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# $S^1$-actions on unitary manifolds and quasi-ample line bundles

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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## § 1. Introduction.

Let $M$ be a compact connected smooth manifold. It is known that there are certain constraints on the topological or differential structure of $M$ if $M$ admits a non-trivial action of a compact connected Lie group. Some of these constraints have similar feature to those arising from the positivity of curvature in a vague sense; cf. [2, 6, 16]. In this paper we shall pursue investigations in this direction.

We restrict our attention to actions of the circle group $S^1$ on $M$ with non-empty fixed point set consisting of only isolated points. A complex line bundle $\xi$ on $M$ with a compatible $S^1$-action will be called fine if the restrictions of $\xi$ at the fixed points are mutually distinct $S^1$-modules. It will be called quasi-ample if it is fine and $c_1(\xi)^n \neq 0$ where $\dim M = 2n$. Quasi-ample line bundle may be regarded as an analogue of ample line bundle in algebraic geometry. We are mainly interested in the case where $M$ is an almost complex manifold such that $c_1(M) = k_0 c_1(\xi)$ where $k_0 > 0$ and $\xi$ is quasi-ample. It will be shown in Theorem 5.1 that the inequalities $k_0 \leqq n+1 \leqq \chi(M)$ hold where $\chi(M)$ denotes the Euler characteristic of $M$. The extreme case $k_0 = n+1 = \chi(M)$ is most interesting. In this case it is shown in Theorem 5.7 that the action resembles linear action on complex projective space. This theorem and Corollary 5.9 can be regarded as an analogue of Kobayashi-Ochiai's theorem in [14]. In a parallel direction with Kobayashi-Ochiai's theorem the case $k_0 = n$ is also treated in Theorem 6.1, Corollary 6.3, Theorem 6.17 and Corollary 6.30.

These main theorems are derived in part from Theorem 4.2 which exhibits certain relations between the tangential representations and the restrictions of $\xi$ at the fixed points. It is formulated for unitary manifolds and it can partly be generalized to Spin$^c$ manifolds; cf.

Theorem 7.3.2. It is particularly useful in the case $n+1=\chi(M)$; we shall return to this subject in a subsequent paper.

The paper is organized as follows. In §2 we recollect basic materials concerning unitary $S^1$-manifolds. Proposition 2.11 is important for applications to almost complex manifolds. In §3 we introduce fine and quasi-ample line bundles and discuss their main properties. Special feature of the case $n+1=\chi(M)$ is stated in Corollary 3.15 which is a consequence of Proposition 3.14 due to Masuda. In §4 we prove Theorem 4.2. §5 and §6 are devoted to the study of almost complex manifolds with $k_0=n+1=\chi(M)$ and $k_0=n$ $(\chi(M)\leqq n+2)$. In §7 various remarks are added.

Results of the present paper were summarized in [9].

## §2. Preliminary results on unitary $S^1$-manifolds.

This section is devoted to recalling basic materials needed later and stating some preliminary results which emerge more or less directly from those materials. Since we are mainly interested in unitary manifolds and especially in almost complex manifolds throughout this paper we shall give statements only for unitary manifolds. Modifications needed for Spin$^c$-manifolds will be summarized in §7.

Let $M$ be a connected closed smooth $S^1$-manifold which has only isolated fixed points. It is well known that the Euler characteristic of $M$, denoted by $\chi(M)$, equals the number of fixed points. We shall assume that there is at least a fixed point and hence $\chi(M)>0$. Then the dimension of $M$ is necessarily even, for the tangent space at a fixed point has a complex structure determined by the isotropy representation of $S^1$. Note that the above complex structure is not canonical. However it is the case if $M$ is unitary $S^1$-manifold. A $C^\infty$ manifold endowed with a complex vector bundle structure on the stable tangent bundle (respectively on the tangent bundle) is called unitary (respectively almost complex) manifold. If a Lie group $G$ acts smoothly on a unitary (respectively almost complex) manifold $M$ and if the differential of each element of $G$ preserves the given complex vector bundle structure then $M$ will be called unitary (respectively almost complex) $G$-manifold. If $M$ is a unitary $S^1$-manifold and $P$ is an isolated fixed point then, by definition, $S^1$ acts linearly on the complex vector space $W=T_PM\oplus \boldsymbol{R}^l$ for some non-negative integer $l$ and $\boldsymbol{R}^l$ coincides with the complex linear subspace of fixed vectors. Thus $W/\boldsymbol{R}^l$ is a complex $S^1$-module without trivial factor and $W/\boldsymbol{R}^l$ is isomorphic to $T_PM$ as real vector space. This defines a canonical complex structure on the tangent space $T_PM$.

A unitary manifold $M$ is oriented in the following way. Let $\tau(M)$ denote the tangent bundle and suppose there is given a complex vector bundle structure on $\tau(M)\oplus l\theta$ where $l\theta$ denotes the product bundle $M\times\boldsymbol{R}^l$. Then $\tau(M)\oplus l\theta$ is oriented as complex vector bundle and $l\theta$ is also oriented in the usual way. These orientation induces an orientation of $\tau(M)$. Now suppose that $M$ is a unitary $S^1$-manifold and $P$ is an isolated fixed point. The tangent space $T_PM$ has two orientations; namely one induced from the orientation of $M$ and the other induced by the complex structure. We shall set

$$\varepsilon(P)=+1 \quad \text{or} \quad -1$$

according as these two orientations coincide or not. The following lemma is obvious.

LEMMA 2.1. *If $M$ is an almost complex $S^1$-manifold and $P$ is an isolated fixed point, then we have*

$$\varepsilon(P)=+1.$$

One of our main tools is the localization theorem for $K_G$-theory. Since we deals mainly with $S^1$-actions having only isolated fixed points on unitary manifolds we shall formulate it only in that case. The character ring $R(S^1)$ of the group $S^1$ is identified with the Laurent polynomial ring $\boldsymbol{Z}[t, t^{-1}]$ where $t$ denotes the standard 1-dimensional $S^1$-module. Let $S$ denote the multiplicatively closed subset consisting of the elements of the form

$$\prod(1-t^{m_j})$$

where the product is a finite product taken over a set of non-zero integers $m_j$. If $X$ is an $S^1$-space then $K_{S^1}(X)$ is an $R(S^1)$-module. Therefore we can consider the localization of $K_{S^1}(X)$ by $S$ which we shall denote by $S^{-1}K_{S^1}(X)$. Note that $R(S^1)$ is canonically embedded in $S^{-1}R(S^1)$.

Let $M$ be a connected closed unitary $S^1$-manifold whose fixed points are all isolated and let $P_1, \cdots, P_\chi$ $(\chi=\chi(M))$ be the fixed points. Then, as noted before, the tangent space $T_{P_i}M$ at $P_i$ is a complex $S^1$-module without trivial factor so that it can be expressed in the form

$$T_{P_i}M=\sum_{k=1}^{n} t^{m_{ik}}\ , \quad \dim M=2n\ ,$$

where $m_{ik}$ are non-zero integers. These $m_{ik}$ will be called the weights of $M$ at $P_i$. We formulate the localization theorem in the following way.

THEOREM 2.2 ([3, 7]). *Let $M$ be as above and let $f_i$ denote the inclusion of $P_i$ into $M$. Then*

$$\sum_i f_i^*\colon S^{-1}K_{S^1}(M) \longrightarrow \sum_i S^{-1}K_{S^1}(P_i)$$

*is an isomorphism and the inverse is given by*

$$\sum_i v_i \longrightarrow \sum_i \frac{f_{i!}(v_i)}{\prod_k (1-t^{-m_{ik}})},$$

where $f_{i!}$ is the Gysin homomorphism of $f_i$.

*Note.* We define the $K$-theory Euler class of a complex line bundle $\xi$ to be $1-\xi^{-1}$. The Thom isomorphism and the Gysin homomorphism are defined accordingly. This convention has the advantage that $p_!(1)$ evaluated at $t=1$ equals the Todd genus $T[M]$ of $M$ where $p\colon M\to P$ (a point); cf. Corollary 2.3 and [4].

For a point $P$ we have a canonical identification $K_{S^1}(P)=R(S^1)$. From the definition of the sign $\varepsilon_i=\varepsilon(P_i)$ it is easy to see that

$$p_!\circ f_{i!}\colon R(S^1)=K_{S^1}(P_i) \longrightarrow K_{S^1}(P)=R(S^1)$$

coincides with the multiplication by $\varepsilon_i$. Hence we obtain

COROLLARY 2.3. *Let $M$ be as above. Then, for $v\in K_{S^1}(M)$, the value $p_!(v)\in R(S^1)\subset S^{-1}R(S^1)$ can be expressed as*

$$p_!(v)=\sum_i \frac{\varepsilon_i f_i^*(v)}{\prod_k (1-t^{-m_{ik}})}.$$

*Note.* In the almost complex case, all signs $\varepsilon_i$ are equal to $+1$.

We apply Corollary 2.3 to elements in $K_{S^1}(M)$ arising from the tangent bundle. Let $\gamma_\lambda\colon K_G(X)\to K_G(X)$ be the operation introduced by Grothendieck (cf. [1]) where $\lambda$ is an indeterminate. If $\xi$ is a $G$ complex line bundle then

$$\gamma_\lambda(\xi-1)=1+(\xi-1)\lambda\ .$$

Let $M$ be as before together with a complex vector bundle structure on $\tau(M)\oplus l\theta$. If we put

$$v=\gamma_\lambda\Big((\tau(M)\oplus l\theta)^*-\frac{1}{2}(2n+l)\Big)\in K_{S^1}(M) \tag{2.4}$$

then we have

$$\begin{aligned} f_i^*(v)&=\gamma_\lambda\Big(\sum_k (t^{-m_{ik}}-1)\Big) \\ &=\prod_k (1-(1-t^{-m_{ik}})\lambda). \end{aligned} \tag{2.5}$$

We set

$$p_i=\text{the number of } \{k;\ m_{ik}>0\}$$

for $1\leqq i\leqq\chi$ and, in the almost complex case,

$$\rho_q=\text{the number of } \{i;\ p_i=q\}$$

for $0\leqq q\leqq n$.

PROPOSITION 2.6. *Let $M$ be as above. Then the following relation holds:*

$$\begin{aligned} \sum_{i=1}^{\chi}\varepsilon_i\prod_{k=1}^{n}\Big(\frac{1}{1-t^{m_{ik}}}-\lambda\Big)&=\sum_{i=1}^{\chi}\varepsilon_i(1-\lambda)^{p_i}(-\lambda)^{n-p_i} \\ &=\sum_{i=1}^{\chi}\varepsilon_i(1-\lambda)^{n-p_i}(-\lambda)^{p_i}. \end{aligned}$$

*In particular, in the almost complex case,*

$$\sum_{i=1}^{\chi}\prod_{k=1}^{n}\Big(\frac{1}{1-t^{m_{ik}}}-\lambda\Big)=\sum_{q=0}^{n}\rho_q(1-\lambda)^q(-\lambda)^{n-q},$$

*and moreover we have $\rho_{n-q}=\rho_q$.*

PROOF. We apply Corollary 2.3 to $v$ given in (2.4). Using (2.5) we have

$$\sum_{i=1}^{\chi}\varepsilon_i\prod_k\Big(\frac{1}{1-t^{-m_{ik}}}-\lambda\Big)=p_!(v)\in R(S^1)=\boldsymbol{Z}[t,\ t^{-1}].$$

We regard each summand $h_i(t)=\prod_k(1/(1-t^{-m_{ik}})-\lambda)$ as a rational function of $t$. It is easy to see that at $t=\infty$ and $t=0$ $h_i(t)$ takes finite values

$$h_i(\infty)=(1-\lambda)^{p_i}(-\lambda)^{n-p_i}$$
$$h_i(0)=(1-\lambda)^{n-p_i}(-\lambda)^{p_i}.$$

Thus the sum $\sum \varepsilon_i h_i(t)$ takes finite values

$$\sum \varepsilon_i(1-\lambda)^{p_i}(-\lambda)^{n-p_i} \quad \text{at} \quad t=\infty$$
$$\sum \varepsilon_i(1-\lambda)^{n-p_i}(-\lambda)^{p_i} \quad \text{at} \quad t=0.$$

But $\sum \varepsilon_i h_i(t)=p_1(v)$ is a Laurent polynomial in $t$ which takes finite values at $t=\infty$ and $t=0$. Therefore it must be a constant and the values at $t=\infty$ and at $t=0$ must coincide. Moreover we can substitute $t^{-1}$ for $t$ to get the same constant. This proves the first part.

In the almost complex case all sign $\varepsilon_i$ are equal to $+1$. Thus the constant takes the form

$$\sum \rho_q(1-\lambda)^q(-\lambda)^{n-q}=\sum \rho_q(1-\lambda)^{n-q}(-\lambda)^q.$$

From this it follows also that $\rho_{n-q}=\rho_q$. Q.E.D.

Comparing the coefficients of $\lambda^j$ in the relation in Proposition 2.6 we can get identities involving elementary symmetric functions of $(1-t^{m_{i1}})^{-1}, \cdots, (1-t^{m_{in}})^{-1}$. In particular we have

COROLLARY 2.7. *The following relations hold*:

(2.8) $$\sum_i \frac{\varepsilon_i}{\prod_k (1-t^{m_{ik}})}=\sum \varepsilon_i \binom{p_i}{n}=\sum \varepsilon_i \binom{n-p_i}{n},$$

(2.9) $$\sum_{i,k} \frac{\varepsilon_i}{1-t^{m_{ik}}}=\sum \varepsilon_i p_i=\sum \varepsilon_i(n-p_i)=\frac{(\sum \varepsilon_i)n}{2}.$$

*In the almost complex case these reduce to*

(2.8)′ $$\sum_i \frac{1}{\prod (1-t^{m_{ik}})}=\rho_n=\rho_0$$

*and*

(2.9)′ $$\sum_{i,k} \frac{1}{1-t^{m_{ik}}}=\sum_q \rho_q q=\frac{n\chi}{2}.$$

REMARK 2.10. (2.8) is the constant term in the expression of Proposition 2.7 and hence is equal to $p_t(1)$. It follows from Note after Theorem 2.2 that

$$T[M]=\sum \varepsilon_i \binom{p_i}{n} = \sum \varepsilon_i \binom{n-p_i}{n}.$$

In the almost complex case we have

$$T[M]=\rho_n=\rho_0.$$

In particular, $T[M]$ is non-negative in the almost complex case. The formula (2.8) is due to Kosniowski [15] in the holomorphic case; cf. also [10].

PROPOSITION 2.11. *Let $M$ be an almost complex $S^1$-manifold. Suppose that the fixed points are all isolated. If $\{m_{ik}\}$ are the weights of $M$ at the fixed point $P_i$, then for each non-zero integer $m$ the number of $(i, k)$ such that $m_{ik}=m$ and the number of $(i, k)$ such that $m_{ik}=-m$ are equal. In particular we have*

$$\sum_{i,k} m_{ik}=0$$

*and $n\chi$ must be even.*

PROOF. This follows easily from (2.9)$'$ by looking at the poles and their multiplicities of the left hand side. Q.E.D.

By a similar argument we obtain the following

PROPOSITION 2.12. *Let $M$ be a unitary $S^1$-manifold having only isolated fixed points. Then for each positive integer $m$ the number of $(i, k)$ such that $|m_{ik}|=m$ is even. In particular*

$$\sum_{i,k} m_{ik}\equiv 0 \quad mod\ 2.$$

Finally by the Atiyah-Singer theorem [4] we have the following expression of the signature of $M$.

PROPOSITION 2.13. *Let $M$ be a unitary $S^1$-manifold with only isolated fixed points. Then*

$$\text{Sign } M=\sum_i \varepsilon_i \prod_k \frac{1+t^{m_{ik}}}{1-t^{m_{ik}}}.$$

## § 3. Quasi-ample line bundles.

In this section $M$ will always denote a connected closed smooth $S^1$-manifold which has only isolated fixed points. We assume that $\chi(M)>0$ as in §2.

Let $\xi$ be a complex line bundle over $M$. $\xi$ will be called admissible if the $S^1$ action on $M$ can be lifted to an action on $\xi$ making $\xi$ an $S^1$ complex line bundle. It is known [11] that $\xi$ is admissible if and only if its first Chern class $c_1(\xi)$ lies in the image of the natural homomorphism

$$H^2_{S^1}(M; \boldsymbol{Z}) \longrightarrow H^2(M; \boldsymbol{Z}).$$

We shall denote by $\mathrm{ad}\, H^2(M)$ the image of the homomorphism and call its elements also admissible. Let $ES^1$ be the universal $S^1$ bundle. Then, the fibration

$$M \xrightarrow{j} M_{S^1} \longrightarrow BS^1,$$

where $M_{S^1}=ES^1\times_{S^1}M$, induces an exact sequence

$$0 \longrightarrow H^2(BS^1; \boldsymbol{Z}) \longrightarrow H^2_{S^1}(M; \boldsymbol{Z}) \longrightarrow \mathrm{ad}\, H^2(M) \longrightarrow 0. \tag{3.1}$$

This follows from the Serre spectral sequence of the fibration and the fact that the fibration admits a cross-section since the action has a fixed point. The exact sequence (3.1) means that the liftings of the given $S^1$ action on $M$ to an admissible line bundle $\xi$ are parametrized by $H^2(BS^1; \boldsymbol{Z})\cong \boldsymbol{Z}$, cf. [11].

