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The maximal subgroups of the sporadic simple group of O'Nan

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§ 0. Introduction

The simple group of O'Nan is one of five or six sporadic simple groups not involved in Monster. (It has not been determined whether J_1 is involved in Monster or not.) This group is defined as a simple group whose Sylow 2-subgroup has the following presentation:

$$\left\langle \begin{matrix} v_1, v_2, v_3, \\ s, t \end{matrix} \;\middle|\; \begin{matrix} v_i^4 = t_2 = 1, \quad [v_i, v_j] = 1 \ (i, j \in \{1, 2, 3\}) \\ s^4 = v_1 v_3, \quad s^t = s^{-1}, \\ v_1^s = v_2, \quad v_2^s = v_3, \quad v_3^s = v_1 v_2^{-1} v_3, \\ v_1^t = v_3^{-1}, \quad v_2^t = v_2^{-1}, \quad v_3^t = v_1^{-1}. \end{matrix} \right\rangle.$$

O'Nan [3] first studied this group and determined much of its local structure as well as its character table. Sims and Andrilli [1] showed the existence and the uniqueness of this group. They also proved that the outer automorphism group of this group is of order 2.

The main results of this paper are the following theorem and corollary.

THEOREM. *The simple group G of O'Nan of order $2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$ has exactly 13 conjugacy classes of maximal subgroups. Their representatives are as follows:*

	Index
(A) *Two 2-local subgroups*	
(i) $(Z_4 \cdot L_3(4)) \rtimes Z_2$	$3^2 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$
(ii) $(Z_4 \times Z_4 \times Z_4) \cdot L_3(2)$	$3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$
(B) *Two 3-local subgroups*	
(iii) $((E_9 \rtimes Z_4) \times A_6) \cdot Z_2$	$2^3 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
(iv) $E_{81} \rtimes (2_-^{1+4} \cdot D_{10})$	$2^3 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
(C) *Nine non-local subgroups*	

(v), (vi)	$L_3(7) \rtimes Z_2$	(2-*classes*)	$2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 31$
(vii)	J_1		$2^6 \cdot 3^3 \cdot 7^2 \cdot 31$
(viii), (ix)	$L_2(31)$	(2-*classes*)	$2^4 \cdot 3^3 \cdot 7^3 \cdot 11 \cdot 19$
(x), (xi)	M_{11}	(2-*classes*)	$2^5 \cdot 3^2 \cdot 7^3 \cdot 19 \cdot 31$
(xii), (xiii)	A_7	(2-*classes*)	$2^6 \cdot 3^2 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$

Moreover (v) *and* (vi), (viii) *and* (ix), (x) *and* (xi), (xii) *and* (xiii) *are* Aut *G-conjugate respectively.*

COROLLARY. Aut G *has exactly* 10 *conjugacy classes of maximal subgroups. Their representatives are as follows:*

		Index
(i)	G	2
(ii)	$(Z_4 \cdot L_3(4)) \cdot E_4$	$3^2 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$
(iii)	$(Z_4 \times Z_4 \times Z_4) \cdot (L_3(2) \times Z_2)$	$3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$
(iv)	$((E_9 \rtimes Z_4) \times A_6) \cdot E_4$	$2^3 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
(v)	$E_{81} \rtimes ((2_-^{1+4} \rtimes D_{10}) \rtimes Z_2$	$2^3 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
(vi)	$7_+^{1+2} \rtimes (D_{16} \times Z_3)$	$2^6 \cdot 3^3 \cdot 5 \cdot 11 \cdot 19 \cdot 31$
(vii)	F_{31}^{30}	$2^9 \cdot 3^3 \cdot 7^3 \cdot 11 \cdot 19$
(viii)	$J_1 \times Z_2$	$2^6 \cdot 3^3 \cdot 7^2 \cdot 31$
(ix)	$PGL_2(9)$	$2^6 \cdot 3^2 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
(x)	$PGL_2(7)$	$2^6 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$

In the above, $A \rtimes B$ and $A \cdot B$ denote split and non-split extensions of A by B respectively. We use Z_n and E_n to denote a cyclic and an elementary abelian group of order n respectively. D_n denotes a dihedral group of order n. We also use F_n^m to denote a Frobenius group with the kernel Z_n and a complement Z_m. For odd prime p, p_+^{1+2n} denotes an extra-special group of order p^{1+2n} of exponent p. 2_-^{1+4} denotes a central product of D_8 and Q_8 with amalgamated centers. We also use S_n and A_n to denote the symmetric and alternating group of degree n. J_1 denotes the first Janko simple group (See [2].) and M_{11} denotes the Mathieu simple group of degree 11.

Recently R. Wilson informed me ([5]) that he has obtained the same result under the assumption that the 3-fold covering group of the simple group of O'Nan has an irreducible representation of dimension 45 over $\boldsymbol{F}_7$.

On this assumption, R. A. Parker has constructed generating matrices. According to R. Wilson, these matrices have been used heavily in his paper together with the use of computer programming. The present work has been completed without such an assumption or use of computers.

Throughout this paper G denotes the simple group of O'Nan.

Now we describe the outline of the proof of the theorem. For a finite simple group, in general, every maximal subgroup is a normalizer of some characteristically simple subgroup — a direct product of isomorphic simple subgroups. Thus the proof is naturally divided into the following several steps.

First step is the determination of the conjugacy classes of abelian characteristically simple subgroups — elementary abelian subgroups — and the normalizers of their representatives. This is treated in § 3.

Second step is to determine which non-abelian characteristically simple groups actually occur as subgroups of G. O'Nan [3] proved the existence of $L_3(7)$ and J_1, but the existence of $L_2(31)$, M_{11} and A_7 was unknown in any published papers, so far as I know. So we must construct these subgroups. This step is one of the most difficult works in a problem of determining the maximal subgroups of a simple group, because such unknown subgroups have considerably large indices and do not have good geometrical meanings. In § 4 and § 5, we construct these subgroups by using the detailed local information obtained in § 2 and § 3, and by examining the relations among some generators. As for M_{11} and $L_2(31)$, the generators can be chosen rather naturally, and in this process of constructions we get an interesting presentation for M_{11} (Lemma 4.2). As for A_7, the generators must be chosen more carefully.

Third step is to determine the conjugacy classes of the non-abelian characteristically simple subgroups obtained in the second step and the normalizers of their representatives. For the determination of maximal subgroups of a simple group (in particular sporadic simple groups), this step is very complicated and is frequently carried out with aid of computers. However for the simple group of O'Nan, the constructions of $L_2(31)$, M_{11} and A_7 in § 4 and § 5 are so explicit that we can dispose of this step in § 6 and § 7 considerably easily by hand. Lemma 6.7 is a key lemma in § 6 and the usual counting argument described in § 1 are useful in § 7.

Finally it is easy to pick up the maximal subgroups of G from the results in first and third steps.

Throughout this paper we use the notations in the theorem to describe isomorphism types of subgroups of G. Furthermore M_{10} denotes the one point stabilizer of M_{11}.

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§ 1. Notations and some counting argument

Throughout this paper G will denote a fixed finite group isomorphic to the simple group of O'Nan. For any subgroup H of G, we use *the "ATLAS" notation* for representing conjugacy classes of H, in which classes of elements of given order are lettered in decending order of the order of their centralizers. For example, see table I and II. In cases where there is a risk of confusion, we will denote these classes of H with the subscript H, for example $(2A)_{A_5}$.

An element of class $(nX)_H$ is called an *$(nX)_H$-element*. If K is a subgroup of H and its isomorphism type is M, then we call K an M-subgroup of H. For example, if $K \leqq H$ and $K \cong A_5$, we say that K is an A_5-subgroup of H. We will also say that a subgroup K of H is of *H-type* (or in case $H=G$, simply *type*) $(lX, mY, nZ)_H$, if there exist an $(lX)_H$-element x and an $(mY)_H$-element y such that xy is an $(nZ)_H$-element and $\langle x, y\rangle = K$.

Now we describe the counting argument used in this paper.

Let H be a subgroup of G and t be a fixed $(nZ)_H$-element. We set $(lX, mY; t)_H = \{(x, y) \in (lX)_H \times (mY)_H \mid xy = t\}$. The subscript is dropped when it is clear which group is meant. The value $^{\#}(lX, mY; t)_H$ can be calculated from the character table of H and is independent of the particular choice of $t \in (nZ)_H$, namely

$$^{\#}(lX, mY; t)_H = \frac{|H|}{|C_H(lX)||C_H(mY)|} \sum_{\chi} \frac{\chi(g)\chi(h)\overline{\chi(t)}}{\chi(1)},$$

where $g \in (lX)_H$, $h \in (mY)_H$, χ runs over all the irreducible characters of G, and $\overline{\chi(t)}$ denotes the complex conjugate of $\chi(t)$. We also denote this value by $^{\#}(lX, mY, nZ)_H$.

Let H be a subgroup of G of G-type (lX, mY, nZ) and t is an $(nX)_G$-element in H. For a subset Q of G containing t, we set

$$(lX, mY; t) \cap Q = \{(x, y) \in (lX)_G \times (mY)_G \mid x, y \in Q \text{ and } xy = t\}.$$

In non-local analyses (§ 6 and § 7) we will often consider the set $(lX, mY; t) \cap (\bigcup_{g \in N_G(t)} H^g)$. If we can show that $(lX, mY; t) \cap (H \cap H^g) = \emptyset$ for $g \in N_G(t)$, then the cardinality of this set is equal to $|(lX, mY; t) \cap H| \times |N_G(t) : N_G(t) \cap N_G(H)|$. We note that if the fusion pattern of H in G is known, then the first factor of this value can be calculated by the character table of H. Indeed, if the H-conjugacy classes in $(lX)_G \cap H$ are $(lX_1)_H, \cdots, (lX_p)_H$ and those of $(mY)_G \cap H$ are $(mY_1)_H, \cdots, (mY_q)_H$, then we have

$$|(lX, mY; t) \cap H| = \sum_{i=1}^{p} \sum_{j=1}^{q} {}^{\#}((lX_i)_H, (mY_j)_H; t)_H.$$

This value turns out to be very useful in order to determine the conjugacy classes of non-abelian simple subgroups of G. (See § 7.)

Table I Conjugacy classes of G

	ATLAS name x	x^2	x^3	$\|C_G(x)\|$	Structure of the centralizer
1	$1A$	$1A$	$1A$	$2^9\cdot3^4\cdot5\cdot7^3\cdot11\cdot19\cdot31$	G
2	$2A$	$1A$	$2A$	$2^9\cdot3^2\cdot5\cdot7$	$(Z_4\cdot L_3(4))\rtimes Z_2$
3	$3A$	$3A$	$1A$	$2^3\cdot3^4\cdot5$	$E_9\times A_6$
4	$4A$	$2A$	$4A$	$2^8\cdot3^2\cdot5\cdot7$	$Z_4\cdot L_3(4)$
5	$4B$	$2A$	$4B$	2^8	
6	$5A$	$5A$	$5A$	$2^2\cdot3^2\cdot5$	$(E_9\rtimes Z_4)\times Z_5$
7	$6A$	$3A$	$2A$	$2^3\cdot3^2$	$E_9\times D_8$
8	$7A$	$7A$	$7A$	$2^2\cdot7^3$	$7_+^{1+2}\rtimes Z_4$
9	$7B$	$7B$	$7B$	7^2	E_{49}
10	$8A$	$4B$	$8A$	2^5	
11	$8B$	$4B$	$8B$	2^5	
12	$10A$	$5A$	$10A$	$2^2\cdot5$	Z_{20}
13	$11A$	$11A$	$11A$	11	Z_{11}
14	$12A$	$6A$	$4A$	$2^2\cdot3^2$	$E_9\times Z_4$
15	$14A$	$7A$	$14A$	$2^2\cdot7$	Z_{28}
16	$15A$	$15B$	$5A$	$3^2\cdot5$	$E_9\times Z_5$
17	$15B$	$15A$	$5A$	$3^2\cdot5$	$E_9\times Z_5$
18	$16A$	$8A$	$16B$	2^4	Z_{16}
19	$16B$	$8A$	$16A$	2^4	Z_{16}
20	$16C$	$8B$	$16D$	2^4	Z_{16}
21	$16D$	$8B$	$16C$	2^4	Z_{16}
22	$19A$	$19B$	$19B$	19	Z_{19}
23	$19B$	$19C$	$19C$	19	Z_{19}
24	$19C$	$19A$	$19A$	19	Z_{19}
25	$20A$	$10A$	$20A$	$2^2\cdot5$	Z_{20}
26	$20B$	$10A$	$20B$	$2^2\cdot5$	Z_{20}
27	$28A$	$14A$	$28A$	$2^2\cdot7$	Z_{28}
28	$28B$	$14A$	$28B$	$2^2\cdot7$	Z_{28}
29	$31A$	$31A$	$31B$	31	Z_{31}
30	$31B$	$31B$	$31A$	31	Z_{31}

§ 2. Preliminary results on local subgroups

In this section we review some notations and results on local subgroups of G from [3]. Furthermore we prove some detailed properties on these subgroups, which will be necessary to later sections.

By the definition of the simple group of O'Nan, G has the following group P as its Sylow 2-subgroup:

$$P=\langle v_1, v_2, v_3, s, t\rangle\ ,$$

$$\text{where}\quad v_1^4=v_2^4=v_3^4=t^2=1,\ s^4=v_1v_3\ ;\quad [v_i, v_j]=1\ \ (i, j\in\{1, 2, 3\})\ ;$$

$$v_1^s=v_2,\ v_2^s=v_3,\ v_3^s=v_1v_2^{-1}v_3\ ;\quad v_1^t=v_3^{-1},\ v_2^t=v_2^{-1},\ v_3^t=v_1^{-1}\ ;\quad \text{and}\quad s^t=s^{-1}\ .$$

Throughout this paper we assume that the letters P, v_1, v_2, v_3, s and t satisfy the above conditions. We set $V=\langle v_1, v_2, v_3\rangle\ (\cong Z_4\times Z_4\times Z_4)$. From § 4 of [3] we have the following.

PROPOSITION 2.1. (O'Nan) (i) *G has one class of involutions.* $C_G(v_1^2v_3^2)=C_G^*(v_1v_2^2v_3^{-1})$ *and* $C_G(v_1v_2^2v_3^{-1})/\langle v_1v_2^2v_3^{-1}\rangle\cong L_3(4)$. *The involutions in* $C_G(v_1^2v_3^2)-C_G(v_1v_2^2v_3^{-1})$ *are conjugate, and they induce on* $C_G(v_1v_2^2v_3^{-1})/\langle v_1v_2^2v_3^{-1}\rangle$ $(\cong L_3(4))$ *unitary automorphisms.*

(ii) *G has two classes of elements of order 4; $4A$ and $4B$. The elements of class $4A$ are conjugate to $v_1v_2^2v_3^{-1}$ and are not squares in G. The elements of class $4B$ are conjugate to v_1v_3. Their centralizers in G are 2-subgroups. The square roots of $v_1^2v_3^2$ of class $4A$ are $v_1v_2^2v_3^{-1}$ and its inverse.*

(iii) *$N_G(V)$ is isomorphic to a non-split extension of $Z_4\times Z_4\times Z_4$ by $L_3(2)$. The set of all $4A$-elements in P generates V.*

Since st is an involution in $C_G(v_1^2v_3^2)-C_G(v_1v_2^2v_3^{-1})$, it follows that the centralizer of st on $C_G(v_1v_2^2v_3^{-1})/\langle v_1v_2^2v_3^{-1}\rangle$ is isomorphic to $U_3(2)\ (\cong E_9\rtimes Z_8)$ from the above proposition. We consider the inverse-image in $C_G(v_1v_2^2v_3^{-1})$ of this centralizer. We let R be a Sylow 3-subgroup of this inverse-image. We set $T=\langle s^2v_3^2, v_1v_2^{-1}, st\rangle$, $K=C_G(R)'$ and $Z=T\cap C_G(K)$.

PROPOSITION 2.2. (O'Nan) (i) *T is a Sylow 2-subgroup of $N_G(R)$.* $N_G(R)=C_G(R)\cdot T$ *and* $|N_G(R):C_G(R)|=8$.

(ii) $K\cong A_6$ *and* $K\cap T=\langle v_1v_2^2v_3^{-1}, st\rangle\cong D_8$. $C_G(R)=R\times K$ *and* $C_G(r)=C_G(R)$ *for any element r in $R^\sharp$. In particular R is a T.I. set in G. Furthermore* $C_G(K)\times K$ *is a subgroup of* $N_G(R)$ *of index 2 and* $N_G(R)/C_G(K)\cong M_{10}$.

(iii) $C_G(K)=R\rtimes Z\cong E_9\rtimes Z_4$. *$Z$ acts on R fixed point freely.*

LEMMA 2.3. *Let b be an element of order 5 in K inverted by $v_1^2v_3^2$. Then we have the following.*

(i) $Z=\langle v_1v_2v_3^2\rangle$.

(ii) $C_G(b)=(R\rtimes Z)\times\langle b\rangle$ *and* $|N_G(b):C_G(b)|=4$.

