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Algebraic cycles on certain abelian varieties and powers of special surfaces[*)]

By Fumio HAZAMA

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Introduction.

In this paper we introduce and exploit the notion of "stable non-degeneracy" of abelian varieties in order to study algebraic cycles on them. Further, using this notion, we show that the Hodge Conjecture holds for certain powers of special surfaces. In this introduction, we explain briefly how we have arrived at the notion of "stable non-degeneracy", and give a discription of the content of this paper.

First we recall the Hodge Conjecture mentioned above:

(0.1) The Hodge Conjecture: For any smooth projective variety X, $H^{2d}(X, \boldsymbol{Q}) \cap H^{d,d}(X) = \{$cohomology classes of algebraic cycles of codimension $d\}$ $\quad (0 \leqq d \leqq \dim X)$.

The elements of the left-hand group are called Hodge cycles. It is known that every algebraic cycle is a Hodge cycle. (0.1) says that the converse is also true. When $d=1$, (0.1) is known to hold (theorem of Lefschetz). From this and the hard Lefschetz theorem, it follows that (0.1) holds for varieties of dimension at most three. But for general higher dimensional varieties, (0.1) remains open. For the present state of knowledge on this conjecture, we refer the reader to Shioda's recent article [19].

We now introduce some notation. For a smooth projective variety X, let $\mathscr{B}^d(X) = H^{2d}(X, \boldsymbol{Q}) \cap H^{d,d}(X)$ and let $\mathscr{B}^*(X)$ denote the Hodge ring $\bigoplus_{d=0}^{\dim X} \mathscr{B}^d(X)$. Let $\mathscr{D}^*(X)$ denote the subring of $\mathscr{B}^*(X)$ generated by divisor classes. Note that if we are able to show $\mathscr{B}^*(X) = \mathscr{D}^*(X)$, then (0.1) holds for X. With the exception of the case treated by Shioda [18], all known cases of (0.1) for abelian varieties are proven

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by proving that $\mathscr{B}^*=\mathscr{D}^*$. Pohlmann [12] investigated abelian varieties of CM-type and proved for them the equivalence of the Hodge Conjecture with the Tate Conjecture. Even for abelian varieties of CM-type, there are many examples for which $\mathscr{B}^* \supsetneqq \mathscr{D}^*$ (see [12], [18] for example). We note that Shioda [18] gives some examples of abelian varieties for which $\mathscr{B}^* \supsetneqq \mathscr{D}^*$ and (0.1) holds.

The main theme of this paper is to study to what extent abelian varieties satisfy the condition $\mathscr{B}^*=\mathscr{D}^*$. Moreover, it will also be natural to consider not only an abelian variety A but also the power A^n. This consideration leads us to the following notion:

DEFINITION (2.2). We say that A is stably non-degenerate, if it satisfies the condition $\mathscr{B}^*(A^n)=\mathscr{D}^*(A^n)$ for all $n \geqq 1$.

We study this notion from the view point introduced by Mumford [8] that the Hodge cycles are characterized as the invariant elements in the cohomology space under the action of the Hodge group (see § 1 below). This view point was exploited systematically by Tankeev [20], [21], [22] to prove the validity of (0.1) for certain abelian varieties. In this paper we give a characterization of stable non-degeneracy and at the same time clarify the reason why his method works well for the abelian varieties considered in his papers mentioned above. Finally we use the notion of stable non-degeneracy as a guide to prove (0.1) for certain powers of algebraic surfaces.

Now we describe the contents of this paper in greater detail. The main objective of PART I is to characterize stably non-degenerate abelian varieties. For this, in § 1, we recall the definition and fundamental properties of the Hodge group of an abelian variety, and then investigate the relation between the Hodge group of a product of abelian varieties and that of each factor. In § 2, we introduce the notion of the "reduced dimension" of an abelian variety (see Definition (2.6) below). Using this, we are able to state the main theorem of PART I as follows:

THEOREM (2.7). *An abelian variety is stably non-degenerate if and only if the rank of its Hodge group over the complex number field is equal to its reduced dimension.*

Since we see that the rank is smaller than or equal to the reduced dimension, we are able to restate this theorem as follows: an abelian variety is stably non-degenerate if and only if the rank of its Hodge group is "as large as possible". This is proved in § 3 in the following manner. First we show that the theorem holds for simple abelian varieties and that the rank of the Hodge group is always smaller than or equal to the reduced dimension for any simple abelian variety. Next we show that, if the rank of the Hodge group of a product of some abelian varieties is as large as possible, then that of each factor is also as large as possible and the representation of the Lie algebra of the Hodge group of the product into the cohomology space is decomposed to the tensor product of each representation of that of each factor. Combining these two arguments, we see that, for any abelian variety, if the rank of the Hodge group is as large as possible, then it is stably non-degenerate. The converse is proved by the fact that the Hodge group of an abelian variety A is the largest subgroup of $GL(H^1(A, \boldsymbol{Q}))$ which leaves invariant the Hodge rings of A^n for all $n \geqq 1$. In § 4, applying the theorem (2.7), we uniformly reprove the Hodge Conjecture for some abelian varieties studied previously from certain view points, and also for their powers. In § 5, we give an example of "stably degenerate" abelian variety A without complex multiplication for which $\mathscr{B}^*(A) = \mathscr{D}^*(A)$ but $\mathscr{B}^*(A^2) \supsetneqq \mathscr{D}^*(A^2)$. This kind of phenomenon is very interesting since such an example is not known when A is of CM-type.

In PART II we show that the investigation of algebraic cycles on stably non-degenerate abelian varieties enables us to understand those on certain kind of projective varieties which are not necessarily abelian varieties. To be more precise, we show the validity of the Hodge Conjecture for the fourfolds which are powers of a Hilbert modular surface satisfying some arithmetic conditions (see (8.2)). This is the main result of PART II. In § 6, we recall the definition of Hilbert modular surface. In § 7, some preliminary lemmas concerning Hodge cycles on a product of projective varieties are proved. In § 8, we formulate the main theorem and prove it. The proof largely depends on Oda's work [11] on the Hodge structure of Hilbert modular surfaces. There he shows that, under some conditions, an essential part of the Hodge structure of a Hilbert modular surface is expressed as a direct sum of tensor products of the Hodge structure of some abelian varieties (see (8.1) and (8.4)). We notice that these abelian varieties are stably non-degenerate and, moreover, have "many real endomorphisms". For abelian varieties having the latter property, we determined the structure of the Hodge ring previously in [2]. In fact, the method of the proof of the main theorem of PART I is obtained by generalizing the one used there. Then the structure of the Hodge ring of a product of Hilbert modular surfaces can be determined by the Künneth formula. On the other hand, we have algebraic cycles on the power of a Hilbert modular surface, arising from Hecke operators and the canonical involution (see § 8). We show that, under the condition mentioned in (8.2), these cycles together with some algebraic cycles given by intersection of divisors generate the Hodge ring.

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Notations.

For a non-singular projective variety X defined over $\boldsymbol{C}$, we define the Hodge ring $\mathscr{B}^*(X)$ by

$$\mathscr{B}^*(X)=\bigoplus_{d=0}^{\dim X}\mathscr{B}^d(X), \quad \text{where}$$
$$\mathscr{B}^d(X)=H^{2d}(X,\boldsymbol{Q})\cap H^{d,d}(X).$$

We call elements of $\mathscr{B}^d(X)$ Hodge cycles of codimension d. We put

$$\mathscr{C}^*(X)=\bigoplus_{d=0}^{\dim X}\mathscr{C}^d(X), \quad \text{where}$$
$$\mathscr{C}^d(X)=\{\text{algebraic cycles of codimension } d\},$$

and

$$\mathscr{D}^*(X)=\bigoplus_{d=0}^{\dim X}\mathscr{D}^d(X), \quad \text{where}$$
$$\mathscr{D}^d(X)=\{\text{algebraic cycles of codimension } d \text{ which is represented as a sum of intersections of divisors}\}.$$

Therefore the following inclusions hold for any X:

$$\mathscr{B}^d(X) \supset \mathscr{C}^d(X) \supset \mathscr{D}^d(X) \qquad (0 \leqq d \leqq \dim X).$$

And the Hodge Conjecture (0.1) is expressed as:

$$\mathscr{B}^*(X) = \mathscr{C}^*(X).$$

Moreover the theorem of Lefschetz to which we refered above says that the equality $\mathscr{B}^1(X) = \mathscr{C}^1(X) = \mathscr{D}^1(X)$ holds for any X.

Further let G (resp. $\mathfrak{g}$) be a group (resp. Lie algebra) and let V be a vector space with G-action (resp. $\mathfrak{g}$-action). Then we denote by $\mathrm{End}_G V$ (resp. $\mathrm{End}_{\mathfrak{g}} V$) the space of G-linear (resp. $\mathfrak{g}$-linear) endomorphisms of V. We denote by $[V]^G$ (resp. $[V]^{\mathfrak{g}}$) the space of G-invariant (resp. $\mathfrak{g}$-invariant) elements in V. We denote by $\mathrm{End}\, A$ the endomorphism ring of an abelian variety A, and put $\mathrm{End}^0 A = \mathrm{End}\, A \otimes \boldsymbol{Q}$.

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PART I

§1. Hodge group.

In this section we recall the definition and the fundamental properties of the Hodge group of an abelian variety (cf. Mumford [8]). Let A be an abelian variety defined over $\boldsymbol{C}$ of dimension g. We put $V = H_1(A, \boldsymbol{Q})$, then $V_{\boldsymbol{R}} = V \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ is given the complex structure induced by the natural isomorphism between $V_{\boldsymbol{R}}$ and the universal covering space of A. Therefore we are given a homomorphism of algebraic groups,

$$\phi\colon T \longrightarrow GL(V)$$

defined over $\boldsymbol{R}$, where T is the compact one-dimensional torus over $\boldsymbol{R}$, i.e. $T_{\boldsymbol{R}}=\{z \in \boldsymbol{C};\ |z|=1\}$, by the formula

$$\phi(e^{i\theta})=\text{the element of } GL(V) \text{ given by multiplying in the complex structure on } V_{\boldsymbol{R}} \text{ by } e^{i\theta}.$$

Note that there is a non-degenerate skew symmetric form $I: V\times V\rightarrow \boldsymbol{Q}$ and that ϕ satisfies the Riemann conditions (a) $\phi(T)\subset Sp(V, I)$, and (b) $I(x, \phi(i)\cdot x)>0$ for all $x\in V$, $x\neq 0$.