Note that if $M'$ is an $S^1$-invariant submanifold, then, for any admissible line bundle $\xi$, its restriction $\xi|M'$ to $M'$ is also admissible. It is also easy to see that if $M$ is a unitary $S^1$-manifold with only isolated fixed points then $c_1(M)\in H^2(M; \boldsymbol{Z})$ is admissible.

LEMMA 3.2. *Let $x\in \mathrm{ad}\, H^2(M)$ and $x'\in H^2(M; \boldsymbol{Z})$ be such that $x=kx'$ for some non-zero integer $k$. Then $x'$ also belongs to* $\mathrm{ad}\, H^2(M)$. *In particular the torsion subgroup of $H^2(M; \boldsymbol{Z})$ is contained in* $\mathrm{ad}\, H^2(M)$.

PROOF. This immediately follows from the fact that the module $\mathrm{ad}\, H^2(M)$ coincides with the kernel of

$$d_2\colon H^2(M; \boldsymbol{Z}) \longrightarrow H^2(BS^1; H^1(M; \boldsymbol{Z}))\cong H^1(M; \boldsymbol{Z})$$

and the module $H^1(M; \boldsymbol{Z})$ is torsion-free. Q.E.D.

COROLLARY 3.3. *Assume $M$ is a unitary $S^1$-manifold. If $x \in H^2(M; \boldsymbol{Z})$ satisfies the relation*

$$c_1(M)=kx \quad mod\ torsion$$

*for some integer $k \neq 0$, then $x \in \mathrm{ad}\, H^2(M)$.*

Let $\{P_i\}_{i=1}^{\chi}$, $\chi=\chi(M)$, be the fixed points. If $\eta$ is an $S^1$ complex line bundle whose underlying bundle is $\xi$, the restriction of $\eta$ at each point $P_i$ determines a complex $S^1$-module which is of the form

$$\eta|P_i=t^{a_i}, \quad a_i \in \boldsymbol{Z}$$

where $t$ is the standard 1-dimensional $S^1$-module. The integer $a_i$ will be called the weight of $\xi$ at $P_i$. If $\eta'$ also corresponds to $\xi$, then by (3.1) we see that

$$\eta|P_i=t^{a_i+a}$$

for some integer $a$. Thus, strictly speaking, the set of weights $\{a_i\}$ of $\xi$ is determined only up to addition of simultaneous constant $a$.

PROPOSITION 3.4. *Let $M$ be an $2n$-dimensional unitary $S^1$-manifold having only isolated fixed points and $\xi$ an admissible line bundle over $M$. We take $\eta$ and $\{a_i\}$ as above. Then, we have*

$$\sum_i \frac{\varepsilon_i t^{la_i}}{\prod (1-t^{-m_{ik}})} \in \boldsymbol{Z}[t, t^{-1}]$$

*for any integer $l$, where $\{m_{ik}\}_{k=1}^{n}$ are the weights of $M$ at the fixed points $P_i$ for each $i$.*

PROOF. $\eta$ belongs to $K_G(M)$. We apply Corollary 2.3 to $\eta^l$ and get

$$\sum_i \frac{\varepsilon_i t^{la_i}}{\prod (1-t^{-m_{ik}})}=p_!(\eta^l) \in R(S^1)=\boldsymbol{Z}[t, t^{-1}]. \qquad \text{Q.E.D.}$$

PROPOSITION 3.5. *Let $M$ and $\xi$ be as in Proposition 3.4. If the weights $\{a_i\}$ of $\xi$ are mutually distinct then the inequality $n+1 \leqq \chi(M)$ must hold.*

PROOF. Take any fixed point $P_i$ and consider the element

$$v=\prod_{j \neq i} (1-\eta^{-1}t^{a_j}) \in K_{S^1}(M).$$

Then, $f_j^*(v)=0$ for all $j$ other than $i$. Hence

$$\varepsilon_i \frac{\prod_{j\neq i}(1-t^{-a_i+a_j})}{\prod_k (1-t^{-m_{ik}})} = f_!(v) \in \boldsymbol{Z}[t, t^{-1}].$$

But this implies the number of factors in the numerator is not less than that of the denominator. Hence $\chi-1\geqq n$. Q.E.D.

LEMMA 3.6. *Let $M$ and $\xi$ be as in Proposition 3.4. Then, for any sequence of $s$ integers $c_1, \cdots, c_s$ with $s\geqq n$, we have*

$$p_!\left(\prod_{i=1}^{s}(1-\eta^{-1}t^{c_i})\right)\Big|_{t=1} = \begin{cases} x^n[M] & \text{if } s=n \\ 0 & \text{if } s>n \end{cases}$$

*where $x=c_1(\xi)\in H^2(M; \boldsymbol{Z})$.*

PROOF. We consider the commutative diagram

$$\begin{array}{ccc} K_{S^1}(M) & \xrightarrow{p_!} & \boldsymbol{Z}[t, t^{-1}] \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\ K(M) & \xrightarrow{p_!} & \boldsymbol{Z} \end{array}$$

where $r$ and $r'$ are induced by the inclusion $\{e\}\to S^1$ and $r'$ coincides with the evaluation at $t=1$. Using this and the Atiyah-Singer formula it follows that

$$\begin{aligned} p_!(\prod(1-\eta^{-1}t^{c_i}))|_{t=1} &= p_!(r\prod(1-\eta^{-1}t^{c_i})) \\ &= p_!(1-\xi^{-1})^s \\ &= \mathcal{T}(M)ch(1-\xi^{-1})^s[M] \end{aligned}$$

where $\mathcal{T}(M)$ is the Todd class of $M$. But the lowest term of $\mathcal{T}(M)ch(1-\xi^{-1})^s$ equals $x^s$ and gives $x^n[M]$ or 0 according to $s=n$ or $s>n$ if evaluated on the fundamental class $[M]$. Q.E.D.

Using the set of weights $\{a_i\}$ of $\xi$ we put

$$\varphi_i(t) = \varepsilon_i \frac{\prod_{j\neq i}(1-t^{a_i-a_j})}{\prod_k (1-t^{m_{ik}})}$$

for each $i$. Since $\varphi_i(t^{-1})=p_!(\prod_{j\neq i}(1-\eta^{-1}t^{a_j}))$, $\varphi_i(t)$ belongs to $\boldsymbol{Z}[t, t^{-1}]$. These functions will play an important role in the sequel.

PROPOSITION 3.7. *If* $n+1=\chi(M)$, *then*

$$\varphi_i(1)=\varepsilon_i\frac{\prod_{j\neq i}(a_i-a_j)}{\prod_k m_{ik}}=x^n[M]$$

*for all* $i$.

PROOF. Since

$$\varphi_i(t^{-1})=p_!\Big(\prod_{j\neq i}(1-\eta^{-1}t^{a_j})\Big)$$

we have

$$\varphi_i(1)=x^n[M]$$

by Lemma 3.6. On the other hand, it is easy to see that

$$\varphi_i(1)=\varepsilon_i\frac{\prod_{j\neq i}(a_i-a_j)}{\prod_k m_{ik}}.$$

COROLLARY 3.8. *If* $x^n\neq 0$ *then* $n+1\leqq\chi$. *Moreover, under the assumption* $n+1=\chi$, *the condition* $x^n\neq 0$ *holds if and only if the* $a_i$ *are mutually distinct.*

PROOF. Assume that $x^n\neq 0$ and $\chi<n+1$. We shall show the $a_i$ are mutually distinct; this contradicts Proposition 3.5. Clearly we may assume $2\leqq\chi$. Taking two distinct fixed points $P_i$ and $P_{i'}$ we apply Lemma 3.6 to

$$f(t)=p_!\Big((1-\eta^{-1}t^{a_{i'}})^{n-(\chi-1)}\prod_{j\neq i}(1-\eta^{-1}t^{a_j})\Big)$$

and get $f(1)=x^n[M]\neq 0$. But

$$f(t)=\varepsilon_i\frac{(1-t^{a_{i'}-a_i})^{n-(\chi-1)}\prod_{j\neq i}(1-t^{a_j-a_i})}{\prod_k(1-t^{-m_{ik}})}.$$

Hence $a_i$ must be different from other $a_j$. This proves the first statement. The second statement follows easily from Proposition 3.7.

Q.E.D.

DEFINITION 3.9. Let $M$ be a connected closed smooth $S^1$-manifold of dimension $2n$ with non-empty fixed point set consisting of only isolated points. An admissible complex line bundle $\xi$ over $M$ will be called fine if the weights $a_i$ are mutually distinct. It will be called quasi-ample if it is fine and $c_1(\xi)^n \neq 0$. Note that if $M$ is unitary $S^1$-manifold and $n+1=\chi(M)$ then a fine line bundle is necessarily quasi-ample by Corollary 3.8.

A typical example of quasi-ample line bundle is the hyperplane bundle (dual bundle of the Hopf bundle) over $n$-dimensional complex projective space $\boldsymbol{CP}^n$ with a linear action having only isolated fixed points. In fact, if $b_0, \cdots, b_n$ are mutually distinct integers then the $S^1$-action on $\boldsymbol{CP}^n$ given by

$$z[z_0, z_1, \cdots, z_n]=[z^{b_0}z_0, \cdots, z^{b_n}z_n], \quad z \in S^1 \subset \boldsymbol{C} \tag{3.10}$$

has isolated fixed points

$$P_i=[0, \cdots, 0, 1, 0, \cdots, 0] \quad (1 \text{ at } i\text{-th factor; } 0 \leqq i \leqq n).$$

Moreover if $\xi$ is the hyperplane bundle then its point (of the total space) is represented as $[z_0, \cdots, z_n, w]$ with relation

$$[\alpha z_0, \cdots, \alpha z_n, \alpha w]=[z_0, \cdots, z_n, w] \qquad \alpha \neq 0.$$

Thus we can lift the action to $\xi$ by the formula

$$z[z_0, \cdots, z_n, w]=[z^{b_0}z_0, \cdots, z^{b_n}z_n, w]$$

and with this lifting the weight of $\xi$ at $P_i$ is $a_i=-b_i$.

We also note here that the weights of $\boldsymbol{CP}^n$ with the above $S^1$-action at $P_i$ are given by

$$\{m_{ik}\}=\{a_i-a_j\}_{j\neq i}. \tag{3.11}$$

PROPOSITION 3.12. *Let $M$ be a smooth $S^1$-manifold as in Definition 3.9 and let $\xi$ be a fine line bundle over $M$. Then there is a canonical ring isomorphism*

$$S^{-1}K_{S^1}(M) \cong S^{-1}R(S^1)[\eta]/\prod_{j=1}^{\chi}(\eta t^{-a_j}-1)$$

*where $\eta$ is an $S^1$-line bundle obtained by lifting the action of $S^1$ on $\xi$ and*

$$\eta|P_i=t^{a_i}.$$

PROOF. Let $f_i\colon P_i \to M$ denote the inclusion. Since $f_i^*(\prod_j(\eta t^{-a_j}-1))=$

0 for all $i$ and $\sum f_i^*$ is an isomorphism by the localization theorem, we get a commutative diagram

$$\begin{array}{ccc} A=S^{-1}R(S^1)[\eta]/\prod(\eta t^{-a_j}-1) & \longrightarrow & S^{-1}K_{S^1}(M) \\ & \searrow^{\varphi} & \cong \Big\downarrow \Sigma f_i^* \\ & & \sum_i S^{-1}R(S^1). \end{array}$$

It suffices to prove $\varphi$ is an isomorphism. For that purpose we set

$$e_i=\frac{\prod_{j\neq i}(1-\eta t^{-a_j})}{\prod_{j\neq i}(1-t^{a_i-a_j})}\in S^{-1}R(S^1)[\eta]$$

and define a homomorphism $\psi\colon \sum_i S^{-1}R(S^1)\to A$ by

$$\psi((v_i))=\sum v_i e_i\ .$$

Using the fact $f_i^*(e_j)=\delta_{ij}$ we see easily that $\varphi\circ\psi$ is the identity map. In particular $\psi$ is monic.

On the other hand, any element $u\in A$ can be written uniquely as a polynomial $h(\eta)$ of degree $\chi-1$ with coefficients in $S^{-1}R(S^1)$. This polynomial $h(\eta)$ is characterized by

$$h(t^{a_i})=v_i,\quad i=1,\cdots,\chi.$$

But $\psi((v_i))$ is such a polynomial and hence must coincides with $h(\eta)$. This proves $\psi$ is epic. Thus $\psi$ and consequently $\varphi$ are isomorphisms. Q.E.D.

Let $S$ denote the set of elements in $H_{S^1}^*(pt;\boldsymbol{Z})$ of the form $m\alpha^\gamma$ with $m\neq 0$ and $\gamma\geqq 1$ and let $S^{-1}H_{S^1}^*(pt;\boldsymbol{Z})$ denote the localization. Then by a similar argument we obtain the following.

PROPOSITION 3.13. *Under the same assumption as in Proposition* 3.12 *there is a canonical isomorphism*

$$S^{-1}H_{S^1}^*(M;\boldsymbol{Z})\cong S^{-1}H_{S^1}^*(pt;\boldsymbol{Z})[y]/\prod(y-a_j\alpha)$$

*where* $y=c_1(\eta)\in H_{S^1}^2(M;\boldsymbol{Z})$ *and* $\alpha=c_1(t)\in H_{S^1}^2(pt;\boldsymbol{Z})=H^2(BS^1;\boldsymbol{Z})$.

*Note.* $y$ maps into $x=c_1(\xi)$ under the canonical homomorphism; cf. (3.1).

If $n+1=\chi$ there is a more precise result about $H_{S^1}^*(M;\boldsymbol{Z})$ due to Masuda [17].

PROPOSITION 3.14. *Let $M$ and $\xi$ be as above. We assume moreover $n+1=\chi$. Then there is a canonical ring isomorphism*

$$H_{S^1}^*(M;\boldsymbol{Q})/H_{S^1}^*(pt;\boldsymbol{Q})\text{-}torsion\cong H_{S^1}^*(pt;\boldsymbol{Q})[y]/\prod(y-a_i\alpha).$$

PROOF. $H_{S^1}^*(M;\boldsymbol{Q})$ divided by $H_{S^1}^*(pt;\boldsymbol{Q})$-torsion is isomorphic to its image in $S^{-1}H_{S^1}^*(M;\boldsymbol{Q})$. Let $u$ be an element in that image. By Proposition 3.13 we may assume that the degree of $u$ is $2q$ and $u$ is of the form

$$u=c_0\alpha^q+c_1\alpha^{q-1}y+\cdots+c_qy^q+c_{q+1}\alpha^{-1}y^{q+1}+\cdots+c_{q+r}\alpha^{-r}y^{q+r}$$

with $q+r\leqq\chi-1=n$ and $c_j\in\boldsymbol{Q}$. Then we have

$$\alpha^ruy^{n-(q+r)}=c_0\alpha^{r+q}y^{n-(q+r)}+\cdots+c_{q+r}y^n.$$

This equality holds not only in $S^{-1}H_{S^1}^{2n}(M;\boldsymbol{Q})$ but in $H_{S^1}^{2n}(M;\boldsymbol{Q})$ since $H_{S^1}^s(M)$ is isomorphic to $\sum_i H_{S^1}^s(P_i)$ which is $H_{S^1}^*(pt)$-torsion free provided that $s\geqq 2n=\dim M$; cf. [5].

Thus we can restrict the above equality to the ordinary cohomology. Assume now $r>0$ and $c_{q+r}\neq 0$. Since $\alpha$ maps to zero by that restriction we get

$$0=c_{q+r}x^n.$$

Since $n+1=\chi$, the line bundle $\xi$ is quasi-ample, i.e. $x^n\neq 0$ by Corollary 3.8. Thus we must have $c_{q+r}=0$ contradicting the assumption. It follows $u$ is of the form

$$u=c_0\alpha^q+c_1\alpha^{q-1}y+\cdots+c_qy^q.$$

This shows that $u$ belongs to $H_{S^1}^*(pt;\boldsymbol{Q})[y]/\prod(y-a_i\alpha)$. Q.E.D.