(iii) *$T\cap N_G(b)$ is a Sylow 2-subgroup of $N_G(b)$ of order 2^4.*

(iv) *$T\cap C_G(b)=Z$ and $T\cap(C_G^*(b)-C_G(b))=\{v_1v_2^{-1}, v_1^{-1}v_2, v_1^2v_3^2, v_2^2v_3^2\}$. For any element $h\in T\cap(N_G(b)-C_G^*(b))$, we have $h^2=v_2^2v_3^2$, $(v_1v_2v_3^2)^h=v_1^{-1}v_2^{-1}v_3^2$ and $(v_1v_2^2v_3^{-1})^h=v_1^{-1}v_2^2v_3$. Furthermore $N_G(K)=N_G(R)=((R\rtimes Z)\times K)\langle h\rangle\cong((E_9\rtimes Z_4)\times A_6)\cdot Z_2$.*

(v) *The square roots of $v_2^2v_3^2$ in $N_G(b)$ are contained in $T\cap N_G(b)$. $v_1^2v_3^2$ is not a square in $N_G(b)$.*

PROOF. (i) By [3] Lemma 5.8, $Z=\langle v_1v_2^{-1}\rangle$ or $\langle v_1v_2v_3^2\rangle$. Since $v_1v_2^{-1}=(stv_1)^2$, $v_1v_2^{-1}$ is a $4B$-element and does not centralize $K(\cong A_6)$ by Lemma 2.1 (ii). Thus $Z=\langle v_1v_2v_3^2\rangle$.

(ii) follows from table I and Proposition 2.2 (iii).

(iii) We set $F=\langle v_1v_2v_3^2, v_1^2v_3^2\rangle$. Then $F(\cong Z_4\times Z_2)$ is a Sylow 2-subgroup of $C_G^*(b)$ by (ii). Suppose that X is a Sylow 2-subgroup of $N_G(b)$ containing F. Then $F\trianglelefteq X$ as $|X:F|=2$. Since $N_G(b)\leqq N_G(R)$, X is a 2-subgroup of $N_G(R)\cap N_G(F)$. We may easily verify that $F\trianglelefteq T$. Since Z acts on R fixed point freely, no element in $R^\#$ normalizes F. Since any element of order 5 in $N_G(R)$ is contained in K, it does not normalize F. Consequently $N_G(R)\cap N_G(F)$ is a 2-subgroup of $N_G(R)$ and so $T=N_G(R)\cap N_G(F)$. Thus $X=T\cap N_G(b)$.

(iv) We set $A=T\cap((R\rtimes Z)\times K)$. Then $A=\langle v_1v_2v_3^2\rangle\times\langle v_1^2v_3^2, st\rangle\cong Z_4\times D_8$. Since A is a subgroup of T of index 2 and $A\cap N_G(b)=F$, it follows from (iii) that $T\cap N_G(b)$ contains an element in $T-A$. We may easily verify that there exist 16 elements of order 8 in $T-A$ and their fourth powers are $v_1^2v_3^2$. Suppose that one of these elements is contained in $N_G(b)$. Then $v_1^2v_3^2$ centralizes b as $|N_G(b):C_G(b)|=4$. This is a contradiction. Thus $T\cap N_G(b)$ contains all the remaining 16 elements in $T-A$. We may also easily verify that the set of these elements coincides with $Fs^{-1}tv_2^2\cup Fs^{-1}tv_1^{-1}v_2^2v_3^{-1}$ and that the squares of elements in this set are $v_2^2v_3^2$. The rest of the assertion follows from direct computations.

(v) A Sylow 2-subgroup of $N_G(b)$ is $R\times\langle b\rangle$-conjugate to $T\cap N_G(b)$. Since $v_2^2v_3^2$ inverts $R\times\langle b\rangle$, the square roots of $v_2^2v_3^2$ in $N_G(b)$ are contained in $T\cap N_G(b)$. Suppose that $v_1^2v_3^2$ has a root in $N_G(b)$. Then $v_1^2v_3^2$ has a root in $T\cap N_G(b)$ since $v_1^2v_2^2$ centralizes R and inverts b. But this contradicts (iv). Thus (v) follows. □

COROLLARY 2.4. (i) *$N_G(b)$ has three conjugacy classes of involutions, represented by $v_1^2v_2^2$, $v_1^2v_3^2$ and $v_2^2v_3^2$.*

(ii) *Suppose that there exists an S_5-subgroup S containing b. Then the involutions in $N_G(b)\cap S'$ are $N_G(b)$-conjugate to $v_2^2v_3^2$. In particular there is no S_5-subgroup which contains an A_5-subgroup of K.*

PROOF. We note that $v_1^2v_2^2$, $v_1^2v_3^2$ and $v_2^2v_3^2$ are not $N_G(b)$-conjugate each other by Lemma 2.3 (v) and the fact that $v_1^2v_2^2 \in C_G(b)$ but $v_1^2v_3^2, v_2^2v_3^2 \notin C_G(b)$. Thus (i) follows from Lemma 2.3 (iv). Since a Sylow 5-normalizer of S_5 is isomorphic to F_5^4, the involutions in $N_G(b) \cap S$ are squares in $N_G(b)$ and are contained in S'. Thus (ii) follows from (i) and Lemma 2.3 (v). □

Suppose that R' is a Sylow 3-subgroup of K normalized by $v_1v_2v_3^{-1}$ and $T_0 = \langle v_1v_2^{-1}, (s^2v_3^2)^2, s^2v_3^2 \cdot st \rangle$. Then by [3] § 5, T_0 normalizes R and R', and there exists an element g such that $g^2 \in T_0$ and $R^g = R'$.

In order to describe the structure of $\langle T_0, g \rangle$ explicitly, we will take a suitable conjugate of $\langle T_0, g \rangle$ in P. By [3] § 5 there exists an element $x \in N_G(V)$ such that $\langle T_0, g \rangle^x = \langle v_1v_2^{-1}, v_1v_3^{-1}, s^2v_1, t \rangle$. We set $R_1 = R^x$, $R_1' = (R')^x$, $K_1 = K^x$, $T_1 = \langle v_1v_2^{-1}, v_1v_3^{-1}, s^2v_1 \rangle$ and $L = \langle s^2v_1^{-1}v_2, v_1v_3^{-1}, v_1^2v_2^2, t \rangle$.

PROPOSITION 2.5. (O'Nan) (i) *$R_1 \times R_1'$ is a Sylow 3-subgroup of G and* $N_G(R_1 \times R_1')/R_1 \times R_1' \cong 2_-^{1+4} \cdot D_{10}$.

(ii) *T_1 normalizes R_1 and R_1', and t interchanges R_1 and R_1'. $\langle T_1, t \rangle$ is a Sylow 2-subgroup of $N_G(R_1 \times R_1')$ with center $\langle v_1^2v_3^2 \rangle$.*

(iii) *We let C be a complement of $N_G(R_1 \times R_1')/R_1 \times R_1'$ containing $\langle T_1, t \rangle$. Then $L \trianglelefteq C$, $C/L \cong D_{10}$ and $L \cong 2_-^{1+4}$. L contains all involutions in $\langle T_1, t \rangle$.*

LEMMA 2.6. (i) *Interchanging R_1 and R_1', if necessary, we may assume that the following holds: $\langle v_1v_2v_3^2 \rangle$ and $\langle v_1^2v_2v_3 \rangle$ act fixed point freely on R_1 and R_1' respectively, $R_1\langle v_1v_2v_3^2 \rangle$ centralizes $R_1'\langle v_1^2v_2v_3 \rangle$, and $R_1'\langle v_1^2v_2v_3 \rangle \leqq K_1$.*

(ii) $\{g \in \langle T_1, t \rangle - L \mid g^2 = v_1^2v_3^2\} = \{(s^2v_1^2v_2^{-1})^{\pm}, (s^2v_2)^{\pm}, (s^2v_1v_2^{-1}v_3)^{\pm}, (s^2v_1v_2v_3^{-1})^{\pm}\}$.

PROOF. Assertion (ii) follows from simple computations.

(i) Since $v_1v_2^2v_3^{-1}$ is an element of $K \cap T_0$ which acts on R' fixed point freely and centralizes R, $(v_1v_2^2v_3^{-1})^x$ is an element of $K_1 \cap T_0^x$ which acts on R_1' fixed point freely and centralizes R_1. We set $k = (v_1v_2^2v_3^{-1})^x$. As $T_0^x \leqq \langle T_1, x \rangle$, $k \in T_1$ by Proposition 2.5 (ii). Since k centralizes R_1, k is of class $4A$. Thus $k \in T_1 \cap V = \langle v_1v_2^{-1}, v_1v_3^{-1} \rangle$. We may easily verify that the elements in $T_1 \cap V - \{(v_1v_2v_3^2)^{\pm}, (v_1^2v_2v_3)^{\pm}\}$ are squares in P. Therefore we have $\{\langle k \rangle, \langle k^t \rangle\} = \{\langle v_1v_2v_3^2 \rangle, \langle v_1^2v_2v_3 \rangle\}$. So k^t centralizes k and $R_1' (= R_1^t)$. Thus the assertion follows. □

In the remainder of this paper, R_1, R_1' and K_1 will denote the subgroups which are defined in the above and satisfy Lemma 2.6 (i).

Finally we note some results on the action of an involution in $\mathrm{Aut}\, G - G$ on G. We will call this element an *outer involution* of G.

LEMMA 2.7. (i) *The outer involutions of G are G-conjugate and their centralizers on G are isomorphic to J_1.*

(ii) *Suppose that an outer involution ϕ of G centralizes an involution j in G. Then ϕ induces on $C_G(j)/Z(C_G(j))$ $(\cong L_3(4))$ an inverse-transpose automorphism.*

(iii) *G has two conjugacy classes of cyclic subgroups of order 16. Any outer involution interchanges these classes.*

(iv) *Suppose that an outer involution ϕ of G acts on R, the E_9-subgroup in Proposition 2.2. Then $C_R(\phi)\cong Z_3$, $\langle K, \phi\rangle \cong PGL_2(9)$ and $C_K(\phi)\cong D_{10}$.*

PROOF. (i) and (ii) follow from [3] §11. (iii) follows from (i) and [3] Lemma 10.12.

(iv) We use the notation in Proposition 2.2. By the assumption ϕ acts on $K=C_G(R)'$ and $R\rtimes Z=C_G(K)$. We may assume that ϕ acts on Z. Then ϕ inverts Z as J_1 has no element of order 4. Suppose that ϕ inverts R. Then $\phi v_1^2v_2^2=\phi^{v_1v_2v_3^2}$ centralizes R as $v_1^2v_2^2$ also inverts R. This implies that J_1 contains an E_9-subgroup, which is a contradiction. So $C_R(\phi)\neq 1$ and $C_R(\phi)\cong Z_3$.

Since there is no outer involution of G centralizing K, $\langle K, \phi\rangle$ is isomorphic to M_{10}, S_6 or $PGL_2(9)$. We have $\langle K, \phi\rangle \not\cong M_{10}$ as there is no involution in $M_{10}-A_6$. Since any involution in S_6-A_6 centralizes an element of order 4 in A_6, $\langle K, \phi\rangle \not\cong S_6$. Thus $\langle K, \phi\rangle \cong PGL_2(9)$ and so $C_K(\phi)\cong D_{10}$. □

§3. Maximal local subgroups

In this section we determine the maximal local subgroups of G. The notation in §2 will be used.

LEMMA 3.1. (i) *G has two classes of four subgroups, represented by $V_1=\langle v_1^2v_3^2, v_1^2v_2^2\rangle$ and $V_2=\langle v_1^2v_3^2, st\rangle$.*

(ii) *$C_G(V_1)=\langle V, s^2, st\rangle$ and $C_G(V_2)=(R\rtimes\langle v_1v_2v_3^2\rangle)\times V_2$.*

PROOF. (i) Let E be a four subgroup of G. Since all involutions in G are conjugate, we may assume that E contains $v_1^2v_3^2$ and so $E\leqq C_G(v_1^2v_3^2)$. We may also assume that $E\leqq P$ as P is a Sylow 2-subgroup of $C_G(v_1^2v_3^2)$. Suppose that $E\leqq C_G(v_1v_2^2v_3^{-1})$. Then we may assume that E contains an involution in the coset $v_1^2v_2^2\langle v_1v_2^2v_3^{-1}\rangle$ since $C_G(v_1v_2^2v_3^{-1})/\langle v_1v_2^2v_3^{-1}\rangle$ $(\cong L_3(4))$ has one class of involutions. Thus $E=\langle v_1^2v_3^2, v_1^2v_2^2\rangle$. Next suppose that $E\not\leqq C_G(v_1v_2^2v_3^{-1})$. We may easily verify that the following holds: the set $P-C_G(v_1v_2^2v_3^{-1})$ coincides with $sV\cup s^{-1}V\cup stV\cup s^{-1}tV$, there are no involutions in $sV\cup s^{-1}V$, and the involutions in $stV\cup s^{-1}tV$ are P-conjugate. Consequently we may take $E=\langle v_1^2v_3^2, st\rangle$ in this case.

Since the centralizer in $L_3(4)$ of an involution is a 2-subgroup, $C_G(v_1v_2^2v_3^{-1})\cap C_G(V_1)=\langle V, s^2\rangle$ and so $C_G(V_1)=\langle V, s^2, st\rangle$. As $R=O_3(C_G(V_2))$ we have $C_G(V_2)\leqq N_G(R)$. Thus $C_G(V_2)=(R\rtimes Z)\times V_2$ by the structure of $N_G(R)$.

In view of the structures of their centralizers V_1 is not conjugate to V_2. □

LEMMA 3.2. (i) *The maximal elementary abelian 2-subgroups of G are isomorphic to* E_8.

(ii) *We set* $W_1=\Omega_1(V)=\langle v_1^2, v_2^2, v_3^2\rangle$, $W_2=\langle v_1^2v_3^2, v_1^2v_2^2, s^2v_1^{-1}\rangle$ *and* $W_3=\langle v_1^2v_3^2, v_1^2v_2^2, st\rangle$. *Then any* E_8*-subgroup of G is conjugate to* W_i *for some* $i\in\{1, 2, 3\}$.

(iii) $C_G(W_1)=V\cong Z_4\times Z_4\times Z_4$, $C_G(W_2)=\langle v_1v_3, v_1^2v_2^2, s^2v_1^{-1}\rangle\cong Z_4\times E_4$ *and* $C_G(W_3)=\langle v_1v_2v_3^2, v_1^2v_3^2, st\rangle\cong Z_4\times E_4$.

PROOF. Let E be a maximal elementary abelian 2-subgroup. We may assume that E contains V_1 or V_2 by Lemma 3.1. We may also assume that $E\leqq P$. Note that $P\leqq N_G(V_1)$. Suppose that $V_1\leqq E$. If $E\leqq V$, then $E=W_1$. So we may assume that $E\cap s^2V\neq\emptyset$ or $E\cap(stV\cup s^{-1}tV)\neq\emptyset$. In the former case E is conjugate to W_2 as the involutions in s^2V are P-conjugate to $s^2v_1^{-1}$. In the latter case we may assume that $V_2\leqq E$ as the involutions in $stV\cup s^{-1}tV$ are P-conjugate to st. Since $E\leqq C_P(V_2)=\langle v_1v_2^{-1}, v_1^2v_3^2, st\rangle$, we have $E=W_3$ in this case. The structures of $C_G(W_i)$ $(i=1, 2, 3)$ are easily determined. □

PROPOSITION 3.3. *Any 2-local subgroup of G is conjugate to a subgroup of one of the following three subgroups*: $C_G(v_1^2v_3^2)$, $N_G(V)$ *and* $N_G(R)$.

PROOF. By Lemma 3.1 and 3.2, any 2-local subgroup of G is conjugate to a subgroup of one of the following subgroups: $C_G(v_1^2v_3^2)$, $N_G(V_1)$, $N_G(V_2)$, $N_G(W_1)$, $N_G(W_2)$ and $N_G(W_3)$. As $V=\langle C_G(V_1)\cap 4A\rangle$ we have $N_G(V_1)\leqq N_G(V)=N_G(W_1)$. Since $\langle C_G(W_2)\cap 4A\rangle=\langle v_1v_2^2v_3^{-1}\rangle$ and $\langle C_G(W_3)\cap 4A\rangle=\langle v_1v_2v_3^2\rangle$. $N_G(W_i)$ are conjugate to subgroups of $C_G(v_1^2v_3^2)$ for $i=2$ or 3. Moreover $N_G(V_2)\leqq N_G(R)$. Therefore the proposition follows. □

PROPOSITION 3.4. *Any 3-local subgroup of G is conjugate to a subgroup of one of the following two subgroups*: $N_G(R)$ *and* $N_G(R_1\times R_1')$.

PROOF. Let E be a 3-subgroup of G. Since the elements of order 3 in G are conjugate, we may assume that $E\cap R_1$ contains a non-trivial element r. Then $E\leqq C_G(r)=C_G(R_1)=R_1\times K_1$. If $E\leqq R_1$, then $C_G(E)=R_1\times K_1$ and so $N_G(E)\leqq N_G(R_1)$. Suppose that $E\not\leqq R_1$. Then we may assume that

$E \leqq R_1 \times R_1'$. Thus $C_G(E) = R_1 \times R_1'$ and so $N_G(E) \leqq N_G(R_1 \times R_1')$. □

For a prime p other than 2 or 3, the structures of maximal p-local subgroups are easily determined. For $p=5$ and 7 the following were obtained in § 2 and [3] § 6 respectively. Note that there exists an $L_2(31)$-subgroup in G as will be shown in § 4.

PROPOSITION 3.5. (i) *The maximal* 5-*local subgroups of* G *are Sylow* 5-*normalizers. They are isomorphic to* $((E_9 \rtimes Z_4) \times D_{10}) \cdot Z_2$ *and conjugate to subgroups of* $N_G(R)$.

