f(t)dt\,,$$

($|I|$ denote the length of I) and the variance $\sigma^2_I(f)$ of f by

$$\sigma^2_I(f)=\frac{1}{|I|}\int_I (f(t)-f_I)^2 dt\,.$$

We will make frequent use of the fact that

$$\sigma^2_I(f)\leqq\frac{1}{|I|}\int_I (f(t)-a)^2 dt\,, \qquad a\in \boldsymbol{R}^1\,.$$

This notation will be used repeatedly in what follows. Of particular importance to us in the next lemma will be the estimate:

$$\text{(A.14)}\qquad P(\sigma_{[0,T]}(B(\cdot))\leqq\varepsilon)\leqq 2^{1/2}\exp(-T/2^7\varepsilon^2)\,,$$

for all $T>0$ and $\varepsilon>0$, where $B(\cdot)$ is a 1-dimensional Brownian motion under P. The estimate (A.14) can be derived by first using Brownian sealing to reduce to the case $T=1$ and then using Wiener's development of Brownian motion on $[0,1]$ as a trigonometric series to compute $\sigma_{[0,1]}(B(\cdot))$ via Parseval's theorem. Details of this argument can be found in Lemma (8.6) of [4].

(A.15) LEMMA. *Suppose that* $\xi(\cdot)$ *is given by*:

$$\xi(T)=\xi_0+\sum_{k=1}^{d}\int_0^T \beta_k(t)d\theta_k(t)+\int_0^T \gamma(t)dt\,, \qquad T\geqq 0$$

where $\xi_0\in \boldsymbol{R}^1$ *and* $\beta:[0,\infty)\times\Theta\to\boldsymbol{R}^{d+1}$ *and* $\gamma:[0,\infty]\times\Theta\to\boldsymbol{R}^1$ *are continuous progressively measurable functions. Given* $M\in[1,\infty)$, *set*

$$A_M=\left\{\theta:\sup_{0\leq t\leq 1}\Big(|\beta(t,\theta)|\vee|\gamma(t,\theta)|\Big)\leqq M\right\}.$$

Then for all $Q, R>0$ *and* $K\in(0,\infty)$:

$$\mathcal{W}\left(\left\{\int_0^1\xi^2(t)dt\leqq Q/K^{10},\ \int_0^1|\beta(t)|^2dt\geqq R/K\right\}\cap A_M\right)$$

$$\leqq 2^{11}K^3\exp\left(-\frac{R^2K^2}{2^{18}(Q+R)M^2}\right).$$

PROOF. Without loss of generality, we assume that $|\beta(t)|\vee|\gamma(t)|\leqq M$, $t\geqq 0$, and that there is an $\varepsilon>0$ such that $|\beta(t)|\geqq\varepsilon$, $t\geqq 0$. In particular, we assume that $A_M=\Theta$.

Given a positive integer N, set $I(k)=[k/N^3,(k+1)/N^3)$ and define

$$E(k)=\left\{\int_{I(k)}\xi^2(t)\,dt\leqq Q/N^{10},\ \int_{I(k)}|\beta(t)|^2dt\geqq R/N^4\right\}.$$

Clearly:

$$\text{(A.16)}\qquad \mathcal{W}\left(\int_0^1\xi^2(t)^2dt\leqq Q/N^{10},\ \int_0^1|\beta(t)|^2dt\geqq R/N\right)$$

$$\leqq\sum_{k=0}^{N3-1}\mathcal{W}(E(k))\,.$$

We therefore must estimate $\mathcal{W}(E(k))$.