COROLLARY 3.15. *Let $M$ be as above with $n+1=\chi$. If $M$ admits a quasi-ample line bundle then the image of*

$$\text{ad } H^2(M)\cong H_{S^1}^2(M;\boldsymbol{Z})/\text{Im } H_{S^1}^2(pt;\boldsymbol{Z})\xrightarrow{\sum f_i^*}\sum H_{S^1}^2(P_i;\boldsymbol{Z})/\varDelta\cong\boldsymbol{Z}^\chi/\varDelta$$

*is infinite cyclic where $\varDelta$ denotes the diagonal.*

PROOF. The image of the composite homomorphism

$$H_{S^1}^2(M;\boldsymbol{Q})\longrightarrow S^{-1}H_{S^1}^2(M;\boldsymbol{Q})\xrightarrow{\sum f_i^*}\sum S^{-1}H_{S^1}^2(P_i;\boldsymbol{Q})\cong\boldsymbol{Q}^\chi$$

has dimension two and is generated by $\sum f_i^*(y)$ and $\sum f_i^*(\alpha)$ by Proposition 3.14. Since $\sum f_i^*(\alpha)$ is nothing but $(\alpha, \cdots, \alpha) \in \varDelta$ the image of $H^2_{S^1}(M; \boldsymbol{Q})$ in $\sum S^{-1}H^2_{S^1}(M; \boldsymbol{Q})/\varDelta$ has dimension 1. This implies the statement of Corollary 3.15. Q.E.D.

Note that the homomorphism ad $H^2(M) \to \boldsymbol{Z}$ of Corollary 3.15 assigns to $x$ the weights $(a_i)$ of the corresponding line bundle $\xi$ with $c_1(\xi)=x$. Corollary 3.15 implies that the weights are either mutually distinct or all equal.

DEFINITION. An $S^1$-manifold satisfying the conditions of Corollary 3.15 will be said to have property $P$. Let $M$ be an $S^1$-manifold having property $P$. A sequence $(a_i) \in \boldsymbol{Z}^\chi$ will be called basic if it generates the image of ad $H^2(M)$ in $\boldsymbol{Z}^\chi/\varDelta$. Thus any sequence $(c_i) \in \boldsymbol{Z}^\chi$ in the image of ad $H^2(M)$ is uniquely written in the form

$$c_i = k_0 a_i + d \qquad \text{for all } i \tag{3.16}$$

with $k_0$, $d \in \boldsymbol{Z}$. An element $x \in \text{ad}\, H^2(M)$ will be called basic if it maps to a basic sequence.

If, moreover, $M$ is a unitary $S^1$-manifold then $c_1(M)$ is admissible and the corresponding sequence is $(\sum_k m_{ik})$ where $\{m_{ik}\}$ are weights of $M$ at $P_i$. Thus if $M$ is an $S^1$-manifold having property $P$ then there are integers $k_0$ and $d$ such that

$$\sum_k m_{ik} = k_0 a_i + d \qquad \text{for all } i \tag{3.17}$$

where $(a_i)$ is a basic sequence. In this case we shall make convention to choose basic sequence so that $k_0 \geqq 0$.

When $n+1<\chi$, not every $x \in \text{ad}\, H^2(M)$ is related to the tangent bundle of $M$ like (3.17) even with rational coefficients. Let $M$ be a unitary $S^1$-manifold and $\xi$ an admissible line bundle over $M$. We shall say $\xi$ or its Chern class $x=c_1(\xi)$ satisfies the condition $D$ if there exist integers $k_0 \geqq 0$ and $d$ such that the relation (3.17) holds for any $i$ where $a_i$ is the weight of $\xi$ at $P_i$.

PROPOSITION 3.18. *Let $M$ be a unitary $S^1$-manifold having only isolated fixed points. If $x \in H^2(M; \boldsymbol{Z})$ is fine and such that $c_1(M)=k_0 x$ mod torsion for some $k_0 \in \boldsymbol{Z}$ then $x$ satisfies the condition $D$.*

PROOF. By definition $c_1(M)=c_1(\tau(M) \oplus l\theta)$ for some $l$.

Therefore

$$c_1(M)=c_1(\wedge^N(\tau(M)\oplus l\theta)) , \quad N=\dim_C(\tau(M)\oplus l\theta).$$

Let $\xi$ be a line bundle with $c_1(\xi)=x$. Since $c_1(M)=k_0x$ mod torsion we have

$$\wedge^N(\tau(M)\oplus l\theta)=\xi^{k_0}\xi_1$$

for some line bundle $\xi_1$ such that $\xi_1^h=1$ for some $h$. By Lemma 3.2 the action on $M$ can be lifted to $\xi$ and $\xi_1$. Let $\eta$ and $\eta_1$ be $S^1$ line bundles with some lifted actions. Then, as $S^1$ line bundles,

$$\wedge^N(\tau(M)+l\theta)=\eta^{k_0}\eta_1 t^{d'}$$

for some $d'\in \boldsymbol{Z}$; cf. (3.1) and [11].

We now claim that $\eta_1=t^{d_1}$ for some $d_1$ as elements of $S^{-1}K_{S^1}(M)$. In fact $\eta_1^h=t^{h'}$ for some $h'$. Applying $f_i^*$ we see $f_i^*(\eta_1)^h=t^{h'}$ in $\boldsymbol{Z}[t, t^{-1}]$ for all $i$. This implies that $f_i^*(\eta_1)=t^{d_1}$ in $\boldsymbol{Z}[t, t^{-1}]$ for all $i$. Since $\sum f_i^*$ is monic we have

$$\eta_1=t^{d_1} \quad \text{in} \quad S^{-1}K_{S^1}(M).$$

Hence, setting $d=d'+d_1$,

$$\wedge^N(\tau(M)\oplus l\theta)=\eta^{k_0}t^d \quad \text{in} \quad S^{-1}K_{S^1}(M).$$

Taking $f_i^*$ on both sides we get

$$\sum m_{ik}=k_0a_i+d. \qquad \text{Q.E.D.}$$

LEMMA 3.19. *Let $M$ an almost complex $S^1$-manifold and $\xi$ an admissible line bundle satisfying the condition D. Then, in* (3.17),

$$d=-\frac{k_0}{\chi}\sum a_i \tag{3.20}$$

*so that we have*

$$\sum_k m_{ik}=\frac{k_0}{\chi}\sum_j (a_i-a_j) \qquad \textit{for all } i.$$

PROOF. By Proposition 2.11 $\sum_{i,k} m_{ik}=0$. On the other hand it follows from (3.17) that

$$\sum_{i,k} m_{ik}=k_0\sum a_i+d\chi.$$

Hence $k_0\sum a_i+d\chi=0$. Q.E.D.

PROPOSITION 3.21. *Let $M$ be a unitary $S^1$-manifold having only isolated fixed points. If $c_1(M)$ is a torsion element then*

$$T[M]=0.$$

PROOF. If the action has no fixed points then $T[M]$, which is equal to $p_1(1)$ evaluated at $t=1$, vanishes by Theorem 2.2 (without assuming $c_1(M)$ is torsion element). Suppose the action has a fixed point. By Proposition 3.18 (in which we can take $k_0=0$) we have

$$\sum m_{ik}=d. \tag{3.22}$$

On the other hand, if we assume $T[M]\neq 0$ then, by Remark 2.10, there exist $i_1$ and $i_2$ such that $p_{i_1}=n$, $p_{i_2}=0$. This means that $m_{i_1k}>0$ for all $k$ and $m_{i_2k}<0$ for all $k$. Therefore

$$\sum m_{i_1k}>0 \quad \text{and} \quad \sum m_{i_2k}<0.$$

This contradicts (3.22). Q.E.D.

REMARK. The Todd genus $T$ for $2n$-dimensional unitary manifold with $n$ odd is divisible by $c_1$; cf. [13]. Thus, if $n$ is odd $T[M]=0$ whenever $c_1(M)$ is a torsion element.

## § 4. The functions $\varphi_i(t)$.

Throughout this section $M$ will be a connected closed unitary $S^1$-manifold of dimension $2n$ having only isolated fixed points $\{P_i\}_{i=1}^{\chi}$, $\chi=\chi(M)>0$. Let $\{m_{ik}\}$ be weights of $M$ around $P_i$. Moreover we suppose there is given a fine complex line bundle $\xi$ over $M$. As before we set

$$\varphi_i(t)=\varepsilon_i\frac{\prod_{j\neq i}(1-t^{a_i-a_j})}{\prod_k(1-t^{m_{ik}})}\in \boldsymbol{Z}[t,t^{-1}]. \tag{4.1}$$

The main goal of this section is to prove

THEOREM 4.2. *Under the above situation there exists a unique sequence $r_0(t), r_1(t), \cdots, r_{\chi-1}(t)$ of elements of $\boldsymbol{Z}[t, t^{-1}]$ such that*

$$\varphi_i(t)=r_0(t)+r_1(t)t^{a_i}+\cdots+r_{\chi-1}(t)t^{(\chi-1)a_i} \qquad \textit{for all } i.$$

*Moreover these $r_s(t)$ satisfy the following properties:*

(1) $$r_0(t)=T[M]=\sum \varepsilon_i\binom{p_i}{n}=\sum \varepsilon_i\binom{n-p_i}{n}.$$

(2) *If* $\xi$ *satisfies the property*

$$D: \qquad \sum_k m_{ik}=k_0a_i+d \qquad \textit{for all } i$$

*where* $k_0\geqq 0$, *then* $k_0\leqq\chi$.

(3) *In* (2) *if* $k_0>0$ *then, setting* $l_0=\chi-k_0$, *we have*

$$r_s(t)=0 \qquad \textit{for all } s>l_0,$$

*and*

$$r_{l_0-s}(t)=(-1)^{\chi-1-n}r_s(t^{-1})t^{-(d+\Sigma a_j)} \qquad \textit{for} \quad s\leqq l_0.$$

*In particular*

$$r_{l_0}(t)=(-1)^{\chi-1-n}r_0t^{-(d+\Sigma a_j)}$$

*where* $r_0=r_0(t)$.

(4) *In* (2), *if* $k_0=0$ *then* $r_0=0$

*and*

$$r_{\chi-s}(t)=(-1)^{\chi-1-n}r_s(t^{-1})t^{-(d+\Sigma a_j)} \qquad \textit{for} \quad 1\leqq s\leqq\chi-1.$$

*Note.* If $M$ is an almost complex $S^1$-manifold then

$$d=-\frac{k_0}{\chi}\sum a_j \quad \text{and} \quad d+\sum a_j=\frac{l_0}{\chi}\sum a_j$$

by (3.20).

PROOF. By Proposition 3.12 and Theorem 2.2 there exists a unique element

$$g(\eta, t)=r_0(t)+r_1(t)\eta+\cdots+r_{\chi-1}(t)\eta^{\chi-1}$$

in $S^{-1}K_{S^1}(M)=S^{-1}R(S^1)[\eta]/\prod_{j=1}^{\chi}(\eta t^{-a_j}-1)$ such that

$$f_i^*(g(\eta, t))=\varphi_i(t).$$

Explicitly $g(\eta, t)$ is given by

$$g(\eta, t)=\sum_i \frac{\prod_{j\neq i}(1-\eta t^{-a_j})}{\prod_{j\neq i}(1-t^{a_i-a_j})}\varphi_i(t)$$

as in the proof of Proposition 3.12. Expanding with respect to $\eta$ and substituting the expression (4.1) of $\varphi_i(t)$ we obtain

$$\text{(4.3)}\qquad r_s(t)=(-1)^s\sum_i \frac{\varepsilon_i \sum_{j_1<\cdots<j_s, j_\nu\neq i} t^{-(a_{j_1}+\cdots+a_{j_s})}}{\prod_k (1-t^{m_{ik}})}$$

$$\in S^{-1}\boldsymbol{Z}[t, t^{-1}], \quad s=0, 1, \cdots, \chi-1.$$

We need to prove $r_s(t)$ actually belongs to $\boldsymbol{Z}[t, t^{-1}]$. Equivalently we shall show that

$$\text{(4.3)}'\qquad \bar{r}_s(t)=r_s(t^{-1})$$

$$=(-1)^s\sum_i \frac{\varepsilon_i \sum_{j_1<\cdots<j_s, j_\nu\neq i} t^{a_{j_1}+\cdots+a_{j_s}}}{\prod_k (1-t^{-m_{ik}})}$$

belongs to $\boldsymbol{Z}[t, t^{-1}]$.

Comparing the expression (4.3)′ with Corollary 2.3 we see that it suffices to show the following

ASSERTION 4.4. *For each* $s=0, \cdots, \chi-1$ *there exists an element* $k_s(\eta, t)\in \boldsymbol{Z}[t, t^{-1}, \eta]$ *such that*

$$f_i^*(k_s(\eta, t))=\sum_{j_1<\cdots<j_s, j_\nu\neq i} t^{a_{j_1}+\cdots+a_{j_s}}$$

*for any* $i$ *where* $f_i^*(k_s(\eta, t))$ *means* $k_s(t^{a_i}, t)\in \boldsymbol{Z}[t, t^{-1}]$. *Then,* $\bar{r}_s(t)$ *is given by*

$$\text{(4.5)}\qquad \bar{r}_s(t)=(-1)^s p_!(k_s(\eta, t)).$$

It is easy to show by calculation that if we define $k_s(\eta, t)$ inductively by the following formula (4.6) then it has desired property.

$$\text{(4.6)}\qquad k_0(\eta, t)=1$$

$$k_1(\eta, t)=\sum (t^{a_j}-\eta)+(\chi-1)\eta$$
$$\vdots$$
$$k_s(\eta, t)=\sum_{j_1<\cdots<j_s}(t^{a_{j_1}}-\eta)\cdots(t^{a_{j_s}}-\eta)$$
$$-\sum_{\nu=1}^{s}\binom{\chi-1-(s-\nu)}{\nu}(-\eta)^{\nu}k_{s-\nu}(\eta, t)$$

for $0\leqq s\leqq\chi-1$. This proves Assertion 4.4 and hence $r_s(t)\in \boldsymbol{Z}[t, t^{-1}]$.

To proceed further it is convenient to use another sequence of polynomials $k_0(\eta, t), \cdots, k_{\chi-1}(\eta, t)$ satisfying (4.5). For that purpose we note the following identity

$$\sum_{j_1<\cdots<j_s, j_\nu\neq i} t^{a_{j_1}+\cdots+a_{j_s}}=t^{\Sigma a_j}\Bigl(t^{-a_i}\sum_{j_1<\cdots<j_{\chi-1-s}, j_\nu\neq i} t^{-(a_{j_1}+\cdots+a_{j_{\chi-1-s}})}\Bigr).$$

Hence, if we set

$$h_r(\eta, t)=k_r(\eta^{-1}, t^{-1})\in \boldsymbol{Z}[t, t^{-1}, \eta^{-1}] \tag{4.7}$$

and

$$\widetilde{k}_s(\eta, t)=t^{\Sigma a_j}(\eta^{-1}h_{\chi-1-s}(\eta, t))\in \boldsymbol{Z}[t, t^{-1}, \eta^{-1}] \tag{4.8}$$

then we have

$$f_i^*(\widetilde{k}_s(\eta, t))=\sum_{j_1<\cdots<j_s, j_\nu\neq i} t^{a_{j_1}+\cdots+a_{j_s}}$$

and

$$\bar{r}_s(t)=(-1)^s p_!(\widetilde{k}_s(\eta, t)). \tag{4.5)$'$}$$

*Note.* $k_s(\eta, t)$ and $\widetilde{k}_s(\eta, t)$ give the same element in $S^{-1}K_{S^1}(M)$.

We need the following

ASSERTION 4.9. *$\widetilde{k}_s(\eta, t)$ is a polynomial of $\eta^{-1}$ with coefficients in $\boldsymbol{Z}[t, t^{-1}]$ of degree $\chi-s$ and without constant term. Moreover the coefficient of $\eta^{-(\chi-s)}$ in $\widetilde{k}_s(\eta, t)$ is equal to*

$$-(-1)^{\chi-s}t^{\Sigma a_j}, \quad 0\leqq s\leqq\chi-1.$$

In fact, $k_s(\eta, t)$ is a polynomial of $\eta$ of degree $s$ and the coefficient of $\eta^s$ is equal to

$$(-1)^s\left\{\binom{\chi}{s}-\left[\binom{\chi-1}{s}+\binom{\chi-2}{s-1}+\cdots+\binom{\chi-s}{1}\right]\right\}=(-1)^s$$

as is easily seen from (4.6). Hence $\tilde{k}_s(\eta, t)$ is a polynomial of $\eta^{-1}$ of degree $1+\chi-1-s=\chi-s$ without constant term and the coefficient of $\eta^{-(\chi-s)}$ is equal to

$$t^{\Sigma a_j}(-1)^{\chi-1-s}$$

by (4.7) and (4.8).

