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On the thin finite simple groups

Dedicated to Professor Hirosi Nagao on his 60th birthday

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Introduction.

The program to classify the finite simple groups has been finished recently and, in particular, simple groups of characteristic 2 type have been classified. (Here, by definition, a group G is of characteristic 2 type if G has even order and $C_L(O_2(L)) \leqq O_2(L)$ for all 2-local subgroups L of G.) The classification is dependent upon the earlier paper of Janko [1] classifying nonsolvable groups in which all 2-local subgroups are solvable and have cyclic Sylow subgroups for all odd primes. Janko's work is further dependent upon the N-group paper [2] in which Thompson classifies nonsolvable groups all of whose local subgroups are solvable. On the other hand, out of the recent investigations of Aschbacher [3] and others, grew several general theorems concerning groups of characteristic 2 type, which are hopefully utilized in revising the earlier papers of Janko, Thompson, and others.

Suggested by some of the recently obtained results, we divide the groups of characteristic 2 type into three classes.

I. The 2-isolated groups of characteristic 2 type.
II. The groups of characteristic 2 type in which some maximal 2-local subgroup has an 'Aschbacher block'.
III. The groups of characteristic 2 type neither of type I nor of type II.

As well known, a group is 2-isolated if and only if it has a strongly embedded subgroup, and groups having a strongly embedded subgroup have been classified by Bender [4]. For the groups of type II, the reader is referred to a survey article by Foote [5]. The 'Aschbacher block' is not yet firmly defined. In this paper, we adopt the definition given in [6] or [7]. We call groups of type III the generic groups of characteristic 2 type. This terminology is partly justified by the fact that a simple group of Lie type and characteristic 2 is a typical group of characteristic 2 type and it is of type III provided its BN-pair rank is greater than 2.

The purpose of this paper is to show how the recently obtained general results are utilized to classify the simple generic groups of characteristic

2 type in which 'sufficiently many' 2-local subgroups are solvable and have cyclic Sylow subgroups for all odd primes. In order to describe the result precisely, we define some notation. Let G be a group of characteristic 2 type, S a Sylow 2-subgroup of G, $\mathcal{M}(S)$ the set of all maximal 2-local subgroups of G containing S, and $Z=\Omega_1(Z(S))$. For each $M\in\mathcal{M}(S)$, we define $V_M=\langle Z^M\rangle$, the subgroup generated by all M-conjugates of Z. If elements M and N of $\mathcal{M}(S)$ satisfy the condition

$$N\leqq C_G(V_N)M\,,$$

let us write

$$N\leqq_{(S)}M\,.$$

It is easy to show that the relation $\leqq_{(S)}$ is a partial order in $\mathcal{M}(S)$. Let $\mathcal{M}^*(S)$ be the set of all maximal elements of $\mathcal{M}(S)$ under $\leqq_{(S)}$. Now, we can state our classification theorem.

THEOREM. *Let G be a nonabelian simple generic group of characteristic 2 type and S a Sylow 2-subgroup of G. Assume that every element of $\mathcal{M}^*(S)\cup\{C_G(\Omega_1(Z(S)))\}$ is solvable and has cyclic Sylow p-subgroups for all odd primes p. Then, under the hypothesis* (H) *stated below, G is isomorphic to* $G_2(2)'$ *or* ${}^2F_4(2)'$.

(H) Let X be a finite group, W a faithful GF(2)X-module, A a nonidentity elementary abelian 2-subgroup of X with $|A|\geqq|W:C_W(A)|$, and K a quasisimple normal subgroup of X with $X=AK$ and $C_X(K)=Z(K)=O(K)$. If K is a section (or subquotient) of some maximal 2-local subgroup of G, then $K\in\mathrm{Chev}(2)-\{(\mathrm{P})\mathrm{SU}_3(2^m), \mathrm{Sz}(2^{2m-1})\mid m\geqq 2\}$ or $K\cong A_n$, $n\geqq 7$.

In the above, Chev(2) denotes the collection of all quasisimple groups L with $O_2(L)=1$ for which $L/Z(L)$ is isomorphic to a simple group of Lie type and characteristic 2. Here, we consider the groups $A_6\cong\mathrm{Sp}_4(2)'$, $\mathrm{SU}_3(3)\cong G_2(2)'$, and ${}^2F_4(2)'$ to be of Lie type and characteristic 2. By a theorem of Aschbacher [17, Th. 1], the hypothesis (H) is satisfied if all nonabelian simple sections of all maximal 2-local subgroups of G are on the following list of the nonabelian simple groups of 'known' type.

1. The alternating groups of degree at least 5.
2. The simple groups of Lie type.
3. The twenty six sporadic simple groups.

Thus, the hypothesis (H) says, in effect, that nonabelian simple sections of maximal 2-local subgroups of G are all of known type. The hypothesis (H) clarifies, however, the properties of the simple groups of known type which we need in this paper.

This paper is designed for the revision program. Therefore, I will give below an outline of the proof of the theorem going into more details than usually required. Let G be a nonabelian simple generic group of characteristic 2 type satisfying the hypothesis (H), $S\in \mathrm{Syl}_2(G)$, $Z=\Omega_1(Z(S))$, and $C=C_G(Z)$. For 2-groups T, let $J(T)$ be the Thompson subgroup generated by the set $\mathcal{A}(T)$ of all elementary abelian subgroups of maximal order and define $K(T)=C_T(\Omega_1(Z(J(T))))$. Furthermore, let $Q(T)$ be the characteristic subgroup of T defined in [7]. The proof may be divided into three parts A, B, and C in view of the nature of the analysis in each part.

(A) Let $N_G(Q(K(S)))\leqq H\in \mathcal{M}(S)$ and $H\leqq_{(S)}M\in \mathcal{M}^*(S)$. Then it directly follows from [6] and [7] that G is generated by C and M and that 2-fusion in G is controlled by $C^G\cup M^G$. The hypothesis (H) is needed here, because we need (H) in [6] (see the concluding remarks of [6]). For each $L\in \mathcal{M}(S)$, let $Q_L=C_S(V_L)$. It is easy to show that V_L is an elementary abelian normal 2-subgroup of L with $O_2(L/C_L(V_L))=1$ and that if $L\in \mathcal{M}^*(S)$ then L is the unique maximal 2-local subgroup of G containing $N_L(Q_L)$. This is probably the most important property of $\mathcal{M}^*(S)$, for it enables us to use the so-called weak closure theory which Aschbacher developed in [3]. The weak closure theory is particularly effective when solvable groups are concerned. Namely, if all elements of $\mathcal{M}^*(S)\cup\{C\}$ are solvable then, using an idea of Aschbacher [18], we can rather quickly show that, for each $L\in \mathcal{M}^*(S)$, there is an involution $t\in L/C_L(V_L)$ such that $|V_L:C_{V_L}(t)|\leqq 4$. This is very strong information about the structure of $L/C_L(V_L)$. Especially, if L has cyclic Sylow subgroups for all odd primes, it readily follows that $L/C_L(V_L)$ has a unique normal dihedral subgroup $D_L/C_L(V_L)$ of order 6 or 10 generated by involutions t such that $|V_L:C_{V_L}(t)|\leqq 4$. Thus, the results in Part A are primarily immediate consequences of the general results concerning groups of characteristic 2 type obtained in [3], [6], and [7].

(B) Now, assume that every element of $\mathcal{M}^*(S)\cup\{C\}$ is solvable and has cyclic Sylow subgroups for all odd primes. First, we prove that if $L\in \mathcal{M}^*(S)$ then L is the unique maximal 2-local subgroup of G containing SD_L. As G is generated by C and M, this shows that $C_{V_M}(D_M)=1$ and hence it follows that V_M is of order 4 or 16. Next, we prove that $\mathcal{M}^*(S)=\{M\}$. Hence if $K\in \mathcal{M}(S)$ then $K\leqq_{(S)}M$, and consequently $V_K=\langle Z^{M\cap K}\rangle$. Using this, we quickly prove that $\mathcal{M}(S)\leqq\{M, C, N\}$, where N is a certain explicitly defined 2-local subgroup. By virtue of the results in Part A, the arguments in this part are not too difficult.

(C) In this part, we obtain precise information about $\mathcal{M}(S)$, using three results concerning pairs (X, Y) of solvable groups having a common 2-subgroup T of odd indices such that no nonidentity subgroup of T is normal both in X and in Y. The first result is an elementary but powerful result probably due to Thompson [2] and concerns the case $X\cap Y\neq T$. The second result is a corollary to Glauberman's 'triple factorization theorem' [8] and concerns the case where both $|X:T|$ and $|Y:T|$ are prime to 3. The third result is a theorem in [9] and concerns the case where both $|X:T|$ and $|Y:T|$ are primes.