Set $V(T)=\int_0^T|\beta(t)|^2dt$, $T\geqq 0$, and put $\tau_k=V(k/N^3)$. Then

$$\int_{I(k)}\xi^2(t)dt=\int_{\tau k}^{\tau k+1}\xi^2\circ V^{-1}(t)/|\beta\circ V^{-1}(t)|^2dt$$

$$\geqq\frac{1}{M^2}\int_{\tau k}^{\tau k+1}\xi^2\circ V^{-1}(t)dt\,.$$

Since for $\theta\in E(k)$ we know that $\tau_{k+1}(\theta)\geqq\tau_k(\theta)+R/N^4$, we now see that

$$\text{(A.17)}\qquad E(k)\subseteqq\left\{\int_{J(k)}\xi^2\circ V^{-1}(t)dt\leqq M^2Q/N^{10},\ \tau_{k+1}\geqq\tau_k+R/N^4\right\},$$

where $J(k)=[\tau_k,\tau_k+R/N^4)$.

Set $B(T)=\xi\circ V^{-1}(T)-\xi_0-\int_0^T\gamma(t)dt,\ T\geqq 0.$ Then:

$$\left(\int_{J(k)}\xi^2\circ V^{-1}(t)dt\right)^{1/2}\geqq\left(\int_{J(k)}\left(\xi_0+\int_0^{k/N^3}\gamma(s)ds+B(t)\right)^2dt\right)^{1/2}$$

$$-\left(\int_{J(k)}\left(\int_{k/N^3}^{V^{-1}(t)}\gamma(s)ds\right)^2dt\right)^{1/2}$$

$$\geqq R^{1/2}/N^2\left(\sigma_{J(k)}(B(\cdot))-\int_{k/N^3}^{V^{-1}(\tau_k+R/N^4)}|\gamma(s)|ds\right).$$

Note that if $\tau_{k+1}(\theta)\geqq\tau_k(\theta)+R/N^4$, then:

$$V^{-1}(\tau_k(\theta)+R/N^4,\theta)\leqq V^{-1}(\tau_{k+1}(\theta),\theta)=(k+1)/N^3.$$

Hence, by (A.17) and the preceding, we now have:

$$E(k)\subseteq\{\sigma_{J(k)}(B(\cdot))\leqq M((Q/R)^{1/2}+1)/N^3\}. \tag{A.18}$$

Observing that $(B(T),\mathscr{B}_{V^{-1}(T)},\mathscr{W})$ is a 1-dimensional Brownian motion and that the τ_k's are $\mathscr{B}_{V^{-1}(\cdot)}$-stopping times, we see that $\sigma_{J(k)}(B(\cdot))$ has the same distribution as $\sigma_{[0,R/N^4)}(B(\cdot))$ and therefore, by (A.14) and (A.18), that

$$\mathscr{W}(E(k))\leqq 2^{1/2}\exp\left(-\frac{R^2N^2}{2^8(Q+R)M^2}\right). \tag{A.19}$$

Combining (A.16) and (A.19), we arrive at

$$\mathscr{W}\left(\int_0^1\xi^2(t)dt\leqq Q/N^{10},\ \int_0^1|\beta(t)|^2dt\geqq R/N\right) \tag{A.20}$$
$$\leqq 2^{1/2}N^3\exp\left(-\frac{R^2N^2}{2^8(Q+R)M}\right)$$

for all $Q,R>0$ and all positive integers N. Finally, given $K\in[1,\infty)$, let N be the smallest integer in $[K,\infty)$ and set $Q'=(N/K)^{10}Q\leqq(1+1/K)^{10}Q\leqq 2^{10}Q$. Then

$$\mathscr{W}\left(\int_0^1\xi^2(t)dt\leqq Q/K^{10},\ \int_0^1|\beta(t)|^2dt\geqq R/M\right)$$

$$\leqq\mathscr{W}\left(\int_0^1\xi^2(t)dt\leqq Q'/N^{10},\ \int_0^1|\beta(t)|^2dt\geqq R/N\right),$$

and so the desired estimate follows immediately from (A.20) with Q' replacing Q. Q.E.D.