Now we shall prove (1) of Theorem 4.2. By (4.6)

$$\bar{r}_0(t)=p_!(k_0(\eta, t))=p_!(1)=\sum_i \frac{\varepsilon_i}{\prod (1-t^{-m_{ik}})}.$$

Then Corollary 2.7 and Note after Theorem 2.2 imply

$$\bar{r}_0(t)=T[M]=\sum \varepsilon_i \binom{p_i}{n}=\sum \varepsilon_i \binom{n-p_i}{n}.$$

In particular $\bar{r}_0(t)$ is a constant and hence $r_0(t)=\bar{r}_0(t)$.

To prove (2) and (3) assume the relation

$$\sum m_{ik}=k_0 a_i+d, \qquad \text{for all } i$$

holds with $k_0 \geqq 0$. We shall first show the following

Assertion 4.10. *If* $k_0>0$, *then*

$$\sum \frac{\varepsilon_i t^{ka_i}}{\prod_k (1-t^{m_{ik}})}=\sum \frac{\varepsilon_i t^{-ka_i}}{\prod_k (1-t^{-m_{ik}})}=0$$

*for* $0<k<k_0$.

To prove this we may replace $t$ by some power $t^r$ $(r\neq 0)$. This corresponds to taking the $r$-fold covering action of the given $S^1$ action on $M$; i.e. $g\in S^1$ acts on $x\in M$ by $g^r x$. With this replacement $m_{ik}$ and $a_i$ get multiplied by $r$ simultaneously. Similarly we may replace the $a_i$ by $a_i+a$. This corresponds to taking another lifting action to $\xi$. With this understood we replace $t$ by $t^{k_0}$ if necessary and may assume that $k_0$ divides $d$. Then, by replacing $a_i$ by $a_i+a$ for some $a$, we may assume $d=0$, i.e.

$$\sum_k m_{ik}=k_0 a_i \qquad \text{for all } i. \tag{4.11}$$

We set

$$h_i(t)=\frac{t^{ka_i}}{\prod(1-t^{m_{ik}})}.$$

If $0<k<k_0$ then we see easily by using (4.11) that the rational function $h_i(t)$ takes value 0 at $t=0$ and $t=\infty$. Then the sum $\sum \varepsilon_i h_i(t)$ is a Laurent polynomial which takes value 0 at $t=0$ and $t=\infty$. Hence $\sum \varepsilon_i h_i(t)$ must vanish identically. This proves Assertion 4.10. By Assertion 4.10 we have

$$p_1(\eta^{-k})=0 \quad \text{for} \quad 0<k<k_0.$$

Hence, by Assertion 4.9, we have

$$p_1(\tilde{k}_s(\eta, t))=0$$

for $\chi-s<k_0$, i.e. $l_0=\chi-k_0<s$. From this and (4.5)$'$ we have $\bar{r}_s(t)=0$ and $r_s(t)=\bar{r}_s(t^{-1})=0$ for $l_0<s$. This proves the first statement of (3).

Assume moreover that $\chi<k_0$. Then $r_s(t)=0$ for $s=0, 1, \cdots, \chi-1$ and hence $\varphi_i(t)=0$. But on the other hand

$$\varphi_i(t)=\varepsilon_i\frac{\prod_{j\neq i}(1-t^{a_i-a_j})}{\prod(1-t^{m_{ik}})}\neq 0.$$

This contradiction proves (2).

Next we shall prove the rest of (3) and (4). Setting $l_0=\chi-k_0$ and using the property $D$ we get

$$\varphi_i(t^{-1})t^{l_0a_i}=(-1)^{\chi-1-n}\varphi_i(t)t^{d+\Sigma a_j}.$$

Therefore, by the first part of (3),

$$\begin{aligned}(4.12)\qquad & r_0(t^{-1})t^{l_0a_i}+r_1(t^{-1})t^{(l_0-1)a_i}+\cdots+r_{l_0-1}(t^{-1})t^{a_i}+r_{l_0}(t^{-1}) \\ &=(-1)^{\chi-1-n}t^{d+\Sigma a_j}(r_0(t)+r_1(t)t^{a_i}+\cdots+r_{l_0-1}(t)t^{(l_0-1)a_i}+r_{l_0}(t)t^{l_0a_i})\end{aligned}$$

for all $i$ in case $k_0>0$. In case $k_0=0$ these identities also hold provided that we make convention that $r_{l_0}(t)=r_0(t)=0$.

We first consider the case $k_0>0$. Since the number of identities is $\chi$ and it is not less than $l_0+1$ which is equal to the number of terms $1, t^{a_i}, \cdots, t^{l_0a_i}$, the coefficients of each $t^{sa_i}$ at the both sides must coincide. Hence we get

$$r_{l_0-s}(t^{-1})=(-1)^{\chi-1-n}r_s(t)t^{d+\Sigma a_j}.$$

If $k_0=0$, then $r_0=T[M]=0$ by (1) and Proposition 3.21. By convention $r_\chi(t)=0$. Thus, by the same argument as above comparing the coefficients of $t^{sa_i}$, we get

$$r_{\chi-s}(t^{-1})=(-1)^{\chi-1-n}r_s(t)t^{d+\Sigma a_j}, \qquad 1\leqq s\leqq \chi-1.$$

This proves (4) and completes the proof of Theorem 4.2.

REMARK 4.13. In Theorem 4.2, the condition (2) can be replaced by

$$c_1(M)=k_0c_1(\xi) \quad \text{mod torsion}$$

in virtue of Proposition 3.18.

REMARK 4.14. We see by simple calculation that

$$\frac{\prod_{j\neq i}(1-t^{a_i-a_j})}{\prod_k (1-t^{m_{ik}})}=\delta_i t^{v_i}\frac{\prod_{j\neq i}(t^{|a_i-a_j|}-1)}{\prod_k (t^{|m_{ik}|}-1)} \tag{4.15}$$

where

$$v_i=\frac{1}{2}\left\{\sum_j (a_i-a_j)-\sum_j |a_i-a_j|-\sum m_{ik}+\sum |m_{ik}|\right\}$$

$$\delta_i=(-1)^{\chi-1-n}\,\mathrm{sgn}\,\frac{\prod_{j\neq i}(a_i-a_j)}{\prod m_{ik}}.$$

Thus, since $\varphi_i(t)\in \boldsymbol{Z}[t, t^{-1}]$, the function

$$\psi_i(t)=\frac{\prod_{j\neq i}(t^{|a_i-a_j|}-1)}{\prod_k (t^{|m_{ik}|}-1)}$$

is a product of cyclotomic polynomials. If $n+1=\chi$, then, by using Proposition 3.7, we have

$$\varphi_i(t)=\mathrm{sgn}(x^n[M])t^{v_i}\psi_i(t). \tag{4.16}$$

The following Proposition gives a topological meaning of the functions $\varphi_i(t)$.

PROPOSITION 4.17. *Let $M$ and $\xi$ be as in Theorem* 4.2. *Then*

$$f_{i!}(1)=\frac{1}{\bar{\varphi}_i(t)}\prod_{j\neq i}(1-\eta^{-1}t^{a_i}) \quad \text{in} \quad S^{-1}K_{S^1}(M)$$

*where $f_i\colon P_i\to M$ is the inclusion and $\bar{\varphi}_i(t)=\varphi_i(t^{-1})$ and $P_i$ is equipped* *with the standard unitary structure of a point.*

PROOF. Elements of $S^{-1}K_{S^1}(M)$ are detected by $\sum_j f_j^*$. Set

$$u=f_{i!}(1) \quad \text{and} \quad v=\frac{1}{\bar{\varphi}_i(t)}\prod_{j\neq i}(1-\eta^{-1}t^{a_j}).$$

If $j\neq i$, then we have $f_j^* u=0$ since $f_{i!}(1)$ has support arbitrarily near $P_i$, and we clearly have $f_j^* v=0$. On the other hand, $f_i^* f_{i!}(1)$ is the Euler class $e(\nu_i)$ of the normal bundle of $f_i$. Hence

$$f_i^* u=f_i^* f_{i!}(1)=\varepsilon_i\prod_k(1-t^{-m_{ik}}),$$

while we clearly have

$$f_i^* v=\varepsilon_i\prod_k(1-t^{-m_{ik}}).$$

Thus we have $u=v$. Q.E.D.

## § 5. Quasi-ample line bundles and theorems of the Kobayashi-Ochiai type.

We continue with the assumption on $M$ made in §4. In this section $\xi$ will be a quasi-ample line bundle satisfying the condition

$$D: \qquad \sum_k m_{ik}=k_0 a_i+d \qquad \text{for all } i$$

with $k_0\geqq 0$. We notice that if $\chi=n+1$ and $c_1(M)^n\neq 0$ then the unitary manifold $M$ has property $P$ by Corollary 3.8; in particular the condition $D$ is satisfied in that case by a basic quasi-ample line bundle $\xi$.

The main goal of this section is Theorem 5.1 and Theorem 5.7.

THEOREM 5.1. *Under the situation as above we have the inequalities*

$$k_0\leqq n+1\leqq\chi.$$

For the proof of Theorem 5.1 it is convenient of introduce a generalization of functions $\varphi_i(t)$. Let $i_1, \cdots, i_s$ be mutually distinct integers, $1\leqq i_\nu\leqq\chi$. For an admissible line bundle $\xi$ we set

$$\varphi_{i_1\cdots i_s}(t)=\sum_\nu \varepsilon_{i_\nu}\frac{\prod_{j\neq i_1,\cdots,i_s}(1-t^{a_{i_\nu}-a_j})}{\prod_k(1-t^{m_{i_\nu k}})}.$$

Then

$$\varphi_{i_1\cdots i_s}(t^{-1})=p_!\Big(\prod_{j\neq i_1,\cdots,i_s}(1-\eta^{-1}t^{a_j})\Big)$$

and hence $\varphi_{i_1\cdots i_s}(t)\in \boldsymbol{Z}[t, t^{-1}]$.

Let $b_1, \cdots, b_s$ be mutually distinct integers. We set

$$\Gamma^k_{b_1\cdots b_s}(t)=\sum_{i=1}^{s}\frac{t^{kb_i}}{\prod_{j\neq i}(1-t^{b_i-b_j})}$$

for $k\in \boldsymbol{Z}$.

LEMMA 5.2. *$\Gamma^k_{b_1\cdots b_s}(t)$ belongs to $\boldsymbol{Z}[t, t^{-1}]$ and satisfies the following properties.*

(1) $\Gamma^0_{b_1\cdots b_s}(t)=1,$

(2) $\Gamma^k_{b_1\cdots b_s}(t)=0$ *for* $0<k<s,$

(3) $\Gamma^s_{b_1\cdots b_s}(t)=(-1)^{s-1}t^{\Sigma b_j},$

(4) $\Gamma^k_{b_1\cdots b_s}(1)=(-1)^{s-1}\binom{k-1}{s-1}.$

PROOF. Consider the linear $S^1$-action on $\boldsymbol{C}P^{s-1}$ given by

$$z[z_1, \cdots, z_s]=[z^{b_1}z_1, \cdots, z^{b_s}z_s].$$

Then Proposition 3.4 applied to this action shows that $\Gamma^k_{b_1\cdots b_s}(t)=p_!(\eta^{-k})\in \boldsymbol{Z}[t, t^{-1}]$; cf. (3.11). Also, formula (2.8) and Assertion 4.10 applied to this action yield (1) and (2); in this case $\sum_{j\neq i}(b_j-b_i)=s(-b_i)+\sum b_j$ and hence $k_0=s$. (3) is proved as follows.

$$\begin{aligned}\Gamma^s_{b_1\cdots b_s}(t)&=\sum_{i=1}^{s}\frac{t^{sb_i}}{\prod_{j\neq i}(1-t^{b_i-b_j})}\\&=(-1)^{s-1}\sum_{i=1}^{s}\frac{t^{sb_i+\Sigma_j(b_j-b_i)}}{\prod_{j\neq i}(1-t^{b_j-b_i})}\\&=(-1)^{s-1}t^{\Sigma b_j}\Gamma^0_{b_1\cdots b_s}(t^{-1})\\&=(-1)^{s-1}t^{\Sigma b_j}\qquad \text{by (1).}\end{aligned}$$

Finally, since $\Gamma^k_{b_1\cdots b_s}(t)=p_!(\eta^{-k})$ we have

$$\Gamma^k_{b_1\cdots b_s}(1)=p_!(\xi^{-k})=\{e^{-k\alpha}\mathscr{T}(CP^{s-1})\}[CP^{s-1}]$$

by the Atiyah-Singer theorem, where $\alpha$ is the canonical generator of $H^2(CP^{s-1}; \boldsymbol{Z})$. Thus $\Gamma^k_{b_1\cdots b_s}(1)$ equals the coefficient of $\alpha^{s-1}$ of the formal power series

$$e^{-k\alpha}\left(\frac{\alpha}{1-e^{-\alpha}}\right)^s.$$

By calculation we see it is equal to $(-1)^{s-1}\binom{k-1}{s-1}$. Q.E.D.

*Note.* It is known that

$$\Gamma^{k+s}_{b_1\cdots b_s}(t)=(-1)^{s-1}t^{\Sigma b_j}\rho_k(t)$$

and

$$\Gamma^{-k}_{b_1\cdots b_s}(t)=\rho_k(t^{-1})$$

for $k\geqq 0$ where

$$\rho_k(t)=\sum_{i_1\leqq\cdots\leqq i_k} t^{b_{i_1}+\cdots+b_{i_k}}.$$

There is a proof of these formulae which uses the Atiyah-Singer formula applied to the above $S^1$-action on $CP^{s-1}$.

PROPOSITION 5.3. *For a fine line bundle $\xi$ we have*

$$\varphi_{i_1\cdots i_s}(t)=r_0+(-1)^{s-1}r_s(t)t^{\Sigma_\nu a_{i_\nu}}+r_{s+1}(t)\Gamma^{s+1}_{a_{i_1}\cdots a_{i_s}}(t)+\cdots+r_{\chi-1}(t)\Gamma^{\chi-1}_{a_{i_1}\cdots a_{i_s}}(t)$$

*where $r_\mu(t)$ are those given in Theorem* 4.2.

PROOF.

$$\begin{aligned}\varphi_{i_1\cdots i_s}(t)&=\sum_{\nu=1}^{s}\varepsilon_{i_\nu}\frac{\prod_{j\neq i_1,\cdots,i_s}(1-t^{a_{i_\nu}-a_j})}{\prod(1-t^{m_{i_\nu k}})}\\ &=\sum_\nu\frac{1}{\prod_{\substack{\mu\neq\nu\\ \mu=1,\cdots,s}}(1-t^{a_{i_\nu}-a_{i_\mu}})}\varphi_{i_\nu}(t)\\ &=\sum_{k=0}^{\chi-1}\sum_\nu\frac{t^{ka_i}}{\prod_{\mu\neq\nu}(1-t^{a_{i_\nu}-a_{i_\mu}})}r_k(t)\\ &=\sum_{k=0}^{\chi-1}\Gamma^k_{a_{i_1}\cdots a_{i_s}}(t)r_k(t)\end{aligned}$$

by Theorem 4.2. We then apply Lemma 5.1 to get the desired form. Q.E.D.

COROLLARY 5.4. *Let $\xi$ be a fine line bundle satisfying condition D with $k_0>0$. If $s>l_0=\chi-k_0$ then*

$$\varphi_{i_1\cdots i_s}(t)=r_0.$$

*For $s=l_0$ we have*

$$\varphi_{i_1\cdots i_s}(t)=r_0\{1-(-1)^{k_0-n-1}t^{-(d+\hat{a}_{i_1\cdots i_s})}\}$$

*where $\hat{a}_{i_1\cdots i_s}=\sum_{j\neq i_1,\cdots,i_s}a_j$.*

PROOF. The first part is clear from Proposition 5.3. For $s=l_0$, using Proposition 5.3 and Theorem 4.2 we see that

$$\begin{aligned}\varphi_{i_1\cdots i_s}(t)&=r_0+(-1)^{l_0-1}r_{l_0}(t)t^{\sum_\nu a_{i_\nu}}\\&=r_0\{1+(-1)^{l_0-1+\chi-1-n}t^{-(d+\sum a_j)+\sum a_{i_\nu}}\}\\&=r_0\{1-(-1)^{k_0-n-1}t^{-(d+\hat{a}_{i_1\cdots i_s})}\}.\end{aligned}$$

LEMMA 5.5. *Let $\xi$ be an admissible line bundle. Then for $s=\chi-n$, we have*

$$\varphi_{i_1\cdots i_s}(1)=x^n[M],\quad x=c_1(\xi).$$

PROOF. This follows immediately from Lemma 3.6.