As to the third result, we can alternatively appeal to the combined work of Goldschmidt [10] and Fan [11]. Nevertheless, in this paper, we will quote [9] rather than [10] and [11] for the following reasons.

i) [9] is much shorter than [10] and [11].

ii) [7] is, in effect, the only outside material needed in [9], while [10] and [11] rely on more outside materials.

iii) [9] is written in the completely standard group-theoretical language, while both [10] and [11] are written in the graph-theoretical language, and thus [9] appears to be easier of access for most readers of the present paper.

Now, using the above three results together with P. Hall's theorems on solvable groups [12, Chap. 6] and weak closure theory, we prove that $\mathcal{M}(S)=\{M, C\}$ with $|M:S|=3$ and $|C:S|=3$ or 5. As $G=\langle M, C\rangle$, no non-identity subgroup of S is normal both in M and in C. Hence if $S^*=(O^2(M)\cap S)(O^2(C)\cap S)$, $M^*=O^2(M)S^*$, and $C^*=O^2(C)S^*$, then the structure of M^* and C^* is described in [9]. As $O^2(G)=G$ and fusion in S is controlled by M and C, the focal subgroup theorem [12, Th. 7.3.4] shows that $S^*=S$. Thus, the structure of C is determined to the extent necessary to quote characterization theorems in terms of the centralizers of involutions [13], [14], [15].

This completes the description of the outline, but Sections 1–6 contain a more complete description of the preliminary results mentioned above. In particular, most results in Part A are contained in those sections. A major part of the proof of the theorem is contained in Section 7, and Parts B and C occupy the equal halves of that section. When treated in full generalities, the results in Sections 1–3 yield further general results on the 2-local structure of groups of characteristic 2 type, which I hope to discuss in the future.

Our notation is standard. Thus, if G is a finite group and X is a subgroup, then $\mathcal{M}(X)$ is the set of all maximal 2-local subgroups of G containing X, X^G is the set of all G-conjugates of X, $\cap X^G$ is the intersection of all subgroups in X^G, and $\langle X^G\rangle$ is the subgroup generated by all subgroups in X^G.

1. Weak closure theory.

In this section, we describe two theorems of Aschbacher [3]. Let G be a group of even order and V an elementary abelian 2-subgroup of G. For 2-subgroups T of G and nonnegative integers n, define $W_n(T, V)$ to be the subgroup generated by all subgroups W of T which are contained in some G-conjugate V^g of V such that

$$|V^g/W| \leqq 2^n.$$

(In [3], $W_n(T, V)$ is defined by the equality $|V^g/W|=2^n$ instead of the inequality above. However, this is not an essential change.) Also, let

$$C_n(T, V)=C_T(W_n(T, V)).$$

The following properties of the $W_n(T, V)$ are direct consequences of the definition.

1.1. $W_n(T, V) \leqq W_{n+1}(T, V)$.

1.2. If $W_n(T, V) \leqq U \leqq T$, *then* $W_n(T, V)=W_n(U, V)$.

1.3. $N_G(T) \leqq N_G(W_n(T, V))$.

Now, let $m_G(V)$ be the minimum rank of $V/C_V(t)$ as t ranges over the set of all involutions of $N_G(V)/C_G(V)$. When $N_G(V)/C_G(V)$ has odd order, we define $m_G(V)$ to be the rank of V. Next, let $r_G(V)$ be the minimum rank of V/W as W ranges over the set of all subgroups W of V such that $C_G(W) \not\leqq N_G(V)$. If such W does not exist (i.e. when $N_G(V)=G$), we define $r_G(V)$ to be the rank of V.

1.4. *Let G be a group of even order and V an elementary abelian 2-subgroup of G. Assume that $m_G(V)>n+1<r_G(V)$ for some positive integer n. Then for any solvable subgroup H of G and any Sylow 2-subgroup T of H, we have*

$$H=C_H(C_{i+1}(T, V))N_H(W_i(T, V)), \qquad 0 \leqq i \leqq n.$$

PROOF. See 6.11 of [3].

1.5. *Let G be a group of characteristic 2 type, M a maximal 2-local subgroup of G, V an elementary abelian normal 2-subgroup of M, and Q a Sylow 2-subgroup of $C_M(V)$. Assume $\mathcal{M}(N_M(Q))=\{M\}$ and $m_G(V)>2$. Then $m_G(V) \leqq r_G(V)$.*

PROOF. See Section 11 of [3].

2. A partial order in $\mathcal{M}(S)$.

In this section, G is a group of characteristic 2 type, $S \in \mathrm{Syl}_2(G)$, and $Z = \Omega_1(Z(S))$. For each $M \in \mathcal{M}(S)$, define

$$V_M = \langle Z^M \rangle ,$$

$$Q_M = C_S(V_M) .$$

2.1. *The following conditions hold.*
(1) $V_M \leqq \Omega_1(Z(O_2(M)))$.
(2) $M = N_G(V_M) \geqq C_G(V_M)$.
(3) $O_2(M/C_G(V_M)) = 1$.
(4) $Q_M \in \mathrm{Syl}_2(C_G(V_M))$ *and* $N_M(S) \leqq N_M(Q_M)$.
(5) $M = N_M(Q_M)C_G(V_M)$.

PROOF. As $Z \leqq C_M(O_2(M)) = Z(O_2(M))$, (1) holds. In particular, V_M is a nonidentity normal 2-subgroup of M and so, as $M \in \mathcal{M}(S)$, (2) holds. Let $X/C_G(V_M) = O_2(M/C_G(V_M))$. Then as $X \leqq SC_G(V_M)$, $Z \leqq C_{V_M}(X)$ and hence we have $V_M \leqq C_{V_M}(X)$, proving (3). (4) is clear because $S \in \mathrm{Syl}_2(M)$ and $C_G(V_M) \lhd M$. Finally, (5) follows from (4) by a Frattini argument.

Now, for elements M and N of $\mathcal{M}(S)$, let us write

$$N \leqq_{(S)} M$$

if they satisfy the following condition.

$$N \leqq C_G(V_N)M .$$

2.2. *The following conditions hold.*
(1) *If* $N \leqq_{(S)} M$, *then* $V_N = \langle Z^{M \cap N} \rangle \leqq V_M$.
(2) $\leqq_{(S)}$ *is a partial order in* $\mathcal{M}(S)$.

PROOF. (1) If $N \leqq_{(S)} M$, then $N = C_G(V_N)(M \cap N)$ by 2.1, so $N \leqq C_G(Z)(M \cap N)$ and $V_N = \langle Z^N \rangle = \langle Z^{M \cap N} \rangle \leqq V_M$.

(2) It is clear that $M \leqq_{(S)} M$ for all $M \in \mathcal{M}(S)$. Suppose $N \leqq_{(S)} M$ and $M \leqq_{(S)} N$. Then $V_M = V_N$ by (1) and so $M = N_G(V_M) = N_G(V_N) = N$ by 2.1. Suppose $N \leqq_{(S)} M$ and $M \leqq_{(S)} L$. Then $N \leqq C_G(V_N)M \leqq C_G(V_N)C_G(V_M)L$. As $C_G(V_M) \leqq C_G(V_N)$ by (1), $N \leqq C_G(V_N)L$ and thus $N \leqq_{(S)} L$.

Now, let $\mathcal{M}^*(S)$ be the set of all maximal elements of $\mathcal{M}(S)$ under $\leqq_{(S)}$.

2.3. *If* $M \in \mathcal{M}^*(S)$, *then* $\mathcal{M}(N_M(Q_M)) = \{M\}$.

PROOF. Suppose $N \in \mathcal{M}(N_M(Q_M))$. Then $N \in \mathcal{M}(S)$ and $M \leqq C_G(V_M)N$ by 2.1, which implies that $M \leqq_{\langle S \rangle} N$. Therefore, $N=M$ by the maximality of M.

2.4. *Assume that some element* $M \in \mathcal{M}^*(S)$ *satisfies* $J(S) \leqq C_G(V_M)$. *Then*

(1) $K(S) \leqq Q_M$,

(2) $\mathcal{M}(N_G(K(S)))=\{M\}$, *and*

(3) $J(S) \not\leqq C_G(V_N)$ *for all* $N \in \mathcal{M}^*(S)-\{M\}$.

PROOF. As $J(S) \leqq C_S(V_M)=Q_M$ and $V_M \leqq \Omega_1(Z(Q_M))$, we have $V_M \leqq \Omega_1(Z(J(S)))$ and so $K(S) \leqq C_S(V_M)=Q_M$. Thus, (1) holds. Furthermore, we have $K(S)=K(Q_M)$ and so $N_G(Q_M) \leqq N_G(K(S))$. Thus, (2) follows from 2.3, and (3) follows from (2).

2.5. *Let* $L \in \mathcal{M}(S)$, U *an elementary abelian normal 2-subgroup of* L, *and* $P=C_S(U)$. *Assume* $\mathcal{M}(N_L(P))=\{L\}$ *and* $m_G(U)>2$. *Assume further that an element* $M \in \mathcal{M}^*(S)$ *is solvable. Then* M *is a unique maximal solvable subgroup of* G *containing* S.