(A.21) LEMMA. *Let $f\in C(\bar{I})$ and let F be a primitive of f on I (i.e.*

$F'=f$ on I). Then

$$\sigma_I^2(F) \geqq \left(\min_I |f|\right)^2 |I|^2/12. \tag{A.22}$$

In particular, if $\sup_{0\leqq s<t\leqq 1} \frac{|f(t)-f(s)|}{|t-s|^{3/8}} \leqq L$ *and* $\int_0^1 |f(t)|^2 dt \geqq \varepsilon^2$, *then, so long as* $\varepsilon \leqq L$:

$$\sigma_{[0,1]}^2(F) \geqq \varepsilon^{10}/4^7 L^8. \tag{A.23}$$

PROOF. To prove (A.22), use the mean value theorem to find a $t_0 \in \operatorname{int} I$ such that

$$F(t_0) = F_I.$$

Then

$$\sigma_I^2(F) = \frac{1}{|I|}\int_I \left(\int_{t_0}^t f(s)ds\right)^2 dt \geqq \left(\min_I |f|\right)^2 \frac{1}{|I|}\int_I (t-t_0^2)dt$$

$$\geqq \left(\min_I |f|\right)^2 \left(\frac{1}{|I|}\int_I t^2 dt - \left(\frac{1}{|I|}\int_I t\,dt\right)^2\right)$$

$$= \left(\min_I |f|\right)^2 |I|^2/12.$$

Turning to (A.23), we choose $t_0 \in (0,1)$ so that $|f(t_0)| \geqq \varepsilon$. Then, $|f(t)| \geqq \varepsilon/2$ for all $t \in [0,1]$ satisfying $L|t-t_0|^{3/8} \leqq \varepsilon/2$. Since $\varepsilon \leqq L$, we conclude that there is an interval $I \subseteq [0,1]$ such that $|I| = (\varepsilon/2L)^{8/3}$ and $\min_I |f| \geqq \varepsilon/2$. Thus, by (A.22):

$$\sigma_{[0,1]}^2(F) \geqq (\varepsilon/2L)^{8/3}\sigma_I^2(F) \geqq \frac{1}{12}(\varepsilon/2L)^8(\varepsilon/2)^2$$

$$\geqq \varepsilon^{10}/4^7 L^8. \qquad \text{Q.E.D.}$$

(A.24) THEOREM. *There exist $C<\infty$ and $\lambda \in (0,\infty)$ such that for all continuous progressively measurable functions $\beta: [0,\infty)\times\Theta \to \boldsymbol{R}^d$, $\tilde{\beta}: [0,\infty)\times\Theta \to \boldsymbol{R}^d$, and $\tilde{\gamma}: [0,\infty)\times\Theta \to \boldsymbol{R}^1$, all ξ_0 and γ_0 in $\boldsymbol{R}^1$, and all $K \in (0,\infty)$:*

$$\mathcal{W}\Bigl(\int_0^1 \xi^2(t)dt \leqq 1/K^{60}, \int_0^1 (|\beta(t)|^2 + |\gamma(t)|^2)dt \geqq 1/K,$$

$$\sup_{0\leqq t\leqq T} |\beta(t)| \vee |\tilde{\beta}(t)| \vee |\gamma(t)| \vee |\tilde{\gamma}(t)| \leqq K^{1/4}\Bigr)$$

$$\leqq C\exp(-\lambda K^{1/2}),$$

where

$$\xi(T) = \xi_0 + \sum_{k=1}^d \int_0^T \beta_k(t)d\theta_k(t) + \int_0^T \gamma(t)dt, \qquad T \geqq 0,$$

and

$$\gamma(T)=\gamma_0+\sum_{k=1}^{d}\int_0^T \tilde{\beta}_k(t)d\theta_k(t)+\int_0^T \tilde{\gamma}(t)dt\,, \qquad T\geqq 0\,.$$

PROOF. Let $K\in[2,\infty)$ be given. Without loss of generality, we assume that $|\beta(t)|\wedge|\tilde{\beta}(t)|\geqq\varepsilon$, $t\geqq 0$, for some $\varepsilon>0$ and that $|\beta(t)|\vee|\tilde{\beta}(t)|\vee|\gamma(t)|\vee|\tilde{\gamma}(t)|\leqq K^{1/4}$, $t\geqq 0$.

