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Modified wave operators with time-independent modifiers

By Hiroshi ISOZAKI and Hitoshi KITADA

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§ 1. Introduction.

In this paper, we introduce a new type of modified wave operators for long-range potential scattering, and prove its completeness.

The long-range scattering problem has been studied by many authors, and it is now in a satisfactory stage at least concerning the existence and completeness of the usual modified wave operators with time-dependent modifier. (For the existence, see Hörmander [6] and the references cited there.) Concerning the completeness there are two types of proof, one by the stationary method and the other by the so-called Enss' time-dependent or geometrical method. For the first type of proof, see Kitada [13], Ikebe and Isozaki [7], [8]. For the second type of proof, see Kitada and Yajima [17], [18], Isozaki [10], Muthuramalingam [20], and the references therein.) However, concerning the scattering amplitude which should be the most important physical quantity in the scattering theory, there seems to be few satisfactory reference up to now except for Agmon's result [1] on the smoothness of the long-range scattering amplitude off diagonal. This paper is the first of a series of papers aiming at studying the long-range scattering amplitude, and, as a first step, is concerned with the proof of the existence and the completeness of a new type of modified wave operators with *time-independent* modifier J. Such types of modified wave operators have been considered by some authors (see e. g. Kako [12] and the references cited there). However, we shall show in subsequent publications our choice of the time-independent modifier opens a new scope to the study of the scattering amplitude.

We consider the Schrödinger operators in $\mathcal{H}=L^2(R^N)$, $N\geqq 1$:

$$(1.1)\qquad \begin{cases} H_0=-\dfrac{1}{2}\Delta=-\dfrac{1}{2}\sum_{j=1}^{N}\partial^2/\partial x_j^2\,, \\ H=H_0+V\,, \end{cases}$$

where the perturbation V is decomposable as $V=V_L(x)+V_S$ with $V_L(x)$

and V_S satisfying the following:

ASSUMPTION.

(L) $V_L(x)$ is a real-valued C^∞ function on R^N such that

$$|\partial_x^\alpha V_L(x)| \leqq C_\alpha \langle x\rangle^{-|\alpha|-\varepsilon} \tag{1.2}$$

for some $0<\varepsilon<1$ and all α, where $\langle x\rangle=\sqrt{1+|x|^2}$.

(S) V_S is a symmetric H_0-compact operator in $\mathcal{H}$ such that the function

$$h(R)=\| V_S(H_0+1)^{-1}\chi_{(|x|\geq R)}\| \tag{1.3}$$

belongs to $L^1([0, \infty))$. Here χ_A is the multiplication operator by the characteristic function of the set A.

Under the above assumption, H is a self-adjoint operator in $\mathcal{H}$ with the domain $\mathcal{D}(H)=\mathcal{D}(H_0)=H^2(R^N)$ (= the Sobolev space of order 2). Hence H generates a unitary group e^{-itH}.

The modified wave operator we shall propose to discuss here is then defined in the form:

$$W_J^\mp(\Gamma)=\text{s-}\lim_{t\to\pm\infty} e^{itH}Je^{-itH_0}E_{H_0}(\Gamma)\,, \qquad \Gamma=[a_0, \infty)\,, \tag{1.4}$$

where $a_0>0$ is an arbitrarily fixed constant, and E_{H_0} and E_H are the spectral measures for H_0 and H. The "modifier" J is time-independent, and will be defined by (3.44) in Section 3 in the form of Fourier integral operators.

We denote by $\mathcal{H}_c(H)$, $\mathcal{H}_{ac}(H)$, and $\mathcal{H}_{sc}(H)$ the continuous, absolutely continuous, and singular continuous spectral subspaces for H. Then our main theorem is the following. We denote the range of an operator T by $\mathcal{R}(T)$.

THEOREM 1.1. *Let the assumptions* (L) *and* (S) *be satisfied. Then the modified wave operators* $W_J^\mp(\Gamma)$ *defined by* (1.4) *exist; are partial isometries in* $\mathcal{H}$; *have the intertwining property; and are complete in the sense that for any* $\Gamma=[a, \infty)$, $a>0$

$$\mathcal{R}(W_J^\mp(\Gamma))=E_H(\Gamma)\mathcal{H}_c(H)\,. \tag{1.5}$$

In particular, the singular continuous spectrum is absent: $\mathcal{H}_{sc}(H)=\{0\}$.

The proof is carried out along the line of the Enss method ([4], [5]), and is similar to that of Kitada and Yajima [18] in the sense that we shall use a compactness argument in its final step (see Section 4). Our definition of the outgoing and ingoing approximate propagators (see Section

3) is essentially the same as in Isozaki [10], in which, using these propagators, he considered the completeness of the usual modified wave operator with time-dependent modifier. Because of the introduction of the time-*in*dependent modifier J, our argument seems to be more transparent than that of [18], e.g. the estimation of the operator norm in $\mathcal{H}=L^2$ of the relevant operators becomes easier (see e.g. Lemma 3.3 and its proof in the appendix). Our approach can also be extended to cover the time-dependent potentials, and we can give another proof of the results in Kitada and Yajima [17], [18]. For the sake of simplicity, we shall not present it here, however. The construction of the phase function φ of J requires more elaborate calculations than in e.g. [17], but due to this phase function φ, we can treat the scattering amplitude for smooth potentials. Our results and the brief sketch of the proof have been announced in Isozaki and Kitada [11].

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§ 2. Classical orbits.

The purpose of this section is to construct a solution φ of the eikonal equation

$$\frac{1}{2}|\nabla_x\varphi(x,\xi)|^2+V_L(x)=\frac{1}{2}|\xi|^2 \tag{2.1}$$

in the outgoing and ingoing regions of the phase space $R_x^N\times R_\xi^N$, namely in the regions where $\cos(x,\xi)=x\cdot\xi/|x||\xi|\geqq\sigma^+(>-1)$ or $\cos(x,\xi)\leqq\sigma^-(<1)$. Such solutions of (2.1) have already been constructed by Isozaki [9], [10] by a direct method. However we present here a new proof, which clarifies a close connection between the solutions of the eikonal equation (2.1) and the Hamilton-Jacobi equation

$$\partial_t\phi(t\,;x,\xi)=\frac{1}{2}|\xi|^2+V(\nabla_\xi\phi(t\,;x,\xi))\,. \tag{2.2}$$

Following the idea of Kitada and Yajima [17], we introduce the time-dependent potential:

$$V_\rho(t,x)=V_L(x)\chi_0(\rho x)\chi_0(\langle\log\langle t\rangle\rangle x/\langle t\rangle)\,,\qquad 0<\rho<1\,, \tag{2.3}$$

where $\chi_0(x)$ is a fixed C^∞ function having the property $0\leqq\chi_0(x)\leqq1$, $\chi_0(x)=1$

for $|x|\geqq 2$ and $=0$ for $|x|\leqq 1$. Then $V_\rho(t, x)$ obviously satisfies

$$|\partial_x^\alpha V_\rho(t, x)|\leqq C_\alpha \rho^{\varepsilon_0}\langle t\rangle^{-l}\langle x\rangle^{-m} \tag{2.4}$$

for any l, $m\geqq 0$, $0<\varepsilon_0<\varepsilon$ with $\varepsilon_0+l+m<|\alpha|+\varepsilon$, where $C_\alpha=C_{\alpha,\varepsilon_0,l,m}$ is independent of t, x and ρ.

Consider the classical orbit $(q, p)(t, s; y, \xi)$ which satisfies the integrated form of the Hamilton's canonical equations

$$\begin{cases} q(t, s)=y+\int_s^t p(\tau, s)d\tau\,, \\ p(t, s)=\xi-\int_s^t \nabla_x V_\rho(\tau, q(\tau, s))d\tau\,. \end{cases} \tag{2.5}$$

This equation has a unique solution verifying the following estimates, which can be proved in a way quite similar to that of Proposition 2.1 in Kitada [14].

PROPOSITION 2.1. *Let ε_0, $\varepsilon_1>0$ be fixed so that $0<\varepsilon_0+\varepsilon_1<\varepsilon$. Then:*

i) *There exist constants C_l $(l=0, 1, 2, \cdots)$ such that for any $(y, \xi)\in R^{2N}$, $\pm t\geqq\pm s\geqq 0$, and α, β*

$$|p(s, t; y, \xi)-\xi|\leqq C_0\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}\,,$$

$$\begin{cases} |\partial_y^\alpha\partial_\xi^\beta[\nabla_y q(s, t; y, \xi)-I]|\leqq C_{|\alpha+\beta|}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}\,, \\ |\partial_y^\alpha\partial_\xi^\beta[\nabla_y p(s, t; y, \xi)]|\leqq C_{|\alpha+\beta|}\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}\,, \end{cases}$$

$$\begin{cases} |\nabla_\xi q(t, s; y, \xi)-(t-s)I|\leqq C_0\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}|t-s|\,, \\ |\nabla_\xi p(t, s; y, \xi)-I|\leqq C_0\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}\,, \end{cases}$$

$$\begin{cases} |\nabla_y q(t, s; y, \xi)-I|\leqq C_0\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}|t-s|\,, \\ |\nabla_y p(t, s; y, \xi)|\leqq C_0\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}\,, \end{cases}$$

$$\begin{aligned} &|\partial_\xi^\alpha[q(t, s; y, \xi)-y-(t-s)p(t, s; y, \xi)]| \\ &\leqq C_{|\alpha|}\rho^{\varepsilon_0}\min(\langle t\rangle^{1-\varepsilon_0}, |t-s|\langle s\rangle^{-\varepsilon_0})\,. \end{aligned}$$

ii) *For $|\alpha+\beta|\geqq 2$ and some $C_{\alpha\beta}$,*

$$\begin{cases} |\partial_y^\alpha\partial_\xi^\beta q(t, s; y, \xi)|\leqq C_{\alpha\beta}\rho^{\varepsilon_0}|t-s|\langle s\rangle^{-\varepsilon_0}\,, \\ |\partial_y^\alpha\partial_\xi^\beta p(t, s; y, \xi)|\leqq C_{\alpha\beta}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}\,. \end{cases}$$

From this, we can prove the following proposition in a way similar to that of Proposition 2.2 in [14].