(1.1) DEFINITION. The Hodge group of A, written $\mathrm{Hg}(A)$, is the smallest algebraic subgroup of $Sp(V, I)$ defined over $\boldsymbol{Q}$ and containing $\phi(T)$.

(1.2) PROPOSITION (Mumford, [8]). *The Hodge group is a connected reductive algebraic group with compact center whose semi-simple part is of Hermitian type.*

To compute the Hodge ring, the following propositions are useful.

(1.3) PROPOSITION ([loc. cit.]). *The endomorphism ring* $\mathrm{End}^0 A$ *of* A *is isomorphic to* $\mathrm{End}_{\mathrm{Hg}(A)} H^1(A, \boldsymbol{Q})$. *The Hodge ring* $\mathscr{B}^*(A)$ *is the subring of* $H^*(A, \boldsymbol{Q})$ *consisting of invariant elements under the action of the Hodge group*: $\mathscr{B}^*(A)=[H^*(A, \boldsymbol{Q})]^{\mathrm{Hg}(A)}$. *Moreover* $\mathrm{Hg}(A)$ *is the largest subgroup of* $GL(V)$ *which leaves invariant the Hodge rings of* A^n *for all* $n\geqq 1$.

(1.4) PROPOSITION ([loc. cit.]). *An abelian variety* A *is of CM-type if and only if* $\mathrm{Hg}(A)$ *is a torus algebraic group.*

(1.5) PROPOSITION (Tankeev [20]). *If* A *is isogenous to a product of abelian varieties of types* I, II, III, *then* $\mathrm{Hg}(A)$ *is semi-simple* (*see* [9] *for the definition of "type" of an abelian variety*).

Here we compute the Hodge ring of an abelian variety such that $\mathrm{Hg}(A)\cong Sp(2g, \boldsymbol{Q})$ ($g=\dim A$).

(1.6) PROPOSITION. *Let* A *be a simple abelian variety of dimension* g *such that* $\mathrm{Hg}(A)\cong Sp(2g, \boldsymbol{Q})$. *Then for any power* A^n ($n\geqq 1$) *of* A, *the Hodge ring* $\mathscr{B}^*(A^n)$ *is generated by* $\mathscr{B}^1(A)$, *i.e.*, $\mathscr{B}^*(A^n)=\mathscr{D}^*(A^n)$.

REMARK. Mattuck proved in [7] that if A is generic, then $\mathscr{B}^*(A)=\mathscr{D}^*(A)$. Since it can be shown that $\mathrm{Hg}(A)\cong Sp(2g, \boldsymbol{Q})$ for a generic A, (1.6) is considered as rephrasing of Mattuck's result.

PROOF OF (1.6). First we note the following

(1.7) LEMMA. *Let V be a vector space over $\boldsymbol{C}$ and let $\mathfrak{g}$ be a Lie algebra over $\boldsymbol{C}$ acting on V. Then for the exact sequence defining $\wedge^n V$:*

$$0 \longrightarrow \ker \pi \longrightarrow \otimes^n V \xrightarrow{\pi} \wedge^n V \longrightarrow 0, \tag{1.7.1}$$

the following sequence is exact:

$$0 \longrightarrow [\ker \pi]^{\mathfrak{g}} \longrightarrow [\otimes^n V]^{\mathfrak{g}} \xrightarrow{\pi} [\wedge^n V]^{\mathfrak{g}} \longrightarrow 0. \tag{1.7.2}$$

PROOF OF (1.7). Consider the map $s: \wedge^n V \to \otimes^n V$ defined by $s(x_1 \wedge \cdots \wedge x_n) = (1/n!) \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$, where S_n denotes the n-th symmetric group. Then it is clear that s gives the splitting of the exact sequence (1.7.1) and that it is compatible with the action of $\mathfrak{g}$. Hence (1.7.2) is exact.

Now we put $V = H^1(A, \boldsymbol{C})$ and $\mathfrak{g} = \mathscr{L}\!ie(\mathrm{Hg}(A)_{\boldsymbol{C}}) \cong \mathfrak{sp}(2g, \boldsymbol{C})$. Since $\mathrm{Hg}(A^n) \cong \mathrm{Hg}(A)$ (see (1.11) below), we must prove that $[\wedge^{i_1} V \otimes \cdots \otimes \wedge^{i_m} V]^{\mathfrak{g}}$, for various $i_1, \cdots, i_m \in \boldsymbol{N}$, are generated by $[\wedge^2 V]^{\mathfrak{g}}$ and $[V \otimes V]^{\mathfrak{g}}$. For this, by (1.7), it suffices to prove that for any $m \geqq 1$, $[\otimes^m V]^{\mathfrak{g}}$ is generated by $[V \otimes V]^{\mathfrak{g}}$. But this is nothing other than the following result which can be found in the classical invariant theory:

(1.8) THEOREM (Weyl [24, Theorem 6.1.A]). *All vector invariants of the symplectic group depending on an arbitrary number of covariant and contravariant vectors, $x, \cdots$ and $\xi, \cdots$, are expressible in terms of the basic invariants of type*

$$[xy], \quad (\xi x), \quad [\xi\eta].$$

Here "vector invariant of a group Γ" means a linear function

$$f: S^{\mu}(V) \otimes S^{\nu}(V) \otimes \cdots \longrightarrow \boldsymbol{C}$$

$$(S^{\mu}(V): \text{symmetric tensor space of degree } \mu)$$

invariant under all substitutions of $A \in \Gamma$, i.e.

$$f(Ax, Ay, \cdots) = f(x, y, \cdots)$$

for any $A \in \Gamma$, $x \in S^{\mu}(V)$, $y \in S^{\nu}(V)$, $\cdots$ (cf. [24, p. 23]). And $[xy]$, (ξx), $[\xi\eta]$ for $x, y \in V$, $\xi, \eta \in V^*$ (= the dual of V) denotes the canonical pairings: e.g.

$$[xy]=(x_1y_{\bar{1}}-x_{\bar{1}}y_1)+\cdots+(x_gy_{\bar{g}}-x_{\bar{g}}y_g),$$

where $x=(x_1, \cdots, x_g, x_{\bar{1}}, \cdots, x_{\bar{g}})$, etc. Thus our proposition (1.6) is proved.

As for the Hodge group of a product of abelian varieties, we have the following propositions.

(1.9) PROPOSITION. *Let A, B be abelian varieties. Then*

$$\mathrm{Hg}(A\times B)\subset \mathrm{Hg}(A)\times \mathrm{Hg}(B).$$

PROOF. Put $V_1=H_1(A, \boldsymbol{Q})$, $V_2=H_1(B, \boldsymbol{Q})$ and $V=H_1(A\times B, \boldsymbol{Q})\cong V_1\oplus V_2$. Then the complex structure on $V_{\boldsymbol{R}}$ is given by the following commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{\ \varphi\ } & GL(V)_{\boldsymbol{R}} \\ & {\scriptstyle \psi_1\times\psi_2}\searrow & \uparrow \\ & & GL(V_1)_{\boldsymbol{R}}\times GL(V_2)_{\boldsymbol{R}}, \end{array}$$

where ψ_i $(i=1, 2)$ denotes the map which gives the complex structure on $V_{i\boldsymbol{R}}$. Therefore we have $\mathrm{Hg}(A\times B)\subset \mathrm{Hg}(A)\times \mathrm{Hg}(B)$.

(1.10) PROPOSITION. *Let A, B be abelian varieties. Then*

$$\mathrm{Hg}(A^n\times B)\cong \mathrm{Hg}(A\times B)$$

for any $n\geqq 1$.

PROOF. Notations being as above, we have the following commutative diagram:

$$\begin{array}{ccl} T & \xrightarrow{\ \varphi\ } & GL(V_1\oplus V_2)_{\boldsymbol{R}} \\ & \searrow & \quad\uparrow \\ & & GL(V_1)_{\boldsymbol{R}}\times GL(V_2)_{\boldsymbol{R}} \\ & {\scriptstyle \psi}\searrow & \quad\downarrow \Delta \\ & & GL(V_1^{\oplus n})_{\boldsymbol{R}}\times GL(V_2)_{\boldsymbol{R}} \\ & \searrow & \quad\downarrow \\ & & GL(V_1^{\oplus n}\oplus V_2)_{\boldsymbol{R}}, \end{array}$$

where φ (resp. ψ) denotes the map which gives the complex structure on $V_{1\boldsymbol{R}}\oplus V_{2\boldsymbol{R}}$ (resp. $V_{1\boldsymbol{R}}^{\oplus n}\oplus V_{2\boldsymbol{R}}$), and Δ is defined by $\Delta(a, b)=(a, \cdots, a, b)$ for $a\in GL(V_1)_{\boldsymbol{R}}$, $b\in GL(V_2)_{\boldsymbol{R}}$. Since Δ is injective and defined over $\boldsymbol{Q}$, we see that $\mathrm{Hg}(A^n \times B) \cong \mathrm{Hg}(A \times B)$.

Noting that the proof above is also valid when $B=(0)$, we have

(1.11) COROLLARY. *For any* $n \geqq 1$,

$$\mathrm{Hg}(A^n) \cong \mathrm{Hg}(A).$$

Moreover $\mathrm{Hg}(A)$ *acts on* $H_1(A^n, \boldsymbol{Q}) \cong H_1(A, \boldsymbol{Q})^{\oplus n}$ *diagonally.*

§2. Main theorem A: Stable non-degeneracy.

In the previous section, we have seen that the Hodge group $\mathrm{Hg}(A)$ of an abelian variety A is characterized as the largest subgroup of $Sp(H^1(A, \boldsymbol{Q}))$ which leaves invariant the Hodge rings $\mathscr{B}^*(A^n)$ for all $n \geqq 1$ (cf. (1.3)). (Recall that the natural representation of the Hodge group in $H_1(A, \boldsymbol{Q})$ is symplectic and, therefore, it is equivalent to the dual representation in $H^1(A, \boldsymbol{Q})$.) Hence if A satisfies the condition

$$\mathscr{B}^*(A^n) = \mathscr{D}^*(A^n) \qquad \text{for all} \quad n \geqq 1, \tag{2.1}$$

then this property must be reflected in the structure of $\mathrm{Hg}(A)$.