Set $E=\left\{\int_0^1\xi^2(t)dt\leqq 1/K^{60} \text{ and } \int_0^1(|\beta(t)|^2+|\gamma(t)|^2)dt\geqq 1/K\right\}$. Then

$$E\subseteq E_1\cup E_2\cup E_3\,, \tag{A.25}$$

where

$$E_1=\left\{\int_0^1\xi^2(t)dt\leqq 1/K^{60} \text{ and } \int_0^1|\beta(t)|^2dt\geqq 1/K^{10}\right\},$$

$$E_2=\left\{\sup_{0\leq s<t\leq 1}\frac{|\gamma(t)-\gamma(s)|}{|t-s|^{3/8}}\geqq 2K^{1/2}\right\},$$

and

$$E_3=E_2^c\cap\left\{\int_0^1\xi^2(t)dt\leqq 1/K^{60},\ \int_0^1|\beta(t)|^2dt\leqq 1/K^{10},\ \int_0^1\gamma^2(t)dt\geqq 1/2K\right\}.$$

Observe that by Lemma (A.15), with $Q=1$ and $R=1/k^4$:

$$\mathcal{W}(E_1)\leqq 2^{11}K^{15}\exp(-K^4/2^{19})\,. \tag{A.26}$$

To estimate $\mathcal{W}(E_2)$, set $\tilde{V}(T)=\int_0^T|\tilde{\beta}(t)|^2dt$ and note that:

$$\frac{|\gamma(t)-\gamma(s)|}{|t-s|^{3/8}}\leqq K^{3/16}\frac{\left|\sum_{k=1}^{d}\int_s^t\tilde{\beta}_k(u)d\theta_k(u)\right|}{|\tilde{V}(t)-\tilde{V}(s)|^{3/8}}+K^{1/4},$$

and so

$$\mathcal{W}(E_2)\leqq\mathcal{W}\left(\sup\frac{\left|\sum_{k=1}^{d}\int_s^t\tilde{\beta}_k(u)d\theta_k(u)\right|}{|\tilde{V}(t)-\tilde{V}(s)|^{3/8}}\geqq K^{5/16}\right).$$

Hence, since $\tilde{V}(1)\leqq K^{1/2}$, Lemma (A.7) yields:

$$\mathcal{W}(E_2)\leqq C_{3/8}\exp(-\lambda_{3/8}K^{1/2})\,. \tag{A.27}$$

We now turn to $\mathcal{W}(E_3)$. Set

$$Y(T)=\sum_{k=1}^{d}\int_0^t\beta_k(t)d\theta_k(t)\,.$$

Then

$$\left(\int_0^1\xi^2(t)dt\right)^{1/2}\geqq\left(\int_0^1\left(\xi_0+\int_0^t\gamma(s)ds\right)^2dt\right)^{1/2}-\left(\int_0^1 Y^2(t)dt\right)^{1/2}$$

$$\geqq \sigma_{[0,1]}\left(\int_0^{\cdot} \gamma(s)ds\right) - \sup_{[0,1]} |Y(t)| .$$

By (A.23), for $\theta \in E_3$:

$$\sigma_{[0,1]}\left(\int_0^{\cdot} \gamma(s,\theta)ds\right) \geqq (1/2^{23})^{1/2}(1/k)^{9/2} .$$

Thus

$$E_3 \subseteqq \left\{\sup_{[0,1]} |Y(t)| \geqq (1/2^{12})(1/K)^{9/2} \text{ and } \int_0^1 |\beta(t)|^2 dt \leqq 1/K^{10}\right\} ;$$

and so, by Lemma (A.7):

$$\text{(A.28)} \qquad \mathcal{W}(E_3) \leqq C_0 \exp(-\lambda_0 K) .$$

Combining (A.25), (A.26), (A.27), and (A.28), we arrive at the desired estimate. Q. E. D.