PROOF. As $r_G(U) \geqq m_G(U)>2$ by 1.5, 1.4 shows that

$$H=C_H(C_{i+1}(T, U))N_H(W_i(T, U)), \qquad i=0,1,$$

for any solvable subgroup H of G and any $T \in \mathrm{Syl}_2(H)$. Thus,

$$M=C_M(C_{i+1}(S, U))N_M(W_i(S, U)) \leqq C_G(Z)N_G(W_i(S, U))$$

for $i=0,1$. Let $W_i=W_i(S, U)$ and $N_i=N_G(W_i)$. Notice that $1 \neq W_i \leqq O_2(N_i)$ as $U \neq 1$. As $S \in \mathrm{Syl}_2(N_i)$ by 1.3, $Z \leqq C_S(O_2(N_i))=Z(O_2(N_i))$. Therefore, $V_M=\langle Z^M \rangle \leqq \langle Z^{N_i} \rangle \leqq Z(O_2(N_i))$ and so $W_i \leqq O_2(N_i) \leqq C_S(V_M)=Q_M$. Thus, $W_i=W_i(Q_M, U)$ by 1.2, and then $N_G(Q_M) \leqq N_G(W_i) \leqq N_G(Z(W_i))$ by 1.3. As $Z(W_i) \neq 1$, 2.3 shows that $N_G(Z(W_i)) \leqq M$ for $i=0,1$. Now, let H be a solvable subgroup of G containing S. Then

$$H=C_H(C_1(S, U))N_H(W_0(S, U)) \leqq C_G(Z(W_1(S, U)))N_G(Z(W_0(S, U)))$$

and so $H \leqq M$. This completes the proof of 2.5.

2.6. *Let* $M \in \mathcal{M}^*(S)$. *Assume that* M *is solvable, that* G *is generated by solvable subgroups containing* S, *and that* $M \neq G$. *Then* $m_G(V_M) \leqq 2$.

PROOF. This is a direct consequence of 2.5 and 2.3.

3. Generic groups of characteristic 2 type.

By definition, a group G of characteristic 2 type is generic if

i) G is not 2-isolated (see [16] for the definition), and
ii) no maximal 2-local subgroup of G has an Aschbacher block (see [6] or [7] for the definition).

Therefore, if G is a generic group of characteristic 2 type, then so are all maximal 2-local subgroups of G.

In this section, G is a generic group of characteristic 2 type satisfying the hypothesis (H), $S\in \mathrm{Syl}_2(G)$, and $Z=\Omega_1(Z(S))$.

3.1. *G is generated by $C_G(Z)$ and $N_G(Q(K(S)))$.*

PROOF. This is a direct consequence of Theorem H of [7].

3.2. *If $G\neq N_G(Z)$, then $N_G(Z)\notin \mathcal{M}^*(S)$.*

PROOF. Suppose $N=N_G(Z)$ is contained in $\mathcal{M}^*(S)$. As $V_N=Z$, $Q_N=C_S(Z)=S$ and so $\mathcal{M}(N_G(S))=\{N\}$ by 2.3. But then $G=N$ by 3.1.

3.3. *Let $N_G(Q(K(S)))\leqq H\in \mathcal{M}(S)$ and $H\leqq_{(S)}M\in \mathcal{M}^*(S)$. Then*
(1) *G is generated by $C_G(Z)$ and M, and*
(2) *Sylow 2-intersections and 2-fusion in G are controlled by $C_G(Z)^G\cup M^G$ (see [6] or [7] for the definition).*

PROOF. By Theorem H of [7], the set $C_G(Z)^G\cup N_G(Q(K(S)))^G$ controls Sylow 2-intersections in G, and so does the set $C_G(Z)^G\cup H^G$. Now, $N_G(S)\leqq N_G(Z)\cap H$ and, consequently, $C_G(Z)$ and H are the only elements of $C_G(Z)^G\cup H^G$ that contain S. Thus, $G=\langle C_G(Z), H\rangle$ by 1.5 of [6]. Now, $H=C_H(V_H)(M\cap H)=C_H(Z)(M\cap H)$. Hence $G=\langle C_G(Z), M\rangle$. Also, $C_G(Z)^G\cup M^G$ controls Sylow 2-intersections in G by 1.9 of [6], and so it controls 2-fusion in G as well by 1.4 of [6].

3.4. *Assume that every element of $\mathcal{M}^*(S)\cup\{C_G(Z)\}$ is solvable and $O_2(G)=1$. Then $m_G(V_M)\leqq 2$ for all $M\in \mathcal{M}^*(S)$, and if $m_G(V_M)=2$ for some $M\in \mathcal{M}^*(S)$ then the following holds.*
(1) *$J(S)\leqq Q_M$.*
(2) *$\mathcal{M}(N_G(K(S)))=\{M\}$.*
(3) *$m_G(V_N)=1$ for all $N\in \mathcal{M}^*(S)-\{M\}$.*

PROOF. The first assertion follows from 2.6 and 3.3. Suppose $m_G(V_M)=2$ for some $M\in\mathcal{M}^*(S)$. Then (1) follows from certain (probably well known) facts on GF(2)-representations. Namely, if $A\in\mathcal{A}(S)$ is not contained in Q_M, then $A\nleqq C_G(V_M)$ and so the maximality of $|A|$ shows that the image of A in $\bar{M}=M/C_G(V_M)$ is contained in the set $\mathcal{P}$ consisting of all nonidentity elementary abelian 2-subgroups X of $\bar{M}$ such that $|X||C_{V_M}(X)|\geqq|Y||C_{V_M}(Y)|$ for all subgroups Y of X. Let $\mathcal{P}^*$ be the set of all minimal elements of $\mathcal{P}$ under the partial order $\prec$ defined by: $X\prec Y$ if and only if $X\leqq Y$ and $|X||C_{V_M}(X)|=|Y||C_{V_M}(Y)|$. Let $X\in\mathcal{P}^*$. Then since $\bar{M}$ is solvable and $O_2(\bar{M})=1$ by 2.1, it follows that $\langle X^{\langle\mathcal{P}^*\rangle}\rangle$ is isomorphic to the dihedral group of order 6 (see for instance 3.1 and 3.3 of [6]). In particular, $|X|=2$ and so $|V_M:C_{V_M}(X)|=2$ by the definition of $\mathcal{P}$, which shows that $m_G(V_M)=1$, a contradiction. Therefore, (1) holds. Finally, (2) follows from (1) and 2.4, and (3) follows from (2) (or (1) and 2.4).

4. GF(2)-representations.

In this section, (G, V) is a pair of a solvable group G of even order with $O_2(G)=1$ and a faithful GF(2)G-module V. Except in 4.1 below, we assume further that

(*) Sylow p-subgroups of G are cyclic for all odd primes p.

Let $m(G, V)$ be the minimum rank of $V/C_V(t)$ as t ranges over the set of all involutions of G. Let $\mathcal{I}(G, V)$ be the set of all involutions t of G for which the rank of $V/C_V(t)$ is equal to $m(G, V)$. Let $\mathcal{P}(G, V)$ be the set of all nonidentity elementary abelian 2-subgroups A of G such that $|A||C_V(A)|\geqq|B||C_V(B)|$ for all subgroups B of A. As usual, D_n and Z_n denote the dihedral group of order n and the cyclic group of order n, respectively. In this section, we consider the case $m(G, V)\leqq 2$. Section 13 of [17] contains a more complete discussion of this situation for an arbitrary group G.

4.1. *If $\mathcal{P}(G, V)$ is nonempty, then $m(G, V)=1$.*

PROOF. We can use the same argument as in the proof of 3.4.1.

4.2. *Assume $m(G, V)\leqq 2$ and let $t\in\mathcal{I}(G, V)$. If X is a t-invariant subgroup of G of odd order with $X=[X, t]\neq 1$, then one of the following holds.*

(1) $|X|=3$ *and* $|[V, X]|=|C_{[V,X]}(t)|^2=4$ *or* 16.

(2) $|X|=5$ *and* $|[V, X]|=|C_{[V,X]}(t)|^2=16$.

PROOF. Assume that X is cyclic and let $W=[V, X]$. Then t inverts X and $C_W(X)=0$, so $|C_W(t)|=|W: C_W(t)|$ and $|W|=|C_W(t)|^2=4$ or 16. As $\langle t, X\rangle$ is faithful on W, $\langle t, X\rangle$ is isomorphic to a subgroup of $\mathrm{GL}_4(2)\cong \mathrm{A}_8$. Inspecting dihedral subgroups of A_8, we obtain $|X|\leqq 5$.

Assume, therefore, that X is not cyclic. As X is solvable, X is a product of a Hall $\{3,5\}'$-subgroup Y, $X_5\in \mathrm{Syl}_5(X)$, and $X_3\in \mathrm{Syl}_3(X)$: $X=YX_5X_3$. We may choose Y, X_5, and X_3 so that they are t-invariant and X_5X_3 is a subgroup. As $[Y, t]$ is generated by elements inverted by t, the last paragraph shows that $[Y, t]=1$. Thus, $X=[X, t]=X_5X_3$. As X_5 and X_3 are cyclic by (*), we have $X_5\lhd X$ (see Section 6). Hence X_3 acts on X_5, and X_3 clearly centralizes the Frattini factor group of X_5. This shows that X_3 centralizes X_5, and therefore X is cyclic, a contradiction.