(2.2) DEFINITION. When an abelian variety A satisfies the condition (2.1), we say that A is stably non-degenerate.

REMARK. The notion of "non-degeneracy" of an abelian variety A is first defined by Kubota when A is simple and of CM-type (cf. [5]). His definition is, in our terminology, that A is "non-degenerate" if $\dim \mathrm{Hg}(A) = \dim A$. Note that $\dim \mathrm{Hg}(A) = \operatorname{rank} \mathrm{Hg}(A)_C$ since $\mathrm{Hg}(A)$ is a torus (cf. (1.4)).

Here are some propositions about our notion "stable non-degeneracy".

(2.3) PROPOSITION. *Let A be an abelian variety and let B be an abelian subvariety of A. Suppose that A is stably non-degenerate. Then B is also stably non-degenerate.*

PROOF. The notion of stable non-degeneracy is invariant under isogeny. More precisely, if A is stably non-degenerate, then any B which is isogenous to A is also stably non-degenerate. Therefore we may assume that $A = B \times B'$ for some abelian variety B'. Then it is clear that if $\mathscr{B}^d(B^n) \supsetneqq \mathscr{D}^d(B^n)$ for some d and n, then $\mathscr{B}^d(A^n) \supsetneqq \mathscr{D}^d(A^n)$.

(2.4) PROPOSITION. *Let m be an integer $\geqq 1$. Then A is stably non-degenerate if and only if A^m is stably non-degenerate.*

PROOF. Only-if part follows from the definition of stable non-degeneracy. If-part follows from Proposition (2.3).

(2.5) PROPOSITION. *Let A_i $(i=1, \cdots, k)$ be abelian varieties and let m_i $(i=1, \cdots, k)$ be positive integers. Then $\prod_{i=1}^k A_i^{m_i}$ is stably non-degenerate if and only if $\prod_{i=1}^k A_i$ is stably non-degenerate.*

PROOF. Only-if part follows from Proposition (2.3). To show the converse we note that

$$\left(\prod_{i=1}^k A_i\right)^{\max\{m_i\}} \supset \prod_{i=1}^k A_i^{m_i}.$$

Then the assertion follows from (2.3) and (2.4).

To state the main theorem, we introduce the following notion of "reduced dimension" of A, which is denoted by rdim A.

(2.6) DEFINITION. When A is a simple abelian variety,

$$\operatorname{rdim} A = \begin{cases} \dim A & \text{if } A \text{ is of type I,} \\ (1/2)\dim A & \text{if } A \text{ is of type II,} \\ \dim A & \text{if } A \text{ is of type III,} \\ (1/d)\dim A & \text{if } A \text{ is of type IV with} \\ & d^2 = [\operatorname{End}^0 A : \operatorname{Cent} \operatorname{End}^0 A]. \end{cases}$$

When A is isogenous to $\prod_{i=1}^k A_i^{m_i}$, where A_i $(1 \leqq i \leqq k)$ are simple and $A_i \not\sim A_j$ for $i \neq j$, we define

$$\operatorname{rdim} A = \sum_{i=1}^k \operatorname{rdim} A_i.$$

Now we are able to state the main theorem:

(2.7) MAIN THEOREM. *An abelian variety A is stably non-degenerate if and only if* $\operatorname{rank} \operatorname{Hg}(A)_C = \operatorname{rdim} A$.

REMARK. In general, we have $\operatorname{rank} \operatorname{Hg}(A)_C \leqq \operatorname{rdim} A$, as is shown in the proof below. Therefore we can restate (2.7) as follows: A is stably non-degenerate if and only if $\operatorname{rank} \operatorname{Hg}(A)_C$ is as large as possible.

§ 3. Proof of Main Theorem A.

By the definition of stable non-degeneracy, and by the propositions (1.10), (2.5), it suffices to prove (2.7) for abelian varieties of the form $A_1\times\cdots\times A_k$ with A_i $(1\leqq i\leqq k)$ simple, $A_i\not\sim A_j$ for $i\neq j$. The following lemma will be used frequently:

(3.1) LEMMA. *Let* $\mathfrak{g}_i$ $(1\leqq i\leqq n)$ *be reductive Lie algebras over* $\boldsymbol{C}$ *and let* $\mathfrak{g}$ *be a reductive subalgebra of* $\mathfrak{g}_1\times\cdots\times\mathfrak{g}_n$. *Let* p_i $(1\leqq i\leqq n)$ *denote the projection of* $\mathfrak{g}_1\times\cdots\times\mathfrak{g}_n$ *to* $\mathfrak{g}_i$. *Suppose that the following conditions are satisfied:*

(3.1.1) $$p_i(\mathfrak{g})=\mathfrak{g}_i \qquad (1\leqq i\leqq n),$$

(3.1.2) $$\operatorname{rank}(\mathfrak{g})=\operatorname{rank}(\mathfrak{g}_1\times\cdots\times\mathfrak{g}_n).$$

Then $\mathfrak{g}=\mathfrak{g}_1\times\cdots\times\mathfrak{g}_n$.

PROOF. It suffices to consider the case $n=2$, since the assertion for the general case follows from this by induction. Put $\mathfrak{a}_0=(p_1|_{\mathfrak{g}})^{-1}(0)=\mathfrak{g}\cap(0\times\mathfrak{g}_2)$, and $\mathfrak{a}=p_2(\mathfrak{a}_0)$. Then $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}_2$. We show that $\mathfrak{a}$ is an ideal of $\mathfrak{g}_2$. By (3.1.1), for any $x_2\in\mathfrak{g}_2$, there exists $x_1\in\mathfrak{g}_1$ such that $(x_1, x_2)\in\mathfrak{g}$. Then $(x_1, x_2)+(0, \mathfrak{a})=(x_1, x_2+\mathfrak{a})\subset\mathfrak{g}$. Therefore

$$\begin{aligned}[(x_1, x_2+\mathfrak{a}), (x_1, x_2+\mathfrak{a})]&=([x_1, x_1], [x_2+\mathfrak{a}, x_2+\mathfrak{a}])\\ &\subset(0, [x_2, \mathfrak{a}]+\mathfrak{a})\subset\mathfrak{g}.\end{aligned}$$

Hence $[x_2, \mathfrak{a}]+\mathfrak{a}\subset\mathfrak{a}$, so we get $[x_2, \mathfrak{a}]\subset\mathfrak{a}$, i.e., $\mathfrak{a}$ is an ideal of $\mathfrak{g}_2$. Now we note that $\mathfrak{a}_0$ is an ideal of $\mathfrak{g}$. For we see that for $(x_1, x_2)\in\mathfrak{g}$ and $a\in\mathfrak{a}$, $[(x_1, x_2), (0, a)]=(0, [x_2, a])\in\mathfrak{g}$, since $[x_2, a]\in\mathfrak{a}$. Therefore considering a decomposition of $\mathfrak{g}$ into $\mathfrak{g}_{(1)}\times\cdots\times\mathfrak{g}_{(m)}\times\mathfrak{g}^{(a)}$, where $\mathfrak{g}_{(i)}$ $(1\leqq i\leqq m)$ is simple and $\mathfrak{g}^{(a)}$ is the center of the reductive Lie algebra $\mathfrak{g}$, we see that for a maximal torus $\mathfrak{t}$ of $\mathfrak{g}$, $\mathfrak{t}\cap\mathfrak{a}_0$ is necessarily a maximal torus of $\mathfrak{a}_0$. By the assumption (3.1.2), $\mathfrak{t}$ is also maximal as a subtorus of $\mathfrak{g}_1\times\mathfrak{g}_2$. Hence we have

$$\mathfrak{a}_0\supset\mathfrak{t}\cap(\{0\}\times\mathfrak{g}_2)=\{0\}\times(\text{a maximal torus of } \mathfrak{g}_2).$$

Thus we see that $\mathfrak{a}=p_2(\mathfrak{a}_0)$ is an ideal of $\mathfrak{g}_2$ and contains a maximal torus of $\mathfrak{g}_2$, so $\mathfrak{a}=\mathfrak{g}_2$. Now for any $x_1\in\mathfrak{g}_1$, we are able to choose $x_2\in\mathfrak{g}_2$ such that $(x_1, x_2)\in\mathfrak{g}$ by the assumption (3.1.1), hence we see that

$$(x_1, \mathfrak{g}_2)=(x_1, x_2+\mathfrak{g}_2)=(x_1, x_2)+(0, \mathfrak{g}_2)\subset\mathfrak{g}.$$

Therefore $\mathfrak{g}=\mathfrak{g}_1\times\mathfrak{g}_2$.

Now we prove the if-part of Theorem (2.7). First we assume that A is simple.

Case 1) A is of type I: In this case $\mathrm{End}^0 A$ is isomorphic to a totally real field over $\boldsymbol{Q}$. Put $e=[\mathrm{End}^0 A:\boldsymbol{Q}]$ and $g=\dim A$. Let $\mathfrak{g}$ denote the Lie algebra of $\mathrm{Hg}(A)_C$ and let V denote the $\boldsymbol{C}$-vector space $H^1(A, \boldsymbol{C})$. Then by Proposition (1.3),

$$\mathrm{End}_{\mathfrak{g}} V\cong\mathrm{End}^0 A\otimes_{\boldsymbol{Q}}\boldsymbol{C}\cong\boldsymbol{C}\oplus\cdots\oplus\boldsymbol{C}\qquad (e \text{ times}).$$

Therefore it follows from Schur's lemma that

$$V\cong V_1\oplus\cdots\oplus V_e,$$

where V_i $(1\leqq i\leqq e)$ are mutually non-isomorphic irreducible $\mathfrak{g}$-modules. Let $\mathfrak{g}_i$ denote the Lie subalgebra of $\mathrm{End}\, V_i$ $(1\leqq i\leqq e)$ which arises from restricting the action of $\mathfrak{g}$ on V to $\mathfrak{g}$-submodule V_i. Then by the definition $\mathfrak{g}\subset\mathfrak{g}_1\times\cdots\times\mathfrak{g}_e$.