PROPOSITION 2.2. *Take $\rho\in(0,1)$ so small that $C_0\rho^{\varepsilon_0}<1/2$ holds for the constant C_0 appearing in Proposition 2.1. Then for $\pm t\geqq\pm s\geqq 0$ there exist diffeomorphisms $x\to \boldsymbol{y}(s,t;x,\xi)$ and $\xi\to\boldsymbol{\eta}(t,s;x,\xi)$ inverse to the diffeomorphisms $y\to x=q(s,t;x,\xi)$ and $\eta\to\xi=p(t,s;x,\eta)$, and they are C^∞ in $(x,\xi)\in R^{2N}$ for each $t,s\in R^1$ and their derivatives $\partial_x^\alpha\partial_\xi^\beta y$ and $\partial_x^\alpha\partial_\xi^\beta\eta$ are C^1 in (t,s,x,ξ). Furthermore they have the following properties. Let ε_0 and ε_1 be as in Proposition 2.1.*

i)
$$\begin{cases} q(s,t;\boldsymbol{y}(s,t;x,\xi),\xi)=x \\ p(t,s;x,\boldsymbol{\eta}(t,s;x,\xi))=\xi\,, \end{cases}$$
$$\begin{cases} \boldsymbol{y}(s,t;x,\xi)=q(t,s;x,\boldsymbol{\eta}(t,s;x,\xi)) \\ \boldsymbol{\eta}(t,s;x,\xi)=p(s,t;\boldsymbol{y}(s,t;x,\xi),\xi)\,. \end{cases}$$

ii) *For any α and β,*
$$\begin{cases} |\partial_x^\alpha\partial_\xi^\beta[\nabla_x\boldsymbol{y}(s,t;x,\xi)-I]|\leqq C_{\alpha\beta}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}\,, \\ |\partial_x^\alpha\partial_\xi^\beta[\nabla_x\boldsymbol{\eta}(t,s;x,\xi)]|\leqq C_{\alpha\beta}\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}\,, \end{cases}$$
$$\begin{cases} |\partial_\xi^\alpha[\boldsymbol{\eta}(t,s;x,\xi)-\xi]|\leqq C_{\alpha\beta}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}\,, \\ |\partial_\xi^\alpha[\boldsymbol{y}(s,t;x,\xi)-x-(t-s)\xi]|\leqq C_\alpha\rho^{\varepsilon_0}\min\{\langle t\rangle^{1-\varepsilon_0},\,|t-s|\langle s\rangle^{-\varepsilon_0}\}\,. \end{cases}$$

iii) *For $|\alpha+\beta|\geqq 2$,*
$$\begin{cases} |\partial_x^\alpha\partial_\xi^\beta\eta(t,s;x,\xi)|\leqq C_{\alpha\beta}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}\,, \\ |\partial_x^\alpha\partial_\xi^\beta y(s,t;x,\xi)|\leqq C_{\alpha\beta}\rho^{\varepsilon_0}\langle t-s\rangle\langle s\rangle^{-\varepsilon_1}\,. \end{cases}$$

Here the constants C_α and $C_{\alpha\beta}$ are independent of ρ,t,s,x and ξ. (In the following, $C, C_{\alpha\beta}, C_{\alpha\beta\gamma}, C_l$, etc. denote the various constants which do not depend on the variables appearing in each formula.)

The following reformulation of Proposition 2.2-i) will be helpful to understand the meaning of the above diffeomorphism: Let $U(t,s)$ be the map which assigns the solution $(q,p)(t,s;x,\eta)$ to the initial data (x,η). Then

$$\begin{array}{ccc} \text{time } s & & \text{time } t \\ \begin{pmatrix} x \\ \boldsymbol{\eta}(t,s;x,\xi) \end{pmatrix} & \xmapsto{U(t,s)} & \begin{pmatrix} \boldsymbol{y}(s,t;x,\xi) \\ \xi \end{pmatrix}. \end{array}$$

Now we define $\phi(t;x,\xi)$ as follows.

$$\phi(t;x,\xi)=u(t;x,\boldsymbol{\eta}(t,0;x,\xi)), \tag{2.6}$$

where

$$u(t;x,\eta)=x\cdot\eta+\int_0^t\{H_\rho-x\cdot\nabla_x H_\rho\}(\tau,q(\tau,0;x,\eta),p(\tau,0;x,\eta))d\tau, \tag{2.7}$$

$$H_\rho(t,x,\xi)=\frac{1}{2}|\xi|^2+V_\rho(t,x). \tag{2.8}$$

Standard calculations show that $\phi(t;x,\xi)$ satisfies the Hamilton-Jacobi equation

$$\begin{cases}\partial_t\phi(t;x,\xi)=\frac{1}{2}|\xi|^2+V_\rho(t,\nabla_\xi\phi(t;x,\xi)), \\ \phi(0;x,\xi)=x\cdot\xi, \end{cases} \tag{2.9}$$

and the relation

$$\begin{cases}\nabla_x\phi(t;x,\xi)=\boldsymbol{\eta}(t,0;x,\xi), \\ \nabla_\xi\phi(t;x,\xi)=\boldsymbol{y}(0,t;x,\xi). \end{cases} \tag{2.10}$$

We now consider the following limit.

DEFINITION 2.3. For any $(x,\xi)\in R^{2N}$,

$$\phi_\pm(x,\xi)=\lim_{t\to\pm\infty}(\phi(t;x,\xi)-\phi(t;0,\xi)). \tag{2.11}$$

Set $\Gamma_\pm(R,d,\sigma_0)=\{(x,\xi)\in R^{2N}|\,|x|\geqq R,\ |\xi|\geqq d,\ \pm\cos(x,\xi)\geqq-\sigma_0\}$ for R, $d>0$ and $\sigma_0\in(0,1)$.

PROPOSITION 2.4. *The limit* (2.11) *exists for any* $(x,\xi)\in R^{2N}$ *and defines a* C^∞ *function* $\phi_\pm(x,\xi)$ *having the following properties: For any* d, $\sigma_0\in(0,1)$ *there exist* $R\gg1$ *and* $0<\rho\ll d$ *such that for any* $(x,\xi)\in\Gamma_\pm(R,d,\sigma_0)$

$$\frac{1}{2}|\nabla_x\phi_\pm(x,\xi)|^2+V_L(x)=\frac{1}{2}|\xi|^2 \tag{2.12}$$

and

$$|\partial_x^\alpha\partial_\xi^\beta(\phi_\pm(x,\xi)-x\cdot\xi)|\leqq C_{\alpha\beta}|\xi|^{-1}\langle x\rangle^{1-|\alpha|-\varepsilon}. \tag{2.13}$$

PROOF. We consider $\phi_+(x,\xi)$ only, since $\phi_-(x,\xi)$ can be treated similarly. We first prove the existence of the limit (2.11). Letting $R(t,x,\xi)=\phi(t;x,\xi)-\phi(t;0,\xi)$, we have by (2.9)

(2.14) $$\partial_t R(t, x, \xi) = \nabla_\xi R(t, x, \xi)\cdot a(t, x, \xi)\ ,$$

where

(2.15) $$a(t, x, \xi) = \int_0^1 (\nabla_x V_\rho)(t, \nabla_\xi \phi(t\,; 0, \xi) + \theta \nabla_\xi R(t, x, \xi))d\theta\ .$$

Since by (2.10)

$$\nabla_\xi R(t, x, \xi) = y(0, t\,; x, \xi) - y(0, t\,; 0, \xi)$$

$$= x\cdot \int_0^1 (\nabla_x y)(0, t\,; \theta x, \xi)d\theta$$

we have by Proposition 2.2-ii)

(2.16) $$|\partial_x^\alpha \partial_\xi^\beta \nabla_\xi R(t, x, \xi)| \leqq C_{\alpha\beta}\langle x\rangle\ .$$

By (2.10) and Proposition 2.2-ii),

$$|\partial_\xi^\beta \nabla_\xi \phi(t\,; 0, \xi)| \leqq C_\beta |t|\ , \qquad |\beta| \neq 0\ .$$

From this, (2.4), (2.15) and (2.16) follows that

(2.17) $$|\partial_x^\alpha \partial_\xi^\beta a(t, x, \xi)| \leqq C_{\alpha\beta}\langle t\rangle^{-1-\varepsilon/2}\langle x\rangle^{|\alpha+\beta|}\ .$$

Thus, by (2.14), (2.16) and (2.17), the limit

(2.18) $$\lim_{t\to\infty} \partial_x^\alpha \partial_\xi^\beta R(t, x, \xi) = \int_0^\infty \partial_x^\alpha \partial_\xi^\beta \{\nabla_\xi R(t, x, \xi)\cdot a(t, x, \xi)\}dt$$

exists, hence $\phi_+(x, \xi) = \lim_{t\to\infty} R(t, x, \xi)$ and $\boldsymbol{\eta}(\infty, 0\,; x, \xi) = \lim_{t\to\infty} \nabla_x \phi(t\,; x, \xi)$ exist and are C^∞.

We next prove (2.12). From the above discussion, the limit

$$\nabla_x \phi_+(x, \xi) = \lim_{t\to\infty} \nabla_x \phi(t\,; x, \xi) = \lim_{t\to\infty} \boldsymbol{\eta}(t, 0\,; x, \xi)$$

$$= \lim_{t\to\infty} p(0, t\,; \boldsymbol{y}(0, t\,; x, \xi), \xi)$$

exists. Thus for $|x|$ large enough, we have

$$\frac{1}{2}|\nabla_x \phi_+(x, \xi)|^2 + V_L(x) = \lim_{t\to\infty} \frac{1}{2}|p(0, t\,; \boldsymbol{y}(0, t\,; x, \xi), \xi)|^2 + V_\rho(0, x)\ .$$

Setting for $0 \leqq s \leqq t < \infty$

$$f_t(s, y, \xi) = \frac{1}{2}|p(s, t\,; y, \xi)|^2 + V_\rho(s, q(s, t\,; y, \xi))\ ,$$

we have by (2.5)

$$\frac{\partial f_t}{\partial s}(s, y, \xi) = \frac{\partial V_\rho}{\partial t}(t, x)\Big|_{t=s,\, x=q(s,t\,;\,y,\xi)}.$$

On the other hand, Proposition 2.2-i) shows that

$$\begin{cases} q(s, t\,; \boldsymbol{y}(0, t\,; x, \xi), \xi) = q(s, 0\,; x, \boldsymbol{\eta}(t, 0\,; x, \xi)), \\ p(s, t\,; \boldsymbol{y}(0, t\,; x, \xi), \xi) = p(s, 0\,; x, \boldsymbol{\eta}(t, 0\,; x, \xi)). \end{cases} \tag{2.19}$$