(A.29) PROOF OF THEOREM (A.6): We have already seen that it suffices to prove (A.13). We now point out that it is enough to check (A.13) when $T=1$. Indeed, by Brownian sealing, the distribution of

$$\frac{1}{T^L}\int_0^t \left(\sum_{\|\alpha\| \leqq L-1} b_\alpha \theta^{(\alpha)}(t)\right)^2 dt$$

under $\mathcal{W}$ coincides with that of

$$\int_0^1 \left(\sum_{\|\alpha\| \leqq L-1} b_\alpha T^{(\|\alpha\|-L+1)/2} \theta^{(\alpha)}(t)\right)^2 dt .$$

But, for $T \in (0,1]$, $\sum_{\|\alpha\| \leqq L-1} b_\alpha^2 T^{(\|\alpha\|-L+1)} \geqq \sum_{\|\alpha\| \leqq L-1} b_\alpha^2$; and so the left hand side of (A.13) is non-decreasing with respect to $T \in (0,1]$. We therefore restrict our attention to the case $T=1$.

We next show that it suffices to prove that there exist $C_L < \infty$ and $\mu_L \in (0,\infty)$ such that

$$\text{(A.30)} \qquad \mathcal{W}\left(\int_0^1 \left(\sum_{\|\alpha\| \leqq L-1} b_\alpha \theta^{(\alpha)}(t)\right)^2 dt \leqq 1/K\right) \leqq C_L \exp(-K^{\mu_L})$$

for all $K \in (0,\infty)$ and $\{b_\alpha : \|\alpha\| \leqq L-1\} \subseteqq \boldsymbol{R}^1$ satisfying $\sum_{\|\alpha\| \leqq L-1} b_\alpha^2 = 1$. To see that (A.30) suffices, let $L \geqq 1$ be given, define $D_L = \text{card}\{\alpha \in \mathcal{A} : \|\alpha\| \leqq L-1\}$, and set $S^{D_L-1} = \{b \in \boldsymbol{R}^{D_L} : \|b\|_{R^{D_L}} = 1\}$. It is easy to see that there is a geometric constant $M_L < \infty$ such that for each $K \in [1,\infty)$ the sphere S^{D_L-1} contains a finite set $\Sigma(K)$ with the properties that $\text{card}(\Sigma(K)) \leqq M_L K^{2D_L} + 1$ and $S^{D_L-1} \subseteqq \bigcup_{c \in \Sigma(k)} B(c, 1/2K^2)$. In particular, we have:

$$\inf_{b \in S^{D_L-1}} \int_0^1 \left(\sum_{\|\alpha\| \leqq L-1} b_\alpha \theta^{(\alpha)}(t)\right)^2 dt$$

$$\geqq \inf_{c\in\Sigma(k)}\int_0^1\Big(\sum_{\|\alpha\|\leqq L-1} c_\alpha\theta^{(\alpha)}(t)\Big)^2 dt - 1/K^2\int_0^1 \sum_{\|\alpha\|\leqq L-1}|\theta^{(\alpha)}(t)|^2 dt\,.$$

Thus

$$\mathcal{W}\Big(\inf_{b\in S^{DL-1}}\int_0^1\Big(\sum_{\|\alpha\|\leqq L-1} b_\alpha\theta^{(\alpha)}(t)\Big)^2 dt\leqq 1/K\Big)$$

$$\leqq \mathcal{W}\Big(\inf_{b\in S^{DL-1}}\int_0^1\Big(\sum_{\|\alpha\|\leqq L-1} b_\alpha\theta^{(\alpha)}(t)\Big)^2 dt\leqq 1/K,\ \sum_{\|\alpha\|\leqq L-1}\int_0^1|\theta^{(\alpha)}(t)|^2dt\leqq K\Big)$$