(ii) *We let* D *be a Sylow* 7-*subgroup of* G. *Then* $D \cong 7_+^{1+2}$, *and* D *has two* $N_G(D)$-*classes of* 7A-*pure* E_{49}-*subgroups. We let* E' *and* E'' *be their representatives. Then* G *has three conjugacy classes of maximal* 7-*local subgroups, represented by* $N_G(D)$, $N_G(E')$ *and* $N_G(E'')$. $N_G(D) \cong 7_+^{1+2} \rtimes (Z_3 \times D_8)$ *and* $N_G(E') \cong N_G(E'') \cong E_{49} \rtimes (SL_2(7) \rtimes Z_2)$. *Furthermore these subgroups are contained in some* $(L_3(7) \rtimes Z_2)$-*subgroup.*

(iii) *The maximal* 11-*local subgroups of* G *are Sylow* 11-*normalizers. They are isomorphic to* F_{11}^{10} *and are contained in* J_1-*subgroups.*

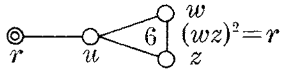
(iv) *The maximal* 19-*local subgroups of* G *are Sylow* 19-*normalizers. They are isomorphic to* F_{19}^{6} *and are contained in* J_1-*subgroups.*

(v) *The maximal* 31-*local subgroups of* G *are Sylow* 31-*normalizers. They are isomorphic to* F_{31}^{15} *and are contained in* $L_2(31)$-*subgroups.*

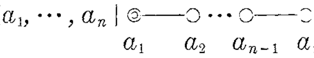
§ 4. Constructions of M_{11} and $L_2(31)$-subgroups

In this section we construct M_{11} and $L_2(31)$-subgroups.

NOTATION 4.1. We use the following diagrams for convenience in order to describe presentations for some groups. Each node ◎ denotes a generator of order 3 and each node ○ denotes an involutive generator. If the product of two generators is of order n, then the corresponding nodes are combined by $(n-2)$-lines or by a simple line to which the order n is attached. The remaining relations are denoted beside the suitable lines. For example, the diagram

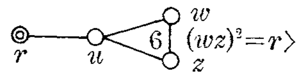


describes the relations $r^3 = u^2 = w^2 = z^2 = (ru)^3 = (rz)^2 = (rw)^2 = (uw)^3 = (uz)^3 = 1$ and $(wz)^2 = r$. (We note that for an involution u and an element r of order 3, $(ru)^3 = 1$ implies $(ur)^3 = 1$ and $(r^{-1}u)^3 = 1$.)

LEMMA 4.2. (i) $\langle a_1, \cdots, a_n \mid$ 

$\rangle$ *is a presentation for* A_{n+2}.

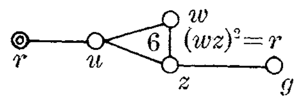
(ii)

$\langle r, u, w, z \mid$ 

$\rangle$

is a presentation for $L_2(11)$.

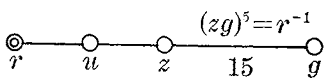
(iii)

$\langle r, u, w, z, g \mid$ 

$\rangle$

is a presentation for M_{11}.

(iv)

$\langle r, u, z, g \mid$ 

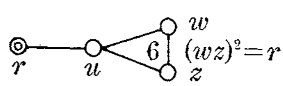
$\rangle$

is a presentation for $L_2(31)$.

PROOF. (i) is well known. (iv) is proved by Perkel [4] (pp. 162–163). (ii) and (iii) follow from the coset enumerations for cosets by $\langle r, u, w\rangle$ $(\cong A_5)$ and $\langle r, u, w, z\rangle$ $(\cong L_2(11))$ respectively.[1] □

Now we start to construct M_{11} and $L_2(31)$-subgroups. Let ϕ be an outer involution and r be an element of order 3 in $C_G(\phi)$. We set $J=C_G(\phi)$, $R=O_3((C_G(r)))$ and $K=C_G(r)'$. We note that $J\cong J_1$, $R\cong E_9$, $K\cong A_6$ and $C_G(r)=R\times K$ (see § 2). Furthermore we identify K with the alternating group on six letters $\{1, 2, \cdots, 6\}$.

By [2], J has $L_2(11)$-subgroups and the elements of order 3 are conjugate in J. Thus by Lemma 4.2 (ii), there exist involutions u, z, w in J which satisfy the following relations:



We set $A=\langle r, u, z\rangle$ and $B=\langle r, u, w\rangle$. Note that A and B are not conjugate in $\langle A, B\rangle$ $(\cong L_2(11))$. Indeed, suppose that $B=A^l$ for some $l\in\langle A, B\rangle$. Then $\langle r, u\rangle$ and $\langle r, u\rangle^{l^{-1}}$ are A_4-subgroups of A $(\cong A_5)$ and so $\langle r, u\rangle=\langle r, u\rangle^{l^{-1}a}$ for some $a\in A$. As $\langle r, u\rangle$ is self-normalizing in $\langle A, B\rangle$, we have $l\in A$. This is a contradiction.

Thus A and B are representatives of J-conjugacy classes of A_5-subgroups of J by [2] Lemma 8.3. Without loss of generality, we may as-

1) M. Kitazume carried out these enumerations by hand.

sume that $C_J(A)=1$ and $C_J(B)\cong Z_2$.

LEMMA 4.3. *Let f be a 4A-element of G such that $C_J(B)=\langle f^2\rangle$. Then we have the following.*

(i) $N_G(B)=(\langle f\rangle\times B)\langle g\rangle$, *where g is an involution such that $f^g=f^{-1}$ and $\langle B, g\rangle\cong S_5$.*

(ii) *For any A_4-subgroup F of B, $N_G(F)=(\langle f\rangle\times F)\langle k\rangle$, where k is an involution such that $f^k=f^{-1}$ and $\langle F, k\rangle\cong S_4$. In particular $N_G(F)\leqq N_G(B)$.*

(iii) $N_G(A)=A$.

PROOF. (i), (ii) By the structure of J_1, $C_J(f^2)\cong Z_2\times A_5$. Then $B\leqq C_G(f^2)'=C_G(f)$ as B is perfect. Since ϕ induces on $C_G(f)/\langle f\rangle$ ($\cong L_3(4)$) an inverse-transpose automorphism, $B\langle f\rangle/\langle f\rangle$ corresponds with the centralizer of ϕ on $C_G(f)/\langle f\rangle$. Now by the structure of Aut $L_3(4)$ and Proposition 2.1 (i), there exists an involution g in $C_G(f^2)-C_G(f)$ which acts on $B\langle f\rangle/\langle f\rangle$ non-trivially. Thus g acts on B ($=\langle B, f\rangle'$) non-trivially, and so we have $\langle B, g\rangle\cong S_5$ and $f^g=f^{-1}$. We may easily verify that, for any A_4-subgroup F of B the centralizer of $F\langle f\rangle/\langle f\rangle$ in $C_G(f^2)/\langle f\rangle$ is trivial. Thus $C_G(F)\cap C_G(f^2)=\langle f\rangle$.

Since g corresponds to a transposition in $\langle B, g\rangle$ ($\cong S_5$), $C_B(g)$ contains an element p of order 3. We write $C_G(p)=R_0\times K_0$, where $R_0\cong E_9$ and $K_0\cong A_6$. We let F be an A_4-subgroup of B containing p. Then $\langle f\rangle\leqq C_G(F)\leqq R_0\times K_0$. As $f^2\in K_0$, $R_0\cap C_G(F)=1$ by the remark in the above paragraph. Thus $C_G(F)\cong Z_4$, D_8 or A_6. Moreover $\langle f\rangle\leqq C_G(F)\leqq K_0$. We note that $\langle f, g\rangle=N_G(f)\cap K_0$ since $\langle f, g\rangle$ is a D_8-subgroup of $C_G(p)$. Suppose that $C_G(F)\cong D_8$ or A_6. Then $\langle f, g\rangle\leqq C_G(F)$ and $\langle g, F\rangle$ ($\cong Z_2\times A_4$) $\leqq\langle g, B\rangle$ ($\cong S_5$), which is a contradiction. Thus $C_G(F)=C_G(B)=\langle f\rangle$.

Since Aut $A_5\cong S_5$, Aut $A_4\cong S_4$ and the A_4-subgroups of A_5 are conjugate, the rest of the assertion follows.

(iii) Suppose that $C_G(A)\neq 1$. Then $C_G(A)\cong Z_2$ or Z_4 since the centralizer of $A\cap B$ ($\cong A_4$) in G is isomorphic to Z_4 by (ii). Since ϕ centralizes A, ϕ normalizes $C_G(A)$ and so $C_J(A)\neq 1$. This contradiction shows that $C_G(A)=1$. Suppose that $A\lneqq N_G(A)$. Then $N_G(A)\cong S_5$. Since ϕ normalizes $N_G(A)$ and centralizes A, we have $N_G(A)\leqq J$ ($\cong J_1$). This is a contradiction. Thus $N_G(A)=A$. □

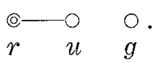
As $(rw)^2=(rz)^2=1$, w and z are contained in $C_G^*(r)$. It follows from the above lemma that w and z have further properties described in the next lemma. Note that $\langle K, \phi\rangle\cong PGL_2(9)$ and $C_K(\phi)\cong D_{10}$ by Lemma 2.7 (iv). We also note that $f\in K$ and $f^2\in C_K(\phi)$ as $f\in C_G(B)\leqq C_G(r)$.

LEMMA 4.4. *$w\in C_G^*(r)\cap C_G(K)$ and $z=rw\times z'$, where z' is an involu-*

tion in $C_K(\phi)-\langle f^2\rangle$.

PROOF. Suppose that y is an involution in $C_G(K)$. Then we have $C_G^*(r)=(R\rtimes\langle y\rangle)\times K$. Since ϕ centralizes r, ϕ acts on K and $R\rtimes\langle y\rangle$. Since w and z invert r and centralize ϕ, we can write $w=qy\times w'$ and $z=q'y\times z'$, where $q, q'\in R$, $w', z'\in C_K(\phi)$ and $(w')^2=(z')^2=1$. Without loss of generality, we may take $q=1$. As $w\in C_G(f)$, $w'\in C_K(\phi)\cap C_G(f)=\langle f^2\rangle$. Since $r=(wz)^2$, $q'=r$ and $(w'z')^2=1$. Suppose that $w'=f^2$. Then we have $z'\in\langle f^2\rangle$. But this implies $[z,f]=1$, and so $A=\langle r,u,z\rangle$ centralizes f. This contradicts Lemma 4.1 (iii). Thus $w'=1$ and z' is an involution in $C_K(\phi)-\langle f^2\rangle$. □

In order to construct M_{11} and $L_2(31)$-subgroups using the presentations in Lemma 4.2, we must consider an involution g which satisfies the following relations:



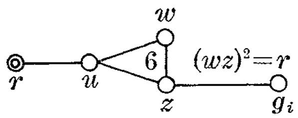
LEMMA 4.5. *We set* $\mathcal{T}=\{g\in G\mid g^2=(ru)^2=(ug)^2=1\}$. *Then we have* $\mathcal{T}=\{w\times k\mid k\in N_K(f)-\langle f\rangle\}$.

PROOF. There exists an involution d such that $f^d=f^{-1}$ and $\langle B,d\rangle\cong S_5$ by Lemma 4.3 (i). We may assume that $r^d=r^{-1}$, $u^d=u$ and $w^d=w$ by Lemma 4.3 (ii). Then $d\in\mathcal{T}$ and $\langle r,u,d\rangle\cong S_4$. Note that $\mathcal{T}$ is the set of involutions which satisfy the same relations as d to r and u, and that an element in $\mathcal{T}$ normalizes $\langle r,u\rangle$. Thus $\mathcal{T}=\{d, df, df^2, df^{-1}\}$ as $N_G(\langle r,u\rangle)=(\langle f\rangle\times\langle r,u\rangle)\langle d\rangle$.

Since an element g in $\mathcal{T}$ inverts r, we can write $g=r_1w\times g'$, where $r_1\in R$, $g'\in K$ and $(g')^2=1$. As w centralizes g, $r_1=1$. Since g inverts f, $g'\in N_K(f)-\langle f\rangle$. Thus the set $\mathcal{T}$ consists of the elements as desired since $|N_K(f)-\langle f\rangle|=4$.

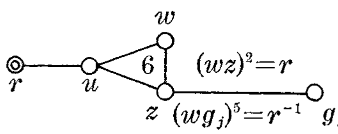
Finally we will examine the relations among the elements r,u,w,z,g for each $g\in\mathcal{T}$. Without loss of generality we may assume that $C_K(\phi)=\langle(13425),(12)(34)\rangle$ and $f^2=(12)(34)$. Then $\langle f\rangle=\langle(1324)(56)\rangle$. By the above lemma, we have $\mathcal{T}=\{w\times(34)(56),\ w\times(13)(24),\ w\times(12)(56),\ w\times(14)(23)\}$. We also have $z=rw\times(35)(24)$, $rw\times(14)(25)$, $rw\times(23)(15)$ or $rw\times(13)(45)$ by Lemma 4.4. Then for any choice of z we may easily verify the following. □

LEMMA 4.6. (i) *For exactly two elements in* $\mathcal{T}$, *say* g_2, *and* g_3, *the following relations hold:*



$(i=2 \text{ or } 3)$.

(ii) *For the remaining two elements in* $\mathcal{T}$, *say* g_1 *and* g_4, *the following relation hold*:



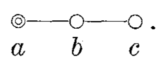
$(j=1 \text{ or } 4)$.

Hence, by Lemma 4.2 (iii) and (iv), we have constructed the desired subgroups.

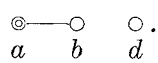
PROPOSITION 4.7. *Under the notations in this section, we have*;
(i) *for each* $i \in \{2, 3\}$, $\langle r, u, z, w, g_i \rangle$ *is an* M_{11}-*subgroup of* G, *and*
(ii) *for each* $j \in \{1, 4\}$, $\langle r, u, z, g_j \rangle$ *is an* $L_2(31)$-*subgroup of* G.

Furthermore we get some results from the above constructions. First we note some general lemma.

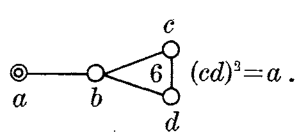
LEMMA 4.8. (i) *Let* X *be a finite group isomorphic to* A_6 *or* $L_2(31)$. *Suppose that* a, b, c *are elements in* X *which satisfy the relations*:



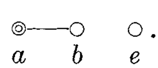
Then there exists an involution d *in* X *which satisfies* $X=\langle a, b, c, d \rangle$ *and the relations*:



(ii) *Let* M *be a finite group isomorphic to* M_{11}. *Suppose that* a, b, c, d *are elements in* M *which satisfy the relations*:



Then there exists an involution e *in* M *which satisfies* $M=\langle a, b, c, d, e \rangle$ *and the relations*:



PROOF. Let A be a finite group isomorphic to A_5. We note that if a, b, c and a', b', c' are elements in A which satisfy the relations; $\underset{a}{\circledcirc}\!-\!\!-\!\underset{b}{\circ}\!-\!\!-\!\underset{c}{\circ}$ and $\underset{a'}{\circledcirc}\!-\!\!-\!\underset{b'}{\circ}\!-\!\!-\!\underset{c'}{\circ}$, then there exists an element $g \in A$ such that $(a')^g = a^{\pm}$ and $(b')^g = b$.

(i) Suppose that $X \cong A_6$. Then there exist elements a', b', c', d' in X which satisfy the relations: $\underset{a'}{\circledcirc}\!-\!\!-\!\underset{b'}{\circ}\!-\!\!-\!\underset{c'}{\circ}\!-\!\!-\!\underset{d'}{\circ}$. Since A_5-subgroups of X are Aut X-conjugate, $\langle a', b', c' \rangle^{\sigma} = \langle a, b, c \rangle$ for some $\sigma \in \mathrm{Aut}\, X$. By the above remark, there exists an element $g \in \langle a, b, c \rangle$ such that $(a')^{\sigma g} = a^{\pm}$ and $(b')^{\sigma g} = b'$. Set $d = (d')^{\sigma g}$. Then the relations $\underset{a}{\circledcirc}\!-\!\!-\!\underset{b}{\circ} \quad \underset{d}{\circ}$ hold. Furthermore $\langle a, b, c, d \rangle = \langle a', b', c', d' \rangle^{\sigma g} = X$. In case of $X \cong L_2(31)$, the same argument holds.

Since $L_2(11)$-subgroups of M_{11} are conjugate, (ii) follows from the analogous argument as above. □

We use the notation mentioned above in this section.

LEMMA 4.9. (i) *There is no A_6-subgroup of G containing B.*

(ii) *There are exactly two A_6-subgroups of G containing A. They are $\langle A, g_2 \rangle$ and $\langle A, g_3 \rangle$. These subgroups are interchanged by ϕ, but are not G-conjugate. Furthermore $\mathrm{N}_G(\langle A, g_2 \rangle) \cong \mathrm{N}_G(\langle A, g_3 \rangle) \cong M_{10}$ and these normalizers are contained in M_{11}-subgroups.*

(iii) *There are exactly two $L_2(31)$-subgroups of G containing A. They are $\langle A, g_1 \rangle$ and $\langle A, g_4 \rangle$. These subgroups are interchanged by ϕ, but are not G-conjugate. Furthermore they are self-normalizing.*

(iv) *There are exactly two M_{11}-subgroups of G containing $\langle A, B \rangle$ ($\cong L_2(11)$). They are $\langle A, B, g_2 \rangle$ and $\langle A, B, g_3 \rangle$. These subgroups are interchanged by ϕ, but are not G-conjugate. Furthermore they are self-normalizing.*

PROOF. We set $\mathcal{I} = \{g \in G \mid g^2 = (rg)^2 = (ug)^2 = 1\}$.

(i) Suppose that C is an A_6-subgroup of G containing B. By Lemma 4.8 (i), there exists an involution $x \in \mathcal{I}$ such that $C = \langle r, u, w, x \rangle$. However for any $x \in \mathcal{I}$ the relations $\underset{r}{\circledcirc}\!-\!\!-\!\underset{u}{\circ}\!-\!\!-\!\underset{w}{\circ} \quad \underset{x}{\circ}$ hold. Thus $C \cong S_5$, which is a contradiction.