Therefore, using Proposition 2.1-i), we can see that for $\cos(x, \xi) \geqq -\sigma_0$

$$\begin{aligned} &|q(s, t\,; \boldsymbol{y}(0, t\,; x, \xi), \xi)| = |q(s, 0\,; x, \boldsymbol{\eta}(t, 0\,; x, \xi))| \\ &\geqq |x + sp(s, 0\,; x, \boldsymbol{\eta}(t, 0\,; x, \xi))| - C_0\rho^{\varepsilon_0}\langle s\rangle^{1-\varepsilon_0} \\ &= |x + sp(s, t\,; \boldsymbol{y}(0, t\,; x, \xi), \xi)| - C_0\rho^{\varepsilon_0}\langle s\rangle^{1-\varepsilon_0} \\ &\geqq c(|x| + s|\xi|) - C_0\rho^{\varepsilon_0}\langle s\rangle^{1-\varepsilon_1} - C_0\rho^{\varepsilon_0}\langle s\rangle^{1-\varepsilon_0} \end{aligned}$$

for some constant $C>0$ and the constant C_0 in Proposition 2.1. Since $|\xi| \geqq d$ and $\operatorname{supp} \frac{\partial V_\rho}{\partial s}(s, x) \subset \{x \mid 1 \leqq \langle \log\langle s\rangle\rangle |x|/\langle s\rangle \leqq 2\}$, we can find some large $S = S_{d,\sigma_0}$ independent of t such that for $s \in [S, t]$

$$\frac{\partial f_t}{\partial s}(s, \boldsymbol{y}(0, t\,; x, \xi), \xi) = 0\,.$$

For $s \in [0, S]$, taking $R = R_S$ large enough, we also have for $|x| \geqq R$ and $\cos(x, \xi) \geqq -\sigma_0$

$$\frac{\partial f_t}{\partial s}(s, \boldsymbol{y}(0, t\,; x, \xi), \xi) = 0\,.$$

Summing up, we have proved that for $(x, \xi) \in \Gamma_+(R, d, \sigma_0)$

$$f_t(s, \boldsymbol{y}(0, t\,; x, \xi), \xi) = \text{constant} \qquad \text{for} \quad 0 \leqq s \leqq t < \infty\,.$$

In particular, we have $f_t(0, \boldsymbol{y}(0, t\,; x, \xi), \xi) = f_t(t, \boldsymbol{y}(0, t\,; x, \xi), \xi)$, hence

$$\frac{1}{2}|p(0, t\,; \boldsymbol{y}(0, t\,; x, \xi), \xi)|^2 + V_\rho(0, x) = \frac{1}{2}|\xi|^2 + V_\rho(t, \boldsymbol{y}(0, t\,; x, \xi))\,.$$

Letting $t \to \infty$ in this equality, we get (2.12) for $(x, \xi) \in \Gamma_+(R, d, \sigma_0)$ if R is sufficiently large.

We finally prove the estimate (2.13). We first consider the derivatives:

$$\partial_\xi^\beta(\phi_+(x,\xi)-x\cdot\xi)=\int_0^\infty \partial_\xi^\beta \partial_t R(t,x,\xi)dt\,. \tag{2.20}$$

Letting $\gamma(t,x,\xi)=\boldsymbol{y}(0,t;x,\xi)-(x+t\xi)$, we get from Proposition 2.2 that for $(x,\xi)\in\Gamma_+(R,d,\sigma_0)$

$$\begin{aligned} &|\nabla_\xi\phi(t;0,\xi)+\theta\nabla_\xi R(t,x,\xi)| \\ &=|\boldsymbol{y}(0,t;0,\xi)+\theta(\boldsymbol{y}(0,t;x,\xi)-\boldsymbol{y}(0,t;0,\xi))| \\ &=|t\xi+\gamma(t,0,\xi)+\theta\{x+t\xi-t\xi+\gamma(t,x,\xi)-\gamma(t,0,\xi)\}| \\ &=|\theta x+t\xi+(1-\theta)\gamma(t,0,\xi)+\theta\gamma(t,x,\xi)| \\ &\geqq c_0(\theta|x|+t|\xi|)-c_1\rho^{\varepsilon_0}\min\{\langle t\rangle^{1-\varepsilon_0},|t|\}\,. \end{aligned} \tag{2.21}$$

Thus there is a large enough $T=T_{d,\sigma_0}>0$ such that for $t\geqq T$ and $(x,\xi)\in\Gamma_+(R,d,\sigma_0)$

$$\langle\nabla_\xi\phi(t;0,\xi)+\theta\nabla_\xi R(t,x,\xi)\rangle^{-1}\leqq C\langle\theta|x|+t|\xi|\rangle^{-1}\,,$$

hence by (2.3), (2.15)

$$|\partial_\xi^\beta a(t,x,\xi)|\leqq C_\beta\int_0^1\langle\theta|x|+t|\xi|\rangle^{-1-\varepsilon}d\theta\,.$$

In view of (2.21), we can also prove this inequality for $0\leqq t\leqq T$ with C_β replaced by another constant $C_{T,\beta}$. From this, (2.14), (2.16), and (2.20) follows that for $(x,\xi)\in\Gamma_+(R,d,\sigma_0)$

$$\begin{aligned} |\partial_\xi^\beta(\phi_+(x,\xi)-x\cdot\xi)| &\leqq C_{T,\beta}\langle x\rangle\int_0^\infty\int_0^1\langle\theta|x|+t|\xi|\rangle^{-1-\varepsilon}d\theta dt \\ &\leqq C_{T,\beta}\langle x\rangle|\xi|^{-1}\int_0^1\langle\theta|x|\rangle^{-\varepsilon}d\theta\leqq C_{T,\beta}\langle x\rangle^{1-\varepsilon}|\xi|^{-1}\,. \end{aligned}$$

For the derivatives $\partial_x^\alpha\partial_\xi^\beta(\phi_+(x,\xi)-x\cdot\xi)$, $|\alpha|\neq 0$, we use the following expression:

$$\begin{aligned} &\nabla_x\phi_+(x,\xi)-\xi \\ &=\lim_{t\to\infty}(\nabla_x\phi(t;x,\xi)-\xi) \\ &=\lim_{t\to\infty}(p(0,t;\boldsymbol{y}(0,t;x,\xi),\xi)-\xi) \\ &=\lim_{t\to\infty}\int_0^t(\nabla_x V_\rho)(\tau,q(\tau,t;\boldsymbol{y}(0,t;x,\xi),\xi))d\tau \\ &=\lim_{t\to\infty}\int_0^t(\nabla_x V_\rho)(\tau,q(\tau,0;x,\boldsymbol{\eta}(t,0;x,\xi)))d\tau \end{aligned} \tag{2.22}$$

where we have used (2.5), (2.19), (2.10) and the existence of the limit (2.18). Since $|(\nabla_x V_\rho)(\tau, x)| \leqq C\langle\tau\rangle^{-1-\varepsilon/2}$ and $\lim_{t\to\infty} \boldsymbol{\eta}(t, 0; x, \xi) \equiv \boldsymbol{\eta}(\infty, 0; x, \xi)$ exists by (2.18), (2.22) is equal to

$$\int_0^\infty (\nabla_x V_\rho)(\tau, q(\tau, 0; x, \boldsymbol{\eta}(\infty, 0; x, \xi)))d\tau .$$

By Proposition 2.1-i), we have

$$\begin{aligned}
&|q(\tau, 0; x, \boldsymbol{\eta}(\infty, 0; x, \xi))| \\
&\geqq |x+\tau p(\tau, 0; x, \boldsymbol{\eta}(\infty, 0; x, \xi))| - C_1\rho^{\varepsilon_0}\langle\tau\rangle^{1-\varepsilon_0} \\
&\geqq |x+\tau\boldsymbol{\eta}(\infty, 0; x, \xi)| - C_0\rho^{\varepsilon_0}\langle\tau\rangle^{1-\varepsilon_1} - C_1\rho^{\varepsilon_0}\langle\tau\rangle^{1-\varepsilon_0},
\end{aligned}$$

and by Proposition 2.2-ii) and the existence of $\boldsymbol{\eta}(\infty, 0; x, \xi) = \lim_{t\to\infty} \boldsymbol{\eta}(t, 0; x, \xi)$

$$|\boldsymbol{\eta}(\infty, 0; x, \xi) - \xi| \leqq C\rho^{\varepsilon_0} .$$

Therefore, taking $\rho > 0$ small enough and $R = R_{d,\sigma_0,\rho}$ large enough such that $\rho \ll d \leqq |\xi|$, we have for $(x, \xi) \in \Gamma_+(R, d, \sigma_0)$

$$\begin{aligned}
&|q(\tau, 0; x, \boldsymbol{\eta}(\infty, 0; x, \xi))| \\
&\geqq |x+\tau\xi| - C\rho^{\varepsilon_0} - C_0\rho^{\varepsilon_0}\langle\tau\rangle^{1-\varepsilon_1} - C_1\rho^{\varepsilon_0}\langle\tau\rangle^{1-\varepsilon_0} \\
&\geqq c_0(|x|+\tau|\xi|) ,
\end{aligned} \tag{2.23}$$

where $c_0 > 0$ is independent of x, ξ and τ. Thus we have by (2.22), (2.23)

$$|\nabla_x\phi_+(x, \xi) - \xi| \leqq C\int_0^\infty \langle|x|+\tau|\xi|\rangle^{-1-\varepsilon}d\tau \leqq C|\xi|^{-1}\langle x\rangle^{-\varepsilon} . \tag{2.24}$$

Higher derivatives can be treated similarly. □

Let $-1 < \sigma_0 < \sigma_1 < 1$, and let $\psi_\pm(\sigma) \in C^\infty([-1, 1])$ satisfy

$$\begin{cases}
0 \leqq \psi_\pm(\sigma) \leqq 1 , \\
\psi_+(\sigma) = \begin{cases} 1 & \text{for} \quad \sigma_1 \leqq \sigma \leqq 1 , \\ 0 & \text{for} \quad -1 \leqq \sigma \leqq (\sigma_0+\sigma_1)/2 , \end{cases} \\
\psi_-(\sigma) = \begin{cases} 0 & \text{for} \quad (\sigma_0+\sigma_1)/2 \leqq \sigma \leqq 1 , \\ 1 & \text{for} \quad -1 \leqq \sigma \leqq \sigma_0 . \end{cases}
\end{cases} \tag{2.25}$$

Set $\chi_\pm(x, \xi) = \psi_\pm(\cos(x, \xi))$, $\cos(x, \xi) = x\cdot\xi/|x||\xi|$, and define $\varphi(x, \xi) = \varphi_{\sigma_0,\sigma_1,d}(x, \xi)$ by

$$\varphi(x,\xi)=\{(\phi_+(x,\xi)-x\cdot\xi)\chi_+(x,\xi)+(\phi_-(x,\xi)-x\cdot\xi)\chi_-(x,\xi)\}\times\chi_0(2\xi/d)\chi_0(2x/R)+x\cdot\xi\,, \tag{2.26}$$

where $d\in(0,1)$ and $R=R_{d,\sigma_0,\sigma_1}$ ($\gg 1$) are the constants specified in Proposition 2.4. Using Proposition 2.4, we can easily obtain the following theorem.