4.3. *Assume $m(G, V)\leqq 2$ and let $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_n$ be the orbits of $\mathcal{I}(G, V)$ under the action of $\langle \mathcal{I}(G, V)\rangle$ by conjugation. Then*

(1) $\langle \mathcal{O}_i\rangle \cong \mathrm{D}_6$ *or* D_{10}, $1\leqq i\leqq n$,

(2) $\langle \mathcal{I}(G, V)\rangle=\langle \mathcal{O}_1\rangle\times\cdots\times\langle \mathcal{O}_n\rangle$,

(3) $n\leqq 2$ *with equality only when* $\langle \mathcal{I}(G, V)\rangle\cong \mathrm{D}_6\times \mathrm{D}_{10}$, *and*

(4) *if* $m(G, V)=1$, *then* $\langle \mathcal{I}(G, V)\rangle \cong \mathrm{D}_6$.

PROOF. Let $\mathcal{J}=\mathcal{I}(G, V)$ and $J=\langle \mathcal{J}\rangle$. For $t\in \mathcal{J}$, let $\mathcal{O}_t$ be the orbit containing t. As $C_J(O(J))\leqq O(J)$, we have $[O(J), t]\cong \mathrm{Z}_3$ or Z_5 by 4.2 and, consequently, $\langle [O(J), t], t\rangle \leqq \langle \mathcal{O}_t\rangle$. If $u\in \mathcal{J}$ is conjugate with t in J, then $[O(J), t]$ and $[O(J), u]$ are conjugate, and so $[O(J), t]=[O(J), u]$ as $F(O(J))$ is cyclic by (*). Conversely, assume $[O(J), t]=[O(J), u]$. Then tu stabilizes the series $O(J)\geqq [O(J), t]\geqq 1$ and, as $C_J(O(J))\leqq O(J)$, it follows that tu has odd order. Thus, u is conjugate to t. Moreover, as $J=O_{2',2}(J)$ (see 6.1), we have $tu\in O(J)$, so $\langle tu\rangle \leqq [O(J), t]$ and $u\in \langle tu, t\rangle \leqq \langle [O(J), t], t\rangle$. We have shown $\langle \mathcal{O}_t\rangle=\langle [O(J), t], t\rangle$, and therefore (1) holds. We have also shown that if $O^2(\langle \mathcal{O}_t\rangle)=O^2(\langle \mathcal{O}_u\rangle)$ then $\langle \mathcal{O}_t\rangle=\langle \mathcal{O}_u\rangle$. As $F(O(J))$ is cyclic, (2) and (3) now immediately follow. (4) is a consequence of (1)–(3) and 4.2.

5. Pairs of groups having a common 2-subgroup.

In this section, G and H are groups having a common nonidentity 2-subgroup S of odd indices.

5.1. *Let P be a nonidentity subgroup of G and H such that $\langle S, P\rangle$ is solvable and $O(\langle S, P\rangle)=1$. Let Q and R be subgroups of $N_G(P)$ and $N_H(P)$, respectively, and assume that both Q and R permute with S. Then P, Q and R normalize a nonidentity normal subgroup of S.*

PROOF. Embedding G and H into their amalgamated product over $\langle S, P\rangle$, we have

$$\langle S, P, Q, R\rangle = \langle S, P\rangle\langle Q, R\rangle$$

and hence P is contained in

$$D = \cap \langle S, P\rangle^{\langle S,P,Q,R\rangle}.$$

Therefore, D is a nonidentity solvable group with $O(D)=1$. Let T be a minimal characteristic subgroup of D. Then T is a 2-subgroup normalized by $\langle S, P, Q, R\rangle$, and hence $T \leqq S$.

By definition, a subgroup Y of a group X is nearly maximal in X if Y is contained in a unique maximal subgroup of X.

5.2. *Assume that G and H are solvable groups with $O(G)=1=O(H)$ and that S is nearly maximal both in G and in H. If $(3, |G||H|)=1$, then some nonidentity subgroup of S is normal both in G and in H.*

PROOF. By the 'triple factorization' theorem of Glauberman [8], there are three nonidentity characteristic subgroups A, B, and C of S such that

$$G = N_G(A)N_G(B) = N_G(B)N_G(C) = N_G(C)N_G(A),$$

$$H = N_H(A)N_H(B) = N_H(B)N_H(C) = N_H(C)N_H(A).$$

As S is nearly maximal in G and H, at least two elements of $\{A, B, C\}$ are normal in G, and the same is true of H. Therefore, some element of $\{A, B, C\}$ is normal both in G and in H.

Before stating the next result, we define some notation and terminology. By the $*$, we denote central products with amalgamated centers, and by $D_8 \sharp D_8$, we denote the group

$$\langle a, b, c \mid a^2 = b^4 = c^4 = (ab)^2 = (ac)^2 = b^{-1}c^{-1}bc = 1\rangle.$$

Now, suppose G and H satisfy the following conditions.

(a) Both $|G:S|$ and $|H:S|$ are odd primes.

(b) No nonidentity subgroup of S is normal both in G and in H.

(c) $C_G(O_2(G)) \leqq O_2(G)$ and $C_H(O_2(H)) \leqq O_2(H)$.

By definition, (G, H) is of $G_2(2)'$-type if $G/O_2(G) \cong H/O_2(H) \cong D_6$, $O_2(G) \cong Z_4 \times Z_4$, and $O_2(H) \cong Z_4 * D_8$; (G, H) is of M_{12}-type if $G/O_2(G) \cong H/O_2(H) \cong D_6$, $O_2(G) \cong D_8 \sharp D_8$, $O_2(H) \cong D_8 * D_8$, and $\langle Z(O_2(G))^H\rangle$ is an abelian 2-group; (G, H) is of ${}^2F_4(2)'$-type if $G/O_2(G) \cong D_6$, $H/O_2(H) \cong F_{20}$ (the Frobenius group of order 20), and there is an H-composition series $O_2(H) = R_0 \geqq R_1 \geqq R_2 \geqq R_3 = 1$ of $O_2(H)$ such that the groups Q_i $(0 \leqq i \leqq 6)$ defined below form a G-composition series of $O_2(G)$.

$$Q_0=O_2(G),\quad Q_1=\langle R_1^G\rangle,\quad Q_2=\cap R_0^G,\quad Q_3=\langle(R_1\cap Q_2)^G\rangle,$$
$$Q_4=\cap R_1^G,\quad Q_5=\langle R_2^G\rangle,\quad \text{and}\quad Q_6=1.$$

Finally, define $S^*=(S\cap O^2(G))(S\cap O^2(H))$, $G^*=O^2(G)S^*$, and $H^*=O^2(H)S^*$. Then by 3.3 of [9], S^* is a nonidentity 2-subgroup of both G^* and H^*, and (G^*, H^*, S^*) satisfies the conditions (a)-(c). Now, we can state the third main result of this section.

5.3. *Assume that G and H satisfy the conditions* (a)-(c) *and that $\Omega_1(Z(S))\leqq Z(H)$. Then (G^*, H^*) is of $G_2(2)'$-type or M_{12}-type or ${}^2F_4(2)'$-type.*

PROOF. See [9]. (This is Theorem A of [9].)

We conclude this section by two more remarks on the pairs (G, H) of $G_2(2)'$-type or M_{12}-type or ${}^2F_4(2)'$-type.

5.4. *Assume that G and H satisfy the conditions* (a)-(c). *Then the following holds.*

(1) *If (G, H) is of $G_2(2)'$-type, then $S\cong Z_4 \operatorname{wr} Z_2$ (the wreath product of Z_4 by Z_2).*

(2) *If (G, H) is of M_{12}-type, then S has the following presentation.*

generators: a, b, c, d;

relations: $a^2=b^4=c^4=d^2=(ab)^2=(ac)^2=b^{-1}c^{-1}bc=(ad)^2=b^{-1}dbd=cbdcd=1.$

Furthermore, there is a noncentral involution x of S such that $C_S(x)\not\leqq Q^h$ for any $h\in H$.

(3) *If (G, H) is of ${}^2F_4(2)'$-type and R_i, Q_i are the same as in the definition of '${}^2F_4(2)'$-type', then $|R_0/R_1|=|R_1/R_2|=16$, $|R_2|=2$, $O_2(H)=R_0\geqq R_1\geqq R_2\geqq R_3=1$ is the upper central series of $O_2(H)$, and $|Q_{i-1}/Q_i|=4$ iff $i=1, 2, 4, 6$.*

PROOF. Let $Q=O_2(G)$, $R=O_2(H)$, $V=\Omega_1(Z(Q))$, and $Z=\Omega_1(Z(S))$.