PROOF OF THEOREM 5.1. We have already shown $n+1\leqq\chi$ in Corollary 3.8. Assume now that $k_0>n+1$. Set $s=\chi-n-1$. Then $s>l_0$ and by Corollary 5.4 we have

$$\varphi_{i_1\cdots i_s}(t)=r_0=\varphi_{i_1\cdots i_s}(1).$$

But

$$\varphi_{i_1\cdots i_s}(t)=\sum\varepsilon_{i_\nu}\frac{\prod_{j\neq i_1\cdots i_s}(1-t^{a_{i_\nu}-a_j})}{\prod(1-t^{m_{ik}})}$$

and each summand takes value 0 at $t=1$ since $\chi-s=n+1>n$. Therefore

$$r_0=\varphi_{i_1\cdots i_s}(1)=0.$$

Next set $s=\chi-n$. Then $s>l_0$ and by Corollary 5.4 we have

$$\varphi_{i_1\cdots i_s}(t)=r_0=0. \tag{5.6}$$

But $\varphi_{i_1\cdots i_s}(1)=x^n[M]$ by Lemma 5.5, and $x^n[M]\neq 0$ because $\xi$ is quasi-ample by assumption. This contradicts (5.6). Therefore we must have $k_0\leqq n+1$. Q.E.D.

It is natural to investigate first the extremal cases $k_0=n+1$ and $n+1=\chi$ among the range $k_0\leqq n+1\leqq\chi$ of Theorem 5.1. Kobayashi-Ochiai's theorem [14] tells us that if there is an ample line bundle $\xi$ on a compact complex manifold $M$ of complex dimension $n$ such that $c_1(M)=(n+1)c_1(\xi)$ then $M$ is biholomorphic to $\boldsymbol{C}P^n$. In our context their assumption corresponds to $k_0=n+1$ (cf. Proposition 3.18) and their conclusion partly corresponds to $n+1=\chi$. Unfortunately the author has no way of proving or disproving this implication $k_0=n+1\Rightarrow n+1=\chi$. Quasi-ampleness doesn't seem strong enough for that purpose; we shall briefly discuss about this point in §7. However if we assume $k_0=n+1=\chi$ from the first and add the assumption that $M$ is almost complex then we can deduce similar conclusions as Kobayashi-Ochiai's theorem.

THEOREM 5.7. *Let $M$ be an almost complex $S^1$-manifold of dimension $2n$ having only isolated fixed points. Assume $\chi(M)=n+1$. If $\xi$ is a quasi-ample line bundle satisfying the condition $D$ with $k_0=n+1$, then $\xi$ is necessarily basic and the weights of $M$ at each fixed point $P_i$ are given by*

$$\{m_{ik}\}=\{a_i-a_j\}_{j\neq i}.$$

*Moreover $M$ is unitary cobordant to $\boldsymbol{C}P^n$. In particular we have*

$$T[M]=1.$$

*Furthermore we have $x^n[M]=1$ where $x=c_1(\xi)$.*

COROLLARY 5.8. *Let $M$ be an almost complex $S^1$-manifold of dimension $2n$ having property $P$ and let $x\in \mathrm{ad}\,H^2(M)$ be such that $x^n\neq 0$ and*

$$c_1(M)=(n+1)x \quad \text{mod torsion}.$$

*Then the same conclusion as in Theorem 5.7 hold.*

COROLLARY 5.9. *Let $M$ be an almost complex $S^1$-manifold (having only isolated fixed points) of dimension $2n$ which has the same cohomology ring over the rationals as $\boldsymbol{C}P^n$. Let $x$ be a generator of $H^2(M;\boldsymbol{Z})$ mod torsion. If*

$$c_1(M)=(n+1)x \quad mod\ torsion$$

*then*

$$c(M)=(1+x)^{n+1} \quad mod\ torsion.$$

*Moreover* $x^n[M]=1$.

The rest of this section is devoted to the proofs of Theorem 5.7, Corollary 5.8 and Corollary 5.9.

First we notice that if the manifold $M$ has property $P$ and quasi-ample line bundle $\xi$ satisfies the condition $D$ with $k_0=n+1$ then $\xi$ is necessarily basic. For otherwise we would have

$$\sum_k m_{ik}=k_0'a_i'+d$$

for a basic sequence $\{a_i'\}$ and with $k_0'>k_0=n+1$, contradicting Theorem 5.1.

Hereafter we shall assume $M$ has property $P$ and $\xi$ is a quasi-ample line bundle such that

$$\sum m_{ik}=k_0a_i+d \qquad \text{for all } i.$$

We set $x=c_1(\xi)$ and $K=x^n[M]$.

LEMMA 5.10. *If* $k_0=n+1$ *then* $K=\pm 1$ *and*

$$\varphi_i(t)=T[M]=K.$$

*In particular, in the almost complex case, we have*

$$\varphi_i(1)=T[M]=1.$$

PROOF. By (1) and (3) of Theorem 4.2 we have

$$\varphi_i(t)=r_0=T[M].$$

Then, using Proposition 3.7, we get

$$\varphi_i(t)=\varphi_i(1)=K.$$

Comparing this with (4.16) in Remark 4.14 in which $\psi_i(t)$ is a product of cyclotomic polynomials, we have

$$\varphi_i(t)=\operatorname{sgn} K.$$

Hence $K=\operatorname{sgn} K$ and $K$ must be equal to $K=\pm 1$. If $M$ is almost complex, then $T[M]\geqq 0$ by Remark 2.10. Hence $K=T[M]$ must be equal to 1. Q.E.D.

REMARK. Using Remark 4.14 we can see that the algebraic condition "$\varphi_i(t)=1$ for all $i$" is equivalent to the following three conditions

(5.11) $$\{|m_{ik}|\}=\{|a_i-a_j|\}_{j\neq i} \qquad \text{for all } i,$$

(5.12) $$\sum_k m_{ik}=\sum (a_i-a_j) \qquad \text{for all } i \quad (\text{i.e. } k_0=n+1),$$

(5.13) $$\varepsilon_i \operatorname{sgn} \prod_k m_{ik}=\operatorname{sgn} \prod_{j\neq i} (a_i-a_j) \qquad \text{for all } i.$$

Therefore the conclusion of Theorem 5.7 states more than "$\varphi_i(t)=1$ for all $i$".

*Note.* In the almost complex case, (5.13) reduces to

(5.13)′ $$\operatorname{sgn} \prod_k m_{ik}=\operatorname{sgn} \prod_{j\neq i} (a_i-a_j).$$

Musin [19] announced Theorem 5.7 in a somewhat weaker form. But there seems to be a gap in his proof. His proof seems to show only $\varphi_i(t)=1$. Also he states the theorem in the unitary category. But in some places he seems to work in the almost complex category forgetting the signs $\varepsilon_i$.

LEMMA 5.14. *Suppose that the condition* (5.11) *is satisfied. Then* $K=\pm 1$ *and* $l_0$ *and* $d+\sum a_j$ *are both even. Moreover* $r_s(t)=0$ *for* $s\neq (l_0/2)$ *and*

$$\varphi_i(t)=Kt^{\{l_0a_i-(d+\Sigma a_j)\}/2}.$$

*Note.* In the almost complex case we know that $d+\sum a_j=(l_0/(n+1))\sum a_j$. Hence $(l_0/2(n+1))\sum a_j\in \boldsymbol{Z}$.

PROOF. $K=\pm 1$ follows from Proposition 3.7. Then, by (4.15) in Remark 4.14 we have

$$\begin{aligned}\varphi_i(t)&=\operatorname{sgn} Kt^{\{\Sigma(a_i-a_j)-\Sigma m_{ik}\}/2}\\&=Kt^{\{l_0a_i-(d+\Sigma a_j)\}/2}.\end{aligned}$$

Here $\{l_0a_i-(d+\sum a_j)\}/2\in \boldsymbol{Z}$.

Assume $l_0$ is odd. Since $r_{l_0-s}(1)=r_s(1)$ by Theorem 4.2 we have

$$\pm 1=K=2(r_0(1)+\cdots+r_{(l_0-1)/2}(1))\in 2\boldsymbol{Z}.$$

This is a contraction. Hence $l_0$ must be even. Therefore $d+\sum a_j$ also must be even. Thus we have an expression

$$\varphi_i(t)=\operatorname{sgn} Kt^{-(d+\Sigma a_j)/2}t^{l_0 a_i/2}.$$

Comparing this with Theorem 4.2 we see that

$$r_s(t)=0 \quad \text{for} \quad s\neq\frac{l_0}{2}$$

and

$$r_{l_0/2}(t)=\operatorname{sgn} Kt^{-(d+\Sigma a_j)/2}.$$ Q.E.D.

COROLLARY 5.15. *Suppose that the condition* (5.11) *is satisfied. If, moreover,* $T[M]\neq 0$ *then* $k_0=n+1$. *In particular, in the almost complex case, if* $\rho_n(=\rho_0)\neq 0$ *then* $k_0=n+1$.

PROOF. $r_0=T[M]\neq 0$ by Remark 2.10 and Theorem 4.2. Therefore $(l_0/2)=0$, i.e. $k_0=n+1$. Q.E.D.

Let $\boldsymbol{Z}_m\subset S^1$ be the subgroup of the $m$-th roots of unity. Each connected component $X$ of the fixed point set of the restricted $\boldsymbol{Z}_m$-action is also a unitary $S^1$-manifold; it is almost complex $S^1$-manifold when $M$ is so. We shall only consider a component $X$ which contains a fixed point $P_i$ of the given $S^1$-action.

LEMMA 5.16. *Let $m$ and $X$ be as above and assume* $\dim X>0$. *If the condition* (5.11) *is satisfied for $M$ then $X$ has property $P$ and also satisfies the condition* (5.11) *with respect to* $\xi|X$. *Moreover if* $P_i\in X$ *then the fixed points lying in $X$ are exactly those $P_j$ such that*

$$a_i-a_j\equiv 0 \quad \mathit{mod}\ m.$$

PROOF. We set $2n'=\dim X$ and $\chi'=\chi(X)$. Let $P_i\in X$. We denote by $\bar{\chi}$ the number of $P_j\in M$ such that $a_i-a_j\equiv 0$ mod $m$.

Obviously $\xi|X$ is a fine line bundle over $X$. Hence

$$n'+1\leqq\chi' \tag{5.17}$$

by Corollary 3.8.

If $P_i, P_j \in X$ then $\eta|P_i$ and $\eta|P_j$ are equivalent as $Z_m$-module since $\eta|X$ is a $Z_m$-line bundle over a connected trivial $Z_m$-space $X$. Hence $a_i - a_j \equiv 0 \mod m$. This implies that

$$\chi' \leqq \bar{\chi}. \tag{5.18}$$

On the other hand the weights of $X$ at $P_i$ are exactly those $m_{ik}$ which are divisible by $m$. Since (5.11) is satisfied this implies that

$$n' = \bar{\chi} - 1. \tag{5.19}$$

Combining (5.17), (5.18) and (5.19) we see that $n'+1 = \chi' = \bar{\chi}$. Q.E.D.

We now proceed to the proof of Theorem 5.7. Thus $M$ will be an almost complex $S^1$-manifold having property $P$ and $\xi$ will be a quasi-ample line bundle such that

$$\sum m_{ik} = \sum (a_i - a_j) \qquad \text{for all } i. \tag{5.20}$$

Recall that the condition (5.11) is satisfied by virtue of Lemma 5.10.

For each fixed point $P_i$ we set

$$A_i = \{a_i - a_j;\ a_i - a_j = m_{ik} \ \text{ for some weight } m_{ik}\}$$
$$B_i = \{a_i - a_j\}_{j \neq i} - A_i.$$

LEMMA 5.21. *$A_i$ and $B_i$ have the following properties:*

(5.22) *$\{m_{ik}\}$ is the disjoint union of $A_i$ and $\{-(a_i - a_j);\ a_i - a_j \in B_i\}$,*

$$\sum_{a_i - a_j \in B_i} (a_i - a_j) = 0, \tag{5.23}$$

(5.24) *the cardinality of $B_i$ is even and $\neq 2$.*

PROOF. (5.22) follows easily from the definition of $A_i$ and $B_i$. By the assumption (5.20) we have

$$\sum_{a_i - a_j \in A_i} (a_i - a_j) + \sum_{a_i - a_j \in B_i} (a_i - a_j) = \sum m_{ik}.$$

From (5.22) it follows that

$$\sum_{a_i - a_j \in A_i} (a_i - a_j) - \sum_{a_i - a_j \in B_i} (a_i - a_j) = \sum m_{ik}.$$

From these two relations we obtain (5.23). The first part of (5.24) follows from (5.13)′.

Finally assume that $B_i$ has exactly two elements $a_i-a_j$ and $a_i-a_{j'}$. Then by (5.23) we have

$$a_i-a_{j'}=-(a_i-a_j).$$

But then (5.11) implies one of them must coincide with some $m_{ik}$ and hence it belongs to $A_i$. This is a contradiction. Hence the cardinality of $B_i$ can not be 2. Q.E.D.

The conclusion to be proven is the claim that $B_i=\emptyset$ for all $i$. Numbering the fixed points as $P_0, \cdots, P_n$ so that

$$a_0<a_1<\cdots<a_{n-1}<a_n,$$

we shall prove that claim by induction starting from $B_n$ down to $B_0$; we might proceed also by induction starting from $B_0$ up to $B_n$.

The claim clearly holds for $n=1$ by (5.20) or (5.24). Thus we assume $n>1$ and the claim is true for manifolds of complex dimension less than $n$.

Claim $(n)$: $B_n=\emptyset$.

This is clear from (5.23) because $a_n-a_j>0$ for all $j\neq n$.

Claim $(n-1)$: $B_{n-1}=\emptyset$.

We devide into two cases $a_n-a_{n-1}=1$ and $a_n-a_{n-1}>1$. First suppose $a_n-a_{n-1}=1$ and $B_{n-1}\neq\emptyset$. Since $a_{n-1}-a_j>0$ for all $j<n-1$, the only negative element in $B_{n-1}$ is $a_{n-1}-a_n$ and there are at least three elements in $B_{n-1}$ by (5.24). Thus

$$\sum_{a_i-a_j\in B_i}(a_i-a_j)>0.$$

But this contradicts (5.23). Hence $B_{n-1}=\emptyset$. Next suppose $m=a_n-a_{n-1}>1$ and $B_{n-1}\neq\emptyset$. Let $X$ be the component of the fixed point set of the restricted $\boldsymbol{Z}_m$-action which contains $P_n$. By Lemma 5.16 $X$ has property $P$ and satisfies (5.11) and $P_{n-1}\in X$. Moreover the fact $B_n=\emptyset$ implies all the weights of $X$ at $P_n$ are positive. Therefore if we denote by $\{m_{ik}(X)\}\subset\{m_{ik}\}$ the weights of $X$ at $P_i\in X$ then, by virtue of (3.17), we have an expression

$$\sum_k m_{ik}(X)=\frac{k_0(X)}{n(X)+1}\sum_{P_j\in X}(a'_i-a'_j), \quad k_0(X)\geqq 0\ ,$$

where $(a'_i)$ are basic sequence for $X$ and $2n(X)=\dim X$. Also $a_i$ can be written in the form

(5.25) $$a_i = ha'_i + a \qquad \text{for all } i.$$

Now the fact $B_n = \emptyset$ implies $m_{nk} > 0$ for all $k$, and hence $m_{nk}(X) > 0$. So we can apply Corollary 5.15 to $X$ and get $k_0(X) = n(X) + 1$ i.e.

$$\sum m_{ik}(X) = \sum_{P_j \in X} (a'_i - a'_j).$$

Then by dimensional induction assumption applied on $X$ we have

$$\{m_{ik}(X)\} = \{a'_i - a'_j\}_{j \neq i}.$$

Comparing this with (5.11), (5.20) and (5.25) we see $h=1$ in (5.25), and hence

(5.26) $$\{m_{ik}(X)\} = \{a_i - a_j\}_{j \neq i, P_j \in X}.$$

This in particular implies that $a_{n-1} - a_n \in A_{n-1}$. Since all other $a_{n-1} - a_j$ are positive we must have $B_{n-1} = \emptyset$ by (5.23).

Claim $(n-2)$: $B_{n-2} = \emptyset$.

It suffices to show that $a_{n-2} - a_n, a_{n-2} - a_{n-1} \in A_{n-2}$. For then we should have $B_{n-2} = \emptyset$ by (5.23). Let $m = a_n - a_{n-2}$ and let $X$ be the component of the fixed point set of the restricted $\boldsymbol{Z}_m$-action containing $P_n$. Then by the same argument as above we see that $P_{n-2} \in X$ and (5.26) is satisfied for this $X$. Thus in particular $a_{n-2} - a_n \in A_{n-2}$.

Next we put $m = a_{n-1} - a_{n-2}$. First consider the case $m > 1$. Let $X$ be the component of the fixed point set of the restricted $\boldsymbol{Z}_m$-action containing $P_{n-1}$. $X$ has property $P$ and satisfies the condition (5.11) and $P_{n-2} \in X$ by Lemma 5.16. If $P_n \in X$ then the same argument as above using $B_n = \emptyset$ shows that (5.26) holds and $a_{n-2} - a_{n-1} \in A_{n-2}$. If $P_n \notin X$, then using $B_{n-1} = \emptyset$ we see that all the $m_{n-1,k}(X)$ are positive. Hence by a similar argument as above we obtain (5.26) and in particular $a_{n-2} - a_{n-1} \in A_{n-2}$.