THEOREM 2.5. *Let the assumption* (L) *be satisfied. Choose* $-1<\sigma_0<\sigma_1<1$ *and* $d>0$ *arbitrarily. Then there exist constants* $R>2$, $0<\rho\ll d$ *and a* C^∞ *function* $\varphi(x,\xi)=\varphi_{\sigma_0,\sigma_1,d}(x,\xi)$ *having the following properties*:

i) *For* $|\xi|\geqq d$, $\cos(x,\xi)\in[-1,\sigma_0]\cup[\sigma_1,1]$ *and* $|x|\geqq R$, φ *solves the eikonal equation*

$$\frac{1}{2}|\nabla_x\varphi(x,\xi)|^2+V_L(x)=\frac{1}{2}|\xi|^2\,. \tag{2.27}$$

ii) *For any* $(x,\xi)\in R^{2N}$ *and* α,β,φ *verifies the estimate*

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x,\xi)-x\cdot\xi)|\leqq C_{\alpha\beta}\langle x\rangle^{1-\varepsilon-|\alpha|}\langle\xi\rangle^{-1}\,. \tag{2.28}$$

In particular, if $|\alpha|\neq 0$,

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x,\xi)-x\cdot\xi)|\leqq C_{\alpha\beta}R^{-\varepsilon_0}\langle x\rangle^{1-\varepsilon_1-|\alpha|}\langle\xi\rangle^{-1} \tag{2.29}$$

for any ε_0, $\varepsilon_1\geqq 0$ *with* $\varepsilon_0+\varepsilon_1=\varepsilon$. *Furthermore*

$$\varphi(x,\xi)=x\cdot\xi \qquad \text{for} \quad |x|\leqq R/2 \quad \text{or} \quad |\xi|\leqq d/2\,. \tag{2.30}$$

iii) *Set*

$$a(x,\xi)=e^{-i\varphi(x,\xi)}\Big(-\frac{1}{2}\Delta+V_L(x)-\frac{1}{2}|\xi|^2\Big)e^{i\varphi(x,\xi)}\,. \tag{2.31}$$

Then we have

$$a(x,\xi)=\frac{1}{2}|\nabla_x\varphi(x,\xi)|^2+V_L(x)-\frac{1}{2}|\xi|^2-\frac{1}{2}i\Delta_x\varphi(x,\xi)\,, \tag{2.32}$$

and the estimate: *For* $|\xi|\geqq d$ *and* $|x|\geqq R$

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\leqq\begin{cases}C_{\alpha\beta}\langle x\rangle^{-1-\varepsilon-|\alpha|}\langle\xi\rangle^{-1}\,, & \cos(x,\xi)\in[-1,\sigma_0]\cup[\sigma_1,1]\,,\\ C_{\alpha\beta}\langle x\rangle^{-\varepsilon-|\alpha|}\,, & \cos(x,\xi)\in[\sigma_0,\sigma_1]\,.\end{cases} \tag{2.33}$$

§ 3. Outgoing and ingoing approximate propagators.

In this section, we introduce the operators of localization $P_\pm$ and approximate propagators $E_\pm(t)$, and investigate the properties of the difference $e^{-itH}P_\pm-E_\pm(t)$.

We begin with defining the Enss phase space decomposition operators. Fix an interval $\Delta=[a, b]$, $0<a<b<\infty$, and let $\gamma_\Delta(\lambda)\in C_0^\infty((0, \infty))$ be such that $0\leqq\gamma_\Delta(\lambda)\leqq 1$, $\gamma_\Delta(\lambda)=1$ for $\lambda\in\Delta=[a, b]$, and $=0$ for $\lambda\leqq a/2$ or $\lambda\geqq 2b$. Choose $\rho_\pm(\sigma)\in C^\infty([-1, 1])$ so that $0\leqq\rho_\pm(\sigma)\leqq 1$; $\rho_+(\sigma)+\rho_-(\sigma)=1$; and $\rho_+(\sigma)=1$ for $1/4\leqq\sigma\leqq 1$, $=0$ for $-1\leqq\sigma\leqq -1/4$. Set $H_0(\xi)=|\xi|^2/2$ and let χ_0 be the function in (2.3).

DEFINITION 3.1. Let

$$p_\pm(\xi, y)=\gamma_\Delta(H_0(\xi))\chi_0(y)\rho_\pm(\cos(\xi, y)) . \tag{3.1}$$

We define $P_\pm$ by

$$P_\pm f(x)=\mathrm{Os}\text{-}\iint e^{i(x-y)\cdot\xi}p_\pm(\xi, y)f(y)dy\,d\!\!{}^-\xi , \qquad f\in L^2(R^N) , \tag{3.2}$$

where $d\!\!{}^-\xi=(2\pi)^{-N}d\xi$ and $\mathrm{Os}\text{-}\iint\cdots$ means the usual oscillatory integral (cf. e.g. Kitada [15] or Kitada and Kumano-go [16]). In the following we always omit the prefix Os- for the sake of simplicity.

From (3.1) we have

$$p_+(\xi, y)+p_-(\xi, y)=\gamma_\Delta(H_0(\xi))\chi_0(y) ,$$

hence

$$P_+ + P_- =\gamma_\Delta(H_0)\chi_0(x) . \tag{3.3}$$

In order to define the approximate propagators, we utilize the solutions $\varphi_+(x, \xi)$ and $\varphi_-(x, \xi)$ of the eikonal equation (2.1). $\varphi_+(x, \xi)\,[\varphi_-(x, \xi)]$ is defined by Theorem 2.5 with $-1<\sigma_0<\sigma_1=-1/2$, $[1/2=\sigma_0<\sigma_1<1]$, $d=\sqrt{a}$. Then they satisfy the eikonal equation (2.27) for $(x, \xi)\in\Gamma_\pm(R, \sqrt{a}, 1/2)$, respectively. Furthermore

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi_\pm(x, \xi)-x\cdot\xi)|\leqq C_{\alpha\beta}\langle\xi\rangle^{-1}\langle x\rangle^{1-\varepsilon-|\alpha|} \tag{3.4}$$

for all α, β;

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi_\pm(x, \xi)-x\cdot\xi)|\leqq C_{\alpha\beta}R^{-\varepsilon/2}\langle\xi\rangle^{-1}\langle x\rangle^{1-\varepsilon/2-|\alpha|} \tag{3.5}$$

for $\alpha\neq 0$; and

$$\varphi_\pm(x, \xi)=x\cdot\xi \qquad \text{for} \quad |x|\leqq R/2 \quad \text{or} \quad |\xi|\leqq\sqrt{a}/2 . \tag{3.6}$$

In what follows, $R=R_a>2$ will be fixed so large that

$$|\nabla_x^t(\nabla_\xi\varphi_\pm(x, \xi))-I|<\frac{1}{2} \tag{3.7}$$

holds, where I is the $N\times N$ identity matrix.

DEFINITION 3.2. We set

$$J_{\pm}f(x)=\iint e^{i(\varphi_{\pm}(x,\xi)-y\cdot\xi)}f(y)dyd\!\!\!{}^{-}\xi\,, \tag{3.8}$$

and

$$\tilde{P}_{\pm}f(x)=\iint e^{i(x\cdot\xi-\varphi_{\pm}(y,\xi))}p_{\pm}(\xi,y)f(y)dyd\!\!\!{}^{-}\xi\,. \tag{3.9}$$

We then define $E_{\pm}(t)$ by

$$E_{\pm}(t)=J_{\pm}e^{-itH_0}\tilde{P}_{\pm}\,. \tag{3.10}$$

For estimating these operators, we prepare a lemma concerning the L^2-boundedness of some integral operators, whose proof will be given in the appendix.

LEMMA 3.3. *Let $\varphi(x,\xi)$ be a real-valued C^∞ function on $R_x^N\times R_\xi^N$ which satisfies*

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x,\xi)-x\cdot\xi)|\leqq C_{\alpha\beta}\langle x\rangle^{1-\varepsilon-|\alpha|} \tag{3.11}$$

for some $\varepsilon>0$ and all α, β, and

$$|\nabla_x^t\nabla_\xi\varphi(x,\xi)-I|<\frac{1}{2}\,, \tag{3.12}$$

where I is the $N\times N$ identity matrix.

i) *Let $p(x,\xi,y)$, $q(\xi,y)$, $r(x,\xi)$ be C^∞ functions such that the norms $|p|$, $|q|$, $|r|$ defined below are finite:*

$$|p|=\max_{|\alpha+\beta+\gamma|\leq M_0}\sup_{x,\xi,y}|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma p(x,\xi,y)|\,,$$

where $M_0=2([N/2]+[5N/4]+2)$. Then the operators P, Q, R defined by

$$\begin{cases} Pf(x)=\iint e^{i(\varphi(x,\xi)-\varphi(y,\xi))}p(x,\xi,y)f(y)dyd\!\!\!{}^{-}\xi\,, \\ Qf(x)=\iint e^{i(x\cdot\xi-\varphi(y,\xi))}q(\xi,y)f(y)dyd\!\!\!{}^{-}\xi\,, \\ Rf(x)=\iint e^{i(\varphi(x,\xi)-y\cdot\xi)}r(x,\xi)f(y)dyd\!\!\!{}^{-}\xi \end{cases} \tag{3.13}$$

are all bounded operators in $L^2(R^N)$ whose operator norms satisfy

$$\|P\|\leqq C_\varphi|p|\,,\quad \|Q\|\leqq C_\varphi|q|\,,\quad \|R\|\leqq C_\varphi|r| \tag{3.14}$$

for some constant C_φ independent of p, q, r.