(3.2) LEMMA. *The representation $\rho_i\colon \mathfrak{g}_i\to\mathrm{End}\, V_i$ is symplectic for any i.*

PROOF. We recall that the Picard number $\rho(A)$ of A is equal to $\dim_C[H^2(A, \boldsymbol{C})]^{\mathfrak{g}}=\dim_C[\wedge^2 V]^{\mathfrak{g}}$ (cf. (1.3)). The space $[\wedge^2 V]^{\mathfrak{g}}$ is decomposed as follows:

$$[\wedge^2 V]^{\mathfrak{g}}\cong\Big(\bigoplus_{i=1}^{e}[\wedge^2 V_i]^{\mathfrak{g}}\Big)\oplus\Big(\bigoplus_{i<j}[V_i\otimes V_j]^{\mathfrak{g}}\Big).$$

Note that the Riemann form on A induces a $\mathfrak{g}$-invariant non-degenerate skew-symmetric form I on V (see §1). Hence we may assume (renumbering if necessary) that there exists an integer s with $0\leqq s\leqq e$, such that

$$V_1^*\cong V_1,\ \cdots,\ V_s^*\cong V_s,$$
$$V_{s+1}^*\cong V_{s+2},\ \cdots,\ V_{e-1}^*\cong V_e,$$

where V_i^* denotes the dual representation space of V_i. Note that the isomorphisms $V_i^*\cong V_i$ $(1\leqq i\leqq s)$ are induced by the skew-symmetric form $I|_{V_i\times V_i}$, hence $\dim_C[\wedge^2 V_i]^{\mathfrak{g}}\geqq 1$. On the other hand, the space of $\mathfrak{g}$-invariant forms on V_i is isomorphic to $\mathrm{Hom}_{\mathfrak{g}}(V_i, V_i^*)\cong\mathrm{Hom}_{\mathfrak{g}}(V_i, V_i)$, which is of dimension one by Schur's lemma. Hence we have $\dim_C[\wedge^2 V_i]^{\mathfrak{g}}=1$

$(1 \leqq i \leqq s)$. When $j \geqq s+1$, $[\wedge^2 V_j]^{\mathfrak{g}}=(0)$, for otherwise there exists an integer $j \geqq s+1$ such that V_j is self-dual, which contradicts to the definition of s. Further $[V_i \otimes V_{i'}]^{\mathfrak{g}}=(0)$ for $i \neq i'$ unless $\{i, i'\}=\{s+1, s+2\}, \cdots, \{e-1, e\}$. For otherwise $\mathrm{Hom}_{\mathfrak{g}}(V_j, V_{i'}) \neq (0)$ for an appropriate j ($j=i$ if $1 \leqq i \leqq s$, and $j=i+1$ or $i-1$ if $s+1 \leqq i \leqq e$), which contradicts to the assumption. Hence we have

$$\dim_C [\wedge^2 V]^{\mathfrak{g}} = s+(e-s)/2=(e+s)/2.$$

On the other hand, it is known that $\rho(A)$ is equal to $[\mathrm{End}^0 A: \boldsymbol{Q}]$ when A is simple of type I (cf. Mumford [9, p. 202]). Hence $(e+s)/2=e$, so that $s=e$. Therefore we obtain

$$[\wedge^2 V]^{\mathfrak{g}} \cong \bigoplus_{i=1}^{e} [\wedge^2 V_i]^{\mathfrak{g}},$$

where $\dim_C [\wedge^2 V_i]^{\mathfrak{g}}=1$ for all i. Thus Lemma (3.2) is proved.

By this lemma, we have

$$\mathrm{rank}\, \mathfrak{g}_i \leqq (1/2) \dim V_i \quad \text{for all } i.$$

On the other hand, by the assumption, we have

$$\mathrm{rank}\, \mathfrak{g} = (1/2) \dim V.$$

Since $\mathfrak{g} \subset \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_e$, this implies that the above inequalities must be equalities for all i and that

$$\mathrm{rank}\, \mathfrak{g} = \mathrm{rank}\, \mathfrak{g}_1 + \cdots + \mathrm{rank}\, \mathfrak{g}_e.$$

Then by Lemma (3.1), we have

$$\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_e.$$

Next we show the following:

(3.3) LEMMA. $\mathfrak{g}_i \cong \mathfrak{sp}(V_i, \boldsymbol{C})$ *for all* i.

PROOF. We recall that $\mathfrak{g}_i$'s are semisimple by (1.5). Hence there exist simple Lie algebras $\mathfrak{g}_i^{(1)}, \cdots, \mathfrak{g}_i^{(a_i)}$ such that

$$\mathfrak{g}_i \cong \mathfrak{g}_i^{(1)} \times \cdots \times \mathfrak{g}_i^{(a_i)}.$$

Then the representation $\mathfrak{g}_i \to \mathrm{End}\, V_i$ must be equivalent to the tensor product representation

$$\mathfrak{g}_i^{(1)}\times\cdots\times\mathfrak{g}_i^{(a_i)}\longrightarrow \mathrm{End}(V_i^{(1)}\otimes\cdots\otimes V_i^{(a_i)})$$

where $V_i^{(j)}$ $(1\leqq j\leqq a_i)$ are irreducible $\mathfrak{g}_i^{(j)}$-modules. Then each representation $\rho_i^{(j)}: \mathfrak{g}_i^{(j)}\to \mathrm{End}\, V_i^{(j)}$ $(1\leqq j\leqq a_i)$ is defined by microweight (cf. Tankeev [21], Serre [16]). Here we recall representations defined by microweight (Bourbaki [1, Ch. VIII, p. 129]):

type	weight	degree	orthogonal: $+1$ symplectic: -1 otherwise : 0
A_n	ω_1	$\binom{n+1}{1}$	0
	ω_2	$\binom{n+1}{2}$	0
	$\vdots$	$\vdots$	$\vdots$
	$\omega_{(n+1)/2}$	$\binom{n+1}{(n+1)/2}$	$(-1)^{(n+1)/2}$
	$\vdots$	$\vdots$	$\vdots$
	ω_n	$\binom{n+1}{n}$	0
B_n	ω_1	2^n	$(-1)^{n(n+1)/2}$
C_n	ω_1	$2n$	-1
D_n	ω_1	$2n$	$+1$
E_6	ω_1	27	0
	ω_6	27	0
E_7	ω_7	56	-1

We fix an i $(1\leqq i\leqq e)$ and put $r_j=\mathrm{rank}\,\mathfrak{g}_i^{(j)}$, $d_j=\dim V_i^{(j)}$ for $1\leqq j\leqq a_j$. Noting that the representation $\rho_i^{(j)}: \mathfrak{g}_i^{(j)}\to \mathrm{End}\, V_i^{(j)}$ must be orthogonal or symplectic (cf. Tankeev [21, 4.8.3]), we see by the table above that $2r_j\leqq d_j$ for each j. Therefore

$$2\cdot \mathrm{rank}\,\mathfrak{g}_i=\sum_{j=1}^{a_i} 2r_j\leqq \sum_{j=1}^{a_i} d_j.$$

On the other hand, by the assumption,

$$2\cdot \mathrm{rank}\,\mathfrak{g}_i=\prod_{j=1}^{a_i} d_j.$$

Hence

$$\prod_{j=1}^{a_i} d_j \leqq \sum_{j=1}^{a_i} d_j.$$

This occurs only if (a) $a_i=1$, or (b) $a_i=2$ and $(d_1, d_2)=(2, 2)$. In the case (b), we see by the table that

$$\mathfrak{g}_i^{(j)} \cong \mathfrak{sl}_2, \quad V_i^{(j)} \cong \boldsymbol{C}^2 \text{ with the standard representation of } \mathfrak{sl}_2.$$

But this does not occur, since in this case the representation $\mathfrak{g}_i=\mathfrak{g}_i^{(1)} \times \mathfrak{g}_i^{(2)} \to \operatorname{End} V_i$ is orthogonal. Hence the case (a) must occur, i.e., $\mathfrak{g}_i$ is simple and the representation $\mathfrak{g}_i \to \operatorname{End} V_i$ is equivalent to the standard representation of $\mathfrak{sp}(V_i, \boldsymbol{C})$. This completes the proof of Lemma (3.3).

Now we note the following lemma which is proved easily:

(3.4) LEMMA. *Let $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) be a Lie algebra over $\boldsymbol{C}$ acting on a complex vector space V_1 (resp. V_2). Then*

$$[V_1 \otimes V_2]^{\mathfrak{g}_1 \times \mathfrak{g}_2} = [V_1]^{\mathfrak{g}_1} \otimes [V_2]^{\mathfrak{g}_2}.$$

We have obtained above $\mathfrak{g}=\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_e$ and $V=V_1 \oplus \cdots \oplus V_e$ where $\mathfrak{g}_i \cong \mathfrak{sp}(V_i, \boldsymbol{C})$, hence by (1.3), (1.6) and this lemma, we see that if A is a simple abelian variety of type I and $\operatorname{rank} \operatorname{Hg}(A)_{\boldsymbol{C}} = \operatorname{rdim} A$, then A is stably non-degenerate.