Next assume $a_{n-1} - a_{n-2} = 1$. Since we have proven $a_{n-2} - a_2 \in A_{n-2}$, only possible negative elements in $B_{n-2}$ is $a_{n-2} - a_{n-1} = -1$. But then $a_{n-2} - a_{n-1} \in B_{n-2}$ would contradicts (5.24). Thus $a_{n-2} - a_{n-1} \in A_{n-2}$. This completes the proof of Claim $(n-2)$.

Now we assume inductively that $B_j = \emptyset$ for $j > s$. Then it is clear now that we can prove that

$$a_s - a_n, a_s - a_{n-1}, \cdots, a_s - a_{s+1} \in A_s$$

and hence $B_s=\emptyset$ by using similar arguments as above. This completes the proof of the main conclusion of Theorem 5.7.

The fact $T[M]=1$ is proved in Lemma 5.10. It remains to prove that $M$ is unitary cobordant to $CP^n$. That amounts to show that $M$ has the same Chern numbers as $CP^n$. Let $s=(s_1, \cdots, s_n, \cdots)$ be the sequence of indeterminates $s_i$. We define an operation $\lambda_s$ on $K_{S^1}(M)$ (and $K(M)$) by setting

$$\lambda_s(\zeta)=\prod_k (1+s_1\zeta_k+s_2\zeta_k^2+\cdots)$$

for a complex vector bundle $\zeta=\zeta_1\oplus\cdots\oplus\zeta_n$ which is a sum of line bundles $\zeta_1, \cdots, \zeta_n$. Then it is clear that the Chern numbers of $M$ can be recovered from

$$p_!(\lambda_s(\tau(M)))\in \boldsymbol{Z}[s_1, s_2, \cdots],$$

where $p_!: K(M)\to \boldsymbol{Z}$. But if we denote by $p_!^G: K_{S^1}(M)\to \boldsymbol{Z}[t, t^{-1}]$ the equivariant Gysin homomorphism then

$$\begin{aligned} (5.27) \qquad p_!(\lambda_s(\tau(M))) &= p_!^G(\lambda_s(\tau(M)))|_{t=1} \\ &= \sum_i \left.\frac{\prod_k (1+s_1t^{m_{ik}}+s_2t^{2m_{ik}}+\cdots)}{\prod_k (1-t^{-m_{ik}})}\right|_{t=1}. \end{aligned}$$

Since the weights $\{m_{ik}\}=\{a_i-a_j\}_{j\neq i}$ are the same as $CP^n$ by (3.11) we get

$$p_!(\lambda_s(\tau(M)))=p_!(\lambda_s(\tau(CP^n))).$$

Hence $M$ has the same Chern numbers as $CP^n$. This completes the proof of Theorem 5.7.

Corollary 5.8 follows from Theorem 5.7 and Proposition 3.18.

To prove Corollary 5.9 we first note that $x$ belongs to $\mathrm{ad}\, H^2(M)$ since $H^1(M; \boldsymbol{Z})=0$ by assumption (cf. [11]) and $x$ is necessarily basic. Then, by Corollary 5.8, $M$ has the same Chern numbers as $CP^n$ and $x^n[M]=1$. On the other hand $M$ has the same rational cohomology ring as $CP^n$ by assumption. It is easy to see from these facts that the Chern classes of $M$ are formally same as those of $CP^n$, that is,

$$c(M)=(1+x)^{n+1} \quad \text{mod torsion}.$$

This completes the proof.

## §6. Quasi-ample line bundles and theorems of the Kobayashi-Ochiai type (continued).

Kobayashi-Ochiai [14] proved that if there is an ample line bundle $\xi$ over a compact complex manifold $M$ of complex dimension $n$ such that $c_1(M)=nc_1(\xi)$ then $M$ is biholomorphic to the complex quadric $Q_n$. We shall give analogous theorems in our context.

We take the following model of $Q_n$ defined by the equation

$$z_0z_1+\cdots+z_nz_{n+1}=0 \quad \text{when} \quad n \quad \text{is even}$$

and

$$z_0z_1+\cdots+z_{n-1}z_n+z_{n+1}^2=0 \quad \text{when} \quad n \quad \text{is odd}$$

in the $(n+1)$-dimensional complex projective space $CP^{n+1}$. Set $h=[n/2]$ and let $b_0, b_1, \cdots, b_h$ be mutually distinct integers. We define an $S^1$ action on $Q_n$ by

$$z[z_0, z_1, \cdots, z_n, z_{n+1}]=[z^{b_0}z_0, z^{-b_0}z_1, \cdots, z^{b_h}z_n, z^{-b_h}z_{n+1}] \quad \text{when } n \text{ is even,}$$

and

$$z[z_0, z_1, \cdots, z_n, z_{n+1}]=[z^{b_0}z_0, z^{-b_0}z_1, \cdots, z^{b_h}z_{n-1}, z^{-b_h}z_n, z_{n+1}] \quad \text{when } n \text{ is odd.}$$

The fixed points are all isolated and given by

$$P_i=[0, \cdots, 0, 1, 0, \cdots, 0] \quad (1 \text{ at } i\text{-th factor})$$

where $0\leqq i\leqq n+1$ when $n$ is even and $0\leqq i\leqq n$ when $n$ is odd. It follows that $\chi(Q_n)=n+2$ when $n$ is even and $\chi(Q_n)=n+1$ when $n$ is odd.

Let $\xi$ be the restriction of the hyperplane bundle on $CP^{n+1}$ to $Q_n$. We lift the $S^1$-action on $Q_n$ to $\xi$ as in Section 3. The weight $a_i$ of $\xi$ at $P_i$ is given by

$$a_{2j}=-b_j, \quad a_{2j+1}=b_j \quad (0\leqq j\leqq h)$$

and the weights of $M$ at $P_i$ are

$$\{m_{ik}\}=\{a_i-a_j\}_{j\neq i,i'}, \quad 0\leqq i\leqq n+1, \quad \text{when} \quad n \quad \text{is even,}$$

and

$$\{m_{ik}\}=\{a_i-a_j\}_{j\neq i,i'}\cup\{a_i\},\quad 0\leqq i\leqq n,\quad \text{when}\quad n\quad \text{is odd,}$$

where for each $i$ we set

$$i'=\begin{cases} i+1 & \text{when}\quad i\quad \text{is even}\\ i-1 & \text{when}\quad i\quad \text{is odd.}\end{cases}$$

Note that $i'$ is characterized by the relation

$$a_i+a_{i'}=0.$$

Returning to our context, $M$ will be a connected closed unitary manifold having only isolated fixed points $P_1,\cdots P_\chi$ $(\chi>0)$ and $\xi$ will be a quasi-ample line bundle over $M$ satisfying condition

$$D:\quad \sum m_{ik}=k_0a_i+d \qquad \text{for all}\ i$$

as before. The following is an analogue of Kobayashi-Ochiai's theorem for odd $n$.

THEOREM 6.1. *Suppose $M$ is an almost complex $S^1$-manifold of dimension $2n$ and $\chi=n+1$. If $k_0=n$ then $n$ is necessarily odd and $\xi$ is basic provided $n>1$. Moreover once $\{a_i\}$ are normalized to fulfil $\sum a_j=0$ then for each $i$ there exists a unique $i'$ such that $a_i+a_{i'}=0$ and the weights $\{m_{ik}\}$ at $P_i$ are given by*

$$\{m_{ik}\}=\{a_i-a_j\}_{j\neq i,i'}\cup\{a_i\}\ .$$

*The almost complex manifold $M$ is unitary cobordant to $Q_n$. Furthermore we have $x^n[M]=2$ where $x=c_1(\xi)$.*

COROLLARY 6.2. *Let $M$ be an almost complex manifold of dimension $2n$ having property $P$ and let $x\in \mathrm{ad}\,H^2(M)$ be such that $x^n\neq 0$ and*

$$c_1(M)=nx \quad \textit{mod torsion.}$$

*Then the same conclusions as Theorem 6.1 hold.*

COROLLARY 6.3. *Let $M$ be an almost complex $S^1$-manifold (having only isolated fixed points) of dimension $2n$ which has the same rational cohomology ring as $CP^n$. Let $x$ be a generator of $H^2(M;\boldsymbol{Z})$ mod torsion. If*

$$c_1(M)=nx \quad \textit{mod torsion}$$

*then $n$ is necessarily an odd integer* $>1$ *and*

$$c(M)=(1+x)^{n+2}(1+2x)^{-1} \quad \textit{mod torsion.}$$

*Moreover* $x^n[M]=2$.

The proof is preceded by several lemmas. For these lemmas $M$ will be a unitary $S^1$-manifold of dimension $2n$ having property $P$ and $\xi$ a quasi-ample line bundle satisfying the condition $D$. We set $x=c_1(\xi)$ and $K=x^n[M]$ as before.

LEMMA 6.4. *If* $k_0=n$ *then*

$$\varphi_i(t)=\operatorname{sgn} K(1+t^{a_i}) \quad \textit{for all} \quad i$$

*provided that we normalize the* $a_i$ *so that*

$$d+\sum a_j=0.$$

*In particular,* $r_0=T[M]=\operatorname{sgn} K$ *and* $K=\pm 2$. *Moreover none of the* $a_i$ *can be zero. In the almost complex case we have*

$$\varphi_i(t)=1+t^{a_i} \quad \textit{for all} \quad i$$

*and* $K=2$, $T[M]=1$.

*Note.* If we replace $a_i$ by $a_i+a$ simultaneously then $d+\sum a_j$ gets replaced by $d+\sum a_j+l_0a$ where $l_0=n+1-k_0$ as before. Hence in case $l_0>0$ we can normalize so that $d+\sum a_j=0$ by taking $l_0$-fold covering action if necessary. Notice $l_0=1$ in the present case $k_0=n$. Moreover notice that if $\varepsilon_i=1$ for all $i$ then $d+\sum a_j=(l_0/(n+1))\sum a_j$.

PROOF. Suppose $k_0=n$. Then by Theorem 4.2 we have

$$\varphi_i(t)=r_0(1+t^{a_i}), \quad r_0=T[M]$$

and also by Remark 4.14 we have

$$\varphi_i(t)=\operatorname{sgn} Kt^{v_i}\psi_i(t)$$

where $\psi_i(t)$ is a product of cyclotomic polynomials. Thus we must have $r_0=\operatorname{sgn} K$ and

$$K=\varphi_i(1)=2r_0=2 \operatorname{sgn} K.$$

Therefore $K=\pm 2$.

Assume that $a_i=0$ for some $i$. Then, for that $i$, we have

$$2=t^{v_i}\psi_i(t)$$

which is a contradiction because $\psi_i(t)$ is a product of cyclotomic polynomials. Q.E.D.

LEMMA 6.5. *Suppose that $k_0=n$. If we normalize the $a_i$ so that $d+\sum a_j=0$ then, for each $i$, there exists a unique $i'$ such that*

$$a_i+a_{i'}=0.$$

*Moreover the weights $\{m_{ik}\}$ at $P_i$ satisfy the following conditions:*

$$\{|m_{ik}|\}=\{|a_i-a_j|\}_{j\neq i,i'}\cup\{|a_i|\} \quad \textit{for all } i \tag{6.6}$$

*and*

$$(\mathrm{sgn}\, K)\varepsilon_i\, \mathrm{sgn} \prod_k m_{ik}=\mathrm{sgn}\prod_{j\neq i}(a_i-a_j) \quad \textit{for all } i. \tag{6.7}$$

*Note.* In the almost complex case, (6.7) reduces to

$$\mathrm{sgn}\prod_k m_{ik}=\mathrm{sgn}\prod_{j\neq i}(a_i-a_j). \tag{6.7$'$}$$

PROOF. From Lemma 6.4 it follows that

$$\varepsilon_i\frac{\prod_{j\neq i}(1-t^{a_i-a_j})}{\prod_k(1-t^{m_{ik}})}=\mathrm{sgn}\, K(1+t^{a_i}).$$

Noting that $a_i\neq 0$ we see that

$$\frac{(1-t^{a_i})\prod_{j\neq i}(1-t^{a_i-a_j})}{(1-t^{2a_i})\prod_k(1-t^{m_{ik}})}=\pm 1.$$

Hence

$$2|a_i|=|a_i-a_{i'}| \quad \text{for some } i' \tag{6.8}$$

and we have

$$\{|m_{ik}|\}=\{|a_i-a_j|\}_{j\neq i,i'}\cup\{|a_i|\}. \tag{6.9}$$

Note that (6.8) is equivalent to either

$$2a_i=a_i-a_{i'}, \quad \text{i.e.} \quad a_{i'}=-a_i \tag{6.8$'$}$$

or

$$(6.8)'' \qquad 2a_i = -(a_i - a_{i'}), \quad \text{i.e.} \quad a_{i'} = 3a_i.$$

We claim that there exists a unique $i'$ such that (6.8)$'$ holds. We prove this by descending induction on absolute values $|a_j|$. If $|a_i| = \max\{|a_j|\}$, then (6.8)$''$ can not occur. For otherwise we would have a contradiction

$$|a_{i'}| = 3|a_i| > \max\{|a_j|\}.$$

Therefore $i'$ which satisfies (6.8) must be unique and (6.8)$'$ holds.

Assume that for each $a_j$ with $|a_j| > |a_i|$ there exists a (necessarily) unique $j'$ such that $a_{j'} = -a_j$ but (6.8)$'$ does not hold for $i'$ which satisfies (6.8). Then $a_{i'} = 3a_i$ by (6.8)$''$.

Set $m = 2|a_i|$. Let $X$ be the component of the fixed point set of the restricted $\boldsymbol{Z}_m$-action containing $P_{i'}$. If we put $\dim X = 2n'$ then $n'$ equals the number of those $m_{i'k}$ which are divisible by $m$; hence, by (6.9), $n'$ is equal to the number of $j \neq i'$ such that $m$ divides $a_{i'} - a_j$. On the other hand if we put $\chi' = \chi(X)$ then $n' + 1 \leqq \chi'$ by Corollary 3.8 and moreover $m$ divides $a_{i'} - a_j$ if $P_j \in X$. Therefore we see that $n' + 1 = \chi'$ and $P_j$ belongs to $X$ if and only if $m$ devides $a_{i'} - a_j$. In particular $P_i \in X$.

We now count the number $\rho$ of $m_{jk}$ which are weights of $X$ and such that $|m_{jk}| = m$. First take $j \neq i, i', i''$. Note that $a_{i'} = 3a_i$ and $a_{i'} + a_{i''} = 0$. Then the number of $m_{jk}$ such that $|m_{jk}| = m$ and that of $m_{j'k}$ such that $|m_{j'k}| = m$ are equal by virtue of (6.9) and the fact that $a_{j'} = -a_j$. Next the number of $m_{i'k}$ such that $|m_{i'k}| = m$ is greater by one than that of $m_{j''k}$ such that $|m_{j''k}| = m$ because there is a weight $m_{i'k}$ such that $|m_{i'k}| = |a_{i'} - a_i|$ while there is no $a_j$ with $a_j = -a_i$ (we note again that $a_{i''} = -a_{i'}$). Finally there are no weights $m_{ik}$ such that $|m_{ik}| = m$ by virtue of (6.9). Therefore the total number $\rho$ is odd. But this contradicts Proposition 2.12. Hence there must be $i'$ satisfying (6.8)$'$.

Once the above claim is proven then (6.6) follows from (6.9). On the other hand (6.7) follows from Remark 4.14. As for (6.7)$'$ we use the fact $K=2$ which is proved in Lemma 6.4. Q.E.D.

LEMMA 6.10. *Suppose that for each $i$ there exists a unique $i'$ such that $a_{i'} = -a_i$ and the condition* (6.6) *is satisfied. Then $K = \pm 2$ and $l_0 - 1$ and $d + \sum a_j$ are both even. Moreover $r_s(t) = 0$ for $s \neq (1/2)(l_0 - 1)$,* $(1/2)(l_0 + 1)$ *and*

$$\varphi_i(t)=\operatorname{sgn} Kt^{-d/2}t^{(l_0-1)a_i/2}(1+t^{a_i}).$$

PROOF. $K=\pm 2$ follows from Proposition 3.7. Then by (4.15) in Remark 4.14 we have

$$\begin{aligned}\varphi_i(t)&=\operatorname{sgn} Kt^{\{l_0a_i-|a_i|-(d+\Sigma a_j)\}/2}(1+t^{|a_i|})\\ &=\operatorname{sgn} Kt^{\{(l_0-1)a_i-d\}/2}(1+t^{a_i})\ .\end{aligned}$$

Here $(1/2)\{(l_0-1)a_i-d\}\in \boldsymbol{Z}$.