ii) *Let $a^{\pm}(x,\xi)$ and $b^{\pm}(\xi,y)$ be C^∞ functions on R^{2N} verifying the following inequalities for some $L\geqq 0$, $-1<\theta_0<\theta_1<1$, $\delta_0>0$, $r\geqq 0$ and $\mu_0>0$:*

$$\text{(A)}\quad \begin{cases} \text{i)} & |\partial_x^\alpha \partial_\xi^\beta a^\pm(x,\xi)| \leqq \begin{cases} C_{\alpha\beta}\langle x\rangle^{-L}, & \pm\cos(x,\xi)\geqq\theta_0, \\ C_{\alpha\beta}\langle \xi\rangle^{\tau}, & \pm\cos(x,\xi)\leqq\theta_0. \end{cases} \\ \text{ii)} & a^\pm(x,\xi)=0 \quad \textit{for} \quad |x|\leqq\delta_0. \end{cases}$$

$$\text{(B)}\quad \begin{cases} \text{i)} & \textit{For any } m\geqq 0 \\ & |\partial_y^\alpha \partial_\xi^\beta b^\pm(\xi,y)| \leqq \begin{cases} C_{\alpha\beta}, & \pm\cos(\xi,y)\geqq\theta_1, \\ C_{\alpha\beta,m}\langle y\rangle^{-m}, & \pm\cos(\xi,y)\leqq\theta_1. \end{cases} \\ \text{ii)} & b^\pm(\xi,y)=0 \quad \textit{for} \quad |\xi|\leqq\mu_0 \quad \textit{or} \quad |y|\leqq\delta_0. \end{cases}$$

Then the operator $L^\pm(t)$ defined for $\pm t\geqq 0$ by

$$L^\pm(t)f(x)=\iint e^{i(\varphi(x,\xi)-t|\xi|^2-\varphi(y,\xi))}a^\pm(x,\xi)b^\pm(\xi,y)f(y)dyd\xi \tag{3.15}$$

satisfies the estimate

$$\|\langle D\rangle^{-\tau}\langle x\rangle^{s_1}L^\pm(t)\langle y\rangle^{s_2}\|\leqq C_{\tau,s_1,s_2}\langle t\mu_0\rangle^{-L+s_1+s_2} \tag{3.16}$$

for $\pm t\geqq 0$ and s_1, $s_2\geqq 0$ with $s_1+s_2\leqq L$.

Using this lemma, we have the following

PROPOSITION 3.4. i) *$P_\pm$, $J_\pm$, $\tilde{P}_\pm$ are bounded operators in $\mathcal{H}=L^2(R^N)$.*
ii) *$E_\pm(t)$ is norm continuous for $t\in R^1$, and*

$$\sup_{t\in R^1}\|E_\pm(t)\|<\infty. \tag{3.17}$$

iii) *$P_+ + P_- -\gamma_\Delta(H_0)$ is compact.*
iv) *$P_\pm^* - P_\pm$ is compact.*
v) *$E_\pm(0)-P_\pm$ is compact.*

PROOF. i) follows from Lemma 3.3-i). ii) (3.17) follows from (3.10) and i). The norm continuity of $E_\pm(t)$ follows from the fact that $\tilde{P}_\pm$ is a bounded operator from $L^2(R^N)$ into $H^m(R^N)$, the Sobolev space of order m, for any $m\geqq 0$. iii) is a consequence of (3.3). iv) We have

$$\begin{aligned} &(P_+^* - P_+)f(x) \\ &=\iint e^{i(x-y)\cdot\xi}(p_+(\xi,x)-p_+(\xi,y))f(y)dyd\xi \\ &=\iint e^{i(x-y)\cdot\xi}(x-y)\cdot\int_0^1 \nabla_x p_+(\xi,y+\theta(x-y))d\theta\, f(y)dyd\xi \end{aligned} \tag{3.18}$$

$$=i\iint e^{i(x-y)\cdot\xi}\int_0^1(\nabla_\xi\cdot\nabla_x p_+)(\xi, y+\theta(x-y))d\theta f(y)dyd\xi\ .$$

Here we take note of the following inequality

$$\int_0^1\langle y+\theta(x-y)\rangle^{-\varepsilon}d\theta\leqq C\min(\langle x\rangle^{-\varepsilon}, \langle y\rangle^{-\varepsilon}) \tag{3.19}$$

for $0<\varepsilon\leqq 1$, which can be proved directly. Then, letting $d(x, \xi, y)=\int_0^1(\nabla_\xi\cdot\nabla_x p_+)(\xi, y+\theta(x-y))d\theta$, we have

$$|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma d(x, \xi, y)|\leqq C_{\alpha\beta\gamma}\langle\xi\rangle^{-1}\langle x\rangle^{-1}\ .$$

Thus $\langle x\rangle\langle D\rangle(P_+^*-P_+)$ is bounded in $\mathscr{H}=L^2$ by Lemma 3.3-i), hence $P_+^*-P_+$ is compact. v) From (3.10) we have

$$(E_\pm(0)-P_\pm)f(x)=\iint e^{i(\varphi_\pm(x,\xi)-\varphi_\pm(y,\xi))}p_\pm(\xi, y)f(y)dyd\xi-P_\pm f(x)\ . \tag{3.20}$$

The phase factor of the integrand can be written as

$$\varphi_\pm(x, \xi)-\varphi_\pm(y, \xi)=(x-y)\cdot\nabla_x\varphi_\pm(x, \xi, y)\ ,$$

$$\nabla_x\varphi_\pm(x, \xi, y)=\int_0^1\nabla_x\varphi_\pm(y+\theta(x-y), \xi)d\theta\ .$$

In view of (3.4), (3.19), one can see that

$$|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma(\nabla_x\varphi_\pm(x, \xi, y)-\xi)|\leqq C_{\alpha\beta\gamma}\langle\xi\rangle^{-1}\langle x\rangle^{-\varepsilon}\ . \tag{3.21}$$

Taking into account of (3.7), we make a change of variable $\eta=\nabla_x\varphi_\pm(x, \xi, y)$. Denoting the inverse of $\xi\to\eta=\nabla_x\varphi_\pm(x, \xi, y)$ by $\nabla_x\varphi_\pm^{-1}(x, \eta, y)$ and its Jacobian by $J(x, \eta, y)=\left|\dfrac{D(\nabla_x\varphi_\pm^{-1})}{D(\eta)}(x, \eta, y)\right|$, we get

$$(E_\pm(0)-P_\pm)f(x)=\iint e^{i(x-y)\cdot\eta}s(x, \eta, y)f(y)dyd\eta\ , \tag{3.22}$$

where

$$s(x, \eta, y)=\{p_\pm(\nabla_x\varphi_\pm^{-1}(x, \eta, y), y)-p_\pm(\eta, y)\}J(x, \eta, y)$$
$$+p_\pm(\eta, y)(J(x, \eta, y)-1)\ .$$

Taking notice of (3.21), we see that

$$|\partial_x^\alpha\partial_\eta^\beta\partial_y^\gamma(\nabla_x\varphi_\pm^{-1}(x, \eta, y)-\eta)|\leqq C_{\alpha\beta\gamma}\langle\eta\rangle^{-1}\langle x\rangle^{-\varepsilon}\ .$$

Therefore $s(x, \eta, y)$ satisfies

$$|\partial_x^\alpha\partial_\eta^\beta\partial_y^\gamma s(x, \eta, y)|\leqq C_{\alpha\beta\gamma}\langle\eta\rangle^{-1}\langle x\rangle^{-\varepsilon}\ .$$

Thus the compactness of $E_{\pm}(0)-P_{\pm}$ follows from the same argument as in iv). □

Next consider the difference $e^{-itH}P_{\pm}-E_{\pm}(t)$, and set

$$G_{\pm}(t)=(D_t+H)E_{\pm}(t)\,, \qquad D_t=-i\partial/\partial t\,, \tag{3.23}$$

$$\begin{cases} G_{\pm,L}(t)=(D_t+H_L)E_{\pm}(t)\,, \\ G_{\pm,S}(t)=V_S E_{\pm}(t)\,, \end{cases} \tag{3.24}$$

where $H_L=H_0+V_L(x)$. Then $G_{\pm}(t)=G_{\pm,L}(t)+G_{\pm,S}(t)$, and

$$e^{-itH}P_{\pm}-E_{\pm}(t)=-ie^{-itH}\int_0^t e^{i\tau H}G_{\pm}(\tau)d\tau+e^{-itH}(P_{\pm}-E_{\pm}(0))\,. \tag{3.25}$$

Without loss of generality, we may and shall assume $V_L(x)=0$ for $|x|\leqq 1$ in the following.

THEOREM 3.5. *$G_{\pm,L}(t)$ is norm-continuous in $t\gtreqless 0$; compact for each t; and satisfies the estimate*

$$\|G_{\pm,L}(t)\|\leqq C\langle t\rangle^{-1-\varepsilon} \tag{3.26}$$

for $\pm t\geqq 0$.

PROOF. We consider the $+$ case only, since the other case can be treated similarly. From (3.10) and (3.24), we have

$$G_{+,L}(t)=(H_LJ_+-J_+H_0)e^{-itH_0}\tilde{P}_+\,. \tag{3.27}$$

The operator $A=H_LJ_+-J_+H_0$ can be expressed as

$$Af(x)=\iint e^{i(\varphi_+(x,\xi)-y\cdot\xi)}a(x,\xi)f(y)dy\,d\!\!{}^-\xi\,, \tag{3.28}$$

where the symbol

$$a(x,\xi)=e^{-i\varphi_+(x,\xi)}\Big(-\frac{1}{2}\Delta+V_L(x)-\frac{1}{2}|\xi|^2\Big)e^{i\varphi_+(x,\xi)} \tag{3.29}$$

satisfies

$$a(x,\xi)=0 \qquad \text{for} \quad |x|\leqq 1 \tag{3.30}$$

and

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\leqq\begin{cases} C_{\alpha\beta}\langle x\rangle^{-1-\varepsilon-|\alpha|}\langle\xi\rangle^{-1}, & \cos(x,\xi)\geqq -1/2\,, \\ C_{\alpha\beta}\langle x\rangle^{-\varepsilon-|\alpha|}, & \cos(x,\xi)\leqq -1/2 \end{cases} \tag{3.31}$$

(see Theorem 2.5-iii)). By Lemma 3.3-i), this and the fact that $\tilde{P}_+$ is bounded from L^2 into H^m for any $m \geqq 0$ imply the norm continuity in t and the compactness of $G_{+,L}(t)$.

We can rewrite (3.27) as

$$G_{+,L}(t)f(x)=\iint e^{i(\varphi_+(x,\xi)-t|\xi|^2/2-\varphi_+(y,\xi))}a(x,\xi)p_+(\xi,y)f(y)dyd\xi . \tag{3.32}$$

In view of (3.1), (3.30) and (3.31), we can see that the symbol $a(x,\xi)p_+(\xi,y)$ satisfies the assumptions (A), (B) of Lemma 3.3-ii). Thus (3.26) immediately follows from Lemma 3.3-ii). □

Next we estimate $G_{\pm,S}(t)$.