(1) Suppose (G, H) is of $G_2(2)'$-type. Then R is non-abelian and so $Z(R)\leqq Q$ by 4.1 of [9]. Pick $g\in G-S$ so that $g^2\in Q$. Then, as $G=\langle R, R^g\rangle$ by 3.6 of [9], we have $Z(R)\cap Z(R^g)=1$ and so $Q=Z(R)\times Z(R^g)$ with $Z(R^g)^g=Z(R)$. Thus, we may pick g so that $g^2=1$, and it follows that $S\cong Z_4 \operatorname{wr} Z_2$.

(2) Suppose (G, H) is of M_{12}-type and define $U=\langle V^H\rangle$. As U is elementary abelian, we have $|U|=8$ and $U\leqq C_G(V)=Q$ by 3.8 of [9], so $Q\cap Q^h=U$ for all $h\in H-S$ by 3.6 of [9]. Let $x\in G-S$ and define $W=R\cap R^x$. Then $|W|=8$ by 3.6 of [9] and $W^2\leqq Z\cap Z^x=1$. Also, as $[W, R]\leqq Z\leqq V\leqq W$ and $G=\langle R, R^x\rangle$, we have $W\lhd G$ and so $W\neq U$. Now, let $T=[Q, O^2(G)]$.

Then $T \cong Z_4 \times Z_4$ and $\{U, W, T \cap R\}$ is the set of the maximal subgroups of $Q \cap R$ containing V. Now, pick $b \in T \cap R - V$ and an involution $d \in C_R(b) - Q$ (this is possible as $R \cong D_8 * D_8$). By Suzuki's lemma, d inverts an element $g \in G$ of order 3. Let $C_W(g) = \langle a \rangle$. Necessarily, $a^d = a$ and $(ab)^2 = 1$. As $C_T(g) = 1$, we have $bb^g b^{g^2} = 1$ and so, if $c = b^g$, then $c^g = b^{-1}c^{-1}$, $bc = cb$, and $(ac)^2 = 1$. Now, $c^d = bcv$ for some $v \in V$. We have $c^{dg} = (bcv)^g = cb^{-1}c^{-1}v^g = b^{-1}v^g$, while $c^{dg} = c^{g^{-1}d} = b^d = b$. Therefore, $v = (b^2)^{g^{-1}} = (b^{-1}c^{-1})^2$ and so $c^d = b^{-1}c^{-1}$. It is now clear that S has the above presentation. Finally, $C_S(a) \not\leqq Q$ and, as $a \notin U$, $a \notin Q^h$ for all $h \in H - S$. Therefore, $C_S(a) \not\leqq Q^h$ for all $h \in H$.

(3) This has been proved in [9, 1.2].

5.5. *Assume that G and H satisfy the conditions* (a)-(c) *and (G, H) is of* $G_2(2)'$-*type or* M_{12}-*type or* ${}^2F_4(2)'$-*type. Then the Frattini factor group of* $[O_2(H), O^2(H)]$ *has order at most* 16.

PROOF. Let $R = O_2(H)$. If (G, H) is of $G_2(2)'$-type, then $R \cong Z_4 * D_8$ and so $[R, O^2(H)]$ is isomorphic to the quaternion group. If (G, H) is of M_{12}-type or ${}^2F_4(2)'$-type, then $[R, O^2(H)] = R$ (see the proof of 3.9 of [9]) and $|R/R^2| = 16$ (when (G, H) is of ${}^2F_4(2)'$-type, this follows from 5.4).

6. Solvable groups.

In this section, G is a solvable group in which all Sylow p-subgroups are cyclic for all odd primes p.

6.1. $G = O_{2,2',2}(G)$.

PROOF. Let $Q = O_3(G)$ and $F/Q = F(G/Q)$. Then F/Q is cyclic and so $G/C_G(F/Q)$ is abelian. As $C_G(F/Q) \leqq F$ and F/Q has odd order, we conclude that $G = O_{2,2',2}(G)$.

6.2. *Let p be the largest odd prime divisor of $|G|$ and let P be a nonidentity p-subgroup of G. Then*

(1) $PO_2(G) \lhd G$,

(2) *P permutes with every $S \in \mathrm{Syl}_2(G)$, and*

(3) *if G is a $\{2, p\}$-group, then $O_2(PS) = O_2(G)$.*

PROOF. (1) Let $\bar{G} = G/O_2(G)$ and $P \leqq X \in \mathrm{Syl}_p(G)$. Then $\bar{X} \in \mathrm{Syl}_p(O(\bar{G}))$ by 6.1. As $O(\bar{G})$ is supersolvable, $\bar{X}$ is the unique element of $\mathrm{Syl}_p(O(\bar{G}))$. Therefore, $\bar{X} \lhd \bar{G}$ and then $\bar{P} \lhd \bar{G}$.

(2) As $PS = PO_2(G)S$ is a subgroup by (1), P permutes with S.

(3) $O_2(PS)$ acts on $O(\bar{G}) = \bar{X}$ and $[\bar{P}, \overline{O_2(PS)}] \leqq \bar{P} \cap \overline{O_2(PS)} = 1$. Therefore,

we have $[O(\bar{G}), \overline{O_2(PS)}]=1$, which shows that $O_2(PS)=O_2(G)$ as $C_{\bar{G}}(O(\bar{G})) \leqq O(\bar{G})$.

7. Thin groups.

In this section, G is a nonabelian simple generic group of characteristic 2 type, $S \in \mathrm{Syl}_2(G)$, and we assume that every element L of $\mathcal{M}^*(S) \cup \{C_G(\Omega_1(Z(S)))\}$ satisfies the following two conditions.

i) L is solvable.

ii) Sylow p-subgroups of L are cyclic for all odd primes p.

Furthermore, we assume that G satisfies the hypothesis (H).

Let $Z=\Omega_1(Z(S))$ and, for each $L \in \mathcal{M}(S)$, define

$$\bar{L}=L/C_L(V_L).$$

Recall from Section 2 that V_L is an elementary abelian normal 2-subgroup of L with $L=N_G(V_L) \geqq C_G(V_L)$ and $O_2(\bar{L})=1$. Now, let $L \in \mathcal{M}^*(S)$. Then $\bar{L}$ has even order by 3.2, and so the pair $(\bar{L}, V_L)$ satisfies the hypothesis of Section 4. As $m(\bar{L}, V_L)=m_G(V_L) \leqq 2$ by 3.4, 4.3 shows that $\bar{L}$ has a normal dihedral subgroup $\bar{D}_L$ of order 6 or 10 generated by elements of $\mathcal{I}(\bar{L}, V_L)$. Let us choose $\bar{D}_L$ so that, if possible, $\bar{D}_L \cong \mathrm{D}_6$. Let $\bar{E}_L=O^2(\bar{D}_L)$ and $\bar{B}_L=C_{\bar{L}}(\bar{D}_L)$. Define D_L, E_L, and B_L to be the complete inverse images of $\bar{D}_L$, $\bar{E}_L$, and $\bar{B}_L$, respectively. Notice that $|[V_L, E_L]|=4$ or 16 by 4.2.

7.1. *If* $L \in \mathcal{M}^*(S)$, *then* $\mathcal{M}(SD_L)=\{L\}$.

PROOF. We distinguish two cases.

Case 1: $\bar{D}_L \cong \mathrm{D}_6$. Let $W=\langle (Z \cap [V_L, E_L])^{D_L} \rangle$. As $[V_L, E_L] \lhd L$, W is a nonidentity D_L-invariant subgroup of V_L. Suppose that W is not normal in L. Then $W \neq [V_L, E_L]$ and, as $C_{[V_L, E_L]}(E_L)=1$, we have $|[V_L, E_L]|=16$, $|W|=4$, and $|C_W(S \cap D_L)|=2$. As $\bar{L}=\bar{D}_L \times \bar{B}_L$ and $S \leqq N_L(W)$, $|B_L : N_{B_L}(W)|$ is odd $\neq 1$ and, moreover, it is not 3 as a Sylow 3-subgroup of L is cyclic. If $x \in B_L - N_{B_L}(W)$, then $W \cap W^x=1$ as $E_L \leqq N_L(W^x)$ and E_L acts irreducibly on W. Thus, $[V_L, E_L]-\{1\}$ is partitioned by five B_L-conjugates of $W-\{1\}$. As $|C_{W^x}(S \cap D_L)|=2$ for each $x \in B_L$, we conclude that $|C_{[V_L, E_L]}(S \cap D_L)|=6$, which is a contradiction. Therefore,