$$+\mathcal{W}\Big(\sum_{\|\alpha\|\leqq L-1}\int_0^1|\theta^{(\alpha)}(t)|^2dt\geqq K\Big)$$

$$\leqq \mathcal{W}\Big(\inf_{c\in\Sigma(k)}\int_0^1\Big(\sum_{\|\alpha\|\leqq L-1} c_\alpha\theta^{(\alpha)}(t)\Big)^2 dt\leqq 2/K\Big)$$

$$+\mathcal{W}\Big(\sup_{0\leqq t\leqq 1}\sum_{\|\alpha\|\leqq L-1}|\theta^{(\alpha)}(t)|^2\geqq K\Big)$$

$$\leqq M_L K^{2DL}\sup_{b\in S^{DL-1}}\mathcal{W}\Big(\int_0^1\Big(\sum_{\|\alpha\|\leqq L-1} b_\alpha\theta^{(\alpha)}(t)\Big)^2 dt< 2/K\Big)$$

$$+L\max_{0\leqq l\leqq L-1}\mathcal{W}\Big(\sup_{0\leqq t\leqq 1}\sum_{\|\alpha\|=l}|\theta^{(\alpha)}(t)|^2\geqq K/L\Big).$$

Noting that, by Lemma (A.12), there is an $\varepsilon_L>0$ such that

$$\sum_{\|\alpha\|=l}|\theta^{(\alpha)}(t)|^2\leqq 1/\varepsilon_L\sum_{\|\alpha\|=l}|\theta^{(\alpha)}(t)|^2$$

for all $t\geqq 0$ and $0\leqq l\leqq L-1$, we see that

$$\mathcal{W}\Big(\sup_{0\leqq t\leqq 1}\sum_{\|\alpha\|=l}|\theta^{(\alpha)}(t)|^2\geqq K/L\Big)\leqq \mathcal{W}\Big(\sup_{0\leqq t\leqq 1}\sum_{\|\alpha\|=l}|\theta^{(\alpha)}(t)|^2\geqq \varepsilon_L K/L\Big)$$

for $0\leqq l\leqq L-1$. Hence, by Theorem (A.5), we see that there exist $B_L<\infty$ and $\lambda_L\in(0,\infty)$ such that

(A.31) $$\max_{0\leqq l\leqq L-1}\mathcal{W}\Big(\sup_{0\leqq t\leqq 1}\sum_{\|\alpha\|=l}|\theta^{(\alpha)}(t)|^2\geqq K\Big)\leqq B_L\exp(-\lambda_L K^{1/2}),\quad K\in(0,\infty)\,.$$

Thus (A.30) implies that

$$\mathcal{W}\Big(\inf_{b\in S^{DL-1}}\int_0^1\Big(\sum_{\|\alpha\|\leqq L-1} b_\alpha\theta^{(\alpha)}(t)\Big)^2 dt\leqq 1/K\Big)$$

$$\leqq C_L M_L K^{2DL}\exp(-(K/2)^{\mu L})$$

$$+LB_L\exp(-\lambda_L K^{1/2}),\quad K\in(0,\infty)\,.$$

Clearly, this means that (A.30) implies the existence of $C_L\leqq\infty$ and $\mu_L\in(0,\infty)$ for which (A.13) holds.

Before turning to the proof of (A.30), we first show that for each $L \geqq 1$ there exist $B_L < \infty$ and $\nu_L \in (0, \infty)$ such that

$$\mathcal{W}\left(\int_0^1 \left(\sum_{\|\alpha\| \leqq L-1} b_\alpha \theta^{(\alpha)}(t)\right)^2 dt \leqq 1/K\right) \leqq B_L \exp(-(1/(1-b_\varnothing^2))^{\nu_L}) \tag{A.32}$$

for all $K \in [16, \infty)$ and $\{b_\alpha : \|\alpha\| \leqq L-1\} \subseteqq \boldsymbol{R}^1$ satisfying $\sum_{\|\alpha\| \leqq L-1} b_\alpha^2 = 1$. To this end, note that