PROPOSITION 3.6. *$G_{\pm,S}(t)$ is norm continuous in t; compact for each fixed t; and satisfies the estimate*

$$\left|\int_0^{\pm\infty}\|G_{\pm,S}(t)\|dt\right|<\infty . \tag{3.33}$$

PROOF. We consider the $+$ case only. It is easy to see that $(H_0+1)E_+(t)$ is bounded and norm continuous in t. Hence $G_{+,S}(t)=V_S(H_0+1)^{-1}\cdot(H_0+1)E_+(t)$ is norm continuous in t and is compact by Assumption (S). Thus we have only to show that

$$\|G_{+,S}(t)\| \tag{3.34}$$

is integrable on $[0,\infty)$. Let $\rho \in C^\infty(R^N)$ be such that $\rho(x)=0$ for $|x|\geqq 2$, and $=1$ for $|x|\leqq 1$. Then (3.34) is majorized by

$$h(\delta t)\|(H_0+1)E_+(t)\|+C\|\rho(x/\delta t)(H_0+1)E_+(t)\| \tag{3.35}$$

for any $\delta>0$, where $h\in L^1([0,\infty))$ is the function appearing in Assumption (S). Thus we have now only to show the existence of $\delta>0$ such that

$$\|\rho(x/\delta t)(H_0+1)E_+(t)\|\leqq C_l\langle t\rangle^{-l} \qquad (t\geqq 0) \tag{3.36}$$

for any $l\geqq 0$. By (3.10) we have

$$(H_0+1)E_+(t)f(x)=\iint e^{i\Phi(x,\xi,y;t)}a(x,\xi,y)f(y)dyd\xi , \tag{3.37}$$

where

$$\begin{cases}\Phi(x,\xi,y;t)=\varphi_+(x,\xi)-t|\xi|^2/2-\varphi_+(y,\xi) , \\ a(x,\xi,y)=\left(1+\dfrac{1}{2}|\nabla_x\varphi_+(x,\xi)|^2-\dfrac{1}{2}i\Delta_x\varphi_+(x,\xi)\right)p_+(\xi,y) .\end{cases} \tag{3.38}$$

Set

$$L=\langle\nabla_\xi\Phi\rangle^{-2}(1-i\nabla_\xi\Phi\cdot\nabla_\xi)\,. \tag{3.39}$$

Since $\cos(\xi, y)\geqq -1/4$ for $(\xi, y)\in \operatorname{supp} p_+$, we have on $\operatorname{supp} p_+(\xi, y)$

$$|t\xi+\nabla_\xi\varphi_+(y, \xi)|\geqq c_0(t|\xi|+|y|)$$

for some constant $c_0>0$ by virtue of (2.26), (2.10) and Proposition 2.2-ii). This together with $\nabla_\xi\Phi=\nabla_\xi\varphi_+(x, \xi)-(t\xi+\nabla_\xi\varphi_+(y, \xi))$ implies by Proposition 2.2 that for $|x|\leqq 2\delta t$ and $(\xi, y)\in \operatorname{supp} p_+$ and for large $t>0$

$$|\nabla_\xi\Phi|\geqq c(|y|+t|\xi|) \tag{3.40}$$

for some constant $c>0$. Using (3.1), (3.4) and (3.40), we see by a direct calculation that

$$|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma({}^tL)^l a(x, \xi, y)|\leqq C_{\alpha\beta\gamma l}\langle|y|+t|\xi|\rangle^{-l} \tag{3.41}$$

for $|x|\leqq 2\delta t$ $(0<\delta\ll 1)$ and $(\xi, y)\in \operatorname{supp} p_+$. Thus, applying Lemma 3.3-i) to the expression:

$$\begin{aligned}\rho(x/\delta t)(H_0+1)E_+(t)f(x)\\ =\iint e^{i(\varphi_+(x, \xi)-\varphi_+(y, \xi))}e^{-it|\xi|^2/2}({}^tL)^l a(x, \xi, y)f(y)dyd\xi\,,\end{aligned} \tag{3.42}$$

we get (3.36) from (3.41). $\square$

Finally we consider the asymptotic behavior of $E_\pm(t)$ as $t\to\pm\infty$. For this purpose we introduce the operators J_a and J: For $a>0$ we define $\varphi_a(x, \xi)$ by (2.26) with $\sigma_0=-1/2$, $\sigma_1=1/2$ and $d=\sqrt{a}$. We also define $\varphi_0(x, \xi)$ by (2.26) with $\sigma_0=-1/2$, $\sigma_1=1/2$ deleting the factor $\chi_0(2\xi/d)$ in (2.26). φ_0 is not smooth at $\xi=0$. Now we define J_a and J by

$$J_af(x)=\iint e^{i(\varphi_a(x, \xi)-y\cdot\xi)}f(y)dyd\xi \tag{3.43}$$

for $f\in L^2(R^N)$, and

$$Jf(x)=\iint e^{i(\varphi_0(x, \xi)-y\cdot\xi)}f(y)dyd\xi \tag{3.44}$$

for f whose Fourier transform $\hat{f}(\xi)=\mathcal{F}f(\xi)$ belongs to $C_0^\infty(R^N-\{0\})$. Note that J_a is bounded in L^2 by Lemma 3.3-i), but J is not. However, if we set $\Gamma_a=[a/2, \infty)$, we see that $JE_{H_0}(\Gamma_a)$ is a well-defined bounded operator. Furthermore

$$JE_{H_0}(\Gamma_a)=J_aE_{H_0}(\Gamma_a)\,. \tag{3.45}$$

This J was used in Section 1 to define the modified wave operator $W_J^{\pm}(\Gamma)$.

LEMMA 3.7. *The operators $J_a^*J_a-I$ and $J_aJ_a^*-I$ are compact in $\mathcal{H}=L^2$.*

PROOF. By (3.43) we have

$$\mathcal{F}(J_a^*J_a-I)\mathcal{F}^{-1}f(\xi)=\iint e^{-i(\varphi_a(y,\xi)-\varphi_a(y,\eta))}f(\eta)dyd\eta-f(\xi)\,. \tag{3.46}$$

Noting (3.7) and $\varphi_a(y,\xi)-\varphi_a(y,\eta)=(\xi-\eta)\cdot\nabla_\xi\varphi_a(\xi,y,\eta)$, $\nabla_\xi\varphi_a(\xi,y,\eta)=\int_0^1\nabla_\xi\varphi_a(y,\eta+\theta(\xi-\eta))d\theta$, we make a change of variable $z=\nabla_\xi\varphi_a(\xi,y,\eta)$. Denoting the associated Jacobian by $J_a(\xi,z,\eta)$, we obtain

$$\mathcal{F}(J_a^*J_a-I)\mathcal{F}^{-1}f(\xi)=\iint e^{-i(\xi-\eta)\cdot z}(J_a(\xi,z,\eta)-1)f(\eta)d\eta dz\,. \tag{3.47}$$

By (2.28) and (3.19), J_a satisfies

$$|\partial_\xi^\alpha\partial_z^\beta\partial_\eta^\gamma(J_a(\xi,z,\eta)-1)|\leqq C_{\alpha\beta\gamma}\langle z\rangle^{-\varepsilon}\langle\xi\rangle^{-1}\,. \tag{3.48}$$

Thus, arguing as in the proof of Proposition 3.4-iv), we see that $\mathcal{F}(J_a^*J_a-I)\mathcal{F}^{-1}$, hence $J_a^*J_a-I$ is compact.

By (3.43) we have

$$(J_aJ_a^*-I)f(x)=\iint e^{i(\varphi_a(x,\xi)-\varphi_a(y,\xi))}f(y)dyd\xi-f(x)\,. \tag{3.49}$$

Thus, interchanging the role of x and ξ in the above discussion, we can similarly show that $J_aJ_a^*-I$ is compact. □

PROPOSITION 3.8. *The following relation holds for any $s\in R^1$:*

$$\underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_0}J_a^*E_\pm(t-s)=e^{isH_0}\tilde{P}_\pm\,. \tag{3.50}$$

PROOF. Since the phase functions $\varphi_\pm(x,\xi)$ and $\varphi_a(x,\xi)$ of $J_\pm$ and J_a coincide around the directions $x/|x|=\pm\xi/|\xi|$, it follows from the stationary phase method (cf. e.g. Hörmander [6]) that

$$\underset{t\to\pm\infty}{\text{s-lim}}\,(J_ae^{-itH_0}-J_\pm e^{-itH_0})f=0 \tag{3.51}$$

for all f such that $\hat{f}\in C_0^\infty(R^N-\{0\})$, hence for all $f\in L^2$. On the other hand, since $\underset{t\to\pm\infty}{\text{w-lim}}\, e^{-itH_0}f=0$, from Lemma 3.7 follows

$$\underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_0}J_a^*J_ae^{-itH_0}f=f\,. \tag{3.52}$$

Now (3.50) follows from (3.51), (3.52) and (3.10). □

§ 4. Proof of Theorem 1.1.

The existence of the limit (1.4) can be proved by the stationary phase method if we note that the operator $HJ-JH_0$ has the form

$$(HJ-JH_0)f(x)=\int e^{i\varphi(x,\xi)}a(x,\xi)\hat{f}(\xi)d\xi\,, \tag{4.1}$$

$a(x,\xi)$ being defined by (2.31). The important fact we utilize here is that $a(x,\xi)$ behaves like $|x|^{-1-\varepsilon}$ as $|x|\to\infty$ if the direction of x is close to $\pm\xi/|\xi|$. (See (2.33).) Since the calculus is routine, we omit the details.

REMARK. It is easy to see by the stationary phase method that our wave operator $W_J^\pm(\Gamma)$ is equal to

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-i\phi(t;0,D_x)}E_{H_0}(\Gamma)\,, \tag{4.2}$$

which is the usual modified wave operator with time-dependent modifier (cf. Hörmander [6]).

We next prove that $W_J^\pm(\Gamma)$ is isometric on $E_{H_0}(\Gamma)\mathcal{H}$. We mimic the argument of Kako [12, p. 141]. By (3.45) we have

$$JE_{H_0}(\Gamma)=J_{2a_0}E_{H_0}(\Gamma)\,,\quad \Gamma=[a_0,\infty)\,,\quad a_0>0\,. \tag{4.3}$$

Thus we have only to show with $c=2a_0$

$$\|W_J^\pm(\Gamma)u\|^2=\lim_{t\to\pm\infty}\|J_ce^{-itH_0}E_{H_0}(\Gamma)u\|^2=\|E_{H_0}(\Gamma)u\|^2\,. \tag{4.4}$$

Letting $v=E_{H_0}(\Gamma)u$, we can write

$$\begin{aligned}\|J_ce^{-itH_0}v\|^2&\\ &=(J_c^*J_ce^{-itH_0}v,\,e^{-itH_0}v)\\ &=((J_c^*J_c-I)e^{-itH_0}v,\,e^{-itH_0}v)+\|v\|^2\,.\end{aligned} \tag{4.5}$$

Since $J_c^*J_c-I$ is compact by Lemma 3.7, the fact that $\operatorname*{w-lim}_{t\to\pm\infty} e^{-itH_0}v=0$ implies (4.4).