$$W \lhd L.$$

Consequently, $C_G(W) \lhd L=N_G(W)$ and so as $C_{D_L}(W)=C_G(V_L)$, we have

$$C_G(W) \leqq B_L.$$

Now, let $K\in\mathcal{M}(SD_L)$ and $K\leqq_{(S)}J\in\mathcal{M}^*(S)$. Then

$$W\leqq V_K\leqq V_J$$

by 2.2, and so

$$C_G(V_J)\leqq C_G(V_K)\leqq C_G(W)\leqq B_L.$$

Now, the pair $(\bar{B}_L, V_L)$ also satisfies the hypothesis of Section 4 provided $\bar{B}_L$ has even order. As $(|\bar{B}_L|, 3)=1$, 4.1 and 4.3 show that $J(S\cap B_L)\leqq C_G(V_L)$ (otherwise, $\mathcal{P}(\bar{B}_L, V_L)$ is nonempty, a contradiction). Thus, if $J(S)\leqq C_G(V_J)$, then $J(S)=J(S\cap B_L)\leqq C_G(V_L)$, so $J=L$ by 2.4 and $K\leqq C_G(V_K)L=L$. Therefore, assume $J(S)\not\leqq C_G(V_J)$. Then $\mathcal{P}(\bar{J}, V_J)$ is nonempty, so $m_G(V_J)=1$ and $\bar{D}_J\cong D_6$ by 4.1 and 4.3. Let $X\in\mathrm{Syl}_3(D_L)$. Then as $X\leqq K\leqq(J\cap K)C_G(V_K)$, there is a Sylow 3-subgroup Y of J such that $X\leqq(Y\cap K)C_G(V_K)$. As $E_L=XC_G(V_L)$ and $E_J=YC_G(V_J)$, we have

$$W=[W, X]\leqq[V_K, Y\cap K]\leqq[V_J, Y]=[V_J, E_J].$$

As $|[V_J, E_J]|=4$ by 4.2, we conclude that $W=[V_J, E_J]$. Therefore, $L=N_G(W)=N_G([V_J, E_J])=J$ and hence $K\leqq L$ as before. This completes the proof of 7.1 in Case 1.

Case 2: $\bar{D}_L\cong D_{10}$. Then $|[V_L, E_L]|=16$ by 4.2. Suppose $L\neq K\in\mathcal{M}(SD_L)$ and let $K\leqq_{(S)}J\in\mathcal{M}^*(S)$. Then as D_L acts irreducibly on $[V_L, E_L]$,

$$[V_L, E_L]=\langle(Z\cap[V_L, E_L])^{D_L}\rangle\leqq V_K\leqq V_J$$

and hence

$$C_G(V_J)\leqq C_G(V_K)\leqq C_G([V_L, E_L])\leqq L.$$

This yields that $J\neq L$ as $K\leqq C_G(V_K)J$. As $m_G(V_L)=2$ by choice of $\bar{D}_L$ and 4.3, we have $m_G(V_J)=1$ by 3.4 and hence $\bar{D}_J\cong D_6$ by 4.3. Let $X\in\mathrm{Syl}_5(D_L)$ and $Y\in\mathrm{Syl}_3(D_J)$. As $X\leqq K\leqq(J\cap K)C_G(V_K)$, there is a Sylow 5-subgroup F of J such that $X\leqq(F\cap K)C_G(V_K)$. Thus,

$$[V_L, E_L]=[V_L, E_L, X]\leqq[V_K, F\cap K]\leqq[V_J, F].$$

As $\bar{J}=\bar{D}_J\times\bar{B}_J$, we have $[Y, F]\leqq C_G(V_J)$ and so F normalizes both $[V_J, Y]$ and $C_{V_J}(Y)$. As $E_J=YC_G(V_J)$, we have $V_J=[V_J, Y]\times C_{V_J}(Y)$ and $|[V_J, Y]|=4$ by 4.2. Thus, we conclude that $[V_J, F]\leqq C_{V_J}(Y)$, and so

$$Y\leqq C_G([V_J, F])\leqq C_G([V_L, E_L])\leqq L.$$

However, this shows that $SD_J=SYC_G(V_J)\leqq L$, which contradicts what we have proved in Case 1. This completes the proof of 7.1.

Now, let $N_G(Q(K(S)))\leqq H\in\mathcal{M}(S)$ and $H\leqq_{(S)}M\in\mathcal{M}^*(S)$. Henceforth, we fix such H and M, and denote V_M, Q_M, D_M, E_M, and B_M by V, Q, D, E, and B, respectively. Furthermore, we define $C=C_G(Z)$.

7.2. *The following conditions hold.*
(1) $V=[V, E]$.
(2) $|V|=|C_V(S\cap D)|^2=4$ *or* 16.
(3) $N_G(Q(K(S)))\leqq M$ (*in fact*, $H=M$).

PROOF. (1) First of all, $V=[V, E]\times C_V(E)$, so we have to prove $C_V(E)=1$. If $C_V(E)\neq 1$, then $C_Z(E)=C_V(E)\cap Z(S)\neq 1$. As $SD=SE\leqq C_G(C_Z(E))\geqq C$ and $\mathcal{M}(SD)=\{M\}$ by 7.1, we conclude that $C\leqq M$. But then $G=M$ by 3.3, which is a contradiction.

(2) This follows from 4.2 and (1).

(3) As $H\leqq_{(S)}M$, $V_H\leqq V$ by 2.2. As $G=\langle C, H\rangle$ by 3.1 and $G\neq N_G(Z)$, $Z\neq V_H$. Thus, if $|V|=4$ then $V_H=V$ and $H=N_G(V_H)=N_G(V)=M$. If $|V|=16$, then $m_G(V)=2$ by 4.2 and (1), and so $H=M$ by 3.4.

7.3. $\bar{D}\cong \mathrm{D}_6$.

PROOF. We assume $\bar{D}\cong \mathrm{D}_{10}$ and argue for a contradiction. As $|V|=16$ by 7.2, $\bar{M}$ is isomorphic to a subgroup of $\mathrm{GL}_4(2)$. As the centralizer of every involution in $\mathrm{GL}_4(2)$ has order $2^6\cdot 3$ or $2^5\cdot 3$, we conclude that $|\bar{B}|=1$ or 3. Also, $|M/DB|\leqq 2$ as $\mathrm{Out}(\mathrm{D}_{10})\cong \mathrm{Z}_2$. Thus, $\mathrm{Syl}_5(\bar{M})=\{\bar{E}\}$ and $\mathrm{Syl}_3(\bar{M})=\{\bar{B}\}$. Now, as $V=\langle Z^E\rangle$, we have $C_B(Z)=C_G(V)=C_E(Z)$. Therefore, $M\cap C=SC_G(V)$. Define $W=C_V(S\cap D)$. Then $|W|=4$ by 7.2. As $SB\leqq N_M(W)$ and $N_E(W)=C_G(V)$, we have $N_M(W)=SB$.

Claim 1: $\mathcal{M}^*(S)=\{M\}$. Assume that $M\neq L\in\mathcal{M}^*(S)$. As $L=\langle C_L(Z), N_L(Q(K(S)))\rangle$ by 3.1, 7.2 shows that $L=\langle C\cap L, M\cap L\rangle$. As $L\neq C$ by 3.2, we conclude that $M\cap L\not\leqq M\cap C=SC_G(V)$. Therefore, there are $p\in\{3, 5\}$ and $X\in \mathrm{Syl}_p(M\cap L)$ such that $|XC_G(V)/C_G(V)|=p$. Now, $m_G(V)=2$ by 4.3 and so, as $L\neq M$, $m_G(V_L)=1$ by 3.4. Thus, $\bar{D}_L\cong \mathrm{D}_6$ and $|[V_L, E_L]|=4$ by 4.3 and 4.2.

Assume that $p=5$. Then $XC_G(V)=E$, so $V=\langle Z^X\rangle\leqq V_L$ and $V=[V, X]\leqq[V_L, X]$. As $V_L=[V_L, E_L]\times C_{V_L}(E_L)$ with $|[V_L, E_L]|=4$ and $[E_L, X]\leqq C_G(V_L)$, we have $[V_L, X]\leqq C_{V_L}(E_L)$ just as in the proof of 7.1. But then $E_L\leqq C_G([V_L, X])\leqq C_G(V)\leqq M$ and $SD_L=SE_L\leqq M$, which contradicts 7.1.

Assume, therefore, that $p=3$. Then $XC_G(V)=B$ and so, as $C_B(W)\leqq C_B(Z)=C_G(V)$, we have $W=\langle Z^B\rangle=\langle Z^X\rangle$ and $W=[W, B]=[W, X]$. Thus, $W\leqq V_L$ and $W\leqq[V_L, X]\leqq[V_L, E_L]$, so we have $W=[V_L, E_L]$, $XC_G(V_L)=E_L$, and $L=N_G([V_L, E_L])=N_G(W)$. Therefore, $V_L=W$. But then $L=D_L=SXC_G(V_L)\leqq(M\cap L)C_G(V_L)$, which implies that $L\leqq_{(S)}M$. With this contradiction, we have proved Claim 1.