Case 2) A is of type II: In this case $\operatorname{End}^0 A$ is isomorphic to an indefinite quaternion algebra over a totally real field F over $\boldsymbol{Q}$. Put $[F:\boldsymbol{Q}]=e$ and $\dim A=g$. It is known that $2e$ divides g (Mumford [9, p. 202]). By (1.3) and Schur's lemma, we have an isomorphism of $\mathfrak{g}$-modules ($\mathfrak{g}$ = the Lie algebra of $\operatorname{Hg}(A)_{\boldsymbol{C}}$):

$$H^1(A, \boldsymbol{C}) \cong (V_{1,1} \oplus V_{1,2}) \oplus \cdots \oplus (V_{e,1} \oplus V_{e,2}),$$

where $V_{i,1}$ and $V_{i,2}$ $(1 \leqq i \leqq e)$ are isomorphic irreducible $\mathfrak{g}$-modules and $V_{i,1} \neq V_{j,1}$ for $i \neq j$. Put $\mathfrak{g}_i$=the projection of $\mathfrak{g}$ to $\operatorname{End}(V_{i,1} \oplus V_{i,2})$ and let $\mathfrak{g}_{i,a}$ be the projection of $\mathfrak{g}$ to $\operatorname{End} V_{i,a}$ for $1 \leqq i \leqq e$, $a=1$ or 2. Then $\mathfrak{g}_{i,1} \cong \mathfrak{g}_{i,2}$ and $\mathfrak{g}_i$=the diagonal of $\mathfrak{g}_{i,1} \times \mathfrak{g}_{i,2}$ $(1 \leqq i \leqq e)$, hence

$$\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_e \cong \mathfrak{g}_{1,1} \times \cdots \times \mathfrak{g}_{e,1}.$$

We show the representations $\mathfrak{g}_{i,1} \to \operatorname{End} V_{i,1}$ are symplectic for all i. Put $V=H^1(A, \boldsymbol{Q})$. Recall that the representation of $\mathfrak{g}$ in V is symplectic. Suppose that for some integer s with $0 \leqq s \leqq e$, we have

$$V_{i,1}^* \cong V_{i,1} \qquad (1 \leqq i \leqq s),$$
$$V_{s+1,1}^* \cong V_{s+2,1}, \cdots, V_{e-1,1}^* \cong V_{e,1}.$$

Then by a similar argument as is used in the Case 1), we have

$$3e = \rho(A) = \dim[\wedge^2 V]^{\mathfrak{g}} \leqq 3s + 4(e-s)/2 = 2e + s.$$

Hence $s=e$ and $\dim[\wedge^2 V_{i,a}]^{\mathfrak{g}}=1$ for all i, a. Thus

$$\operatorname{rank} \mathfrak{g} \leqq \sum_{i=1}^{e} \operatorname{rank} \mathfrak{g}_{i,1} \leqq \sum_{i=1}^{e} (1/2)\dim V_{i,1} = g/2.$$

Hence by the assumption that $\operatorname{rank} \mathfrak{g} = \operatorname{rdim} A = g/2$ and by (3.1), we have

$$\mathfrak{g} \cong \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_e$$

and

$$\operatorname{rank} \mathfrak{g}_{i,1} = (1/2)\dim V_{i,1}.$$

By a similar argument in the Case 1), we see that

$$\mathfrak{g}_{i,1} \cong \mathfrak{sp}(V_{i,1}, \boldsymbol{C}),$$

hence, A is stably non-degenerate when A is a simple abelian variety of type II with $\operatorname{rank} \operatorname{Hg}(A)_{\boldsymbol{C}} = \operatorname{rdim} A$.

Case 3) A is of type III: Put $D = \operatorname{End}^0 A$ and $F = \operatorname{Cent} D$. Then, by definition, F is a totally real field over $\boldsymbol{Q}$ and D is a definite quaternion division algebra over F. Put $e = [F:\boldsymbol{Q}]$. Then it is known that $2e$ divides $g = \dim A$ and $2e \neq g$ (cf. Mumford [9, p. 202]). Since $D \otimes_{\boldsymbol{Q}} \boldsymbol{C} \cong M_2(\boldsymbol{C}) \times \cdots \times M_2(\boldsymbol{C})$ (e times), (1.3) and Schur's lemma implies that

$$H^1(A, \boldsymbol{C}) \cong (V_{1,1} \oplus V_{1,2}) \oplus \cdots \oplus (V_{e,1} \oplus V_{e,2}),$$

where $V_{i,1}$ and $V_{i,2}$ are isomorphic irreducible $\mathfrak{g}$-modules for each i and $V_{i,1} \not\cong V_{j,1}$ for $i \neq j$. Let $\mathfrak{g}$, $\mathfrak{g}_i$, $\mathfrak{g}_{i,a}$ be as above in the Case 2). Note that $\rho(A)=e$. We show that the representation $\mathfrak{g}_{i,1} \rightarrow \operatorname{End} V_{i,1}$ are orthogonal for all i. Since $V^* = V$, we may assume that there exists an integer s with $0 \leqq s \leqq e$ such that

$$V_{i,1}^* \cong V_{i,1} \qquad (1 \leqq i \leqq s),$$
$$V_{s+1,1}^* \cong V_{s+2,1}, \cdots, V_{e-1,1}^* \cong V_{e,1}.$$

Then we have

$$e=\rho(A)=\dim[\wedge^2 V]^{\mathfrak{g}}\geqq s+4(e-s)/2=2e-s.$$

Hence $s=e$, i.e., $V_{i,1}^*\cong V_{i,1}$ and $[\wedge^2 V_{i,1}]^{\mathfrak{g}}=(0)$ for all i. Thus the representations $\mathfrak{g}_{i,1}\to\mathrm{End}\, V_{i,1}$ are orthogonal for all i. So we have

$$[\wedge^2 H^1(A, \boldsymbol{C})]^{\mathfrak{g}}\cong\bigoplus_{i=1}^{e}[V_{i,1}\otimes V_{i,1}]^{\mathfrak{g}}.$$

On the other hand,

$$[\wedge^{\dim V_{i,1}} V_{i,1}]^{\mathfrak{g}}\neq(0)$$

since any irreducible one-dimensional representation of semi-simple Lie algebra is trivial. Since the determinant is independent of the inner product, we see that for any abelian variety of type III, $\mathscr{B}^*(A)\supsetneqq\mathscr{D}^*(A)$ and that

$$\begin{aligned}\mathrm{rank}\,\mathrm{Hg}(A)_{\boldsymbol{C}}&\leqq(1/2)\dim A\\&=(1/2)\mathrm{rdim}\,A\\&<\mathrm{rdim}\,A.\end{aligned}$$

Case 4) A is of type IV: In this case $\mathrm{End}^0 A$ is isomorphic to a division algebra D with center K, a CM-field. Put $[D:K]=d^2$ and $[K:\boldsymbol{Q}]=e$. It is known that d^2e divides $2g$ and that e is an even integer (cf. Mumford [9, p. 202]). Put $\mathfrak{g}=\mathscr{L}ie(\mathrm{Hg}(A)_{\boldsymbol{C}})$, then we see by (1.3) that

$$(*)\qquad \mathrm{End}_{\mathfrak{g}}\, H^1(A, \boldsymbol{C})\cong M_d(\boldsymbol{C})\times\cdots\times M_d(\boldsymbol{C})\qquad (e\ \text{times}).$$

Hence by Schur's lemma we have an isomorphism of $\mathfrak{g}$-modules

$$H^1(A, \boldsymbol{C})\cong(V_{1,1}\oplus\cdots\oplus V_{1,d})\oplus\cdots\oplus(V_{e,1}\oplus\cdots\oplus V_{e,d}),$$

where $V_{i,j}$ are irreducible $\mathfrak{g}$-modules such that for fixed i, $V_{i,j}$ are isomorphic for all j, and $V_{i,j}\not\cong V_{i',j}$ for $i\neq i'$. Let $\mathfrak{g}_{i,j}$ denote the projection of $\mathfrak{g}$ to $\mathrm{End}\, V_{i,j}$ and let $\rho_{i,j}$: $\mathfrak{g}_{i,j}\to\mathrm{End}\, V_{i,j}$ be the natural representation. Then we may assume that

$$\begin{aligned}&V_{1,j}^*\cong V_{1,j},\ \cdots,\ V_{s,j}^*\cong V_{s,j},\\&V_{s+1,j}^*\cong V_{s+2,j},\ \cdots,\ V_{e-1,j}^*\cong V_{e,j}\qquad(1\leqq j\leqq d),\end{aligned}$$

holds for some integer s with $0\leqq s\leqq e$. Further we assume that $\rho_{i,j}$

is orthogonal for $1\leqq i\leqq s_1$, symplectic for $s_1+1\leqq i\leqq s_1+s_2=s$. Then we have

$$\begin{aligned}\dim_{C}[\wedge^2 V]^{g}&=s_1d(d-1)/2+s_2d(d+1)/2+(e-s)d^2/2\\&=(s_2-s_1)d/2+ed^2/2.\end{aligned}$$

On the other hand, it is known that $\dim_{C}[\wedge^2 V]^{g}=ed^2/2$ (cf. Mumford [9, p. 202]). Hence we have $s_1=s_2=s/2$. Here we need a little more argument than in Case 1)$\sim$3). The above isomorphism $(*)$ is obtained from the isomorphism $\mathrm{End}_{\mathrm{Hg}(A)} H^1(A, \boldsymbol{Q})\cong D$ by tensoring $\boldsymbol{C}$ over $\boldsymbol{Q}$. Since D is a division algebra, the representation ρ of $\mathrm{Hg}(A)$ on $H^1(A, \boldsymbol{Q})$ is $\boldsymbol{Q}$-irreducible. Hence it is a direct sum of the form $\rho=m\sum\rho_1^{\tau}$, where m is a positive integer, ρ_1 an $\bar{\boldsymbol{Q}}$-(hence $\boldsymbol{C}$-) irreducible sub-representation of ρ, and τ are conjugations over $\bar{\boldsymbol{Q}}$. Moreover, by definition, this decomposition is compatible with the preceding one to $V_{i,j}$'s. Hence if one of $V_{i,j}$ has $\mathfrak{g}$-invariant orthogonal (resp. symplectic) form, then all the others have $\mathfrak{g}$-invariant orthogonal (resp. symplectic) form. (Note that $\mathrm{Hg}(A)\subset Sp(H^1(A, \boldsymbol{Q}), I)$ where the Riemann form I is defined over $\boldsymbol{Q}$.) Therefore we have $s_1=s_2=0$, so $s=0$. Hence, $V_{1,j}^*\cong V_{2,j}, \cdots, V_{e-1,j}^*\cong V_{e,j}$ for all j. Then, since $\mathfrak{g}\subset\mathfrak{g}_{1,1}\times\mathfrak{g}_{3,1}\times\cdots\times\mathfrak{g}_{e-1,1}$, we have

$$\begin{aligned}\mathrm{rank}\,\mathfrak{g}&\leqq\mathrm{rank}\,\mathfrak{g}_{1,1}+\cdots+\mathrm{rank}\,\mathfrak{g}_{e-1,1}\\&\leqq\dim V_{1,1}+\cdots+\dim V_{e-1,1}\\&=g/d=\mathrm{rdim}\,A.\end{aligned}$$

Hence if $\mathrm{rank}\,\mathfrak{g}=\mathrm{rdim}\,A$, then we have $\mathrm{rank}\,\mathfrak{g}_{i,1}=\dim V_{i,1}$ for all i, and

$$\mathrm{rank}\,\mathfrak{g}=\mathrm{rank}\,\mathfrak{g}_{1,1}+\cdots+\mathrm{rank}\,\mathfrak{g}_{e-1,1}.$$