(ii) Suppose that C is an A_6-subgroup of G containing A. By Lemma 4.8 there exists an involution $x \in \mathcal{I}$ such that $C = \langle r, u, z, x \rangle$. Since A_6 has no $L_2(31)$-subgroup, we have $x = g_2$ or g_3 by Lemma 4.6.

Since ϕ fixes A, ϕ acts on $\mathcal{I}$. So ϕ acts on $\{g_2, g_3\}$ and $\{g_1, g_4\}$. Thus $g_2^{\phi} = g_3$ and $g_1^{\phi} = g_4$ as $\mathrm{C}_G(\phi)$ ($\cong J_1$) has no A_6 nor $L_2(31)$-subgroup. In particular

$\langle A, g_2\rangle^{\phi}=\langle A, g_3\rangle$.

Suppose that $\langle A, g_2\rangle^k=\langle A, g_3\rangle$ for some $k\in G$. Then A and A^k are A_5-subgroups in $\langle A, g_3\rangle$. Since $\mathrm{N}_G(\langle A, g_3\rangle)\cap\langle A, B, g_3\rangle\cong M_{10}$, there exists an element h in this M_{10}-subgroup such that $A^{kh}=A$. Then we have $kh\in \mathrm{N}_G(A)=A$ by Lemma 4.2. Thus $k\in\mathrm{N}_G(\langle A, g_3\rangle)$. This implies that ϕ normalizes $\langle A, g_2\rangle=\langle A, g_3\rangle$. Then ϕ centralizes $\langle A, g_3\rangle$ $(\cong A_6)$, which is a contradiction. Therefore $\langle A, g_2\rangle$ and $\langle A, g_3\rangle$ are not G-conjugate.

Since $\mathrm{C}_G(\langle A, g_3\rangle)=\mathrm{C}_G(A)=1$, $\mathrm{N}_G(\langle A, g_3\rangle)\cong M_{10}$ or $\mathrm{Aut}\, A_6$. Suppose that $\mathrm{N}_G(\langle A, g_3\rangle)\cong\mathrm{Aut}\, A_6$. Then there exists an S_6-subgroup S containing A. We may easily verify that $S=\langle A, x\rangle$ for some involution $x\in\mathcal{T}$. But this contradicts Lemma 4.6. Thus $\mathrm{N}_G(\langle A, g_3\rangle)\cong M_{10}$.

(iii) Suppose that L is an $L_2(31)$-subgroup of G containing A. Since $L_2(31)$ has no A_6-subgroup, we have $L=\langle A, g_1\rangle$ or $\langle A, g_4\rangle$ by the same argument as in (ii). As $g_1^{\phi}=g_4$ we have also $\langle A, g_1\rangle^{\phi}=\langle A, g_4\rangle$.

Suppose that $\langle A, g_1\rangle^k=\langle A, g_4\rangle$ for some $k\in G$. Then $k\phi$ normalizes $\langle A, g_1\rangle$. We set $L=\langle A, g_1\rangle$. Since any element of order 31 is self-centralizing, $\mathrm{C}_G(L)=1$. Thus $\mathrm{N}_G(L)=L$ as G has no element of order 32. Consequently $\mathrm{N}_{\mathrm{Aut}\, G}(L)\cong PGL_2(31)$. Then for any element h of order 16 in L and for any outer involution ϕ in $\langle L, k\phi\rangle$, $\langle h\rangle$ and $\langle h\rangle^{\phi}$ are L-conjugate. This contradicts Lemma 2.7 (iii). Therefore $\langle A, g_1\rangle$ and $\langle A, g_4\rangle$ are not G-conjugate. Thus (iii) is proved.

(iv) Suppose that M is an M_{11}-subgroup of G containing $\langle A, B\rangle$. By Lemma 4.8 (ii), there exists an involution $x\in\mathcal{T}$ such that $\langle A, B, x\rangle=M$. Since M_{11} has no $L_2(31)$-subgroup, we have $x=g_2$ or g_3 by Lemma 4.6. As $g_2^{\phi}=g_3$ we have also $\langle A, B, g_2\rangle^{\phi}=\langle A, B, g_3\rangle$. Since M_{11} has one class of A_6-subgroup, $\langle A, B, g_2\rangle$ and $\langle A, B, g_3\rangle$ are not G-conjugate by (ii). Since $\mathrm{C}_G(M)=1$ and $\mathrm{Aut}\, M_{11}\cong M_{11}$, M is self-normalizing. □

§ 5. A construction of an A_7-subgroup

In this section R_1, R_1', K_1, T_1, C and L have the same meanings as in § 2.

We first classify A_4-subgroups of G.

PROPOSITION 5.1. *G has four conjugacy classes of A_4-subgroups, represented by $F_i\,(i=1, \cdots, 4)$ which satisfy the following.*

(i) $\mathrm{N}_{K_1}(F_1)\cong S_4$ *and* $\mathrm{N}_G(F_1)=(R_1\rtimes\langle v_1v_2v_3^2\rangle)\times\mathrm{N}_{K_1}(F_1)$.

(ii) $\mathrm{N}_G(F_2)=(R_1\times F_2)\langle k\rangle$, *where k is an involution inverting R_1. Furthermore* $\langle F_2, k\rangle\cong S_4$.

(iii) F_2 *and* F_3 *are* Aut G*-conjugate.*

(iv) F_4 *centralizes some outer involution.* $\mathrm{N}_G(F_4)=(\langle f\rangle\times F_4)\langle g\rangle$, *where*

$|f|=4$, $|g|=2$, $f^g=f^{-1}$ *and* $\langle F_4, g\rangle \cong S_4$.

PROOF. Let F_1 be an A_4-subgroup of K_1 ($\cong A_6$) such that $N_{K_1}(F_1)\cong S_4$. We may assume that F_1 contains a non-trivial element p in R_1', a Sylow 3-subgroup of K_1. We let u be an involution in F_1.

Note that $R_1\langle v_1v_2v_3^2\rangle \leqq C_G(F_1)\leqq C_G(p)=R_1'\times K_1^t$ and $C_G(F_1)\cap R_1'\leqq C_G(u)\cap R_1'=1$. Consequently $C_G(F_1)$ is isomorphic to a subgroup of K_1^t ($\cong A_6$) which contains a Sylow 3-normalizer. Thus $C_G(F_1)=R_1\langle v_1v_2v_3^2\rangle$ or K_1^t. Since the centralizers of four subgroups in G are $\{2, 3\}$-subgroups by Proposition 3.1, we have $C_G(F_1)=R_1\langle v_1v_2v_3^2\rangle$. Then $N_G(F_1)=(R_1\langle v_1v_2v_3^2\rangle)\times N_{K_1}(F_1)$ as $\mathrm{Aut}\,A_4\cong S_4$.

There exists an involution $v\in N_{K_1}(F_1)$ such that $u^v=u$, $p^v=p^{-1}$. We let r be a non-trivial element in R_1. We set $F_2=\langle u, rp\rangle$. As $[r, K_1]=1$, we have $u^2=(rp)^3=(u\cdot rp)^3=1$. Thus $F_2\cong A_4$. By the same argument as above, $C_G(F_2)$ is isomorphic to a proper subgroup of K_1^t which contains a Sylow 3-subgroup. Thus $R_1\leqq C_G(F_2)\leqq R_1\rtimes\langle v_1v_2v_3^2\rangle$. Since $v_1v_2v_3^2$ acts fixed point freely on R_1 ($\ni r$), we have $C_G(F_2)=R_1$. Thus $N_G(F_2)=(R_1\times F_1)\langle v_1^2v_2^2v\rangle$ as $\langle F_2, v_1^2v_2^2v\rangle\cong S_4$.

There exists an outer involution ϕ centralizing r, since the elements of order 3 of G are conjugate. Suppose that $F_2^\phi=F_2^g$ for some $g\in G$. Then $g\phi$ is an element in $\mathrm{Aut}\,G-G$ which normalizes F_2. Since $S_4\cong\mathrm{Aut}\,A_4\cong N_G(F_2)/C_G(F_2)$, there exists an outer involution ψ centralizing F_2. Consequently ψ acts on $R_1=C_G(F_2)$ and centralizes some non-trivial element $q\in R_1$ by Lemma 2.7 (iv). But this implies that $C_G(\psi)$ ($\cong J_1$) contains $\langle q\rangle\times F_2$ ($\cong Z_3\times A_4$), which is a contradiction. Thus F_2 and F_2^ϕ are not G-conjugate. We set $F_3=F_2^\phi$.

The existence of an A_4-subgroup which satisfies (iv) is already shown in Lemma 4.3 (ii).

It follows from the above remark and the structures of their centralizers, that F_i ($i=1, \cdots, 4$) are not G-conjugate. We will use the counting argument in §1 in order to prove that any A_4-subgroup of G is conjugate to F_i for some $i=1, \cdots, 4$. We denote a conjugate of F_i containing p also by F_i for $i\in\{1, \cdots, 4\}$. Since $\langle a, b\,|\,a^2=b^3=(ab)^3=1\rangle$ is a presentation for A_4, we have $(2A, 3A\,;p)\cap F_i\cap F_i^g=\emptyset$ whenever $F_i^g\neq F_i$ for $g\in N_G(p)$. Thus by the structures of $N_G(F_i)$ ($i=1, \cdots, 4$) we have

$$\Big|(2A, 3A\,;p)\cap\Big(\bigcup_{i=1}^{4}\bigcup_{g\in N_G(p)}F_i^g\Big)\Big|$$

$$={}^\sharp(2A, 3A, 3A)_{A_4}\times|N_G(p)|\times\Big(\frac{1}{6\cdot 9\cdot 4}+\frac{2}{6\cdot 9}+\frac{1}{6\cdot 4}\Big)=|C_G(p)|/2\,.$$

On the other hand we get ${}^\sharp(2A, 3A\,;p)=1620=|C_G(p)|/2$ by the character table of G. Therefore the set of all A_4-subgroups of G containing p coincides with

$$\left\{\langle x, y\rangle \,\middle|\, (x, y)\in(2A, 3A\,;p)\cap\left(\bigcup_{i=1}^{4}\bigcup_{g\in \mathrm{N}_G(p)} F_i^g\right)\right\}.$$

This completes the proof of the proposition. □

Now we start to construct an A_7-subgroup of G. We identify $K_1\,(\cong A_6)$ with the alternating group on six letters $\{1,\cdots,6\}$. Without loss of generality we may assume that $R_1'=\langle(123), (456)\rangle$ and $v_1^2v_2v_3=(14)(2536)$ through this identification. We set $u=v_1^2v_2^2\times(14)(56)$. Note that u centralizes $v_1v_2v_3^2$ and inverts $v_1^2v_2v_3$.

LEMMA 5.2. *There exists an element* $h\in \mathrm{N}_G(R_1\times R_1')$ *such that* $h^2=v_1^2v_3^2=v_1^2v_2^2\times(23)(56)$, $(uh)^6=1$ *and* $h\notin \mathrm{N}_G(R_1)$.

PROOF. For a subset X in $\mathrm{C}_G(v_1^2v_3^2)$, $\bar{X}$ denotes the image of X in $\mathrm{C}_G(v_1^2v_3^2)/\langle v_1v_2^2v_3^{-1}\rangle$ by the natural projection. We identify $\overline{\mathrm{C}_G(v_1v_2^2v_3^{-1})}$ with $L_3(4)$ and represent its elements as matrices in $SL_3(4)$ modulo the multiplication of the scaler matrices. Furthermore θ denotes the unitary automorphism of $L_3(4)$ defined by;

$X^\theta=J\,{}^t\bar{X}^{-1}J$ modulo the multiplication of the scalar matrices for $X=(\xi_{i,j})_{1\leq i,j\leq 3}\in SL_3(4)$, where $J=\begin{bmatrix} & & 1\\ & 1 & \\ 1 & & \end{bmatrix}$ and ${}^t\bar{X}=(\xi_{j,i}^2)_{1\leq i,j\leq 3}$. Note that $\overline{\mathrm{C}_G(v_1^2v_3^2)}=L_3(4)\langle\theta\rangle$.

As $\mathrm{Z}(C)=\langle v_1^2v_3^2\rangle$, $C\leqq \mathrm{C}_G(v_1^2v_3^2)$. We may easily verify that $\langle T_1, t\rangle$, a Sylow 2-subgroup of C, is contained in $\mathrm{C}_G(v_1v_2^2v_3^{-1})$. Moreover $u\in \mathrm{C}_G(v_1^2v_3^2)$.

We will divide the proof into several steps.

Step 1. *We may assume that* $\bar{u}=\theta$, $L=\left\{\begin{bmatrix}1 & & \\ 0 & 1 & \\ \beta & \gamma & 1\end{bmatrix}\,\middle|\,\beta,\gamma\in \boldsymbol{F}_4\right\}$ *and* $\overline{\langle T_1, t\rangle}=\left\langle\begin{bmatrix}1 & & \\ \alpha & 1 & \\ 0 & 0 & 1\end{bmatrix}, \bar{L}\right\rangle$, *where* α *is some element in* $\boldsymbol{F}_4^\sharp$.

Proof of Step 1. Since $\overline{v_1^2v_2^2}$ is contained in the center of $\bar{P}$, a Sylow 2-subgroup of $\overline{\mathrm{C}_G(v_1^2v_2^2)}$, we may assume that $\overline{v_1^2v_2^2}=\begin{bmatrix}1 & & \\ 0 & 1 & \\ 1 & 0 & 1\end{bmatrix}$ and $\bar{P}=\left\{\begin{bmatrix}1 & & \\ * & 1 & \\ * & * & 1\end{bmatrix}\right\}\langle\theta\rangle$. Note that $\bar{L}$ is an E_{16}-subgroup of $\bar{P}\cap\overline{\mathrm{C}_G(v_1v_2^2v_3^{-1})}$ containing $\overline{v_1^2v_2^2}$, and that $\overline{\langle T_1, t\rangle}$ is a subgroup of $\bar{P}\cap\overline{\mathrm{C}_G(v_1v_2^2v_3^{-1})}$ which contains $\bar{L}$ as a subgroup of index 2. Thus we may assume that $\bar{L}=\left\{\begin{bmatrix}1 & & \\ 0 & 1 & \\ * & * & 1\end{bmatrix}\right\}$ and $\overline{\langle T_1, t\rangle}=\left\{\begin{bmatrix}1 & & \\ \alpha & 1 & \\ * & * & 1\end{bmatrix}\,\middle|\,\alpha\in \boldsymbol{F}_2\right\}$.

Since $\overline{v_1^2v_2v_3}$ and $\overline{v_1v_2v_3^2}$ are square roots of $\overline{v_2^2v_3^2}=\overline{v_1^2v_2}$ in $\overline{\langle T_1, t\rangle}$, they are of the shape $\begin{bmatrix}1 & & \\ 1 & 1 & \\ * & 1 & 1\end{bmatrix}$. We note that the centralizer in $L_3(4)$ of elements of this shape is $\left\{\begin{bmatrix}1 & & \\ \xi & 1 & \\ \eta & \xi & 1\end{bmatrix} \middle| \xi, \eta\in \boldsymbol{F}_4\right\}$. Suppose that $\bar{u}\in L_3(4)$. Then $\bar{u}$ centralizes $\overline{v_1v_2v_3^2}$ and $\overline{v_1^2v_2v_3}$, which is a contradiction. Thus $\bar{u}\notin L_3(4)$. So $\bar{u}$ is contained in the coset $(\bar{P}\cap\overline{C_G(v_1v_2^2v_3^{-1})})\theta$ since $\bar{u}\in C_{L_3(4)\langle\theta\rangle}\overline{(v_1^2v_2^2)}=\bar{P}$. As all involutions in $(\bar{P}\cap L_3(4))\theta$ are $\bar{P}\cap L_3(4)$-conjugate to θ, the assertions follow.

Step 2. *Under the assumptions of step* 1, *we may also assume that* $\bar{C}=\left\langle \bar{L}, \begin{bmatrix}1 & & \\ \alpha & 1 & \\ & & 1\end{bmatrix}, \begin{bmatrix}1 & \omega^2\alpha^{-1} & 0\\ \alpha & \omega & 0\\ 0 & 0 & 1\end{bmatrix}\right\rangle$, *where* $\langle\omega\rangle=\boldsymbol{F}_4^{\#}$.

Proof of Step 2. C is contained in the parabolic subgroup $N_{L_3(4)}(\bar{L})$ of $L_3(4)$. Since $N_{L_3(4)}(\bar{L})/\bar{L}\cong A_5$, there are 8 elements of order 5 in $N_{L_3(4)}(\bar{L})/\bar{L}$ which are inverted by the involution $\begin{bmatrix}1 & & \\ \alpha & 1 & \\ 0 & 0 & 1\end{bmatrix}\bar{L}$. They form two cyclic subgroups of order 5. We may easily verify that these cyclic subgroups correspond to $\left\langle\begin{bmatrix}1 & \omega^2\alpha^{-1} & 0\\ \alpha & \omega & 0\\ 0 & 0 & 1\end{bmatrix}, \bar{L}\right\rangle$ and $\left\langle\begin{bmatrix}1 & \omega\alpha^{-1} & 0\\ \alpha & \omega & 0\\ 0 & 0 & 1\end{bmatrix}, \bar{L}\right\rangle$, where $\langle\omega\rangle=\boldsymbol{F}_4^{\#}$. Replacing α by a suitable element in $\boldsymbol{F}_4$, if necessary, we may assume that the former case holds. Thus $\bar{C}$ is of the desired shape as $\bar{C}/\bar{L}\cong D_{10}$.