We shall prove that $l_0-1$ is even; then automatically $d$ would be even. For that purpose we may assume all the $a_i$ are even by taking double covering action if necessary. Then $d$ must be even. Setting $a_i=2c_i$ we can write $\varphi_i(t)$ as

$$\varphi_i(t)=\operatorname{sgn} Kt^{-d/2}(t^{(l_0-1)c_i}+t^{(l_0+1)c_i}). \tag{6.11}$$

On the other hand by Theorem 4.2 we have

$$\varphi_i(t)=r_0+r_1(t)t^{2c_i}+r_2(t)t^{4c_i}+\cdots+r_{l_0}(t)t^{2l_0c_i}$$

where

$$r_{l_0-s}(t)=r_s(t^{-1})t^{-(d+\Sigma a_j)}=r_s(t^{-1})t^{-d}.$$

Comparing this with (6.11) we see that $l_0$ is odd. Moreover we see that $r_s(t)=0$ for $s\neq(l_0-1)/2$, $(l_0+1)/2$ and

$$r_{(l_0-1)/2}(t)=r_{(l_0+1)/2}(t)=\operatorname{sgn} Kt^{-d/2}. \qquad \text{Q.E.D.}$$

COROLLARY 6.12. *In Lemma 6.10 if we assume moreover that $T[M]\neq 0$ then $l_0=1$ i.e. $k_0=n$. In particular, in the almost complex case, if $\rho_n(=\rho_0)\neq 0$ then $k_0=n$.*

PROOF. Since $r_0=T[M]\neq 0$ we must have $l_0-1=0$. Q.E.D.

LEMMA 6.13. *We continue with the situation of Lemma 6.10. Let $X$ be a component of the restricted $\boldsymbol{Z}_m$-action which contains a fixed point $P_i$ of the given $S^1$-action where $m$ is an integer $>1$. We set $\dim X=2n'$ and $\chi(X)=\chi'$. We assume $n'>0$. Then $X$ is of one of the following three types:*

(i) $\chi'=n'+1$ *and (5.11) is satisfied.*

(ii) $\chi'=n'+1$ *and (6.6) is satisfied.*

(iii) $\chi'=n'+2$ *and*

$$\{|m_{ik}|\}=\{|a_i-a_j|\}_{j\neq i,i',P_j\in X} \quad \textit{for all } P_i\in X. \tag{6.14}$$

*Moreover $m$ is even in case* (iii) *and if $X'$ is the component of $\boldsymbol{Z}_{m'}$-action containing $X$ where $m=2m'$ then $X'$ is of type* (ii).

PROOF. Let $P_i \in X$. $n'$ is equal to the number of $k$ such that $m \mid |m_{ik}|$. Note that $|m_{ik}|=|a_i-a_j|$ for $j\neq i, i'$ or $|m_{ik}|=|a_i|$. We divide into three cases:

(i) $m \nmid 2a_i$.

(ii) $m|a_i$.

(iii) $m|2a_i$ but $m \nmid a_i$.

By a similar argument as in the proof of Lemma 5.16 we see easily that in case (i) $\chi'=n'+1$ and (5.11) is satisfied while in case (ii) $\chi'=n'+1$ and (6.6) is satisfied.

In case (iii) we see similarly that the absolute values of weights of $X$ at $P_i$ are

$$W=\{|a_i-a_j|;\ j\neq i, i', m \mid |a_i-a_j|\}$$

and $n'$ is the cardinality of $W$. Note that if $|a_i-a_j| \in W$ then $|a_i-a_{j'}| \in W$ where $a_{j'}=-a_j$. There are two possibilities: $\chi'=n'+2$ or $\chi'=n'+1$. Note that $\chi'=n'+2$ if and only if all the $P_j$ such that $m \mid |a_i-a_j|$ belong to $X$.

Assume that $\chi'=n'+1$. Then there exists just one $P_j$ (which may be $P_{i'}$) which does not belong to $X$ but $m \mid |a_i-a_j|$. Let $X'$ be the component of the fixed point set of the restricted $\boldsymbol{Z}_m$-action containing this $P_j$. By the same reasoning as for $P_i$ we see that $\chi(X')=n'+2$ or $\chi(X')=n'+1$. Moreover if $\chi(X')=n'+2$ then $P_i$ must belong to $X'$ which is absurd. If $\chi(X')=n'+1$ then $P_i$ is the only $P_k$ such that $m \mid |a_j-a_k|$ or equivalently $m \mid |a_i-a_k|$ and $k\neq j$. Since $X\cap X'$ is empty this implies that $X=\{P_i\}$ which contradicts the assumption $\dim X>0$. Therefore $\chi'$ must equal $n'+2$. Moreover (6.14) follows from the form of $W$.

Finally, in case (iii), it is clear that $m$ can be written as $m=2m'$ and $m'$ divides $a_i$. Moreover the component of the fixed point of $\boldsymbol{Z}_{m'}$-action containing $X$ is of type (ii). Q.E.D.

We are now ready to prove Theorem 6.1. The proof will proceed in a manner parallel to that of Theorem 5.7. Thus we shall only give a sketch of proof and mainly stress the points where we need modifications or additional arguments.

So suppose $M$ satisfies all the assumptions of Theorem 6.1. In particular we are in the almost complex case. By Lemmas 6.4 and 6.5 $n$ is necessarily odd and $K=2$. Assume that $\xi$ is not basic. Then there are integers $r>1$ and $d_1$ such that

$$a_i = ra'_i + d_1$$

for any $i$ where $\{a'_i\}$ is a basic sequence. Then

$$\sum m_{ik} = nra'_i + d',$$

and $nr \leqq n+1$ by Theorem 5.1. This is impossible for $n>1$. Thus $\xi$ is basic provided $n>1$.

We normalize $\{a_i\}$ so that $\sum a_j = 0$. Then since we are in the almost complex case, $d + \sum a_j = (l_0/(n+1)) \sum a_j = 0$ and, by Lemma 6.5, there exists a unique $i'$ for each $i$ such that $a_{i'} = -a_i$. We denote by $A_i$ the subset of

$$C_i = \{a_i - a_j\}_{j \neq i, i'} \cup \{a_i\}$$

consisting of those $a_i - a_j$ (or $a_i$) which are equal to some $m_{ik}$. Note that this definition is licit since $a_i - a_j \neq a_i - a_k$ if $j \neq k$ and $a_i - a_j \neq a_i$. We also set $B_i = C_i - A_i$. $B_i$ satisfies the following conditions:

(6.15) $$\sum_{c_{ik} \in B_i} c_{ik} = 0,$$

(6.16) the cardinality of $B_i$ is even and $\neq 2$.

The proof is similar to that of (5.23) and (5.24).

We shall prove $B_i = \varnothing$ for all $i$. We proceed by induction on $n = (1/2) \dim M$. We know $n$ is odd. Setting $2h = n+1$ we rename the fixed points as $P_h, \cdots, P_1, P_{-1}, \cdots, P_{-h}$ so that

$$a_h > \cdots > a_1 \quad \text{and} \quad a_{-i} = -a_i.$$

Note that $i' = -i$ with this renaming.

The case $n=1$ is clear from (6.16). We assume $n>1$ and the statement is true for manifolds of complex dimension less than $n$. The proof will proceed by descending induction starting from $P_h$ down to $P_1$, then by ascending induction starting from $P_{-h}$ up to $P_{-1}$.

Claim ($h$): $B_h = \varnothing$.

This is clear from (6.15) because $c_{hk} > 0$ for all $k$; cf. proof of Claim ($n$) in §5.

Claim ($h-1$): $B_{h-1} = \varnothing$.

We devide into two cases $a_h - a_{h-1} \leqq 2$ and $a_h - a_{h-1} > 2$. If $a_h - a_{h-1} = 1$ or 2 then $a_{h-1} - a_h \in A_{h-1}$ by virtue of (6.16) and then $B_{h-1} = \varnothing$ by virtue of (6.15); cf. proof of Claim ($n-1$) in §5.

Assume $m=a_h-a_{h-1}>2$. We consider the component $X$ of the fixed point set of the restricted $\boldsymbol{Z}_m$-action containing $P_h$. By Lemma 6.13 three types are possible for $X$. If $X$ is of type (i) then, by using the fact $B_h=\emptyset$ and Corollary 5.15, we see that $k_0(X)=n(X)+1$. We then apply Theorem 5.7 to conclude that $a_{h-1}-a_h \in A_{h-1}$. Since $a_{h-1}-a_h$ is the only negative element in $C_{h-1}$ the set $B_{h-1}$ must be empty by (6.15).

If $X$ is of type (ii) then we deduce that $k_0(X)=n(X)$ from the fact $B_h=\emptyset$ and Corollary 6.12 in a similar way to the proof of Theorem 5.7. Then from the dimensional induction assumption applied to $X$ it follows that $a_{h-1}-a_h \in A_{h-1}$ and hence $B_{h-1}=\emptyset$.

If $X$ is of type (iii) then we replace $X$ by $X'$ described in Lemma 6.13 which is of type (ii). Then by the same argument as above we see $B_{h-1}=\emptyset$. Thus Claim $(h-1)$ is proved.

We now assume inductively that $B_j=\emptyset$ for $j>s\geqq 1$. Then we show

$$a_s-a_h,\ a_s-a_{h-1},\ \cdots,\ a_s-a_{s+1} \in A_s$$

by similar arguments to those used in the proof of Theorem 5.7. Namely assuming inductively we have proved $a_s-a_j \in A_s$ for $j>s'$ we set $m=a_{s'}-a_s$. In case $m=1$ or 2 we use (6.16) to deduce $a_s-a_{s'} \in A_s$. In case $m>2$ we consider the component $X$ of the fixed point set the restricted $\boldsymbol{Z}_m$-action containing $P_{s'}$. If $X$ is of type (iii) then we replace it by $X'$ of type (ii) given in Lemma 6.13. Then using all the induction assumptions applied on $X$ (or $X'$) and using Theorem 5.7 in case $X$ is of type (i) we deduce that $a_s-a_{s'} \in A_s$ in a similar manner to the proof of Claim $(h-1)$.

The proof of the fact $B_j=\emptyset$ for $j<0$ is entirely similar. We start from $j=-h$ and proceed by induction.

Finally we see that $M$ is unitary cobordant to $Q_n$ by a similar argument to the proof of Theorem 5.7 using (5.27). The relevant fact is that the weights $\{m_{ik}\}$ of $M$ are the same as that of $Q_n$ with the standard action. This completes the proof of Theorem 6.1.

Corollary 6.2 follows from Theorem 6.1 and Proposition 3.18.

Corollary 6.3 is a consequence of the fact that $x$ belongs to ad $H^2(M)$ and $M$ has the same Chern number as $Q_n$ and $x^n[M]=2$. Note that $Q_n$ has the same rational cohomology ring as $\boldsymbol{C}P^n$ when $n$ is odd and $x'^n[Q_n]=2$ for a suitable generator $x' \in H^2(Q_n;\boldsymbol{Z})$. Moreover

$$c(Q_n)=(1+x')^{n+2}(1+2x')^{-1}.$$

Then it is easily seen the Chern classes of $M$ are formally same as those of $Q_n$. This completes the proof of Corollary 6.3.

We turn to the case where $\chi=n+2$ and $k_0=n$. In this case $l_0=\chi-k_0=2$. Hence

$$\varphi_i(t)=r_0+r_1(t)t^{a_i}+r_2(t)t^{2a_i} \qquad \text{for all } i.$$

THEOREM 6.17. *Suppose $M$ is an almost complex $S^1$-manifold of dimension $2n$ and $\chi=n+2$. If $k_0=n$ and $r_1(t)=0$ then $n$ is necessarily even and $\xi$ is basic. Moreover once $\{a_i\}$ are normalized to fulfil $\sum a_j=0$ then for each $i$ there exists a unique $i'$ such that $a_i+a_{i'}=0$ and the weights $\{m_{ik}\}$ at $P_i$ are given by*

$$\{m_{ik}\}=\{a_i-a_j\}_{j\neq i,i'}.$$

*The almost complex manifold $M$ is unitary cobordant to $Q_n$. Furthermore we have $x^n[M]=2$ where $x=c_1(\xi)$.*

REMARK 6.18. The condition $k_0=n$ can be replaced by the following: There exists a fine $x\in \operatorname{ad} H^2(M)$ such that

$$c_1(M)=nx \quad \text{mod torsion.}$$

The proof of Theorem 6.17 is almost parallel to that of Theorem 6.1. So we shall only give indication. Instead of Lemmas 6.4, 6.5, 6.10, Corollary 6.12, Lemma 6.13 we use the following lemmas. Let $M$ be a unitary $S^1$-manifold of dimension $2n$ and $\xi$ a quasi-ample line bundle satisfying the condition $D$. We set $x=c_1(\xi)$ and $K=x^n[M]$.

LEMMA 6.19. *If $\chi=n+2$, $k_0=n$ and $r_1(t)=0$ then*

$$\varphi_i(t)=\operatorname{sgn} K(1-t^{2a_i}) \qquad \textit{for all } i$$

*provided that we normalize the $a_i$ so that*

$$d+\sum a_j=0.$$

*In particular $r_0=T[M]=\operatorname{sgn} K$ and $K=\pm 2$. Moreover none of the $a_i$ can be zero. In the almost complex case we have*

$$\varphi_i(t)=1-t^{2a_i} \qquad \textit{for all } i$$

*and $K=2$. $T[M]=1$.*

LEMMA 6.20. *Suppose that $\chi=n+2$ and $k_0=n$. If we normalize the $a_i$ so that $d+\sum a_j=0$ then, for each $i$, there exists a unique $i'$ such that*

$a_i+a_{i'}=0$. *Moreover the weights* $\{m_{ik}\}$ *satisfy the following conditions*

$$(6.21) \qquad \{|m_{ik}|\}_{k=1}^{n}=\{|a_i-a_j|\}_{j\neq i,i'} \qquad \text{for all } i,$$

*and*

$$(6.22) \qquad (\operatorname{sgn} K)\varepsilon_i \operatorname{sgn}\prod_k m_{ik}=\operatorname{sgn}\prod_{j\neq i}(a_i-a_j) \qquad \text{for all } i.$$

LEMMA 6.23. *Suppose that* $\chi=n+2$ *and that there exists* $i'$ *for each* $i$ *such that* $a_{i'}=-a_i$ *and* (6.21) *is satisfied. Then* $l_0$ *and* $d+\sum a_j$ *are both even. Moreover* $r_s(t)=0$ *for* $s\neq l_0/2-1$, $l_0/2$ *and*

$$\varphi_i(t)=\pm t^{-(1/2)(d+\sum a_j)}t^{((1/2)l_0-1)a_i}(1-t^{2a_i}).$$

COROLLARY 6.24. *In Lemma* 6.23 *if we assume moreover that* $T[M]\neq 0$ *then* $l_0=2$ *i.e.* $k_0=n$, *and* $r_1(t)=0$. *In particular, in the almost complex case, if* $\rho_n(=\rho_0)\neq 0$ *then* $k_0=n$ *and* $r_1(t)=0$.

LEMMA 6.25. *We continue with the situation of Lemma* 6.23. *Let* $X$ *be a component of the fixed point set of the restricted* $Z_m$*-action containing a fixed point* $P_i$ *of the given* $S^1$*-action. We set* $\dim X=2n'$ *and* $\chi(X)=\chi'$. *We assume* $n'>0$. *Then* $X$ *is either of the following two types*:

(i) $\chi'=n'+1$ *and* (5.11) *is satisfied.*

(ii) $\chi'=n'+2$ *and* (6.21) *is satisfied.*

First we prove Lemma 6.19. If $\chi=n+2$, $k_0=n$ and $d+\sum a_j=0$ then by Theorem 4.2 we have

$$\varphi_i(t)=r_0(1-t^{2a_i}).$$

From this and Remark 4.14 we see that $r_0=\pm 1$. On the other hand

$$K=\varphi_{ij}(1)=2r_0$$

by Proposition 5.3 and Lemma 5.5 for $i\neq j$. Therefore

$$K=\pm 2 \quad\text{and}\quad T[M]=r_0=\operatorname{sgn} K.$$

This completes the proof of Lemma 6.19.

The proof of Lemma 6.20 is similar to that of Lemma 6.5. From Lemma 6.19 we see that

$$\frac{\prod_{j\neq i}(1-t^{a_i-a_j})}{(1-t^{2a_i})\prod(1-t^{m_{ik}})}=\pm 1.$$

Hence it follows that for each $i$ there exists $i'$ such that $2|a_i|=|a_i-a_{i'}|$ and

$$\{|m_{ik}|\}=\{|a_i-a_j|\}_{j\neq i,i'}.$$

To show that there exists a unique $i'$ for each $i$ such that $a_{i'}=-a_i$ we can argue in an entirely similar manner to the argument used in the proof of Lemma 6.5.