Therefore we see by (3.1) that

$$\mathfrak{g}=\mathfrak{g}_{1,1}\times\mathfrak{g}_{3,1}\times\cdots\times\mathfrak{g}_{e-1,1}.$$

Next we determine the representation $\rho_{i,1}\colon\mathfrak{g}_{i,1}\to\mathrm{End}\,V_{i,1}$ $(i=1,\cdots,e-1)$. Let $\mathfrak{g}_{i,1}\cong\mathfrak{g}_{i,1}^{(0)}\times\mathfrak{g}_{i,1}^{(1)}\times\cdots\times\mathfrak{g}_{i,1}^{(t)}$, where $\mathfrak{g}_{i,1}^{(0)}$ denotes the center of the reductive Lie algebra $\mathfrak{g}_{i,1}$ and $\mathfrak{g}_{i,1}^{(j)}$ denotes simple component $(1\leqq j\leqq t)$. Then according to this decomposition, $V_{i,1}$ must be decomposed as

$$V_{i,1}\cong V_{i,1}^{(0)}\otimes V_{i,1}^{(1)}\otimes\cdots\otimes V_{i,1}^{(t)},$$

where each $V_{i,1}^{(j)}$ $(0\leqq j\leqq t)$ is an irreducible $\mathfrak{g}_{i,1}^{(j)}$-module of dimension d_j. Then we have

$$\operatorname{rank} \mathfrak{g}_{i,1} = \sum_{j=0}^{t} \operatorname{rank} \mathfrak{g}_{i,1}^{(j)} \leqq d_0 + \sum_{j=1}^{t} (d_j - 1),$$

since $\operatorname{rank} \mathfrak{g}_{i,1}^{(j)} \leqq d_j - 1$ $(1 \leqq j \leqq t)$ by the table of simple Lie algebras and its representations (Bourbaki [1, Ch. VIII, p. 214]). On the other hand, by the above equality $\operatorname{rank} \mathfrak{g}_{i,1} = \dim V_{i,1}$ $(i=1, 3, \cdots, e-1)$, we have

$$\operatorname{rank} \mathfrak{g}_{i,1} = \prod_{j=1}^{t} d_j.$$

Hence

$$\prod_{j=0}^{t} d_j \leqq \sum_{j=0}^{t} d_j - t.$$

Since $d_0 \geqq 1$ and $d_j \geqq 2$ $(1 \leqq j \leqq t)$, we see that $t=1$, $d_0=1$ and d_1 is an integer $\geqq 2$. Moreover we have $\operatorname{rank} \mathfrak{g}_{i,1}^{(0)} = 1$ and $\operatorname{rank} \mathfrak{g}_{i,1}^{(1)} = d_1 - 1$. Hence we see that

$$\mathfrak{g}_{i,1}^{(1)} \cong \mathfrak{sl}_{d_1}$$

and the representation $\rho_{i,1}^{(1)}: \mathfrak{g}_{i,1}^{(1)} \to \operatorname{End} V_{i,1}^{(1)}$ is equivalent to the standard representation of $\mathfrak{sl}_{d_1}$ in $\boldsymbol{C}^{d_1}$. Then everything is reduced to the following:

(3.5) Lemma. *Let* $\mathfrak{g}_1 = \mathfrak{sl}_m$, $V_1 = \boldsymbol{C}^m$ *and let* $\rho: \mathfrak{g} = \mathfrak{g}_1 \otimes \boldsymbol{C} \to \operatorname{End} V_1$ *be the standard representation of* $\mathfrak{sl}_m$ *in* $\boldsymbol{C}^m$ *tensored by the scalar action of* $\boldsymbol{C}$ *on* $\boldsymbol{C}^m$. *Let* V_1^* *denote the dual representation space of* V_1. *Put* $V = V_1 \oplus V_1^*$. *Then* $[\wedge^{2d}(\underbrace{V \oplus \cdots \oplus V}_{n \text{ times}})]^{\mathfrak{g}}$ $(n \geqq 1, 1 \leqq d \leqq mn)$ *are generated by* $[\wedge^2(\underbrace{V \oplus \cdots \oplus V}_{n \text{ times}})]^{\mathfrak{g}}$.

Proof. First we show that

$$[\wedge^2 V_1]^{\mathfrak{g}} = [\wedge^2 V_1^*]^{\mathfrak{g}} = (0).$$

When $m > 2$, $\wedge^2 \boldsymbol{C}^m$ is one of the fundamental representation space of $\mathfrak{sl}_m$, hence $[\wedge^2 V_1]^{\mathfrak{g}} \subset [\wedge^2 \boldsymbol{C}^m]^{\mathfrak{sl}_m} = (0)$. Similarly we have $[\wedge^2 V_1^*]^{\mathfrak{g}} = (0)$. When $m=2$, $\wedge^2 \boldsymbol{C}^2$ is one-dimensional space on which $\mathfrak{sl}_2$ acts trivially, but each $t \in \mathscr{L}ie(G_m)_C$ acts on it by multiplication of $2t$. Therefore in any case $[\wedge^2 V_1]^{\mathfrak{g}} = [\wedge^2 V_1^*]^{\mathfrak{g}} = (0)$. Hence we see that $[\wedge^2(V^{\oplus n})]^{\mathfrak{g}}$ is an n^2-dimensional space generated by various $[V_1 \otimes V_1^*]^{\mathfrak{g}}$'s. Next we compute $[\wedge^{2d}(V^{\oplus n})]^{\mathfrak{g}}$. We claim that

$$[(\wedge^{d_1} V_1 \otimes \cdots \otimes \wedge^{d_\alpha} V_1) \otimes (\wedge^{e_1} V_1^* \otimes \cdots \otimes \wedge^{e_\beta} V_1^*)]^{\mathfrak{g}} = (0)$$

unless

$$\sum_{i=1}^{\alpha} d_i = \sum_{j=1}^{\beta} e_j.$$

This follows from the fact that $t \in \mathscr{Lie}(G_m)_C$ acts by $x \mapsto tx$ on V_1 and by $x \mapsto -tx$ on V_1^*. Here we recall the following proposition of classical invariant theory:

(3.6) PROPOSITION (Weyl [24, 2.6.A, 2.14.A]). *Let $V = \boldsymbol{C}^m$ be given the natural action of $\mathfrak{sl}_m$, and let V^* be its dual. Then $[V^{\otimes k} \otimes V^{*\otimes l}]^{\mathfrak{sl}_m}$ is generated by tensor products of the following three types of elements:*

1) $[x^1, \cdots, x^m] = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) x^1_{\sigma(1)} \otimes \cdots \otimes x^m_{\sigma(m)}$, *where* $x^i \in V$ $(1 \leqq i \leqq m)$,
2) $[\xi^1, \cdots, \xi^m] = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \xi^1_{\sigma(1)} \otimes \cdots \otimes \xi^m_{\sigma(m)}$, *where* $\xi^i \in V^*$ $(1 \leqq i \leqq m)$,
3) $(x, \xi) = \sum_{i=1}^{m} x_i \xi_i e^i$, *where* $x \in V$, $\xi \in V^*$ *and* $\{e^1, \cdots, e^m\}$ *denotes the standard basis of* $V = \boldsymbol{C}^m$.

Moreover the following relation holds for them:

$$[x^1, \cdots, x^m][\xi^1, \cdots, \xi^m] = \det\begin{pmatrix} (x^1, \xi^1) & \cdots & (x^1, \xi^m) \\ \vdots & & \vdots \\ (x^m, \xi^1) & \cdots & (x^m, \xi^m) \end{pmatrix}.$$

By this proposition and by Lemma (1.7), we see that $[\wedge^{2d}(V^{\oplus n})]^{\mathfrak{g}}$ is generated by $[\wedge^2(V^{\oplus n})]^{\mathfrak{g}}$ for all d, n.

Thus the if-part of Theorem (2.7) is proved for simple abelian varieties. For general abelian varieties, we proceed as follows. As we remarked earlier, we are able to assume that $A = \prod_{i=1}^k A_i$, where A_i $(1 \leqq i \leqq k)$ are simple abelian varieties with $A_i \not\sim A_j$ for $i \neq j$. But we have seen in the course of the proof above that $\operatorname{rank} \operatorname{Hg}(A_i)_C \leqq \operatorname{rdim} A_i$ for all i. Hence we have

$$\begin{aligned} \operatorname{rank} \operatorname{Hg}(A)_C &\leqq \sum_{i=1}^{k} \operatorname{rank} \operatorname{Hg}(A_i)_C \\ &\leqq \sum_{i=1}^{k} \operatorname{rdim} A_i \\ &= \operatorname{rdim} A. \end{aligned}$$

Therefore if $\operatorname{rank} \operatorname{Hg}(A)_{\boldsymbol{C}}=\operatorname{rdim} A$, then all the inequalities above must be equalities. Hence

$$\operatorname{rank} \operatorname{Hg}(A_i)_{\boldsymbol{C}}=\operatorname{rdim} A_i \quad \text{for all } i$$

and

$$\operatorname{rank} \operatorname{Hg}(A)_{\boldsymbol{C}}=\sum_{i=1}^{k} \operatorname{rank} \operatorname{Hg}(A_i)_{\boldsymbol{C}}.$$

So, by (3.1) and (1.9), we obtain

$$\operatorname{Hg}(A)=\prod_{i=1}^{k} \operatorname{Hg}(A_i)$$

(recall that the Hodge group is connected). Hence by (3.3), it suffices to prove the if-part when A is simple. But this is already done.