We set $\bar{a}=\begin{bmatrix}1 & & \\ \alpha & 1 & \\ 0 & 0 & 1\end{bmatrix}$, $\bar{y}=\begin{bmatrix}1 & \omega^2\alpha^{-1} & 0\\ \alpha & \omega & 0\\ 0 & 0 & 1\end{bmatrix}$, and $\mathcal{S}=\{g\in\langle T_1, t\rangle-L \mid g_2=v_1^2v_3^2\}$. We note that $\bar{\mathcal{S}}$ consists of involutions in $\overline{\langle T_1, t\rangle}-\bar{L}$ and $|\bar{\mathcal{S}}|=4$ by Lemma 2.6 (ii). Consequently $\bar{\mathcal{S}}$ is the set of all involutions in $\overline{\langle T_1, t\rangle}-\bar{L}$. Thus we may take $a\in\mathcal{S}$. We may also assume that y is of order 5 in C as $|\bar{y}|=5$.

Step 3. *Set* $h=a^y$. *Then* h *satisfies the conditions in the lemma.*

Proof of Step 3. Since $a\in\mathcal{S}$, $h^2=(a^2)^y=v_1^2v_3^2\in Z(C)$.

As $\bar{y}^{\bar{a}}=\bar{y}^{-1}$ we have $\bar{h}=\bar{a}\bar{y}^2$. So we can write $h=lay^2$, where $l\in\langle v_1v_2^2v_3^{-1}\rangle$. Suppose that $l\notin\langle v_1^2v_3^2\rangle$. Then $v_1v_2^2v_3^{-1}=\{h(ay^2)^{-1}\}^{\pm}$ and C contains this element. It follows that $\langle v_1v_2^2v_3^{-1}\rangle\times\langle T_1, t\rangle\leqq C$, which is a contradiction. Thus $l\in\langle v_1^2v_3^2\rangle$. In particular l, $a\in T_1$ (note that $\mathcal{S}\subseteqq T_1$ by Lemma 2.6 (ii).) and la normalizes R_1 and R_1'. Suppose that $h\in N_G(R_1)$. Then $y^2\in N_G(R_1)$. This implies that y, an element of order 5, is contained in $K_1\leqq C$. But $K_1\cap C\leqq K_1\cap N_G(R_1\times R_1')=N_{K_1}(R_1')\cong E_9\rtimes Z_4$. This contradiction shows that $h\notin N_G(R_1)$.

We may easily verify that $\bar{u}\bar{h}$ is of order 6 by simple matrix computations. Thus $(uh)^6 \in \langle v_1 v_2^2 v_3^{-1} \rangle$. Since $4A$-elements are not squares in G, we have $(uh)^6 \in \langle v_1^2 v_3^2 \rangle$. Suppose that $(uh)^6 = v_1^2 v_3^2$. Then $(uh)^3$ is an element of order 4 which centralizes the element $(uh)^4$ of order 3. So $(uh)^3$ is a $4A$-element. As $(uh)^3$ is a square root of $v_1^2 v_3^2$, $(uh)^3 = (v_1 v_2^2 v_3^{-1})^{\pm}$ by Proposition 2.1 (ii). This implies that $\bar{u}\bar{h}$ is of order 3, which is a contradiction. Thus $(uh)^6 = 1$ and all the conditions are verified. □

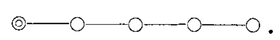
In the remainder of this section we use h to denote an element which satisfies the conditions in Lemma 5.2. As $h \in \mathrm{N}_G(R_1 \times R_1')$, we can write $r^h = r_1 \times r_1'$ for $r \in R_1^\#$, where $r_1 \in R_1$ and $r_1' \in R_1'$. Since $h \notin \mathrm{N}_G(R_1)$, we have $r_1' \neq 1$. Since this holds for any $r \in R_1^\#$, the correspondence $r \to r_1'$ is a bijection from $R_1^\#$ onto $(R_1')^\#$. Thus there exists the unique element $r \in R_1^\#$ such that $r^h = r_1 \times (123)$. In the remainder of this section we use r and r_1 to denote these elements. We set $q = r^h$.

LEMMA 5.3. $r_1 \neq 1$.

PROOF. Suppose that $r_1 = 1$. Then $R_1^h = R_1'$ as R_1 and R_1' are T. I. sets. So h acts on $\{R_1, R_1'\}$ since $h^2 = v_1^2 v_3^2$ inverts $R_1 \times R_1'$. As is shown in the proof of step 3 in Lemma 5.2, we can write $h = lay$, where $l, a \in T_1$ and y is an element of order 5. Since T_1 normalizes R_1 and R_1', h normalizes R_1 and R_1'. This is a contradiction. □

Now we set $a_1 = r$, $a_2 = (h^2 u)^{huq^{-1}}$, $a_3 = h^2 = v_1^2 v_2^2 \times (23)(56)$, $a_4 = (h^2)^{q^{-1}} = r_1^{-1} v_1^2 v_2^2 \times (12)(56)$ and $a_5 = u = v_1^2 v_2^2 \times (14)(56)$.

PROPOSITION 5.4. *The elements a_i $(i = 1, \cdots, 5)$ satisfy the following relations*:



In particular $\langle a_1, a_2, a_3, a_4, a_5 \rangle$ is an A_7-subgroup of G.

PROOF. As $h^2 u = (14)(23)$, we have $|a_1| = 3$ and $|a_i| = 2$ $(i = 2, \cdots, 5)$. Since a_i $(i = 2, \cdots, 5)$ invert r, $a_1 a_i$ are involutions $(i = 2, \cdots, 5)$. As h^2 centralizes u, it follows that $a_3 a_5$ and $a_2 a_4$ $(= (h^2 u)^{huq^{-1}} (h^2)^{huq^{-1}} = u^{huq^{-1}})$ are involutions. The elements $a_3 a_4$ $(= r_1 \times (123))$ and $a_4 a_5$ $(= r_1^{-1} \times (124))$ are of order 3.

By a direct computation in $R_1 \langle v_1 v_2 v_3^2 \rangle \times K_1$, we may easily verify that $u^{qu} = (h^2)^{q^{-1}}$. So $u^{quh^{-1}} = (h^2)^{q^{-1}h^{-1}} = (h^2)^{h^{-1}r^{-1}} = r^{-1} v_1^2 v_2^2 \times (23)(56)$ as $q^{-1} = h^{-1} r^{-1} h$. Thus we have $(a_2 a_5)^{quh^{-1}} = h^2 u \cdot u^{quh^{-1}} = r^{-1} v_1^2 v_2^2 \times (14)(56)$, and so $a_2 a_5$ is an involution.

As $r^{quh^{-1}} = r^{uh^{-1}} = (r^{-1})^{h^2 h} = q$, we have $(a_1 a_2)^{quh^{-1}} = r_1 \times (123) \cdot (14)(23) =$

$r_1\times(134)$. Thus a_1a_2 is of order 3.

Finally we will verify that a_2a_3 is of order 3. As $(uh)^6=1$ and $[h^2, u]=1$, we have $uhuhu=h^{-1}uh^{-1}uh^{-1}uh^{-1}=h^{-1}uh\cdot u\cdot h^{-1}uh=u^{huh^{-1}}$. Thus $(uh^2)^{hu}=uhuhu=u^{huh^{-1}}$. As is shown in the above paragraph, $\langle r, (uh^2)^{huq^{-1}}\rangle\cong A_4$. So $(uh^2)^{huq^{-1}}\cdot r^{-1}$ is of order 3. Now we have $\{(uh^2)^{huq^{-1}}\cdot r^{-1}\}^{qhq^{-1}}=\{(uh^2)^{hu}\}^{hq^{-1}}\cdot q^{-1}=\{u^{huh^{-1}}\}^{hq^{-1}}\cdot q^{-1}=u^{huq^{-1}}\cdot q^{-1}=(uh^2)^{huq^{-1}}\cdot(h^2)^{q^{-1}}\cdot q^{-1}=a_2a_3$. Thus $a_2\cdot a_3$ is of order 3. □

We set $F=\langle a_1, a_2\rangle$, $D=\langle a_1, a_2, a_3\rangle$ and $E=\langle a_1, a_2, a_3, a_4, a_5\rangle$.

LEMMA 5.5. (i) $C_G(F)=\langle r_1(142), p(356)\rangle\ (\cong E_9)$ *for some* $p\in R_1$. $N_G(F)=(F\times C_G(F))\langle u\rangle$.

(ii) $N_G(D)=N_E(D)\cong S_5$.

(iii) *There exist exactly two* A_6*-subgroups of* G *which contain* D. *They are* $\langle D, a_4\rangle$ *and* $\langle D, a_5\rangle$.

(iv) *Suppose that* S *is an* A_7*-subgroup of* G *such that* $N_S(D)\cong S_5$. *Then* $S=E$ *and* $N_G(E)=E$.

PROOF. (i) Since $a_4a_5=r_1^{-1}\times(124)\in C_G(F)$, $C_G(F)\cong E_9$ or $E_9\rtimes Z_4$ by Proposition 5.1. We have $C_G(F)\leqq C_G(r)=R_1\times K_1$. Suppose that $C_G(F)\cap R_1$ contains a non-trivial element x. Then $F\leqq C_G(x)=C_G(r)$, and so $r\in Z(F)$. This contradiction shows that $C_G(F)\cap R_1=1$ and that $C_G(F)$ is isomorphic to an E_9 or $E_9\rtimes Z_4$-subgroup of K_1 which contains (124). Suppose that $C_G(F)\cong E_9\rtimes Z_4$. We let y be an involution in $C_G(F)$. Then y inverts $r_1\cdot(142)$ $(\in O_3(C_G(F)))$. As $y\in K_1$, this implies $r_1=1$. This contradicts Lemma 5.3. Thus $C_G(F)\cong E_9$ and $C_G(F)=\langle r_1(142), p(356)\rangle$ for some $p\in R_1$. Since $\langle F, u\rangle\cong S_4\cong \mathrm{Aut}\,A_4$, we have $N_G(F)=(F\times C_G(F))\langle u\rangle$.

(ii) By (i) we have $C_G(D)\leqq C_G(F)\ (\cong E_9)$. We set $R_0=C_G(F)$. By Proposition 5.1, R_0 is conjugate to R_1. Thus $C_G(R_0)=R_0\times K_0$, where $K_0=C_G(R_0)'\cong A_6$. Suppose that $C_G(D)$ contains a non-trivial element x. Then $D\leqq C_G(x)=R_0\times K_0$, and so $D\leqq K_0$. Since $N_E(D)=\langle D, u\rangle\cong S_5$, this contradicts Corollary 2.4 (ii). Thus $C_G(D)=1$ and $N_G(D)=N_E(D)\cong S_5$.

(iii), (iv) We set $\mathcal{U}=\{g\in G\mid g^2=(gr)^2=(ga_2)^3=1\}$. Then $u\in\mathcal{U}$ and so $\mathcal{U}=C_G(F)u$ by (i). Thus $\mathcal{U}=\{g_i\mid i=1,\cdots,9\}$, where $g_1=u$, $g_2=a_4=r_1^{-1}v_1^2v_2^2\times(12)(56)$, $g_3=g_2^u$, $g_4=pv_1^2v_2^2\times(14)(36)$, $g_5=g_4^u$, $g_6=r_1^{-1}pv_1^2v_2^2\times(12)(36)$, $g_7=g_6^u$, $g_8=r_1pv_1^2v_2^2\times(24)(36)$ and $g_9=g_8^u$. Now we may easily verify that; $g_1\cdot a_3=uh^2$ is an involution, $g_i\cdot a_3$ $(i=2$ or $3)$ is of order 3, $g_i\cdot a_3$ $(i=4$ or $5)$ is of order 4 or 12, and $g_i\cdot a_3$ $(i=6,\cdots,9)$ is of order 5 or 15.

Suppose that A is an A_6-subgroup containing D. Then there exists an involution $g\in\mathcal{U}$ such that $A=\langle D, g\rangle$ by Lemma 4.8. Since the centralizer in A of r is an E_9-subgroup, we have $g=a_4$ or a_4^u from the above paragraph.

Suppose that S is an A_7-subgroup such that $N_S(D)\cong S_5$. Since S_5-

subgroups in A_7 are conjugate, it follows from the analogous argument as in Lemma 4.8 that there exist involutions $g, k \in \mathcal{U}$ such that $\langle g, k\rangle \cong S_3$ and $S=\langle D, g, k\rangle$. Since the centralizer in S of r is a $Z_3\times A_4$-subgroup, $a_3\cdot g$ and $a_3\cdot k$ are of order 2, 3 or 6. Thus $\{g, k\}\subseteq\{a_4, a_4^u, u\}$ from the previous paragraph. Therefore $S=E$.

By the structures of centralizers of elements in G, we have $\mathrm{C}_G(E)=1$. Suppose that $\mathrm{N}_G(E)\cong S_7$. Then we have $\mathrm{N}_G(E)\cap \mathrm{C}_G(D)\cong Z_2$. But this contradicts (ii). Thus $\mathrm{N}_G(E)=E$. □

§ 6. Non-local analyses for $A_5, A_6, A_7, L_2(11), M_{11}$ and $L_2(31)$

In § 6 and § 7 we classify the conjugacy classes of non-abelian characteristically simple subgroups of G and determine the structures of normalizers of their representatives.

By the classification of finite simple groups, the following is the list of non-abelian characteristically simple groups whose orders divide $2^9\cdot 3^4\cdot 5\cdot 7^3\cdot 11\cdot 19\cdot 31$, the order of G.

$A_n \quad (n=5, \cdots, 9)$;
$L_2(q) \quad (q=7, 8, 11, 19, 31$ and $32)$;
$L_2(7)\times L_2(7), L_2(8)\times L_2(8), L_2(7)\times L_2(7)\times L_2(7)$;
$L_3(4), L_3(7), U_3(3), U_3(8), U_4(2), Sp_6(2)$;
M_{11}, M_{12}, J_1 and M_{22}.

Note that $A_9, L_2(8), L_2(8)\times L_2(8), L_2(19), U_3(8), U_4(2)$ and $Sp_6(2)$ have elements of order 9 and that $U_3(3), M_{12}$ have non-abelian Sylow 3-subgroups. Since Sylow 3-subgroups of G are isomorphic to E_{81}, there is no subgroup of G isomorphic to any one of these groups. Since G has no E_{16}-subgroup by Proposition 3.2, there is no subgroup of G isomorphic to $L_2(32)$, $A_8\cong L_4(2)$, $L_3(4)$, M_{22}, $L_2(7)\times L_2(7)$ nor $L_2(7)\times L_2(7)\times L_2(7)$.

On the other hand G has $L_3(7)$ and J_1-subgroups by [3]. So G has also $L_2(7)$, $L_2(11)$ and A_5-subgroups. G has M_{11} and $L_2(31)$-subgroups as is shown in § 4. So G has A_6-subgroups. Finally G has A_7-subgroups by § 5. Thus we have;

PROPOSITION 6.1. *The complete list of non-abelian characteristically simple groups which actually occur as subgroups in G is as follows*:

$A_5, L_2(7), A_6, L_2(11), A_7, M_{11}$,
$L_2(31), J_1, L_3(7)$.

First we classify the A_5-subgroups of G. In the next lemma we use R, K, T and Z to denote the subgroups in Proposition 2.1. Furthermore we let ϕ be an outer involution and F_i $(i=1, \cdots, 4)$ be the A_4-subgroups in Proposition 5.1.

PROPOSITION 6.2. *G has five conjugacy classes of A_5-subgroups of G, represented by B_i $(i=1, \cdots, 5)$ which satisfy the following.*

(i) *$B_1 \leqq K$ and $\mathrm{N}_G(B_1)=(R \rtimes Z) \times B_1 \cong (E_9 \rtimes Z_4) \times A_5$. Any A_4-subgroup of B_1 is conjugate to F_1.*

(ii) *$\mathrm{N}_G(B_2)=(\langle f\rangle \times B_2)\langle g\rangle \cong (Z_4 \times A_5) \rtimes Z_2$, where $|f|=4$, $|g|=2$, $f^g=f^{-1}$ and $\langle B_2, g\rangle \cong S_5$. Any A_4-subgroup of B_2 is conjugate to F_2. Furthermore $B_2 \leqq \mathrm{C}_G(\phi)$.*

(iii) *$B_3 \leqq \mathrm{C}_G(\phi)$, $\mathrm{N}_G(B_3)=B_3$ and $\langle B_2, B_3\rangle \cong L_2(11)$. Any A_4-subgroup of B_3 is conjugate to F_2.*

(iv) *$\mathrm{N}_G(B_4) \cong S_5$. There exists an A_7-subgroup of G which contains B_4. Any A_4-subgroup of B_4 is conjugate to F_3.*

(v) *$B_5=B_4^\phi$.*

PROOF. Let B_1 be an A_5-subgroup of K and F be an A_4-subgroup of B_1. Then we have $R \rtimes Z \leqq \mathrm{C}_G(B_1) \leqq \mathrm{C}_G(F)$. By Proposition 5.1, F is conjugate to F_1 and $\mathrm{C}_G(B_1)=\mathrm{C}_G(F)=R \rtimes Z$. In particular $\mathrm{N}_G(B_1) \leqq \mathrm{N}_G(R)$. By Proposition 2.2 (ii), $\mathrm{N}_G(R)/R \rtimes Z \cong M_{10}$. Since M_{10} has no S_5-subgroup, $\mathrm{N}_G(B_1)/R \rtimes Z \not\cong M_{10}$. Thus $\mathrm{N}_G(B_1)=(R \rtimes Z) \times B_1$.