$$\left(\int_0^1 \left(\sum_{\|\alpha\| \leqq L-1} b_\alpha \theta^{(\alpha)}(t)\right)^2 dt\right)^{1/2} \geqq |b_\varnothing| - \left(\int_0^1 \left(\sum_{1 \leqq \|\alpha\| \leqq L-1} b_\alpha \theta^{(\alpha)}(t)\right)^2 dt\right)^{1/2}$$

$$\geqq |b_\varnothing| - (1-b_\varnothing^2)^{1/2} \sup_{0 \leqq t \leqq 1} \left(\sum_{1 \leqq \|\alpha\| \leqq L-1} |\theta^{(\alpha)}(t)|^2\right)^{1/2}.$$

Assuming that $|b_\varnothing| \geqq 1/2$, we see that for $K \in [16, \infty)$:

$$\mathcal{W}\left(\int_0^1 \left(\sum_{\|\alpha\| \leqq L-1} b_\alpha \theta^{(\alpha)}(t)\right)^2 dt \leqq 1/K\right)$$

$$\leqq \mathcal{W}\left(\sup_{0 \leqq t \leqq 1} \sum_{1 \leqq \|\alpha\| \leqq L-1} |\theta^{(\alpha)}(t)|^2 \geqq 1/16(1-b_\varnothing^2)\right).$$

Thus, by (A.31), we see that there exist $B_L < \infty$ and $\nu_L \in (0, \infty)$ for which (A.32) holds whenever $K \in [16, \infty)$ and $|b_\varnothing| \geqq 1/2$. Clearly, after adjusting B_L, the same inequality extends to all $\{b_\alpha : \|\alpha\| \leqq L-1\}$ with $\sum_{\|\alpha\| \leqq L-1} b_\alpha^2 = 1$.

We now prove (A.30) by induction on $L \geqq 1$. Obviously there is nothing to prove when $L=1$. Assuming that (A.30) holds for all $1 \leqq L \leqq L_0$, let $\{b_\alpha : \|\alpha\| \leqq L_0\} \subseteqq \boldsymbol{R}^1$ satisfying $\sum_{\|\alpha\| \leqq L_0} b_\alpha^2 = 1$ be given and set

$$\xi(T) = \sum_{\|\alpha\| \leqq L_0} b_\alpha \theta^{(\alpha)}(T),$$

$$\xi_k(T) = \sum_{\substack{1 \leqq \|\alpha\| \leqq L_0 \\ \alpha_* = k}} b_\alpha \theta^{(\alpha')}(T), \qquad 0 \leqq k \leqq d,$$

and

$$\xi_{0,k}(T) = \sum_{\substack{\|\alpha\| \leqq L_0,\ |\alpha| \geqq 2 \\ \alpha_* = 0,\ (\alpha')_* = k}} b_\alpha \theta^{(\alpha'')}(T), \qquad 0 \leqq k \leqq d,$$

where $\alpha'' = (\alpha')'$. Defining

$$\beta(T) = \begin{pmatrix} \xi_1(T) \\ \vdots \\ \xi_d(T) \end{pmatrix},$$

$$\tilde{\beta}(T) = \begin{pmatrix} \xi_{0,1}(T) \\ \vdots \\ \xi_{0,d}(T) \end{pmatrix},$$

$$\gamma(T) = \xi_0(T),$$

and

$$\tilde{\gamma}(T)=\xi_{0,0}(T)\,,$$

we have:

$$\xi(T)=\xi_0+\sum_{k=1}^{d}\int_0^T\beta_k(t)d\theta_k(t)+\int_0^T\gamma(t)dt\,,\qquad T\geqq 0\,,$$

and

$$\gamma(T)=\gamma_0+\sum_{k=1}^{d}\int_0^T\tilde{\beta}_k(t)d\theta_k(t)+\int_0^T\tilde{\gamma}(t)dt\,,\qquad T\geqq 0\,,$$

where

$$\xi_0=b_\varnothing$$

and

$$\gamma_0=\begin{cases} b_0 & \text{if } L_0\geqq 3\\ 0 & \text{if } L_0=2\,.\end{cases}$$