The proof of Lemma 6.23 and Corollary 6.24 is similar to that of Lemma 6.10 and Corollary 6.12.

To prove Lemma 6.25 we devide into two cases: (i) $m\nmid 2a_i$ and (ii) $m|2a_i$. Then the rest of proof is similar to that of Lemma 6.13.

Once the above lemmas are proved the proof of Theorem 6.17 proceeds like that of Theorem 6.1. The details are left to the reader.

There are examples in which $\chi=n+2$, $k_0=n$ but $r_1(t)\neq 0$. They are given in §7. A sufficient condition for $r_1(t)=0$ to hold was given in Corollary 6.24. Here we shall give another one.

PROPOSITION 6.26. *Let $M$ be a unitary $S^1$-manifold of dimension $2n$ having only isolated fixed points with $\chi(M)=n+2$. Assume that $M$ can be embedded as a unitary $S^1$-submanifold into a unitary $S^1$-manifold $M'$ of dimension $2n+2$ having only isolated fixed points with $\chi(M')=n+2$ and that there exists a quasi-ample line bundle $\xi'$ over $M'$ satisfying the condition $D$:*

$$\sum m'_{ik}=(n+1)a_i+d' \qquad \textit{for all } i, \tag{6.27}$$

*where the summation is extended over the weights $\{m'_{ik}\}$ of $M'$ at the fixed point $P_i$ and $a_i$ is the weight of $\xi'$ at $P_i$. Then the fixed points of $M$ and $M'$ coincide and the weight of $M'$ normal to $M$ at $P_i$ is of the form $a_i+d_1$ where $d_1$ is independent of $i$. The line bundle $\xi=\xi'|M$ is quasi-ample and it satisfies the condition $D$:*

$$\sum m_{ik}=na_i+d \qquad \textit{for all } i.$$

*Moreover $r_1(t)=0$.*

PROOF. Since the number of fixed points of $M$ and $M'$ are both equal to $n+2$ they have the same fixed points. We may suppose that the $a_i$ are normalized so that $d'+\sum a_j=0$. Then, by Lemma 6.5 applied to $M'$, we see that there exists $i'$ for each $i$ such that $a_{i'}=-a_i$ and (6.6) is satisfied. We know also that $\xi'$ is basic; cf. the proof of

Theorem 6.1 where we did not need the almost complex assumption for that fact.

Now the normal bundle $\nu$ of $M$ in $M'$ is an $S^1$-line bundle in a natural way. Its weight $a'_i$ at $P_i$ is the weight of $M'$ normal to $M$. $a'_i$ can be written in the form

$$a'_i = ra_i + d_1 \tag{6.28}$$

since $\{a_i\}$ is a basic sequence. We claim in fact that $a'_i = a_i$, i.e. $r=1$, $d_1=0$. To prove this let $i_0$ be such that $a_{i_0}$ takes the maximum value among the $a_i$. Then all the $|m'_{i_0k}|$ and $|m'_{i'_0k}|$ are less than $2|a_{i_0}|$ by (6.6). In particular $|a'_{i_0}|$, $|a'_{i'_0}| < 2|a_{i_0}|$. Hence $|r|$ can not exceed 1.

On the other hand from (6.27) and (6.28) we get

$$\sum m_{ik} = (n+1-r)a_i + d' - d_1$$

where the summation is extended over the weights of $M$ at $P_i$. Thus $r$ cannot be equal to $-1$ by Theorem 5.1. The possibility $r=0$ is excluded by Remark 7.2.1 of the next section. Therefore $r=1$.

Assume now $d_1 \neq 0$, say $d_1 > 0$. Then by looking at (6.6) for $i=i_0$ we see that there exists $j$ such that $a_j = -d_1$. But then $a'_j = a_j + d_1 = 0$ would be a weight of $M'$ at $P_j$ which is a contradiction. If $d_1 < 0$ then looking at (6.6) for $i = i'_0$ leads also to a contradiction. Thus $d_1 = 0$.

We know by Lemma 6.4 that

$$\varepsilon_i \frac{\prod_{j\neq i} (1-t^{a_i-a_j})}{\prod (1-t^{m_{ik}})} = \pm(1+t^{a_i}).$$

From this and the above claim $a'_i = a_i$ it follows that

$$\varphi_i(t) = \varepsilon_i \frac{\prod_{j\neq i} (1-t^{a_i-a_j})}{\prod (1-t^{m_{ik}})} = \pm(1-t^{2a_i}).$$

This implies that $k_0 = n$ and $r_1(t) = 0$. Q.E.D.

As an immediate consequence of Proposition 6.26 and Theorem 6.17 we obtain

COROLLARY 6.29. *If in Proposition* 6.26 *we assume moreover that* $M'$ *is an almost complex* $S^1$*-manifold and* $M$ *is an almost complex* $S^1$*-submanifold of* $M'$ *then the weights of* $M$ *are*

$$\{m_{ik}\}=\{a_i-a_j\}_{j\neq i,i'}$$

*and M is unitary cobordant to* $Q_n$ *(n even).*

*We also have*

COROLLARY 6.30. *Let* $M'$ *be an almost complex* $S^1$*-manifold of complex dimension* $n+1$ *having only isolated fixed points. Suppose that* $M'$ *has the same rational cohomology ring as* $CP^{n+1}$ *and that*

$$c_1(M')=(n+1)x' \quad \textit{mod torsion}$$

*for a generator* $x'$ *of* $H^2(M'; \boldsymbol{Z})$ *mod torsion. If M is an almost complex* $S^1$*-submanifold of* $M'$ *of complex codimension* 1 *and with* $\chi(M)=n+2$. *Then the Chern classes of M are formally the same as those of* $Q_n$, *i.e.*

$$c(M)=(1+x)^{n+2}(1+2x)^{-1} \quad \textit{mod torsion}$$

*where* $x=j^*x'$ *(*$j: M\to M'$ *is the inclusion).*

PROOF. By Corollary 6.3 we have

$$c(M')=(1+x')^{n+3}(1+2x')^{-1} \quad \text{mod torsion.}$$

Since the complex codimension of $M$ in $M'$ is equal to 1 the Euler class $x=e(\nu)$ of the normal bundle is the restriction of some element in $H^2(M'; \boldsymbol{Z})$ and hence

$$x=e(\nu)=j^*(rx') \quad \text{mod torsion.}$$

But since the weight of $\nu$ at $P_i$ is equal to $a_i$, $r$ must equal 1. Therefore

$$\begin{aligned} c(M)&=j^*(c(M'))c(\nu)^{-1} \\ &=j^*((1+x')^{n+3}(1+2x')^{-1})(1+x)^{-1} \quad \text{mod torsion} \\ &=(1+x)^{n+2}(1+2x)^{-1} \quad \text{mod torsion.} \end{aligned}$$

## § 7. Concluding remarks.

**7.1.** In §6 before Proposition 6.26 we mentioned examples in which $\chi=n+2$, $k_0=n$ and $r_1(t)\neq 0$. Here we exhibit some of them. They come from actions on Hirzebruch surfaces [12]. Let $l$ be an integer $\geqq 0$ and $\sum_l$ the hypersurface in $CP^2\times CP^1$ defined by

$$z_1w_1^l-z_2w_2^l=0$$

where $([z_0, z_1, z_2], [w_1, w_2]) \in CP^2 \times CP^1$. Define an $S^1$ action on $\sum_l$ by

$$z([z_0, z_1, z_2][w_1, w_2]) = ([z_0, z^{b+lb'}z_1, z^b z_2], [w_1, z^{b'}w_2])$$

where $z \in S^1$ and $b, b'$ are integers. If we assume $bb' \neq 0$, $b+lb' \neq 0$ then the fixed points are all isolated and they are

$$P_1 = ([1, 0, 0], [1, 0]),\ P_2 = ([0, 0, 1], [1, 0]),$$
$$P_3 = ([1, 0, 0], [0, 1]),\ P_4 = ([0, 1, 0], [0, 1]).$$

In particular $\chi(\sum_l) = 4$. The weights $\{m_{ik}\}$ of $\sum_l$ at $P_i$ $(i=1, 2, 3, 4)$ are, in this order,

$$\{b, b'\},\ \{-b, b'\},\ \{b+lb', -b'\},\ \{-(b+lb'), -b'\}.$$

Let $\xi_1$ and $\xi_2$ be the pull-backs of the hyperplane bundles over $CP^2$ and $CP^1$ respectively. The $S^1$-action can be lifted to $\xi_1$ and $\xi_2$ as in §3 so that the weights at $P_i$ $(i=1, 2, 3, 4)$ are

$$0,\ -b,\ 0,\ -(b+lb') \quad \text{for} \quad \xi_1$$

and

$$0,\ 0,\ -b',\ -b' \quad \text{for} \quad \xi_2.$$

Set $\xi' = \xi_1^2 \xi_2^{-(l-2)}$ and $\xi = \xi_1 \xi_2^{-(l-2)/2}$ if $l$ is even. The weights $\{a_i'\}$ of $\xi'$ at $P_i$ $(i=1, 2, 3, 4)$ are

$$0,\ -2b,\ (l-2)b',\ -(2b+(l+2)b')$$

so that $\xi'$ is fine generically if $l \neq \pm 2$, and we have

$$\sum_k m_{ik} = a_i' + b + b' \qquad \text{for all } i.$$

Thus in case $l$ is even the weight $a_i = a_i'/2$ of $\xi$ at $P_i$ satisfies

$$\sum_k m_{ik} = 2a_i + b + b'.$$

In this case the functions $\varphi_i(t)$ are given by

$$\varphi_i(t) = 1 + r_1(t)t^{a_i} - t^{b+b'}t^{2a_i}$$

where

$$r_1(t) = \frac{(1-t^{lb'/2})(1-t^{-(b+lb'/2)})}{1-t^{b'}} t^{b+b'}.$$

The verification is left to the reader. Thus we always have $r_1(t)=0$ if $l=0$, but $r_1(t)\neq 0$ in general for $l\neq 0$. Note that $\sum_0=Q_2=CP^1\times CP^1$.

**7.2.** In Kobayashi-Ochiai's theorem the assumption $k_0=n+1$ implies $\chi=n+1$ and the assumption $k_0=n$ implies $\chi=n+1$ or $\chi=n+2$. In our context the conditions for the Euler characteristic $\chi$ are included in the assumption. We do not know those additional assumptions are really needed. But we shall prove the following Remark which was used in the proof of Proposition 6.26.

REMARK 7.2.1. Let $M$ and $\xi$ be as in §5. Suppose $k_0=n+1$ then $\chi=n+2$ cannot occur.

PROOF. We may assume that the $a_i$ are normalized to fulfil $d+\sum a_j=0$. Assume $k_0=n+1$ and $\chi=n+2$. Since $l_0=\chi-k_0=1$ we have

$$\varepsilon_i\frac{\prod_{j\neq i}(1-t^{a_i-a_j})}{\prod(1-t^{m_{ik}})}=\varphi_i(t)=r_0(1-t^{a_i})$$

by Theorem 4.2. It follows that, for each $i$, there exists $i'$ such that $|a_i-a_{i'}|=|a_i|$, i.e. $a_{i'}=0$ or $a_{i'}=2a_i$. But if $a_{i'}=0$ for some $i$ then we would have $\varphi_{i'}(t)=0$ which is a contradiction. Hence $a_{i'}=2a_i$ for all $i$. But this is also a contradiction because $\{a_i\}$ is a finite set of mutually distinct integers. Thus if $k_0=n+1$ then $\chi\neq n+2$. Q.E.D.

When $n=2$ we shall prove the following:

REMARK 7.2.2. Suppose $M$ is almost complex of complex dimension 2. If $k_0=3$ then the following relations hold:

$$x^2[M]=T[M]=\mathrm{Sign}\, M$$

where $x=c_1(\xi)$. In particular

$$c_1(M)^2=3c_2(M)=3\chi(M)>0.$$

PROOF. Set $s=\chi-n=\chi-2$. Then by Corollary 5.4 and Lemma 5.5 we have

$$\sum_{\nu=1}^{s}\frac{\prod_{j\neq i_1,\cdots,i_s}(a_{i_\nu}-a_j)}{\prod m_{i_\nu k}}=\varphi_i(1)=r_0=T[M]=x^2[M]\ .$$

If we apply Lemma 5.5 to the complex line bundle $\Lambda_C^2\tau(M)$ instead of

$\xi$ we obtain

$$\sum_{\nu=1}^{s} \frac{\prod_{j\neq i_1,\cdots,i_s} (d_{i_\nu}-d_j)}{\prod m_{i_\nu k}} = c_1(M)^2[M],$$

where $d_i = \sum_k m_{ik} = 3a_i + d$. Hence

$$c_1(M)^2[M] = 9x^2[M] = 9T[M].$$

Then using

$$T[M] = \frac{c_1(M)^2 + c_2(M)}{12}[M]$$

$$\text{Sign } M = \frac{c_1(M)^2 - 2c_2(M)}{3}[M]$$

we obtain the desired result. Q.E.D.

*Note.* It is proved in [8] that if $M$ is an almost complex manifold of complex dimension 2 which admits a non-trivial almost complex $S^1$-action then the equality $c_1(M)^2 = 3c_2(M)$ holds. It is known [18] that a compact complex surface satisfying $c_1^2 = 3c_2 > 0$ is either biholomorphic to $CP^2$ or covered by the unit ball.

**7.3.** Part of results in §2, §3 and §4 can be generalized to Spin$^c$ $S^1$-manifolds. We shall briefly indicate necessary modifications in Spin$^c$ case. The reader is referred to [8] for materials concerning $S^1$-actions on Spin$^c$ manifolds.

Let $M$ be a closed connected smooth $S^1$-manifold with an invariant Riemannian metric. Let $p: Q \to M$ denote the tangential orthonormal frame bundle. There is a canonical lifting of $S^1$-action on $Q$. If $w_2(M)$ comes from an integral class then there exists a Spin$^c$ structure $P \to Q$ of $Q$. $P \to Q$ can be regarded as an $S^1$-bundle. If the first Chern class $c_1$ of the $S^1$-bundle $P \to Q$ belongs to ad $H^2(Q)$ then there is a lifting of the $S^1$-action on $Q$ to the Spin$^c$-structure $P$ and it induces an $S^1$-action on the $S^1$-bundle $L = P/\text{Spin}(n) \to M$.

Now assume that the fixed points of the $S^1$-action on $M$ are all isolated. For each fixed point $P_i$ the linear isotropy representation of $S^1$ decomposes the tangent space $T_{P_i}M$ into a sum of weight spaces

$$T_{P_i}M = \sum t^{m_{ik}}.$$

However the weights $m_{ik}$ are well-determined only up to sign. For simplicity's sake we shall make convention that all the weights $m_{ik}$ be positive.

Let $\omega_i$ be the weight of $L$ with the induced $S^1$-action at $P_i$, i.e. $L|P_i=t^{\omega_i}$. Then $\omega_i-\sum_k m_{ik}$ is an even integer. We set

$$\lambda_i=(\omega_i-\sum_k m_{ik})/2.$$

The formula corresponding to Corollary 2.3 in this case reads as follows:

$$p_!(v)=\sum_i \frac{\varepsilon_i t^{\lambda_i} f_i^*(v)}{\prod (1-t^{-m_{ik}})}. \tag{7.3.1}$$

Next let $\xi$ be a fine complex line bundle. We take an $S^1$ complex line bundle $\eta$ whose underlying bundle is $\xi$. Then $\eta|P_i=t^{a_i}$ as before. We set

$$\varphi_i(t)=\varepsilon_i t^{-\lambda_i}\frac{\prod_{j\neq i}(1-t^{a_i-a_j})}{\prod_k (1-t^{m_{ik}})}.$$

Then, by (7.3.1), we have

$$\varphi_i(t^{-1})=p_!\Big(\prod_{j\neq i}(1-\eta^{-1}t^{a_j})\Big)\in \boldsymbol{Z}[t, t^{-1}].$$

It follows that the same conclusion $n+1\leqq\chi(M)$ as in Proposition 3.5 holds in this case too. Moreover the first part of Theorem 4.2 still holds in Spin$^c$ case. Namely we have:

THEOREM 7.3.2. *Let $M$ be a connected closed smooth $S^1$-manifold having only isolated fixed points. Assume that the $S^1$-action can be lifted to a* Spin$^c$*-structure $P$ on $M$. If $\xi$ is a fine line bundle over $M$ then the associated function $\varphi_i(t)$ can be expressed uniquely in the form*

$$\varphi_i(t)=r_0(t)+r_1(t)t^{a_i}+\cdots+r_{\chi-1}(t)t^{(\chi-1)a_i}$$

*where $r_0(t), \cdots, r_{\chi-1}(t)\in \boldsymbol{Z}[t, t^{-1}]$ are independent of $i$.*

The proof is essentially the same as that of Theorem 4.2.

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