Claim 2: $\mathcal{M}(S)\leqq\{M, C, N_G(W)\}$. Let $K\in\mathcal{M}(S)$. Then $K\leqq_{(S)}M$ by

Claim 1 and so

$$V_K=\langle Z^{M\cap K}\rangle=\langle Z^{(M\cap K)C_G(V)}\rangle$$

by 2.2. If $M\cap K\leqq SC_G(V)$, then $V_K=Z$. If $M\cap K\nleqq SC_G(V)$ and $M\cap K\leqq SB$, then $V_K=\langle Z^B\rangle=W$. If $E\leqq(M\cap K)C_G(V)$, then $V_K=\langle Z^E\rangle=V$. Thus, $K=N_G(V_K)\in\{N_G(Z), N_G(W), M\}$. If $N_G(Z)\neq C$, then $Z=W$. Therefore, we have proved Claim 2.

Hereafter, we will frequently use the following fact. If $S\leqq X\leqq L\in\mathcal{M}(S)$, then $O(X)=1$. This is because $O(X)\leqq C_L(O_2(L))\leqq O_2(L)$.

Claim 3 : $\mathcal{M}(S)=\{M, C\}$. Assume that $N=N_G(W)$ is a member of $\mathcal{M}(S)$ and $N\neq C$. Then $V_N=W$ and $N\neq SC_G(W)$, so $N/C_G(V_N)\cong D_6$. As $N\leqq_{(S)}M$ by Claim 1, 3 divides $|M\cap N|$ and N is solvable (but Sylow 3-subgroups of N may not be cyclic). Recall that $M\cap N=SB$ and so $|M:M\cap N|=5$.

Let P and F be a Sylow 3-subgroup and a Sylow 5-subgroup of M, respectively, chosen so that SP, SF, and PF are all subgroups. As $(|M:M\cap N|, 6)=1$, we can choose P and F so that $P\leqq M\cap N$. Then we can choose $T\in \mathrm{Syl}_3(N)$ so that $P\leqq T$ and $ST=TS$. As P and F are cyclic, $F\lhd PF$ and so $X=\Omega_1(P)$ acts on F by conjugation. As the group of order 15 is cyclic, we conclude that $[F, X]=1$. Now, let Y be a subgroup of $N_T(X)$ permuting with S. Then 5.1 shows that there is an element $K\in\mathcal{M}(S)$ containing F and Y. As F is contained neither in C nor in N, Claim 2 shows $K=M$, and thus $Y\leqq T\cap M=P$.

Assume that 3 divides $|N:M\cap N|$. Then $T\nleqq P$ and so X is not normal in T. In particular, T is nonabelian and, as $C_T(V_N)$ is a cyclic maximal subgroup of T, T has the following presentation

$$\langle x, y \mid x^{3^{m-1}}=y^3=1,\ x^y=x^{1+3^{m-2}}\rangle$$

for some integer $m\geqq 3$ [12, Th. 5.4.4]. Let $T_0=T\cap O_{2,3}(ST)$. Then $\langle x^3\rangle=Z(T)<T_0$ as $C_{ST}(T_0)\leqq O_{2,3}(ST)$. If $Z(T)$ permutes with S, then $Z(T)\leqq P$ and so $X\leqq Z(T)$, which is a contradiction. Thus, $Z(T)$ does not permute with S, which shows that $T_0\neq T$ and T_0 is abelian of rank 2. As $\Omega_1(T)\cong Z_3\times Z_3$, this forces $X\leqq T_0$ and so $T_0\leqq P$, which is a contradiction as P is cyclic. Therefore, $(|N:M\cap N|, 3)=1$.

Now, let p be a prime divisor of $|N:M\cap N|$ and pick a Hall $\{2, p\}$-subgroup P^* of $SC_G(V_N)$ containing S. Let F^* be a Hall $\{2, 5\}$-subgroup of M containing S. As $p\neq 3$ and both a Sylow p-subgroup of P^* and a Sylow 5-subgroup of F^* are cyclic, 5.2 shows that there is an element of $\mathcal{M}(S)$ containing P^* and F^*. This contradicts Claim 2 as $C\ngeqq F^*\nleqq N$ and $P^*\nleqq M$. Thus, $\mathcal{M}(S)\leqq\{M, C\}$. As $C\nleqq M$ by 3.3, we have proved Claim 3.

Now, we conclude the proof of 7.3. As $|M:M\cap C|=5$ or 15 and $\mathcal{M}(S)=\{M, C\}$, the argument in the last paragraph shows that $|C:M\cap C|$ is a power of 3. Assume $|M\cap C:S|$ is divided by $p\in\{3,5\}$ and let X_p be a Hall $\{2,p\}$-subgroup of $M\cap C$ containing S. Let X be a Hall $\{2,5,p\}$-subgroup of M containing X_p, X_5 a Hall $\{2,5\}$-subgroup of X containing S, and $X_{5,p}$ a Hall $\{5,p\}$-subgroup of X. Define $F=X_5\cap X_{5,p}$ and $P=X_p\cap X_{5,p}$. Then $F\in \mathrm{Syl}_5(M)$, $P\in \mathrm{Syl}_p(M\cap C)$, F permutes with S, and F centralizes $P_0=\Omega_1(P)$ as the group of order 15 is cyclic. Let Y be a Hall $\{2,3,p\}$-subgroup of C containing X_p, and Y_3 a Hall $\{2,3\}$-subgroup of Y containing S. Then as $P_0O_2(Y)\lhd Y$ by 6.2, $Y=N_Y(P_0)O_2(Y)$ and so $N_{Y_3}(P_0)$ contains a Sylow 3-subgroup T of C, which necessarily permutes with S. We can now use 5.1 and conclude that F and T are contained in some element K of $\mathcal{M}(S)=\{M, C\}$. However, as $F\nleqq C$ and $T\nleqq M$, we have $M\neq K\neq C$, a contradiction. Therefore, $(|M\cap C:S|, 15)=1$. Now, let X^* be a Hall $\{2,5\}$-subgroup of M containing S, and T^* a Sylow 3-subgroup of C which permutes with S. Then $Y^*=S\Omega_1(T^*)$ is a subgroup by 6.2. As $(|M\cap C:S|, 15)=1$ and $\mathcal{M}(S)=\{M, C\}$, X^* and Y^* satisfy the hypotheses of 5.3 with respect to the common 2-subgroup S. However, $|X^*:S|=5$, $|Y^*:S|=3$, and $Z\leqq Z(Y^*)$, which is impossible by 5.3. With this contradiction, we have established 7.3.

7.4. *The following conditions hold.*
(1) $M/C_G(V)\cong \mathrm{D}_6$.
(2) $M\cap C=SC_G(V)$ *and* $|M:M\cap C|=3$.
(3) $\mathcal{M}(S)=\{M, C\}$.

PROOF. (1) As $\bar{D}\cong \mathrm{D}_6$ by 7.3, $\bar{M}=\bar{D}\times\bar{B}$ and it suffices to prove $\bar{B}=1$. As $|V|\leqq 16$ by 7.2, $\bar{M}$ is isomorphic to a subgroup of $\mathrm{GL}_4(2)$. As the centralizer of every involution in $\mathrm{GL}_4(2)$ is a $\{2,3\}$-group and as Sylow 3-subgroups of $\bar{M}$ are cyclic, $\bar{B}$ is a 2-group. Now, $O_2(\bar{M})=1$ by 2.1 and thus $\bar{B}=1$ as required.

(2) This follows from (1) as $M\nleqq C$ by 3.2 or 3.3.

(3) Let $N\in\mathcal{M}(S)$ and assume $N\nleqq C$. As $N=\langle C\cap N, M\cap N\rangle$ by 3.1 and 7.2, we have $M\cap N\nleqq M\cap C$ and so, as $M\cap C=SC_G(V)$ is maximal in M by (2), we conclude that $M=(M\cap N)C_G(V)$. This shows $M\leqq_{(S)}N$ and so $N=M$ by the maximality of M. As $C\nleqq M$ by 3.3, we have proved $\mathcal{M}(S)=\{M, C\}$.

7.5. *Let p and r be odd prime divisors of $|M\cap C|$ and $|C:M\cap C|$, respectively. Then $p\neq r$ and a Hall $\{p,r\}$-subgroup of C is a Frobenius group whose kernel is a Sylow r-subgroup.*

PROOF. Let X_p be a Hall $\{2, p\}$-subgroup of $M\cap C$ containing S, X a Hall $\{2, p, r\}$-subgroup of C containing X_p, X_r a Hall $\{2, r\}$-subgroup of X containing S, and $X_{p,r}$ a Hall $\{p, r\}$-subgroup of X. Define $P=X_p\cap X_{p,r}$ and $R=X_r\cap X_{p,r}$. Then $P\in \mathrm{Syl}_p(M\cap C)$, $R\in \mathrm{Syl}_r(C)$, and R permutes with S. Recall from 7.4 that $|M:M\cap C|=3$. Let Y be a Hall $\{2, 3, p\}$-subgroup of M containing X_p, and Y_3 a Hall $\{2, 3\}$-subgroup of Y containing S. As $PO_2(Y)\lhd Y$ by 6.2, we have $Y=N_Y(P)O_2(Y)$ and so $N_{Y_3}(P)$ contains a Sylow 3-subgroup T of M, which necessarily permutes with S and normalizes $P_0=\Omega_1(P)$. If R normalizes P_0, then $\langle R, T\rangle$ is contained in some $K\in \mathcal{M}(S)$ by 5.1. However, this contradicts 7.4 as $R\not\leqq M$ and $T\not\leqq C$. Therefore, R does not normalize P_0. Then $p<r$, $R\lhd X_{p,r}$, and P_0 acts nontrivially on $\Omega_1(R)$. This implies that $X_{p,r}$ is a Frobenius group with kernel R.