The intertwining property can be proved in the same way as in the short-range case.

Next we prove the completeness (1.5). We treat $W_J^+(\Gamma)$ only. From the intertwining property, it follows that $\mathcal{R}(W_J^+(\Gamma))\subset E_H(\Gamma)\mathcal{H}_{ac}(H)$. Thus we have only to prove $E_H(\Gamma)\mathcal{H}_c(H)\subset\mathcal{R}(W_J^+(\Gamma))$. Consider $\phi\in\mathcal{H}_c(H)$ such that $E_H(\Delta)\phi=\phi$ for some $\Delta=[a,b]\subset\Gamma$, $0<a<b<\infty$. Then by the Ruelle [22]-Amrein-Georgescu [2] theorem, there exists a sequence $\{t_n\}\to\infty$ $(n\to\infty)$

such that

$$\text{(4.6)} \qquad \underset{n\to\infty}{\text{w-lim}}\, e^{-it_nH}\phi=0\,.$$

For $t\geqq t_n$, we have

$$\text{(4.7)} \qquad \begin{aligned} &e^{-it_nH}\phi-e^{i(t-t_n)H}E_+(t-t_n)e^{-it_nH}\phi \\ &=(\gamma_\Delta(H)-P_+)e^{-it_nH}\phi+(P_+-e^{i(t-t_n)H}E_+(t-t_n))e^{-it_nH}\phi\,. \end{aligned}$$

We first show that the first term of (4.7) converges to zero strongly as $n\to\infty$. Since $\tilde{\gamma}_\Delta(\tau)\equiv(2\pi)^{-1}\int_{-\infty}^{\infty}e^{i\tau\lambda}\gamma_\Delta(\lambda)d\lambda\in\mathcal{S}(R^1)$,

$$\begin{aligned} \gamma_\Delta(H_0)-\gamma_\Delta(H)&=\int_{-\infty}^{\infty}(e^{-i\tau H_0}-e^{-i\tau H})\tilde{\gamma}_\Delta(\tau)d\tau \\ &=\int_{-\infty}^{\infty}i\int_0^{\infty}e^{i(\sigma-\tau)H}Ve^{-i\sigma H_0}d\sigma\tilde{\gamma}_\Delta(\tau)d\tau \end{aligned}$$

is approximated in operator norm by a sequence of compact operators in $\mathcal{H}$ by the assumption on V, hence is compact. Therefore $(P_+-\gamma_\Delta(H))+P_-=(P_++P_--\gamma_\Delta(H_0))+(\gamma_\Delta(H_0)-\gamma_\Delta(H))$ is compact by Proposition 3.4-iii). Thus by (4.6) the first term in (4.7) is asymptotically equal to $P_-e^{-it_nH}$ as $n\to\infty$. But

$$\text{(4.8)} \qquad \begin{aligned} \|P_-e^{-it_nH}\phi\|^2&=((P_-^*-P_-)P_-e^{-it_nH}\phi,\, e^{-it_nH}\phi) \\ &\quad+((P_--e^{-it_nH}E_-(-t_n))P_-e^{-it_nH}\phi,\, e^{-it_nH}\phi) \\ &\quad+(P_-e^{-it_nH}\phi,\, E_-(-t_n)^*\phi)\,. \end{aligned}$$

By Proposition 3.4-iv), the first summand on the RHS converges to zero as $n\to\infty$. By (3.25), Proposition 3.4-v), Theorem 3.5 and Proposition 3.6, we see that the limits

$$\text{(4.9)} \qquad \lim_{t\to\pm\infty}(P_\pm-e^{itH}E_\pm(t))=K_\pm$$

exist in the operator norm in $\mathcal{H}$ and $K_\pm$ are compact. Thus the second summand on the RHS of (4.8) converges to zero as $n\to\infty$. The third term in (4.8) is bounded by

$$\text{(4.10)} \qquad C\|E_-(-t_n)\|\,\|\chi_{(|x|\geqq R)}\phi\|+C\|\chi_{(|x|<R)}E_-(-t_n)\|\,\|\phi\|$$

for any $R>0$. Similarly to the proof of (3.36), we see that for $t\geqq 0$, $R>0$ and any $l\geqq 1$

$$\|\chi_{(|x|<R)}E_-(-t)\|\leqq C_{lR}\langle t\rangle^{-l}\,,$$

which implies that the second summand of (4.10) converges to zero as $n\to\infty$ for any fixed $R>0$. Since $R>0$ is arbitrary, this means that the third term of (4.8) converges to zero as $n\to\infty$. Summing up, we have proved

$$\lim_{n\to\infty}\|P_-e^{-it_nH}\phi\|=0\,, \tag{4.11}$$

hence

$$\lim_{n\to\infty}\|[P_+-\gamma_\Delta(H)]e^{-it_nH}\phi\|=0\,. \tag{4.12}$$

For the second term on the RHS of (4.7), by (4.9) and (4.6), we have

$$\lim_{n\to\infty}\lim_{t\to\infty}\|[P_+-e^{i(t-t_n)H}E_+(t-t_n)]e^{-it_nH}\phi\|=0\,. \tag{4.13}$$

Thus from (4.7), (4.12) and (4.13) we get

$$\begin{aligned}\lim_{n\to\infty}\limsup_{t\to\infty}&\|e^{itH_0}J_a^*[e^{-itH}\phi-E_+(t-t_n)e^{-it_nH}\phi]\|\\ &\leqq C\lim_{n\to\infty}\lim_{t\to\infty}\|e^{-it_nH}\phi-e^{i(t-t_n)H}E_+(t-t_n)e^{-it_nH}\phi\|=0\,.\end{aligned} \tag{4.14}$$

$e^{itH_0}J_a^*E_+(t-t_n)e^{-it_nH}\phi$ converges to $Z(t_n)\phi=e^{it_nH_0}\tilde{P}_+e^{-it_nH}\phi$ strongly as $t\to\infty$ by Proposition 3.8. Thus

$$\lim_{n\to\infty}\limsup_{t\to\infty}\|e^{itH_0}J_a^*e^{itH}\phi-Z(t_n)\phi\|=0\,, \tag{4.15}$$

which, by the Cauchy criterion for the convergence, implies the existence of the limit

$$\operatorname*{s-lim}_{t\to\infty} e^{itH_0}J_a^*e^{-itH}\phi\equiv\Omega^a\phi\,. \tag{4.16}$$

From this, (4.6) and Lemma 3.7 we get

$$W_{J_a}^+\Omega^a\phi=\phi\,, \tag{4.17}$$

where $W_{J_a}^+$ is defined by (1.4) with $J=J_a$. Hence, remembering (3.45) and the definitions of J_a and J and using the intertwining property of $W_{J_a}^+$, we have

$$W_J^+E_{H_0}(\Delta)\Omega^a\phi=W_{J_a}^+E_{H_0}(\Delta)\Omega^a\phi=E_H(\Delta)W_{J_a}^+\Omega^a\phi=\phi\,. \tag{4.18}$$

This implies that $\phi\in\mathcal{R}(W_J^+E_{H_0}(\Delta))\subset\mathcal{R}(W_J^+E_{H_0}(\Gamma))$. The proof is complete. $\square$

REMARK. In the above we have assumed (1.2) for all α. But this is redundant. It suffices to assume (1.2) up to a certain finite order depending only on the space dimension, independent of $0<\varepsilon<1$.

Appendix. PROOF OF LEMMA 3.3.

i) We first prove the L^2-boundedness of P. Taking notice of (3.12), and the formula $\varphi(x, \xi)-\varphi(y, \xi)=(x-y)\cdot\nabla_x\varphi(x, \xi, y)$, $\nabla_x\varphi(x, \xi, y)=\int_0^1\nabla_x\varphi(y+\theta(x-y), \xi)d\theta$, we make a change of variable: $\eta=\nabla_x\varphi(x, \xi, y)$ in Pf. Then, denoting the inverse mapping of $\xi\to\eta=\nabla_x\varphi(x, \xi, y)$ by $\nabla_x\varphi^{-1}(x, \eta, y)$ and its Jacobian by $J(x, \eta, y)$, we can rewrite P in the form of a pseudodifferential operator:

$$Pf(x)=\iint e^{i(x-y)\cdot\eta}p(x, \nabla_x\varphi^{-1}(x, \eta, y), y)J(x, \eta, y)f(y)dyd\eta\,.$$

By (3.11) and (3.12) the symbol $q(x, \eta, y)=p(x, \nabla_x\varphi^{-1}(x, \eta, y), y)J(x, \eta, y)$ is easily seen to satisfy

$$|\partial_x^\alpha\partial_\eta^\beta\partial_y^\gamma q(x, \eta, y)|\leqq C_\varphi|p|\,,\qquad |\alpha+\beta+\gamma|\leqq M_0$$

for some constant C_φ independent of p. Thus the estimate for P in (3.14) follows from Calderón-Vaillancourt theorem [3]. (3.14) for Q and R is reduced to (3.14) for P if we take note of the formula

$$\begin{cases} Q^*Qf(x)=\iint e^{i(\varphi(x,\xi)-\varphi(y,\xi))}\overline{q(\xi, x)}q(\xi, y)f(y)dyd\xi\,, \\ RR^*f(x)=\iint e^{i(\varphi(x,\xi)-\varphi(y,\xi))}r(x, \xi)\overline{r(y, \xi)}f(y)dyd\xi \end{cases}$$

and

$$|\overline{q(\xi, x)}q(\xi, y)|\leqq|q|^2\,,\qquad |r(x, \xi)\overline{r(y, \xi)}|\leqq|r|^2\,.$$

ii) We consider the case $t\geqq 0$ and $r=0$ only dropping the superscript $+$. The other case can be treated similarly. Without loss of generality we may assume $a(x, \xi)=0$ for $|\xi|\leqq\mu_0/2$.