It follows from Lemma 4.3 that there exist A_5-subgroups B_2, B_3 which satisfy (ii) and (iii). By Lemma 5.5 there exists an A_5-subgroup B_4 which satisfies (iv). Suppose that B_4 and B_4^ϕ are G-conjugate for an outer involution ϕ in (ii). Then there exists an element $\psi \in \mathrm{Aut}\, G-G$ which normalizes B_4. As $\mathrm{N}_G(B_4) \cong S_5 \cong \mathrm{Aut}\, A_5$, we may assume that ψ is an outer involution of G centralizing B_4. Since all outer involutions are G-conjugate and J_1 has exactly two conjugacy classes of A_5-subgroups, the A_5-subgroup B_4 of $\mathrm{C}_G(\psi)$ is conjugate to B_2 or B_3. This contradicts the structures of their normalizers. Thus B_4 and $B_5=B_4^\phi$ are not G-conjugate.

It follows from the above remark and the structures of their normalizers that $B_1, B_2, \cdots, B_5$ are not G-conjugate each other. We also use B_i to denote a conjugate of B_i which contains the element b of order 5 in Lemma 2.3 for each $i \in \{1, \cdots, 5\}$. Since $\langle x, y \mid x^2=y^3=(xy)^5=1\rangle$ is a presentation for A_5, it follows from the counting argument in §1 that

$$\left|(2A, 3A\,; b) \cap \Big(\bigcup_{i=1}^{5} \bigcup_{g \in \mathrm{N}_G(b)} B_i^g\Big)\right|$$

$$=5 \cdot |\mathrm{N}_G(b)| \cdot \frac{1}{5} \cdot \left(\frac{1}{9 \cdot 4 \cdot 2}+\frac{1}{4 \cdot 4}+\frac{1}{2}+\frac{2}{4}\right)=755\,.$$

On the other hand we have $^\sharp(2A, 3A\,; b)=755$ by the character table of G.

Therefore $B_1, \cdots, B_5$ are representatives of conjugacy classes of A_5-subgroups of G. $\square$

In this section we use $\mathcal{A}_i$ to denote the conjugacy class of A_5-subgroups, represented by B_i for $i=1, \cdots, 5$. We also use b to denote the element of order 5 in Lemma 2.3.

LEMMA 6.3. (i) *Suppose that A is an A_5-subgroup in $\mathcal{A}_1$ which contains b. Then $A \leqq K$. In particular the involutions in $\mathrm{N}_A(b)$ are $\langle b \rangle$-conjugate to $v_1^2 v_3^2$.*

(ii) *Suppose that A is an A_5-subgroup in $\mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5$ which contains b. Then the involutions in $\mathrm{N}_A(b)$ are $\mathrm{N}_G(b)$-conjugate to $v_2^2 v_3^2$.*

PROOF. (i) Since $(E_9 \rtimes Z_4 \cong) \mathrm{C}_G(A) \leqq \mathrm{C}_G(b) = (R \rtimes Z) \times \langle b \rangle$, we have $\mathrm{C}_G(A) = R \rtimes Z$. Thus $A \leqq \mathrm{C}_G(R) = R \times K$, and so $A \leqq K$.

(ii) Since there exists an S_5-subgroup which contains each A_5-subgroup in $\mathcal{A}_2 \cup \mathcal{A}_4 \cup \mathcal{A}_5$, the assertion holds for this A_5-subgroup by Corollary 2.4 (i). There exist A_5-subgroups $A \in \mathcal{A}_2$ and $B \in \mathcal{A}_3$ such that $\langle A, B \rangle \cong L_2(11)$. Thus we may take $\mathrm{N}_A(b) = \mathrm{N}_{\langle A, B \rangle}(b) = \mathrm{N}_B(b)$. So the assertion also holds for each A_5-subgroup in $\mathcal{A}_3$. $\square$

In the remainder of this section we treat proper simple subgroups of G which contain A_5-subgroups properly. They are isomorphic to A_6, A_7, $L_2(11)$, M_{11}, $L_2(31)$ and M_{11} by Proposition 6.1. We summalize some properties on A_5-subgroups of these groups in the next lemma. We may easily verify these properties from the structure constants $^{\sharp}(2A, 3A, 5A)$ in each of these groups. As for J_1-subgroups, the assertion (v) follows from the structures of $\mathrm{C}_G(\phi)\,(\cong J_1)$ and the A_5-subgroup B in Lemma 6.3 (ii).

LEMMA 6.4. *Let X be a subgroup of G which contains b.*

(i) *If $X \cong A_6$ or $L_2(11)$, there exist exactly two A_5-subgroups, say A and B, of X containing b. We have $\langle A, B \rangle = X$ and $\mathrm{N}_A(b) = \mathrm{N}_B(b) = \mathrm{N}_X(b)$.*

(ii) *If $X \cong A_7$, there exist exactly three A_5-subgroups, say A, B and C, of X containing b. We may assume that $\mathrm{N}_X(A) \cong S_5$, $\langle A, B \rangle \cong \langle A, C \rangle \cong A_6$, $X = \langle B, C \rangle$ and $\mathrm{N}_A(b) = \mathrm{N}_B(b) = \mathrm{N}_C(b)$. Furthermore B and C are $\mathrm{N}_X(b)$-conjugate.*

(iii) *If $X \cong M_{11}$, there exist exactly three A_5-subgroups, say A, B and C, of X containing b. We may assume that $\mathrm{N}_X(A) \cong S_5$, $\langle A, B \rangle \cong \langle A, C \rangle \cong L_2(11)$, $\langle B, C \rangle \cong A_6$ and $\mathrm{N}_A(b) = \mathrm{N}_B(b) = \mathrm{N}_C(b)$. Furthermore B and C are $\mathrm{N}_X(b)$-conjugate, and $\langle B, C, \mathrm{N}_X(b) \rangle \cong M_{10}$.*

(iv) *If $X \cong L_2(31)$, there exist exactly six A_5-subgroups of X containing b. These are split into two orbits under the action of $\mathrm{C}_X(b)\,(\cong Z_{15})$. For any A_5-subgroup A in one orbit, there exists a unique A_5-subgroup*

B in the other orbit such that $\mathrm{N}_A(b)=\mathrm{N}_B(b)$ *and* $X=\langle A, B\rangle$.

(v) *If* $X\cong J_1$, *there exist exactly nine* A_5-*subgroups of* X *containing* b. *We may assume that they are* $A, A^r, A^{r^{-1}}, B, B^y, B^r, B^{ry}, B^{r^{-1}}$ *and* $B^{r^{-1}y}$, *where* $\mathrm{C}_X(b)=\langle r, y\rangle\times\langle b\rangle$, $\mathrm{N}_X(A)=\langle y\rangle\times A$, $\mathrm{N}_X(B)=B$, *and* $\langle A, B\rangle\cong L_2(11)$. *Furthermore* $\langle A, B\rangle\cong\langle A, B^y\rangle\cong L_2(11)$ *and* $\langle A, A^r\rangle=\langle A, A^{r^{-1}}\rangle=\langle B, B^g\rangle=\langle A, B^h\rangle=X$ *for any* $g\in\langle r, y\rangle^{\#}$ *and for any* $h\in\{r, ry, r^{-1}, r^{-1}y\}$.

LEMMA 6.5. *Let A be an* A_5-*subgroup in* $\mathcal{A}_1$ *which contains b. Then the following hold.*

(i) *K is the unique* A_6-*subgroup containing A.*

(ii) *There is no* $L_2(11)$, M_{11}, A_7 *nor* $L_2(31)$-*subgroup of G containing A.*

PROOF. Suppose that X is a subgroup of G which contains A and is isomorphic to A_6, $L_2(11)$, A_7 or $L_2(31)$. Then there exists an A_5-subgroup B of X such that $\mathrm{N}_A(b)=\mathrm{N}_B(b)$ and $\langle A, B\rangle=X$ by Lemma 6.4. Consequently $B\in\mathcal{A}_1$ by Lemma 6.3 (i), (ii). Thus we have $A, B\leqq K$, which implies (i). Since any A_5-subgroup of M_{11} is contained in some $L_2(11)$-subgroup by Lemma 6.3 (ii), (ii) follows. $\square$

LEMMA 6.6. *Let A be an* A_5-*subgroup in* $\mathcal{A}_2$ *such that* $\mathrm{N}_A(b)=\langle b, v_2^2v_3^2\rangle$. *We also let ϕ be an outer involution of G centralizing A. Then the following hold.*

(i) $\mathrm{C}_G(A)=Z=\langle v_1v_2v_3^2\rangle$.

(ii) *$T\cap\mathrm{N}_G(b)$ is a ϕ-invariant Sylow 2-subgroup of* $\mathrm{N}_G(A)\cap\mathrm{N}_G(b)$.

(iii) *For any square root h of $v_2^2v_3^2$ in* $\mathrm{N}_G(b)$, *we have* $h^\phi=h(v_1v_2v_3^2)^{\pm}$.

PROOF. As $(Z_4\cong)\,\mathrm{C}_G(A)=\mathrm{C}_G(b)\cap\mathrm{C}_G(v_2^2v_3^2)=Z$, (i) follows. Suppose that h is a square root of $v_2^2v_3^2$. Then $h\in T\cap\mathrm{N}_G(b)$ by 2.3 (v), and so the set of square roots of $v_2^2v_3^2$ coincides with $hZ\cup h^{-1}Z$. Since $\mathrm{N}_G(A)$ contains an S_5-subgroup, $\mathrm{N}_G(A)\cap\mathrm{N}_G(b)$ contains a square root of $v_2^2v_3^2$. Thus $\langle Z, h\rangle=T\cap\mathrm{N}_G(b)\leqq\mathrm{N}_G(A)\cap\mathrm{N}_G(b)$. Since ϕ acts on AZ/Z trivially, ϕ also acts on $\mathrm{N}_G(A)/Z$ ($\cong S_5$) trivially. Thus $h^\phi\in hZ$. Since the outer involutions ϕ and $\phi v_1v_2v_3^2$ do not centralize the element h of order 4, we have $h^\phi\neq h$ and $h^\phi\neq hv_1^2v_2^2$. Thus (iii) follows. Furthermore $\langle Z, h\rangle=T\cap\mathrm{N}_G(b)$ is ϕ-invariant, and so (ii) follows. $\square$

The next lemma is a key lemma for classifications of A_6, $L_2(11)$, A_7 and $L_2(31)$-subgroups of G.

LEMMA 6.7. *Let ϕ be an outer involution of G which centralizes b and normalizes* $T\cap\mathrm{N}_G(b)$ *(Lemma 6.6). We also let g be an element in* $\mathrm{N}_{\mathrm{Aut}\,G}(b)=(R\times\langle b\rangle)(T\cap\mathrm{N}_G(b))\langle\phi\rangle$.

Suppose that A and B are (not necessary distinct) A_5-subgroups in $\mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5$ which contain b. Assume that $N_A(b) = N_B(b) = \langle b, v_2^2 v_3^2 \rangle$ and that $\langle A, B^g \rangle$ is a proper subgroup of G containing A properly. Then one of the following (i), (ii) *or* (iii) *holds.*

(i) $\langle A, B^g \rangle \cong J_1$.

(ii) $\langle A, B^g \rangle \cong A_6, A_7$ *or* $L_2(11)$. *There exists an element* $k \in (T \cap N_G(b))\langle \phi \rangle$ *such that* $B^g = B^k$.

(iii) $\langle A, B^g \rangle \cong L_2(31)$. *There exists an element* $k \in (T \cap N_G(b))\langle \phi \rangle$ *such that* $A = B^k$ *or* $\langle A, B^g \rangle = \langle A, B^k \rangle$.

PROOF. We set $X = \langle A, B^g \rangle$. As A and B are perfect, X is perfect. Suppose that X is contained in some local subgroup. It follows from § 3 and Proposition 6.2 that $A, B \in \mathcal{A}_2$ and that all maximal local subgroups containing X are centralizers of involutions. We may assume that A is the A_5-subgroup in Lemma 6.6. By Sylow's theorem we have $B^g = A^x$ for some $x \in N_G(b)$. Since $\langle b \rangle (T \cap N_G(b))$ normalizes A, we may take $x \in R$. As $A \neq B^g$, $x \neq 1$. Since the involutions centralizing A and A^x are $v_1^2 v_2^2$ and $(v_1^2 v_2^2)^x = v_1^2 v_2^2 x$ respectively, there is no involution of G centralizing $X = \langle A, A^x \rangle$. This is a contradiction. Thus there is no local subgroup of G containing X.

Suppose that M is a maximal subgroup of G containing X. Then minimal normal subgroups of M are non-abelian characteristically simple groups and their centralizers in G are trivial by the above paragraph. Thus we have $M' \cong A_6, A_7, L_2(11), M_{11}, L_2(31)$ or J_1 by Proposition 6.1 and the structures of the automorphism groups of simple groups appearing in this proposition. Since X is a perfect subgroup in these group and is generated by two A_5-subgroups containing b, we have $X \cong A_6, A_7, L_2(11), L_2(31)$ or J_1 by Lemma 6.4.

Suppose that $X \cong A_6, A_7$ or $L_2(11)$. Then $N_{B^g}(b) = N_A(b)$ by Lemma 6.4. By the hypothesis we have $N_A(b) = N_B(b) = \langle b, v_2^2 v_3^2 \rangle$. Thus we have $v_2^2 v_3^2 \cdot (v_2^2 v_3^2)^g = [v_2^2 v_3^2, g] \in N_{B^g}(b)$ $(\cong D_{10})$. Now we can write $g = rk$, where $r \in R$ and $k \in (T \cap N_G(b))\langle \phi \rangle$. Since the squares of elements in $(T \cap N_G(b)) - C_G^*(b)$ are $v_2^2 v_3^2$, we have $v_2^2 v_3^2 \in Z((T \cap N_G(b))\langle \phi \rangle)$. As $v_2^2 v_3^2$ inverts R, we have $[v_2^2 v_3^2, g] = [v_2^2 v_3^2, r]^k \cdot [v_2^2 v_3^2, k] = (r^{-1})^k$. Thus $r = 1$ and $g = k \in (T \cap N_G(b))\langle \phi \rangle$.

Suppose that $X \cong L_2(31)$. Then there exists an element $r \in O_3(C_X(b))$ such that $A = B^{gr}$ or $N_A(b) = N_{B^{gr}}(b)$ by Lemma 6.4 (iv). So by the same argument as above, there exists an element $k \in (T \cap N_G(b))\langle \phi \rangle$ such that $B^{gr} = B^k$. □

LEMMA 6.8. (i) *There is no $PGL_2(31)$-subgroup in* Aut G.
(ii) *There is no $PGL_2(11)$-subgroup in* Aut G.

PROOF. (i) Since G has no element of order 32, G has no $PGL_2(31)$-subgroup. By the proof of Lemma 4.9 (iii), there is no $L_2(31)$-subgroup of G normalized by an outer involution.

(ii) Suppose that X is a $PGL_2(11)$-subgroup of G. By the character table of $PGL_2(11)$, we have $^{\sharp}(2A, 4A, 5A)_X=10$. Since $(4A)_X$-elements centralize elements of order 3 in X, they are of class $4A$ in G. But it follows from the character table of G that $^{\sharp}(2A, 4A, 5A)_G=10$ and any pairs in $(2A, 4A; b)$ generate K. This implies that X has an A_6-subgroup, which is a contradiction.

Next suppose that X is an $L_2(11)$-subgroup of G and $\langle X, \phi\rangle \cong PGL_2(11)$ for some outer involution ϕ. Then there exists an element $x\in\langle X, \phi\rangle - X$ of order 12. Thus x^3 is an element of order 4 in $\mathrm{Aut}\, G-G$ centralizing the element x^4 of order 3. We can write $\mathrm{C}_{\mathrm{Aut}\,G}(x^4)=(R_0\times K_0)\langle\phi_0\rangle$ $(\cong(E_9\times A_6)\rtimes Z_2)$ for some outer involution ϕ_0. By Lemma 2.7 (iv) we have $\langle K_0, \phi\rangle\cong PGL_2(9)$. Since Sylow 2-subgroups of $PGL_2(9)$ are D_{16}-subgroups, the elements of order 4 in $\mathrm{C}_{\mathrm{Aut}\,G}(x^4)$ are squares and are contained in K_0 $(\leqq G)$. This is a contradiction. □

LEMMA 6.9. *Let $A\in\mathcal{A}_2$ be an A_5-subgroup in Lemma 6.6. Then we have the following.*

(i) *Suppose that B is an A_5-subgroup in $\mathcal{A}_2$ which contains b. If $\langle A, B\rangle$ is a proper subgroup of G containing A properly, then $\langle A, B\rangle\cong J_1$.*

(ii) *Suppose that B is an A_5-subgroup in $\mathcal{A}_3$ which contains b. If $\langle A, B\rangle$ is a proper subgroup of G, then $\langle A, B\rangle\cong J_1$ or $L_2(11)$. Furthermore if $\langle A, B\rangle\cong\langle A, C\rangle\cong L_2(11)$ for an A_5-subgroup C in $\mathcal{A}_3$ containing b, then $\langle A, B\rangle$ and $\langle A, C\rangle$ are $\mathrm{N}_G(b)$-conjugate.*

(iii) *Suppose that B is an A_5-subgroup in $\mathcal{A}_4\cup\mathcal{A}_5$ which contains b. Then $\langle A, B\rangle=G$ or $\cong J_1$. (In fact the latter case does not occur.)*

PROOF. (i) There exists an element $g\in\mathrm{N}_G(b)$ such that $B=A^g$. Suppose that $\langle A, B\rangle\not\cong J_1$. Then we have $B=A^k$ or $\langle A, B\rangle=\langle A, A^k\rangle$ for some $k\in T\cap\mathrm{N}_G(b)$ by Lemma 6.7. Since A is normalized by $T\cap\mathrm{N}_G(b)$, we have $A=B$. This is a contradiction. We note that $\langle A, A^r\rangle\cong J_1$ for some $r\in\mathrm{C}_G(b)$ by Lemma 6.4 (v).