7.6. $(|C:M\cap C|, 15)\neq 1$.

PROOF. Let $R=O_2(C)$ (we retain this notation in the rest of this section). There are two cases to consider.

Case 1 : $R\not\leqq Q$. Then as $|S:Q|=2$ by 7.4, we have $S=QR$ and so $N_C(Q)\leqq N_G(S)$. Recall from Section 2 that $M=N_G(Q)C_G(V)$. Hence if X is a Hall $\{2, 3\}$-subgroup of $N_G(Q)$ containing S, then $|X:X\cap C|=3$ by 7.4. As $N_G(S)\leqq N_G(Z)=C$ by 7.4, we have $O_2(X)=Q$, while $O_2(X\cap C)=S$ as $N_C(Q)\leqq N_G(S)$. Thus, $X\cap C=S$ and $|X:S|=3$ by 6.2. Now, let r be a prime divisor of $|C:M\cap C|$, and Y_r a Sylow r-subgroup of C permuting with S. Then $Y=S\Omega_1(Y_r)$ is a subgroup by 6.2, and $Y\not\leqq M$ by 7.5. As $\mathcal{M}(S)=\{M, C\}$ by 7.4, no nonidentity subgroup of S is normal both in X and in Y. Thus, we can conclude by 5.3 that $r=3$ or 5, proving 7.6 in Case 1.

Case 2 : $R\leqq Q$. Let $N=\cap(M\cap C)^C$. Suppose an odd prime p divides $|N|$, and let $P\in \mathrm{Syl}_p(N)$. Then $C=N_C(P)N$, and so if r is a prime divisor of $|C:M\cap C|$ then r divides $|N_C(P)|$. As this is impossible by 7.5, N must be a 2-group, which implies that $N=R$.

As $R\leqq Q=C_S(V)$, we have $V\leqq C_C(R)=Z(R)$ and so $U=\langle V^C\rangle$ is contained in $\Omega_1(Z(R))$. As $R\leqq C_C(U)\leqq C_C(V)\leqq M\cap C$, the remark in the last paragraph shows that $C_C(U)=R$. Therefore, $m_G(U)\leqq 2$ by 2.5. Then by 4.3, C/R has a normal subgroup X/R isomorphic to D_6 or D_{10}. As $X\not\leqq M\cap C$ by the last paragraph, we conclude that $|X(M\cap C):M\cap C|=3$ or 5, proving 7.6 in Case 2.

7.7. $M\cap C=S$.

PROOF. By 7.5 and 7.6, any prime divisor p of $|M\cap C:S|$ must divide $3-1$ or $5-1$. Therefore, $M\cap C=S$.

7.8. *One of the following holds.*
(1) $|M:S|=|C:S|=3$.
(2) $|M:S|=3$ *and* $|C:S|=5$.

PROOF. First of all, 7.4 and 7.7 show that $|M:S|=3$. Let r be a prime divisor of $|C:S|$ and pick a Sylow r-subgroup X_r of C permuting with S. Define $X=\Omega_1(X_r)S$. Then X is a subgroup by 6.2, and $X\not\leq M$ by 7.5 or 7.7. So, as $M\in\mathcal{M}(S)$, no nonidentity subgroup of S is normal both in M and in X, and we can appeal to 5.3. Let $S^*=(S\cap O^2(M))(S\cap O^2(X))$, $M^*=O^2(M)S^*$, and $X^*=O^2(X)S^*$. Then (M^*, X^*) is of $\mathrm{G}_2(2)'$-type or M_{12}-type or ${}^2\mathrm{F}_4(2)'$-type, and consequently, $r=3$ or 5. If $r=3$ then the number n of the noncentral 2-chief factors of M is clearly equal to 2, while if $r=5$ then $n=4$ by 5.4. This implies that $|C:S|$ is a power of r, so $O_2(X)=R$ $(=O_2(C))$ and $\Omega_1(X_r)R\lhd C$ by 6.2. Now, as $X^*\lhd X$, we may deduce as follows:

$$[R, O^2(X)]\leq R\cap X^*=O_2(X^*),$$
$$[R, O^2(X)]=[R, O^2(X), O^2(X)]=[O_2(X^*), O^2(X)],$$
$$[O_2(X^*), O^2(X^*)]=[R, O^2(X)]=[R, O^2(\Omega_1(X_r)R)].$$

This shows that $Y=[O_2(X^*), O^2(X^*)]$ is normal in C. Now, $|Y:Y^2|\leq 16$ by 5.5. Therefore, Y does not admit a nontrivial automorphism of order r^2, and it follows that $|C:S|=r$.

7.9. $(S\cap O^2(M))(S\cap O^2(C))=S$.

PROOF. Suppose false. Then M and C, respectively, have subgroups M_0 and C_0 of index 2 such that $S\cap M_0=S\cap C_0$. Now, $N_G(S)\leq M\cap N_G(Z)$ by 7.2 and, in particular, M and C are the only elements of $M^G\cup C^G$ that contain S. Therefore, M and C control fusion in S (in the usual sense) by 3.3, and so $S\cap G'\leq S\cap M_0$ by the focal subgroup theorem. This contradicts the simplicity of G.

7.10. $G\cong \mathrm{G}_2(2)'$ *or* ${}^2\mathrm{F}_4(2)'$.

PROOF. First of all, no nonidentity subgroup of S is normal in M and C as $C\not\leq M$. So by 7.8, 7.9, and 5.3, (M, C) is of $\mathrm{G}_2(2)'$-type or M_{12}-type or ${}^2\mathrm{F}_4(2)'$-type.

Case 1: (M, C) is of $\mathrm{G}_2(2)'$-type. Then by 5.4, G satisfies the hypothesis of Fong's theorem [13, Th. 2], and therefore $G\cong\mathrm{G}_2(2)'$.

Case 2: (M, C) is of M_{12}-type. Then S has the presentation given in 5.4 and, in particular, S contains a self-centralizing cyclic subgroup Y of order 8 all of whose generators are conjugate in S (let, say, $Y=\langle cd\rangle$ in the notation of 5.4). Also, S has a noncentral involution x such that $C_S(x)\not\leqq Q^c$ for any $c\in C$. We wish to show that x does not fuse to the central involution of S. Suppose it does, and let $C_S(x)\leqq T\in \mathrm{Syl}_2(C_G(x))$. Then $T\in \mathrm{Syl}_2(G)$ and $S\cap T=C_S(x)$. As $M^G\cup C^G$ controls Sylow 2-intersections in G by 3.3, there are Sylow 2-subgroups $S_0, S_1, \cdots, S_n$ of G and elements $X_1, \cdots, X_n$ of $M^G\cup C^G$ such that $S_0=S$, $S_n=T$, $\langle S_{i-1}, S_i\rangle\leqq X_i$ $(1\leqq i\leqq n)$, and $C_S(x)\leqq S_i$ $(1\leqq i\leqq n)$. We may assume $S_{i-1}\neq S_i$ and $X_{i-1}\neq X_i$ for each i. Thus, $S_{i-1}\cap S_i=O_2(X_i)$. As $C_S(x)\not\leqq Q$, we have $X_1=C$ and so $C_S(x)<R$. Hence, $n\geqq 2$ and if $S_1=S^c$ $(c\in C)$ then $X_2=M^c$. However, this shows $C_S(x)\leqq Q^c$, a contradiction. So x does not fuse to the central involution of S, as desired. Now, we appeal to a theorem of Brauer and Fong [14] and conclude that $G\cong \mathrm{M}_{12}$ (we can alternatively appeal to a theorem of W. J. Wong [19] after determining the structure of C and showing that G has precisely two classes of involutions). However, M_{12} is eliminated because it is not of characteristic 2 type.

Case 3 : (M, C) is of ${}^2\mathrm{F}_4(2)'$-type. Then by 5.4, G satisfies the hypothesis of Parrott's theorem [15], and therefore $G\cong {}^2\mathrm{F}_4(2)'$.

7.10 completes the proof of the main theorem of this paper.

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Added in proof.

Recently, it has been shown that the method of [9] yields a result stronger than is stated in Section 5. By the result, we can significantly improve Section 7; we can avoid using Glauberman's theorem [8] and shorten the proofs of 7.3 and 7.5-7.8. For details, the reader is referred to

Gomi, K. and Y. Tanaka, On pairs of groups having a common 2-subgroup of odd indices, to appear in Sci. Papers College Arts and Sciences Univ. Tokyo.