We take θ_0' and θ_1' so that $-1<\theta_0<\theta_0'<\theta_1'<\theta_1<1$, and choose C^∞ functions $\rho_0^\pm(\sigma)$, $\rho_1^\pm(\sigma)$ on $[-1, 1]$ such that for $j=0, 1$

$$\text{(A. 1)}\qquad \begin{cases} \rho_j^+(\sigma)+\rho_j^-(\sigma)=1\,, \qquad 0\leqq\rho_j^\pm(\sigma)\leqq 1\,, \\ \rho_j^+(\sigma)=\begin{cases} 1 & \text{for } \sigma\geqq\max(\theta_j, \theta_j') \\ 0 & \text{for } \sigma\leqq\min(\theta_j, \theta_j')\,, \end{cases} \end{cases}$$

and set

$$\text{(A. 2)} \qquad \begin{cases} a_\pm(x, \xi) = a(x, \xi)\rho_0^\pm(\cos(x, \xi)), \\ b_\pm(\xi, y) = b(\xi, y)\rho_1^\pm(\cos(\xi, y)). \end{cases}$$

Then $a_\pm$ and $b_\pm$ satisfy

$$\text{(A. 3)} \qquad \begin{cases} |\partial_x^\alpha \partial_\xi^\beta a_+(x, \xi)| \leqq C_{\alpha\beta}\langle x\rangle^{-L}, \\ |\partial_y^\alpha \partial_\xi^\beta b_-(\xi, y)| \leqq C_{\alpha\beta, m}\langle y\rangle^{-m}, \qquad m \geqq 0, \\ a_-, b_+ \in \mathcal{B}^\infty(R^{2N}), \end{cases}$$

$$\text{(A. 4)} \qquad \begin{cases} a_-(x, \xi) = 0 \qquad \text{for} \quad \cos(x, \xi) \geqq \theta_0', \\ b_+(\xi, y) = 0 \qquad \text{for} \quad \cos(\xi, y) \leqq \theta_1', \end{cases}$$

where $\mathcal{B}^\infty(R^{2N})$ denotes the space of all C^∞ functions with bounded derivatives. We decompose $a(x, \xi)b(\xi, y)$ as

$$\text{(A. 5)} \qquad a(x, \xi)b(\xi, y) = (a_+b_+ + a_+b_- + a_-b_+ + a_-b_-)(x, \xi, y),$$

and accordingly decompose $L(t)$ as

$$\text{(A. 6)} \qquad L(t) = L_{++}(t) + L_{+-}(t) + L_{-+}(t) + L_{--}(t).$$

We estimate each term on the RHS.

1° $L_{++}(t)$: Let $\phi(t; y, \xi) = t|\xi|^2/2 + \varphi(y, \xi)$ and set

$$\text{(A. 7)} \qquad P = \langle \nabla_\xi \phi\rangle^{-2}(1 - i\nabla_\xi\phi \cdot \nabla_\xi).$$

On the support of $a_+(x, \xi)b_+(\xi, y)$, we have

$$\text{(A. 8)} \qquad \langle \nabla_\xi \phi\rangle^{-1} \leqq C\langle |y| + t|\xi|\rangle^{-1}$$

by (3.11), (A. 4) and $t \geqq 0$. Noting $Pe^{-i\phi} = e^{-i\phi}$, we have by integration by parts

$$\text{(A. 9)} \qquad L_{++}(t)f(x) = \iint e^{i\Phi(x, \xi, y; t)}\{e^{-i\varphi(x, \xi)}({}^tP)^l[e^{i\varphi(x, \xi)}a_+(x, \xi)b_+(\xi, y)]\}f(y)dyd\xi$$

for any $l \geqq 0$, where

$$\text{(A. 10)} \qquad \Phi(x, \xi, y; t) = \varphi(x, \xi) - t|\xi|^2/2 - \varphi(y, \xi).$$

The function in $\{\ \}$ of (A. 9) is a finite sum of the functions of the form $a_j^l(x, \xi)b_j^l(\xi, y; t)$, and each a_j^l and b_j^l satisfies

$$\text{(A. 11)} \qquad \begin{cases} |\partial_x^\alpha \partial_\xi^\beta a_j^l(x, \xi)| \leqq C_{\alpha\beta}\langle x\rangle^{l-L}, \\ |\partial_y^\alpha \partial_\xi^\beta b_j^l(\xi, y; t)| \leqq C_{\alpha\beta}\langle |y| + t|\xi|\rangle^{-l} \end{cases}$$

by (A)-i) of Lemma 3.3-ii) and (A. 8). Hence letting

$$\text{(A. 12)} \qquad \begin{cases} A_j^l f(x) = \iint e^{i(\varphi(x,\xi) - y\cdot\xi)} a_j^l(x,\xi) f(y) dy d\xi \,, \\ B_j^l(t) f(x) = \iint e^{i(x\cdot\xi - \varphi(y,\xi))} b_j^l(\xi, y\,;\, t) f(y) dy d\xi \,, \end{cases}$$

we have

$$\text{(A. 13)} \qquad L_{++}(t) = \sum_j A_j^l e^{-itH_0} B_j^l(t) \,.$$

(A. 11) shows that for $0 \leqq m \leqq l$

$$\text{(A. 14)} \qquad \begin{cases} \|\langle x\rangle^{L-l} A_j^l\| < \infty \,, \\ \|B_j^l(t)\langle y\rangle^m\| \leqq C\langle t\mu_0\rangle^{m-l} \,. \end{cases}$$

Thus we get

$$\text{(A. 15)} \qquad \|\langle x\rangle^{L-l} L_{++}(t)\langle y\rangle^m\| \leqq C\langle t\mu_0\rangle^{m-l}$$

for any $m \in [0, l]$ and any integer $l \geqq 0$. Interpolating this with respect to l, we finally get

$$\text{(A. 16)} \qquad \|\langle x\rangle^{s_1} L_{++}(t)\langle y\rangle^{s_2}\| \leqq C\langle t\mu_0\rangle^{-L+s_1+s_2}$$

for s_1, $s_2 \geqq 0$ with $s_1 + s_2 \leqq L$.

2° $L_{--}(t)$: This case is reduced to 1° by considering $\langle x\rangle^{s_2} L_{--}(t)^* \langle y\rangle^{s_1}$ and making a change of variable $\xi \to -\xi$ in the expression of $L_{--}(t)^*$ similar to (A. 9). Then we can obtain

$$\text{(A. 17)} \qquad \|\langle x\rangle^{s_1} L_{--}(t)\langle y\rangle^{s_2}\| \leqq C_k\langle t\mu_0\rangle^{-k}$$

for s_1, $s_2 \geqq 0$ and any $k \geqq 0$.

3° $L_{+-}(t)$: We argue in a way similar to 1°. In this case, however, we must replace the estimate (A. 8) of $\nabla_\xi \phi$ by

$$\text{(A. 18)} \qquad \langle \nabla_\xi \phi(t\,;\, y, \xi)\rangle^{-1} \langle y\rangle^{-1} \leqq C\langle t\xi\rangle^{-1} \,.$$

Integrating by parts in $L_{+-}(t)$ by the use of P in (A. 7), we get (A. 9) with b_+ replaced by b_-, the function in { } of (A. 9) being of the same form as in 1°. By (A. 18) and (A. 3), we have

$$\text{(A. 19)} \qquad \begin{cases} |\partial_x^\alpha \partial_\xi^\beta a_j^l(x,\xi)| \leqq C_{\alpha\beta}\langle x\rangle^{l-L} \,, \\ |\partial_y^\alpha \partial_\xi^\beta b_j^l(\xi, y\,;\, t)| \leqq C_{\alpha\beta, m}\langle t\xi\rangle^{-l} \langle y\rangle^{2l+|\beta|-m} \end{cases}$$

for $0 \leqq l \leqq L$ and any $m \geqq 0$. Hence we have

(A. 20)
$$\begin{cases} \|\langle x\rangle^{L-l}A_j^l\|<\infty \\ \|B_j^l(t)\langle y\rangle^k\|\leqq C\langle t\mu_0\rangle^{-l} \end{cases}$$

for $0\leqq l\leqq L$ and any $k\geqq 0$, where A_j^l and $B_j^l(t)$ are defined by (A. 12). Thus by (A. 13) we get

(A. 21)
$$\|\langle x\rangle^{L-l}L_{+-}(t)\langle y\rangle^k\|\leqq C\langle t\mu_0\rangle^{-l}$$

for $0\leqq l\leqq L$ and $k\geqq 0$. By an interpolation, we obtain

(A. 22)
$$\|\langle x\rangle^{s_1}L_{+-}(t)\langle y\rangle^{s_2}\|\leqq C\langle t\mu_0\rangle^{-L+s_1}$$

for $0\leqq s_1\leqq L$ and $s_2\geqq 0$.

$4°$ $L_{-+}(t)$: Let Φ be as in (A. 10). On the support of $a_-(x,\xi)b_+(\xi,y)$, we have

(A. 23)
$$\begin{aligned}\langle\nabla_\xi\Phi\rangle^{-1}&\leqq C\langle|x-y|+t|\xi|\rangle^{-1}\\ &\leqq C\langle|x|+|y|+t|\xi|\rangle^{-1}\end{aligned}$$

by (A. 4), (3.11) and $t\geqq 0$. Thus integrating by parts in $L_{-+}(t)f$ by the use of the differential operator

(A. 24)
$$Q=\langle\nabla_\xi\Phi\rangle^{-2}(1-i\nabla_\xi\Phi\cdot\nabla_\xi),$$

we get for any integer $l\geqq 0$

(A. 25)
$$L_{-+}(t)f(x)=\iint e^{i(\varphi(x,\xi)-\varphi(y,\xi))}q^l(x,\xi,y;t)f(y)dyd\xi.$$

Here

(A. 26)
$$q^l(x,\xi,y;t)=e^{-it|\xi|^2/2}({}^tQ)^l(a_-b_+)(x,\xi,y)$$

satisfies by (A. 23)

(A. 27)
$$|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma q^l(x,\xi,y;t)|\leqq C_{\alpha\beta\gamma}\langle t\xi\rangle^{M_0}\langle x\rangle^{-l/3}\langle y\rangle^{-l/3}\langle t\xi\rangle^{-l/3}$$

for any α,β,γ with $|\alpha+\beta+\gamma|\leqq M_0=2([N/2]+[5N/4]+2)$ and for any integer $l\geqq 0$. Thus by Lemma 3.3-i), we obtain the estimate

(A. 28)
$$\|\langle x\rangle^{s_1}L_{-+}(t)\langle y\rangle^{s_2}\|\leqq C_{s,k}\langle t\mu_0\rangle^{-k}$$

for any s_1, s_2, $k\geqq 0$.

(A. 6), (A. 16), (A. 17), (A. 22) and (A. 28) complete the proof of Lemma 3.3-ii). □

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Hiroshi Isozaki
Department of Mathematics
Faculty of Science
Kyoto University
Kyoto
606 Japan

Hitoshi Kitada
Department of Mathematics
College of General Education
University of Tokyo
Meguro, Tokyo
153 Japan