Only-if part: First we note that the proof of the last assertion of (1.3) uses only the reductivity of $\operatorname{Hg}(A)$ (cf. [2, 3.1]). Hence the following version of it also holds:

(3.7) PROPOSITION. *$\operatorname{Hg}(A)_{\boldsymbol{C}}$ is the largest subgroup of $GL(H^1(A, \boldsymbol{C}))$ which leaves invariant $\mathscr{B}^*(A^n)\otimes\boldsymbol{C}$ for all $n\geqq 1$.*

On the other hand, as we see in the proof of the necessity, the structure of $\mathscr{D}^*(A^n)\otimes\boldsymbol{C}$ is uniquely determined by the isogeny decomposition of A. Therefore if $\mathscr{B}^*(A^n)=\mathscr{D}^*(A^n)$ for all $n\geqq 1$, then the structure of $\mathscr{B}^*(A^n)\otimes\boldsymbol{C}$ is uniquely determined by the isogeny decomposition of A. Now suppose that $\operatorname{rank} \operatorname{Hg}(A)_{\boldsymbol{C}}<\operatorname{rdim} A$, but $\mathscr{B}^*(A^n)=\mathscr{D}^*(A^n)$ for all $n\geqq 1$. By the proof of necessity we see that there exists a reductive Lie algebra $\mathfrak{g}'\subset GL(H^1(A, \boldsymbol{C}))$ such that $\operatorname{rank} \mathfrak{g}'=\operatorname{rdim} A$, $[\wedge^2(V^{\oplus n})]^{\mathfrak{g}'}\cong \mathscr{B}^1(A^n)\otimes\boldsymbol{C}$ and $[\wedge^{2d}(V^{\oplus n})]^{\mathfrak{g}'}$ is generated by $[\wedge^2(V^{\oplus n})]^{\mathfrak{g}'}$. Then by the assumption, we have $[\wedge^{2d}(V^{\oplus n})]^{\mathfrak{g}}=[\wedge^{2d}(V^{\oplus n})]^{\mathfrak{g}'}$. This implies by (3.7) that $\mathfrak{g}=\mathfrak{g}'$. This contradicts to the assumption that $\operatorname{rank} \mathfrak{g}<\operatorname{rdim} A=\operatorname{rank} \mathfrak{g}'$. Thus Theorem (2.7) is proved completely.

§ 4. Examples.

Applying our theorem (2.7), we can uniformly reprove the Hodge Conjecture for abelian varieties of the following type and also for their powers.

(4.1) The Hodge Conjecture for a power E^n $(n\geqq 1)$ of an elliptic curve E (Tate [23], Murasaki [10]):

PROOF. The Hodge group of E is isomorphic to $SL_2(\boldsymbol{Q})$ (resp. one-dimensional torus) when E is not of CM-type (resp. is of CM-type) (cf. Hazama [2]), hence we have $\operatorname{rank} \operatorname{Hg}(E)_C = 1 = \operatorname{rdim} E$ in both cases. Then, it follows immediately from (2.7) that $\mathscr{B}^*(E^n) = \mathscr{D}^*(E^n)$ for all $n \geqq 1$.

(4.2) The Hodge Conjecture for a power A^n $(n \geqq 1)$ of a simple prime-dimensional abelian variety A of CM-type (Tankeev [22], Ribet [14]):

PROOF. It is shown in the articles cited above that the equality $\dim \operatorname{Hg}(A) = \dim A$ holds for such A. Since $\operatorname{rank} \operatorname{Hg}(A) = \dim A$ (recall that $\operatorname{Hg}(A)$ is a torus when A is of CM-type (1.4)) and $\operatorname{rdim} A = \dim A$ (see the definition of reduced dimension (2.6)), the assertion follows from (2.7).

(4.3) The Hodge Conjecture for $J_1(N)^n$ $(N \geqq 1, n \geqq 1)$ where $J_1(N)$ denotes the Jacobian variety of the modular curve $X_1(N)$ (Hazama [2]):

PROOF. The Hodge group of $J_1(N)$ is computed in [loc. cit.], and the equality $\operatorname{rank} \operatorname{Hg}(J_1(N)_C) = \operatorname{rdim} J_1(N)$ follows immediately from that computation. Hence we can apply (2.7) to obtain the Hodge Conjecture for $J_1(N)^n$ $(n \geqq 1)$.

§ 5. An example of stably degenerate abelian variety.

We call an abelian variety *stably degenerate* if it is not stably non-degenerate. In this section we prove the following:

(5.1) PROPOSITION. *There exists a simple four-dimensional abelian variety A which enjoys the following properties*:

a) *A is of type* I,
b) *A is stably degenerate,*
c) $\mathscr{B}^*(A) = \mathscr{D}^*(A)$ (*in particular the Hodge Conjecture holds for A*),
d) $\mathscr{B}^*(A^2) \supsetneqq \mathscr{D}^*(A^2)$.

REMARK. Note that, if A is stably degenerate, then our theorem (2.7) claims the existence of an integer $n \geqq 1$ such that $(*)$ $\mathscr{B}^*(A^n) \supsetneqq \mathscr{D}^*(A^n)$. We denote by $i(A)$ the smallest integer n which satisfies the condition $(*)$, and call it the index of degeneracy of A. Proposition

(5.1) says that the index of degeneracy $i(A)$ is not always equal to one. On the other hand, it is not known whether there exists a stably degenerate abelian variety A of CM-type with $i(A)\geqq 2$. In other words, the index of degeneracy of every known stably degenerate abelian varieties of CM-type is equal to one. On this subject we refer to our recent article [3].

PROOF OF (5.1). Let k be a totally real number field of degree three, and let B be a quaternion algebra over k such that $B\otimes_{\boldsymbol{Q}}\boldsymbol{R}\cong M_2(\boldsymbol{R})\times\boldsymbol{H}\times\boldsymbol{H}$, where $M_2(\boldsymbol{R})$ denotes the algebra of all the 2×2 matrices over $\boldsymbol{R}$, and $\boldsymbol{H}$ denotes the Hamilton's quaternion algebra. Let $G=SL(1, B)$, and put $\boldsymbol{G}=\mathrm{Res}_{k/\boldsymbol{Q}}(G)$. $\boldsymbol{G}$ is an algebraic group defined over $\boldsymbol{Q}$ and $\boldsymbol{G}_{\boldsymbol{Q}}\cong G_k=B_1^\times=\{x\in B^\times;\ \nu(x)=1\}$, where ν denotes the reduced norm, and $\boldsymbol{G}_{\boldsymbol{R}}\cong SL_2(\boldsymbol{R})\times SU_2\times SU_2\subset SL_2(\boldsymbol{C})\times SL_2(\boldsymbol{C})\times SL_2(\boldsymbol{C})\cong\boldsymbol{G}_{\boldsymbol{C}}$. Then it follows from Kuga [6] that there exists a family of abelian varieties π: $V\to U$ attached to a symplectic representation ρ of $\boldsymbol{G}$ of degree 8 defined over $\boldsymbol{Q}$ such that:

1) V and U are projective varieties and $U\cong\Gamma\backslash\boldsymbol{G}_{\boldsymbol{R}}/\{$a maximal compact subgroup$\}$, for a Zariski-dense arithmetic subgroup Γ of $\boldsymbol{G}_{\boldsymbol{R}}$,

2) for a generic point $P\in U$ over a field of definition for V, U, π, the fiber A_P is a four-dimensional abelian variety and

$$(5.1.1)\qquad [H^{2k}(A_P)]^{\pi_1(U,P)}=\mathscr{B}^k(A_P)\qquad(1\leqq k\leqq 4),$$

$$(5.1.2)\qquad \pi_1(U, P)\cong\Gamma.$$

Note that the equality (5.1.1) follows from the "Condition Inner" [6, 1.4.10 and the final remark]. Further note that, if the "Condition Inner" is satisfied for ρ, then it is also satisfied for any direct sum $\rho\oplus\cdots\oplus\rho$ (n times, $n\geqq 1$). Hence we have

$$(5.1.3)\quad [H^{2k}(A_P^n)]^{\pi_1(U,P)}=\mathscr{B}^k(A_P^n)\quad\text{for any } n\geqq 1\ \text{ and }\ k\in\{1,\cdots,4n\}.$$

Noting that $\Gamma\cong\pi_1(U, P)$ is Zariski-dense in $\boldsymbol{G}$, we see by (5.1.3) that for any $n\geqq 1$, $k\in\{1,\cdots,4n\}$, $[H^{2k}(A_P^n)]^{\boldsymbol{G}}=\mathscr{B}^k(A_P^n)$. By (1.3) this implies that $\mathrm{Hg}(A_P)\supset\boldsymbol{G}$. Since we see by (5.2.3) below that $\dim\mathscr{B}^1(A_P)=1$, A_P is simple and of type I (cf. Mumford [9, p. 202]). Hence A_P has the property a). The properties c) and d) are reduced to the following by (5.1.3). Note that d) implies by (2.7) that A_P is stably degenerate, i.e., that A_P has the property b). We also note that $\mathrm{rank}\,\mathrm{Hg}(A_P)_{\boldsymbol{C}}=3$ since $3=\mathrm{rank}\,\boldsymbol{G}_{\boldsymbol{C}}\leqq\mathrm{rank}\,\mathrm{Hg}(A_P)_{\boldsymbol{C}}<\mathrm{rdim}\,A_P=4$.

(5.2) LEMMA. *Let* $\mathfrak{g}=\mathfrak{sl}_2(\boldsymbol{C})\times\mathfrak{sl}_2(\boldsymbol{C})\times\mathfrak{sl}_2(\boldsymbol{C})$ *and let* $\rho=\alpha\otimes\beta\otimes\gamma\colon\mathfrak{g}\to\operatorname{End}V$, $V=\boldsymbol{C}^2\otimes\boldsymbol{C}^2\otimes\boldsymbol{C}^2$ *be the tensor product of the standard representations. Then*

(5.2.1) $[\wedge^{2d}V]^{\mathfrak{g}}$ *is generated by* $[\wedge^2 V]^{\mathfrak{g}}$ $(d=1, 2, 3, 4)$,

(5.2.2) $[\wedge^4(V\oplus V)]^{\mathfrak{g}}$ *is not generated by* $[\wedge^2(V\oplus V)]^{\mathfrak{g}}$.

PROOF OF (5.2). The first equality (5.2.1) is already proved in Tankeev [20], Kuga [6, 2.2.2]. To prove (5.2.2), we recall the computation done in [6, Lemma 2.2.1 (3)]: let α_k denote the symmetric tensor representation $S_k\circ\alpha$ of α and let $\mathbf{1}$ denote the trivial representation. Further we write $\alpha\beta\gamma$ for $\alpha\otimes\beta\otimes\gamma$, etc. Then

(5.2.3) $\wedge^2(\alpha\beta\gamma)\cong\wedge^6(\alpha\beta\gamma)\cong\alpha_2\beta_2\oplus\beta_2\gamma_2\oplus\gamma_2\alpha_2\oplus\mathbf{1}$,

(5.2.4) $\wedge^3(\alpha\beta\gamma)\cong\wedge^5(\alpha\beta\gamma)\cong\alpha_3\beta\gamma\oplus\alpha\beta_3\gamma\oplus\alpha\beta\gamma_3\oplus\alpha\beta\gamma$,

(5.2.5) $\wedge^4(\alpha\beta\gamma)\cong\alpha_4\oplus\beta_4\oplus\gamma_4\oplus\alpha_2\beta_2\gamma_2\oplus\alpha_2\beta_2\oplus\beta_2\gamma_2\oplus\gamma_2\alpha_2\oplus\mathbf{1}$.