Given $K\in[16,\infty)$, we proceed as follows. If $1-b_\varnothing^2\leqq 1/K^{1/2}$, then, by (A.32):

$$\mathcal{W}\left(\int_0^1\left(\sum_{\|\alpha\|\leqq L_0}b_\alpha\theta^{(\theta)}(t)\right)^2 dt\leqq 1/K\right)\leqq B_{L_0+1}\exp(-K^{\nu L_0+1/2})\,.$$

Thus, we assume that $1-b_\varnothing^2\geqq 1/K^{1/2}$. Referring to the notation just introduced, define

$$E_1=\left\{\int_0^1\xi^2(t)dt\leqq 1/K^{60},\ \int_0^1|\beta(t)|^2+|\gamma(t)|^2dt\geqq 1/K,\right.$$
$$\left.\sup_{0\leqq t\leqq 1}|\beta(t)|\vee|\tilde{\beta}(t)|\vee|\gamma(t)|\vee|\tilde{\gamma}(t)|\leqq K^{1/4}\right\}$$

$$E_2=\left\{\sup_{0\leqq t\leqq 1}|\beta(t)|\vee|\tilde{\beta}(t)|\vee|\gamma(t)|\vee|\tilde{\gamma}(t)|\geqq K^{1/4}\right\}$$

and

$$E_3=\left\{\int_0^1(|\beta(t)|^2+|\gamma(t)|^2)dt<1/K\right\}\,.$$

By Theorem (A.24),

$$\mathcal{W}(E_1)\leqq C\exp(-\lambda K^{1/2})\,.$$

Moreover, by (A.31), we can find $M_{L_0}<\infty$ and $\nu\in(0,\infty)$ such that:

$$\mathcal{W}(E_2)\leqq M_{L_0}B_{L_0}\exp(-\lambda_{L_0}K^{\nu})\,.$$

To estimate $\mathcal{W}(E_3)$, set $\rho=1-b_\varnothing^2$. Then $\rho\geqq 1/k^{1/2}$ and

$$\sum_{k=0}^{d}\sum_{\substack{1\leqq\|\alpha\|\leqq L_0\\ \alpha_*=k}}b_\alpha^2=\rho\,.$$

Thus there is a $k_0\in\{0,\cdots,d\}$ such that

$$N_{k_0}\equiv\sum_{\substack{1\leqq\|\alpha\|\leqq L_0\\ \alpha_*=k_0}}b_\alpha^2\geqq\rho/(d+1)\geqq 1/(d+1)K^{1/2}\,.$$

Since $|\beta(t)|^2+|\gamma(t)|^2 \geqq \xi_{k_0}^2(t)$, we have that:

$$\mathcal{W}(E_3) \leqq \left(\int_0^1 \xi_{k_0}^2(t)dt \leqq 1/K\right).$$

At the same time, by induction hypothesis:

$$\begin{aligned}\mathcal{W}\left(\int_0^1 \xi_{k_0}^2(t)dt \leqq 1/K\right) &= \mathcal{W}\left(\frac{1}{N_{k_0}}\int_0^1 \xi_{k_0}^2(t)dt \leqq 1/N_{k_0}K\right) \\ &\leqq C_{L_0} \exp(-(N_k K)^{\mu L_0}) \\ &\leqq C_{L_0} \exp(-(K^{1/2}/d+1)^{\mu L_0}).\end{aligned}$$

Combining this with the estimates already obtained on $\mathcal{W}(E_1)$ and $\mathcal{W}(E_2)$, we conclude that there exist $C_{L_0+1}<\infty$ and $\mu_{L_0+1} \in (0, \infty)$ such that the desired estimate holds for all $K \in [16, \infty)$. Q. E. D.

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