(ii) There exists an A_5-subgroup $C\in\mathcal{A}_3$ such that $\langle A, C\rangle\cong L_2(11)$. We can write $B=C^g$ for some $g\in\mathrm{N}_G(b)$. Suppose that $\langle A, B\rangle\not\cong J_1$. Then there exists an element $k\in T\cap\mathrm{N}_G(b)$ such that $\langle A, B\rangle=\langle A, C^k\rangle=\langle A, C\rangle^k\cong L_2(11)$ by Lemma 6.7.

(iii) Since there is no A_6-subgroup containing A by Lemma 4.9 (i), we have $\langle A, B\rangle\not\cong A_6$. Suppose that $\langle A, B\rangle\cong L_2(11)$. Then A_4-subgroups of A and B are conjugate in $\langle A, B\rangle$, which contradicts Propositions 5.1 and

6.2. Thus $\langle A, B\rangle \not\cong L_2(11)$. Since A and B are not G-conjugate $\langle A, B\rangle \not\cong A_7$ by Lemma 6.4. Finally suppose that $\langle A, B\rangle \cong L_2(31)$. We may assume that $N_A(b)=N_B(b)=\langle b, v_2^2v_3^2\rangle$ by Lemma 6.4 (iv) and 6.7. As $N_G(B)\cong S_5$, $N_G(B)\cap N_G(b)$ contains a square root h of $v_2^2v_3^2$. Since $h\in T\cap N_G(b)$ by Lemma 2.3 (v), h normalizes $\langle A, B\rangle$. As $C_G(\langle A, B\rangle)=1$, $\langle A, B, h\rangle \cong PGL_2(31)$. This contradicts Lemma 6.8. Thus $\langle A, B\rangle \not\cong L_2(31)$. Then the assertion follows from Lemma 6.7. □

LEMMA 6.10. *Let $A\in\mathcal{A}_3$ be an A_5-subgroup which contains b. Then we have the following.*

(i) *Suppose that $B\in\mathcal{A}_3$ be an A_5-subgroup which contains b. If $\langle A, B\rangle$ is a proper subgroup of G which contains A properly, then $\langle A, B\rangle \cong A_6$, $L_2(31)$ or J_1. Furthermore if $\langle A, B\rangle \cong L_2(31)$, then A and B are $O_3(C_{\langle A, B\rangle}(b))$-conjugate.*

(ii) *Suppose that $B\in\mathcal{A}_4\cup\mathcal{A}_5$ be an A_5-subgroup which contains b. Then $\langle A, B\rangle$ is not isomorphic to $L_2(11)$ nor A_7.*

PROOF. We may assume that $N_A(b)=\langle b, v_2^2v_3^2\rangle$.

(i) We let h be a root of $v_2^2v_3^2$ in $T\cap N_G(b)$. Suppose that $\langle A, B\rangle \not\cong J_1$. Then by Lemma 6.7, $\langle A, B\rangle \cong L_2(31)$ and $B=A^r$ for some $r\in O_3(C_{\langle A, B\rangle}(b))$, or there exists an element $k\in T\cap N_G(b)$ such that $\langle A, B\rangle = \langle A, A^k\rangle$. In the latter case $\langle A, B\rangle$ is $N_{\mathrm{Aut}\,G}(b)$-conjugate to one of $\langle A, A^{v_1^2v_2^2}\rangle$, $\langle A, A^{v_1v_2v_3^2}\rangle$, $\langle A, A^h\rangle$ or $\langle A, A^{hv_1^2v_2^2}\rangle$ by Lemma 6.6 (iii). As $v_1^2v_2^2\in C_G(\phi)$, $\langle A, A^{v_1^2v_2^2}\rangle = C_G(\phi) \cong J_1$ by Lemma 6.4 (v).

We set $X=\langle A, A^{v_1v_2v_3^2}\rangle$. Since ϕ inverts $v_1v_2v_3^2$, X is normalized by the outer involution $\psi=\phi v_1v_2v_3^2$. Thus $X\not\cong L_2(31)$ and $X\not\cong L_2(11)$ by Lemma 6.8. Suppose that $X\cong A_7$. Then $\langle X, \psi\rangle \cong S_7$ and ψ centralizes some element of order 4 in X. This contradiction shows that $X\not\cong A_7$.

Next we set $X=\langle A, A^h\rangle$. As $h^2=v_2^2v_3^2\in A$, h normalizes X. Thus $X\not\cong L_2(11)$ and $X\not\cong L_2(31)$ by Lemma 6.8. Suppose that $X\cong A_7$. There exists an A_5-subgroup C of X such that $N_X(C)\cong S_5$, $N_C(b)=N_A(b)=N_{A^h}(b)$, and $\langle C, A\rangle \cong \langle C, A^h\rangle \cong A_6$. By Proposition 6.1 and Lemma 6.9 we have $C\in\mathcal{A}_4\cup\mathcal{A}_5$. However it follows from Lemma 4.9 (ii) that, for any A_6-subgroup Y containing an A_5-subgroup in $\mathcal{A}_3$, all A_5-subgroups of Y are of class $\mathcal{A}_3$. This contradiction shows that $X\not\cong A_7$.

The same proof as above holds for $X=\langle A, A^{hv_1^2v_2^2}\rangle$. Thus the assertion follows from Lemma 6.6.

(ii) As A and B are not conjugate, $\langle A, B\rangle \not\cong A_7$. Suppose that $\langle A, B\rangle \cong L_2(11)$. Then A_4-subgroups of A and B are conjugate, which contradicts Proposition 6.2. □

LEMMA 6.11. *Let A and B are A_5-subgroups in $\mathcal{A}_4\cup\mathcal{A}_5$ which contain*

b. *Suppose that* $\langle A, B\rangle$ *is a proper subgroup of* G *which contains* A *properly, and that* $\mathrm{N}_A(b)=\langle b, v_2^2v_3^2\rangle$. *Then one of the following* (i), (ii) *or* (iii) *holds.*

(i) $\langle A, B\rangle\cong J_1$ *(In fact this case does not occur.).*

(ii) $\langle A, B\rangle\cong L_2(31)$ *and* A *is* $\mathrm{O}_3(\mathrm{C}_{\langle A, B\rangle}(b))$*-conjugate to* B.

(iii) $\langle A, B\rangle$ *is* $\mathrm{N}_{\mathrm{Aut}\,G}(b)$*-conjugate to* $\langle A, A^{v_1v_2v_3^2}\rangle$, $\langle A, A^{\phi}\rangle$ *or* $\langle A, A^{\phi v_1^2v_2^2}\rangle$, *where* ϕ *is the outer involution in Lemma* 6.6.

PROOF. Since $\mathrm{N}_G(A)\cong S_5$, we have $\mathrm{N}_G(A)\cap\mathrm{N}_G(b)=\langle b, h\rangle$ where $h^2=v_2^2v_3^2$. Then $h\in T\cap\mathrm{N}_G(b)$ and $h^{\phi}=h(v_1v_2v_3^2)^{\pm}$ by Lemma 6.6. By Lemma 6.7, $\langle A, B\rangle\cong J_1$, $\langle A, B\rangle\cong L_2(31)$ and A is $\mathrm{O}_3(\mathrm{C}_{\langle A, B\rangle}(b))$-conjugate to B, or $\langle A, B\rangle=\langle A, B^k\rangle$ for some $k\in(T\cap\mathrm{N}_G(b))\langle\phi\rangle$. In the last case $\langle A, B\rangle$ is $\mathrm{N}_{\mathrm{Aut}\,G}(b)$-conjugate to one of $\langle A, A^{v_1^2v_2^2}\rangle$, $\langle A, A^{v_1v_2v_3^2}\rangle$, $\langle A, A^{\phi}\rangle$ or $\langle A, A^{\phi v_1^2v_2^2}\rangle$.

Thus we have only to show that $\langle A, A^{v_1^2v_2^2}\rangle\cong J_1$ or G. We set $X=\langle A, A^{v_1^2v_2^2}\rangle$. Suppose that $X=A$. Then $v_1^2v_2^2$ centralizes A since $v_1^2v_2^2$ centralizes b. So we have $A\lneqq X$. We note that h normalizes A and $A^{v_1^2v_2^2}$ since h centralizes $v_1^2v_2^2$. As $\mathrm{C}_G(X)=1$, it follows from the structures of $\mathrm{Aut}\,A_6$, $PGL_2(11)$, $PGL_2(31)$ and S_7 that $\langle X, h\rangle\cong S_6$. But this implies that there exists an involution in G centralizing an A_4-subgroup of A. This contradicts Proposition 6.1. Therefore $X\cong J_1$ or $X=G$ by Lemma 6.7. □

We are now in a position to classify the conjugacy classes of A_6, A_7, $L_2(11)$, $L_2(31)$ and M_{11}-subgroups of G and to determine the normalizers of their representatives.

PROPOSITION 6.12. (i) *G has two conjugacy classes of $L_2(31)$-subgroups. They are* $\mathrm{Aut}\,G$*-conjugate. The $L_2(31)$-subgroups are self-normalizing.*

(ii) *G has one conjugacy class of $L_2(11)$-subgroups. The $L_2(11)$-subgroups are self-normalizing and are contained in the centralizers of outer involutions.*

(iii) *G has two conjugacy classes of M_{11}-subgroups. They are* $\mathrm{Aut}\,G$*-conjugate. The M_{11}-subgroups are self-normalizing.*

(iv) *G has two conjugacy classes of A_7-subgroups. They are* $\mathrm{Aut}\,G$*-conjugate. The A_7-subgroups are self-normalizing.*

(v) *G has four conjugacy classes of A_6-subgroups, represented by K_1, K_2, K_3 and K_4 such that;*

$$\mathrm{N}_G(K_1)\cong((E_9\rtimes Z_4)\times A_6)\cdot Z_2\,,$$

$$\mathrm{N}_G(K_2)\cong M_{10}\cong\mathrm{N}_G(K_3)\,,$$

$$\mathrm{N}_G(K_4)=K_4\cong A_6\,.$$

Furthermore K_2 *and* K_3 *are* Aut G*-conjugate and are contained in* M_{11}*-subgroups.* K_4 *is contained in an* A_7*-subgroup and* $\mathrm{N}_{\mathrm{Aut}\,G}(K_4)\cong PGL_2(9)$.

PROOF. (i), (ii), (iii) By Lemmas 6.4, 6.5, 6.7, 6.9 and 6.10 we have the following: Any $L_2(31)$-subgroup of G is generated by a suitable A_5-subgroup in $\mathcal{A}_3$ and a suitable A_5-subgroup in $\mathcal{A}_4\cup\mathcal{A}_5$. Any $L_2(11)$-subgroup of G is generated by a suitable A_5-subgroup in $\mathcal{A}_2$ and a suitable A_5-subgroup in $\mathcal{A}_3$. Thus (i) follows from Lemma 4.9 (iii), and (ii) follows from Lemma 6.8 (ii), 6.9 (ii). Furthermore (iii) follows from (ii) and Lemma 4.9 (iv).

(iv) Suppose that X is an A_7-subgroup of G, and that A is an A_5-subgroup of X such that $\mathrm{N}_X(A)\cong S_5$. Then $A\in\mathcal{A}_2\cup\mathcal{A}_4\cup\mathcal{A}_5$. Since any A_4-subgroup of A centralizes an element of order 3 in X, $A\in\mathcal{A}_4\cup\mathcal{A}_5$. Thus (iv) follows from Lemma 5.5 (iv).

(v) We note that any A_5-subgroups in A_7-subgroups of G are contained in $\mathcal{A}_4\cup\mathcal{A}_5$ by (iv) and Lemma 6.9∼6.11. Thus by Lemma 6.11 one of the subgroups $\langle A, A^{v_1v_2v_3^2}\rangle$, $\langle A, A^{\phi}\rangle$ and $\langle A, A^{\phi v_1^2 v_2^2}\rangle$ in this lemma is isomorphic to A_7. Then there exists an A_7-subgroup in G normalized by some outer involution. But this implies that this outer involution centralizes an element of order 4 in G, which is a contradiction. Thus $\langle A, A^{v_1v_2v_3^2}\rangle\cong A_7$.

Then the structure of A_7 shows that $\langle A, A^{\phi}\rangle\cong A_6$ or $\langle A, A^{\phi v_1^2 v_2^2}\rangle\cong A_6$. Suppose that $\langle A, A^{\psi}\rangle\cong A_6$ where $\psi=\phi$ or $\phi v_1^2 v_2^2$. We set $X=\langle A, A^{\psi}\rangle$. Assume that $X\lneqq \mathrm{N}_G(X)$. As A and A^{ψ} are not G-conjugate by Proposition 6.2, we have $\mathrm{N}_G(X)\cong S_6$. This implies that an A_4-subgroup of an A_5-subgroup of X centralizes an involution, which contradicts Proposition 6.2. Thus $\mathrm{N}_G(X)=X$. Since ψ centralizes b, $\mathrm{N}_{\mathrm{Aut}\,G}(X)=\langle X, \psi\rangle\cong PGL_2(9)$.

By Lemma 5.5 there exists a unique A_7-subgroup E such that $\mathrm{N}_E(A)\cong S_5$ for any A_5-subgroup A in $\mathcal{A}_4$. There exist exactly two A_6-subgroups of G containing A by Lemma 5.5 (ii). So they are contained in E and are $\mathrm{N}_E(A)$-conjugate. Thus A_6-subgroups containing A are G-conjugate to X. By symmetry A_6-subgroups containing A^{ψ} $(\in\mathcal{A}_5)$ are also G-conjugate to X. Therefore A_6-subgroups containing an A_5-subgroup in $\mathcal{A}_4\cup\mathcal{A}_5$ are G-conjugate to X.

By Lemma 4.9 (ii) A_6-subgroups containing an A_5-subgroup in $\mathcal{A}_3$ are G-conjugate to K_2 or K_3, where $K_2=\langle A, g_2\rangle$ and $K_3=\langle A, g_3\rangle$ are the A_6-subgroups in this lemma. Furthermore there is no A_6-subgroup containing an A_5-subgroup in $\mathcal{A}_2$ by Lemma 4.9 (i), and A_6-subgroups containing an A_5-subgroup in $\mathcal{A}_1$ are conjugate to K by Lemma 6.3 (i). Hence the proof of (v) is completed. □

§ 7. Non-local analyses for $L_3(7)$, J_1 and $L_2(7)$

In this section we determine the conjugacy classes of $L_2(7)$, J_1, $L_3(7)$-subgroups and the structures of normalizers of their representatives.

PROPOSITION 7.1. *G has two conjugacy classes of $L_3(7)$-subgroups. They are* Aut *G-conjugate. The normalizers in G of $L_3(7)$-subgroups are isomorphic to the split extension of $L_3(7)$ by an inverse-transpose automorphism.*

PROOF. Let L_0 be an $L_3(7)$-subgroup of G and D be a Sylow 7-subgroup of L_0. Then D is a Sylow 7-subgroup of G isomorphic to 7_+^{1+2}. Since $N_{L_0}(D)$ is a maximal parabolic subgroup of L_0, we have $L_0=\langle N_{L_0}(D), N_{L_0}(E)\rangle$ for any E_{49}-subgroup E of D whose non-trivial elements are L_0-conjugate.

By [3] Lemma 6.4, E_{49}-subgroups of D whose non-trivial elements are G-conjugate are divided into two $N_G(D)$-conjugacy classes represented by E' and E'', say. Thus L_0 is a subgroup of $L=\langle N_G(D), N_G(E')\rangle$ or $L_1=\langle N_G(D), N_G(E'')\rangle$. By [3] § 10 L and L_1 are isomorphic to the split extension of $L_3(7)$ by an inverse-transpose automorphism, and they are interchanged by some outer involution. Since G has no element of order 21, we have $C_G(L_0)=1$ and $N_G(L_0)/L_0\cong Z_2$. Thus the proposition follows. □

In the remainder of this section $L_3(7)^*$ denotes the extension of $L_3(7)$ by an inverse-transpose automorphism. The G-fusion pattern of $L_3(7)$-subgroups are shown in Table II below by suitably ordering the classes $7B, 7C, 7D$ of $L_3(7)$. As for the elements of order 8 or 16 we note [3] Lemma 10.13.

LEMMA 7.2. *All subgroups of type $(2A, 3A, 7A)$ are isomorphic to $L_2(7)$. They are G-conjugate. For an $L_2(7)$-subgroup X of this type we have $N_G(X)=(\langle f\rangle\times X)\langle a\rangle$, where $|f|=4$, $|a|=2$, $f^a=f^{-1}$ and $\langle X, a\rangle\cong PGL_2(7)$.*

PROOF. Let L be an $L_3(7)^*$-subgroup of G and u be an involution of L inducing on L' $(\cong L_3(7))$ an inverse-transpose automorphism. Then we have $C_L(u)=\langle u\rangle\times M$, where $M\cong PGL_2(7)$. An element x of order 7 in M is a $7A$-element as it centralizes the involution u. Thus M' is an $L_2(7)$-subgroup of type $(2A, 3A, 7A)$. Since M' is a perfect subgroup of $C_G(u)$, M' centralizes a $4A$-element f which is a root of u. Thus $C_G(M')=\langle f\rangle$. Since $N_G(x)\cong 7_+^{1+2}\rtimes(Z_3\times D_8)$, we have $f^a=f^{-1}$ for some involution a in $N_M(x)$.

Suppose that $M'=\langle g, h\rangle$ for $(g, h)\in(2A, 3A\,; x)$. The $C_G(x)$-orbit of $(2A, 3A\,; x)$ containing (g, h) is of length 7^3 as $C_G(M')=\langle f\rangle$. By the character table of G we have ${}^*(2A, 3A, 7A)_G=7^3$, and so all pairs in $(2A, 3A\,; x)$

Table II G-fusion of an $L_3(7)$-subgroup

	ATLAS name in $L_3(7)$ x	$\lvert C_{L_3(7)}(x) \rvert$	x in G
1	$1A$	$2^5\cdot 3^2\cdot 7^3\cdot 19$	$1A$
2	$2A$	$2^5\cdot 3\cdot 7$	$2A$
3	$3A$	$2^2\cdot 3^2$	$3A$
4	$4A$	2^4	$4B$
5	$6A$	$2^2\cdot 3$	$6A$
6	$7A$	$2\cdot 7^3$	$7A$
7	$7B$	7^2	$7A$
8	$7C$	7^2	$7B$
9	$7D$	7^2	$7B$
10	$8A=(16A)^2=(16B)^2$	2^4	$8A$ or $8B$
11	$8B=(16C)^2=(16D)^2$	2^4	$8A$ or $8B$
12	$16A$	2^4	$16A$ or $16C$
13	$16B=(16A)^9$	2^4	$16A$ or $16C$
14	$16C=(16A)^3$	2^4	$16B$ or $16D$
15	$16D=(16A)^{11}$	2^4	$16B$ or $16D$
16	$19A$	19	$19A$
17	$19B=(19A)^{-1}$	19	$19A$
18	$19C=(19A)^2$	19	$19B$
19	$19D=(19A)^{-2}$	19	$19B$
20	$19E=(19A)^4$	19	$19C$
21	$19F=(19A)^{-4}$	19	$19C$

are $C_G(x)$-conjugate. Thus the assertion follows. □

LEMMA 7.3. *Any element of order 7 which is contained in an A_7-subgroup of G is of class 7B.*

PROOF. Suppose that A is an A_7-subgroup of G and x is an element in A of order 7. There exist two $L_2(7)$-subgroups L_1 and L_2 of A which contain x. Assume that x is a $7A$-element. By the proof of Lemma 7.2 there exists an element $g\in C_G(x)$ such that $L_1^g=L_2$. We set $C_G(L_1)=\langle f\rangle$ $(\cong Z_4)$. We note that $C_G(x)=D\rtimes\langle f\rangle$, where $D\,(\cong 7_+^{1+2})$ is a Sylow 7-subgroup of G containing x in its center. Thus we may take $g\in D$. As $D\trianglelefteq N_G(x)$, we have $t^{-1}\cdot t^g=[t, g]\in\langle L_1, L_2\rangle\cap D=\langle x\rangle$ for an element t of order 3 in $N_{L_1}(x)$. So t acts on $\langle x, y\rangle\,(\cong E_{49})$. Since G has no element of order 21, there exists an element $h\in\langle x, g\rangle-\langle x\rangle$ such that $h^t=h^2$. As $x\in L_1$, $L_2=L_1^g=L_1^h$. Then by the same argument as above we have $h^{-1}=[t, h]\in\langle L_1, L_2\rangle\cap C_G(x)=\langle x\rangle$, which is a contradiction.

It follows from the character table and the list of maximal subgroups of J_1 ([2]) that J_1 is of type $(2A, 3A, 7A)_{J_1}$. Thus any element of order 7 in a J_1-subgroup of G is a $7B$-element by Lemma 7.2.

We note that any element of order 7 in the 2-local subgroup $N_G(V)$ (see §2) is also a $7B$-element. Indeed, suppose that it is not so. Then $N_G(V)$ has a subgroup of type $(2A, 3A, 7A)$ since we have $^{\sharp}(2A, 3A, 7A)_{N_G(A)} = 4\times 7$ by the character table of $N_G(V)$. But it follows from Lemma 7.2 that $N_G(V)$ is a split extention V by $L_2(7)$, which is a contradiction. □

In the remainder of this section c denotes a fixed $7B$-element of $N_G(V)$. Since the centralizer on G of an outer involution is a J_1-subgroup, there exists an outer involution of G which centralizes c by the above remark. Let ϕ be an outer involution centralizing c.

We set $C_G(c)=\langle c, d\rangle\ (\cong E_{49})$ and $N_G(c)=\langle c, d\rangle\langle f\rangle\ (\cong E_{49}\rtimes Z_6)$. We may assume that ϕ centralizes f since Sylow 7-normalizers of J_1 are isomorphic to F_7^6. We set $t=f^2$. Since t acts fixed point freely on $\langle c, d\rangle$, we may assume that $c^t=c^2$ and $d^t=d^2$.

We use A and L to denote an A_7-subgroup and an $L_3(7)^*$-subgroup of G containing c respectively. We may assume that $N_G(c)\cap N_G(V)=\langle c, t\rangle\ (\cong F_7^3)$, $N_G(c)\cap A=\langle c, t\rangle\ (\cong F_7^3)$, and $N_G(c)\leqq L$.

Furthermore we set $\mathscr{X}=(2A, 3A\,; c)$, $\mathscr{X}_1=\mathscr{X}\cap\left(\bigcup_{g\in N_G(c)} N_G(V)^g\right)$, $\mathscr{X}_2=\mathscr{X}\cap\left(\bigcup_{g\in N_G(c)} C_G(\phi)^g\right)$, $\mathscr{X}_3=\mathscr{X}\cap L$, $\mathscr{X}_4=\mathscr{X}\cap L^{\phi}$ and $\mathscr{X}_5=\mathscr{X}-\bigcup_{i=1}^{4}\mathscr{X}_i$.

LEMMA 7.4. (i) $|\mathscr{X}_1|=8\times 7^2$. *For any pair* $(g, h)\in\mathscr{X}_1$ *we have* $\langle g, h\rangle\not\cong L_2(7)$.

(ii) $|\mathscr{X}_2|=7\times 7^2$. *All pairs in* $\mathscr{X}_2$ *generate self-normalizing* J_1-*subgroups of* G. *They are conjugate.*

(iii) $|\mathscr{X}_3|=|\mathscr{X}_4|=7^2$. *For any pair* $(g, h)\in\mathscr{X}_3\cup\mathscr{X}_4$ *we have* $\langle g, h\rangle\cong L_2(7)$, $N_G(\langle g, h\rangle)\cong PGL_2(7)$. $N_G(\langle g, h\rangle)$ *is a subgroup of* L *or* L^{ϕ}. *Furthermore* $\langle g, h\rangle$ *and* $\langle g, h\rangle^{\phi}$ *are not* G-*conjugate.*

PROOF. (i) We set $N=N_G(V)$. From the character table of N we have $|\mathscr{X}\cap N|=4\times 7$. Since $N/V\ (\cong L_2(7))$ does not split, $\langle g, h\rangle\not\cong L_2(7)$ for any $(g, h)\in\mathscr{X}\cap N$.

Suppose that $(a, b)\in\mathscr{X}\cap N\cap N^g$ for some $g\in N_G(c)$. Then $\langle a, b\rangle$ is a perfect subgroup of $N\cap N^g$ which intersects V and V^g non-trivially. We set $W=\Omega_1(V)$. Since c acts transitively on $W^{\sharp}$ and $(W^{\sharp})^g$, we have $W\trianglelefteq\langle a, b\rangle$ and $W^g\trianglelefteq\langle a, b\rangle$. As N/V is simple, W and W^g are contained in V. Thus $W=W^g$ or $g\in N_G(W)=N$. Consequently $|\mathscr{X}_1|=|\mathscr{X}\cap N|\cdot|N_G(c): N_N(c)|=8\times 7^2$.

(ii) From the character table of J_1 we have $|\mathscr{X}\cap C_G(\phi)|=7^2$. Since the only subgroup of J_1 of type $(2A, 3A, 7A)_{J_1}$ is J_1 itself, we have $|\mathscr{X}_2|=$

$|\mathcal{X}\cap C_G(\phi)||N_G(c):N_G(c)\cap C_G(\phi)|=7\times 7^2$. Since $\mathrm{Aut}\,J_1\cong J_1$, the remaining assertions immediately follow.

(iii) Since any pair in $\mathcal{X}$ generates a perfect subgroup of G, we have $\mathcal{X}\cap L=\mathcal{X}\cap L'$. We note that the centralizer on $L_3(7)$ of an inverse-transpose involution is isomorphic to $PGL_2(7)$, and that any element of order 7 in this group is non-central in $L_3(7)$ as ${}^{\sharp}(2A, 3A, 7A)_{L_3(7)}=0$. Since $L_3(7)$ has three classes $7B, 7C$ and $7D$ of non-central element of order 7 and these are fused in $PGL_3(7)$, for any non-central element of order 7 there exists a $PGL_2(7)$-subgroup of $L_3(7)$ containing it.

Consequently there exists a pair $(a, b)\in\mathcal{X}\cap L'$ such that $\langle a, b\rangle\cong L_2(7)$. By the character table of $L_3(7)$ we have $|\mathcal{X}\cap L'|=7^2=|C_G(c)|$. Thus $C_G(c)$ $(=C_L(c)=C_{L^{\phi}}(c))$ acts on $\mathcal{X}\cap L'$ and $\mathcal{X}\cap(L')^{\phi}$ regularly. So we have $\langle x, y\rangle\cong L_2(7)$ for any $(x, y)\in\mathcal{X}_3\cup\mathcal{X}_4$. As $C_G(x, y)=1$ we have $N_G(\langle x, y\rangle)\cong PGL_2(7)$. Furthermore this normalizer is contained in L' or $(L')^{\phi}$.

If $\langle x, y\rangle$ and $\langle x, y\rangle^{\phi}$ are G-conjugate, then there exists an element in $\mathrm{Aut}\,G-G$ which normalizes $\langle x, y\rangle$. But as $N_G(\langle x, y\rangle)\cong PGL_2(7)\cong\mathrm{Aut}\,L_2(7)$, this implies that there exists an outer involution of G which centralizes $\langle x, y\rangle\,(\cong L_2(7))$. Since J_1 has no $L_2(7)$-subgroup, this is a contradiction. Thus $\langle x, y\rangle$ and $\langle x, y\rangle^{\phi}$ are not G-conjugate. □

LEMMA 7.5. *Suppose that $M=\langle a, b\rangle$ is an $L_2(7)$-subgroup of G for some pair $(a, b)\in\mathcal{X}_3\cup\mathcal{X}_4\cup\mathcal{X}_5$. Then for any $g\in C_G(c)$ we have $\langle M, M^g\rangle\not\cong A_7$ and $\langle M, M^{\phi g}\rangle\not\cong A_7$.*

PROOF. Suppose that $\langle M, M^g\rangle\cong A_7$ for some $g\in C_G(c)$. We may assume that $N_M(c)=\langle c, t\rangle$. Since $C_G(c)=\langle c, d\rangle$, we may take $g=d^i$ for some $i\in\{0,\cdots, 6\}$. Then we have $d^i=(t^{-1})^g\cdot t=[g, t]\in\langle M, M^g\rangle\cap C_G(c)=\langle c\rangle$. This implies $g=1$, which is a contradiction.

Next suppose that $\langle M, M^{\phi g}\rangle\cong A_7$ for some $g\in C_G(c)$. Since the outer involution ϕ acts on $C_G(c)$ and $C_G(\phi)$ has no E_{49}-subgroup, there exists a non-trivial element e such that $e^{\phi}=e^{-1}$ and $\langle c, e\rangle=C_G(c)$. We write $g=c^ie^j$ for some $i, j\in\{0,\cdots, 6\}$. Then we have $(\phi g)^2=g^{\phi}\cdot g=c^{2i}\in M\cap M^{\phi g}$. Thus ϕg interchanges M and $M^{\phi g}$, and so ϕg induces an outer automorphism of $\langle M, M^{\phi g}\rangle\,(\cong A_7)$ which centralizes the element c of order 7. This implies that there exists an outer involution of G centralizing an A_7-subgroup of G, which is a contradiction. □

By Lemma 7.5 we have $\langle a, b, g, h\rangle\not\cong A_7$ for any two pairs (a, b) and (g, h) in $\mathcal{X}_3\cup\mathcal{X}_4$. Thus there exists an pair $(a, b)\in\mathcal{X}_5$ such that $\langle a, b\rangle$ is an $L_2(7)$-subgroup of A. We set $Q=\langle a, b\rangle$. We let M be the other $L_2(7)$-subgroup of A containing c. Thus $M=\langle g, h\rangle$ for some $(g, h)\in\mathcal{X}_3\cup\mathcal{X}_4$ by

Lemmas 7.4, 7.5.

LEMMA 7.6. (i) *$\mathcal{X}_5$ is divided into two $C_G(c)$-orbits. They are interchanged by ϕ. In particular any pair in $\mathcal{X}_5$ generates an $L_2(7)$-subgroup.*

(ii) *If $\langle Q^g, M\rangle \cong A_7$ for some $g \in C_G(c)$, then we have $Q^g = Q$.*

(iii) *The $L_2(7)$-subgroups of G which are generated by the pairs in $\mathcal{X}_5$ are $N_G(c)$-conjugate. They are self-normalizing in G, but their normalizers in* Aut G *are isomorphic to $PGL_2(7)$.*

PROOF. (i) Since $C_G(Q)=1$, the $C_G(c)$-orbit $\mathcal{X}'_5$ in $\mathcal{X}_5$ containing (a, b) is of length 7^2. By the character table of G we have $|\mathcal{X}| = {}^{\sharp}(2A, 3A, 7B)_G = 19 \times 7^2$, and so $|\mathcal{X}_5| = 2 \times 7^2$ by Lemma 7.4. Suppose ϕ acts on $\mathcal{X}'_5$. Then ϕ fixes some pair in $\mathcal{X}'_5$ and $C_G(\phi)$ contains an $L_2(7)$-subgroup, which is a contradiction. Thus ϕ interchanges $\mathcal{X}'_5$ and $\mathcal{X}''_5 = \mathcal{X}_5 - \mathcal{X}'_5$.

(ii) As $C_G(c) = \langle c, d\rangle$ we may assume that $g = d^i$ for some $i \in \{0, \cdots, 6\}$. By the structure of A_7 we have $N_M(c) = N_Q(c) = N_{Q^g}(c) = \langle c, t\rangle$ $(\cong F_7^3)$. Thus we have $d^i = [t, g^{-1}] = t^{-1} \cdot t^{g^{-1}} \in M \cap C_G(c) = \langle c\rangle$. So $g = 1$.

(iii) Since $N_G(M) \cong PGL_2(7)$ by Lemma 7.4, we may assume that $N_G(M) \cap N_G(c) = \langle c, f\rangle$. Then $\langle Q, M\rangle^f = \langle Q^f, M\rangle \cong A_7$. Suppose that $Q^f = \langle x, y\rangle$ for some pair $(x, y) \in \mathcal{X}'_5$. Then $Q^f = Q^g$ for some $g \in C_G(c)$ and so we have $\langle Q, M\rangle^f = \langle Q^f, M\rangle = \langle Q, M\rangle$ by (ii). Thus $\langle Q, M, f\rangle$ is an S_7-subgroup of G, which contradicts Proposition 6.12 (iv). Thus $Q^f = \langle x, y\rangle$ for some pair $(x, y) \in \mathcal{X}''_5$, and so the $L_2(7)$-subgroups which are generated by the pairs in $\mathcal{X}_5$ are $N_G(c)$-conjugate.

Since f and ϕ interchange $\{\langle u, v\rangle \mid (u, v) \in \mathcal{X}'_5\}$ and $\{\langle x, y\rangle \mid (x, y) \in \mathcal{X}''_5\}$, the outer involution $f\phi$ acts on the set $\{\langle u, v\rangle \mid (u, v) \in \mathcal{X}'_5\}$ of cardinality 7. Then $f\phi$ normalizes some $L_2(7)$-subgroup in this set. Thus $N_{\mathrm{Aut}\, G}(Q)$ contains a $PGL_2(7)$-subgroup, and so $N_{\mathrm{Aut}\, G}(Q) \cong PGL_2(7)$ and $N_G(Q) = Q$ since $C_{\mathrm{Aut}\, G}(Q) = 1$ and $PGL_2(7) \cong \mathrm{Aut}\, L_2(7)$. □

Summarizing the results obtained in the above lemmas, we have the following proposition. Hence we have determined the conjugacy classes of all non-abelian characteristically simple subgroups of G and the normalizers (in G and in Aut G) of their representatives.

PROPOSITION 7.7. (i) *G has one conjugacy class of J_1-subgroups. They are the centralizers on G of outer involutions and are self-normalizing in G.*

(ii) *G has four conjugacy classes of $L_2(7)$-subgroups represented by L_1, L_2, L_3, L_4. These representatives satisfy the following:*

$N_G(L_1) = (\langle f\rangle \times L_1) \rtimes \langle a\rangle$, *where* $|f| = 4$, $|a| = 2$, $f^a = f^{-1}$ *and* $\langle L_1, a\rangle \cong$

$PGL_2(7)$.

$N_G(L_2)$ *is a* $PGL_2(7)$*-subgroup of some* $L_3(7)$*-subgroup of* G. L_2 *and* L_3 *are* Aut G*-conjugate.*

$N_G(L_4)=L_4$ *and* $N_{\mathrm{Aut}\,G}(L_4)\cong PGL_2(7)$. L_4 *is contained in some* A_7*-subgroup of* G.

Furthermore the elements of order 7 contained in L_1 *are 7A-elements but those contained in* L_i *are 7B-elements for any* $i\in\{2, 3, 4\}$.

§ 8. Conclusion

PROOF OF THE THEOREM. By the results in § 3, § 6 and § 7, any maximal subgroup of G is conjugate to one of the subgroups given in the theorem. Inspecting the orders and the structures of these subgroups, we may easily verify that there is no inclusion-relation between any two of them.

PROOF OF THE COROLLARY. Any maximal subgroup of Aut G is G, the centralizer of an outer involution ($\cong Z_2\times J_1$), or a normalizer in Aut G of a characteristically simple subgroup of G. By the results in § 3, § 6 and § 7, we can easily determine which characteristically simple subgroup of G has a normalizer in Aut G not contained in G.

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