In order to avoid confusion, we take two copies V_1, V_2 of $V\cong\boldsymbol{C}^2$ and consider $[\wedge^2(V_1\oplus V_2)]^{\mathfrak{g}}$ (resp. $[\wedge^4(V_1\oplus V_2)]^{\mathfrak{g}}$) instead of $[\wedge^2(V\oplus V)]^{\mathfrak{g}}$ (resp. $[\wedge^4(V\oplus V)]^{\mathfrak{g}}$). Then $[\wedge^2(V_1\oplus V_2)]^{\mathfrak{g}}$ is decomposed as follows:

$$[\wedge^2(V_1\oplus V_2)]^{\mathfrak{g}}\cong[\wedge^2 V_1]^{\mathfrak{g}}\oplus[V_1\otimes V_2]^{\mathfrak{g}}\oplus[\wedge^2 V_2]^{\mathfrak{g}},$$

hence its dimension is equal to three. (Note that $[V_1\otimes V_2]^{\mathfrak{g}}\cong\operatorname{Hom}_{\mathfrak{g}}(V_1^*, V_2)\cong\operatorname{Hom}_{\mathfrak{g}}(V_1, V_2)$.) On the other hand,

$$[\wedge^4(V_1\oplus V_2)]^{\mathfrak{g}}\cong[\wedge^4 V_1]^{\mathfrak{g}}\oplus[(\wedge^3 V_1)\otimes V_2]^{\mathfrak{g}}\oplus[(\wedge^2 V_1)\otimes(\wedge^2 V_2)]^{\mathfrak{g}}$$
$$\oplus[V_1\otimes(\wedge^3 V_2)]^{\mathfrak{g}}\oplus[\wedge^4 V_2]^{\mathfrak{g}}.$$

The first summand $[\wedge^4 V_1]^{\mathfrak{g}}$ is one-dimensional by (5.2.5) (note that its dimension is equal to the constant term of the right side of (5.2.5)). For the second, we compute as follows, using the Clebsh-Gordan formula; $\alpha_m\otimes\alpha_n\cong\alpha_{m+n}\oplus\alpha_{m+n-2}\oplus\cdots\oplus\alpha_{|m-n|}$ (here, $\alpha_1=\alpha$, $\alpha_0=\mathbf{1}=$the trivial representation):

$$\begin{aligned}(\wedge^3 V_1)\otimes V_2&\cong(\alpha_3\beta\gamma\oplus\alpha\beta_3\gamma\oplus\alpha\beta\gamma_3\oplus\alpha\beta\gamma)\otimes(\alpha\beta\gamma)\\&\cong(\alpha_4\oplus\alpha_2)\otimes(\beta_2\oplus\mathbf{1})\otimes(\gamma_2\oplus\mathbf{1})\\&\quad\oplus(\alpha_2\oplus\mathbf{1})\otimes(\beta_4\oplus\beta_2)\otimes(\gamma_2\oplus\mathbf{1})\end{aligned}$$

$$\oplus(\alpha_2\oplus 1)\otimes(\beta_2\oplus 1)\otimes(\gamma_4\oplus\gamma_2)$$
$$\oplus(\alpha_2\oplus 1)\otimes(\beta_2\oplus 1)\otimes(\gamma_2\oplus 1).$$

Hence $\dim[(\wedge^3 V_1)\otimes V_2]^g=1$. For the third summand,

$$(\wedge^2 V_1)\otimes(\wedge^2 V_2)\cong(\alpha_2\beta_2\oplus\beta_2\gamma_2\oplus\gamma_2\alpha_2\oplus\mathbf{1})$$
$$\otimes(\alpha_2\beta_2\oplus\beta_2\gamma_2\oplus\gamma_2\alpha_2\oplus\mathbf{1}).$$

Hence $\dim[(\wedge^2 V_1)\otimes(\wedge^2 V_2)]^g=4$. But, since $[\wedge^2(V_1\oplus V_2)]^g\cong[\wedge^2 V_1]^g\oplus[V_1\otimes V_2]^g\oplus[\wedge^2 V_2]^g$ and each summand of the right side is one-dimensional, the dimension of the subspace of $[(\wedge^2 V_1)\otimes(\wedge^2 V_2)]^g$ generated by $[\wedge^2(V_1\oplus V_2)]^g$ under the exterior product is at most two. Thus the assertion (5.2.2) is proved. This completes the proof of Proposition (5.1).

PART II

§6. Hilbert modular surface.

In this section we recall the definition of Hilbelt modular surface. Fundamental references are Hirzebruch-Van de Ven [4] and Oda [11].

Let F be a real quadratic field and let O_F denote the integer ring of F. For any element α of F, we denote by α' the conjugate of α over $\boldsymbol{Q}$. The imbedding $F\hookrightarrow\boldsymbol{R}\oplus\boldsymbol{R}$, obtained by a mapping $\alpha\to(\alpha,\alpha')$, induces an imbedding $SL_2(F)\hookrightarrow SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$ of the special linear group $SL_2(F)$ with entries in F into the self-product of $SL_2(\boldsymbol{R})$. Let H be the complex upper half plane on which $SL_2(\boldsymbol{R})$ acts by the following formula:

$$g(z)=(az+b)/(cz+d),\quad\text{for}\quad z\in H,\ g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\boldsymbol{R}).$$

The product $SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$ acts factorwise on the product $H\times H$. Then we obtain an action of $SL_2(F)$ on $H\times H$ by composing the action with the imbedding $SL_2(F)\hookrightarrow SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$. The subgroup $SL_2(O_F)$ of $SL_2(F)$, which is a discrete subgroup of $SL_2(\boldsymbol{R})\times SL_2(\boldsymbol{R})$, acts properly discontinuously on $H\times H$. We denote by Γ the group $SL_2(O_F)/\{\pm 1\}$. Then the quotient analytic space $S=\Gamma\backslash(H\times H)$ is called a Hilbert modular surface. We denote by $S_2(SL_2(O_F))$ the space of cusp forms of weight two with respect to $SL_2(O_F)$. An element f of $S_2(SL_2(O_F))$ is called a primitive form if f is a common eigenform of all Hecke operators [11, Ch. 1, §0].

§7. Preliminaries for §8.

The following lemmas will be used in §8:

(7.1) LEMMA. *Let X, Y be non-singular projective varieties. For any $\alpha\in H^{2p}(X, \boldsymbol{Q})$ and $\beta\in H^{2q}(Y, \boldsymbol{Q})$, if $\alpha\otimes\beta\in\mathscr{B}^{p+q}(X\times Y)$ and $\alpha\in\mathscr{B}^{p}(X)$, then $\beta\in\mathscr{B}^{q}(Y)$.*

PROOF. Consider the Hodge decomposition of β in $H^{2q}(Y, \boldsymbol{C})\cong\bigoplus_{i=0}^{2q}H^{i,q-i}(Y)$. Then the assumption implies that β is purely of type (q, q).

In particular we have

(7.2) COROLLARY. *Let X, Y be non-singular projective surfaces. For any $\alpha\in H^{2}(X, \boldsymbol{Q})$ and $\beta\in H^{2}(Y, \boldsymbol{Q})$, if $\alpha\otimes\beta\in\mathscr{B}^{2}(X\times Y)$ and $\alpha\in\mathscr{B}^{1}(X)$, then $\beta\in\mathscr{C}^{1}(X)$.*

PROOF. This follows from (7.1) and Lefschetz' theorem which says that $\mathscr{B}^{1}(X)=\mathscr{C}^{1}(X)$ for every non-singular projective variety X.

§8. Main Theorem B: Algebraic cycles on a power of Hilbert modular surface.

For the rest of this paper, we assume that the class number of F is one and that there exists a unit ε in the integer ring O_F with negative norm $N_{F/\boldsymbol{Q}}(\varepsilon)=\varepsilon\varepsilon'=-1$. To state the main theorem, we must recall some of Oda's results in [9]. He defined a rational polarized Hodge structure $H^2_{sp}(S, \boldsymbol{Q})$ as a sub-Hodge structure of $W_2H^2(S, \boldsymbol{Q})$ ($W.$ denotes the weight filtration on the mixed Hodge structure $H^2(S, \boldsymbol{Q})$. Let $\mathscr{H}$ denote the ring of Hecke operators and let $\mathscr{H}_0$ be the subring of the endomorphism ring $\mathrm{End}(H^2_{sp}(S, \boldsymbol{Q}))$ of the Hodge structure $H^2_{sp}(S, \boldsymbol{Q})$, generated by the identity and the images of the elements of $\mathscr{H}$ over $\boldsymbol{Q}$. Let Ξ be a maximal subset of the set of all primitive forms of $S_2(SL_2(O_F))$ such that any two elements of Ξ are not companions mutually ([11, Ch. 1, 2.6]). Then

$$\mathscr{H}_0\cong\bigoplus_{f\in\Xi}K_f \quad\text{and}\quad H^2_{sp}(S, \boldsymbol{Q})\cong\bigoplus_{f\in\Xi}H_f,$$

where K_f $(f \in \Xi)$ is a totally real number field and H_f $(f \in \Xi)$ is a polarized rational Hodge structure on which K_f acts as endomorphisms of rational Hodge structure ([loc. cit., Ch. I, §2 and §3]). One of his main results is:

(8.1) THEOREM ([loc. cit., Th. 7.2]). *Notation being as above, we have an isomorphism of Hodge structures*

$$H_f \cong H^1(A_f^1, \boldsymbol{Q}) \bigotimes_{K_f} H^1(A_f^2, \boldsymbol{Q}),$$

where A_f^i $(i=1, 2)$ *are abelian varieties with* $K_f \subset \mathrm{End}^0 A_f^i$. *Moreover the above isomorphism is compatible with* K_f*-action.*

Now our main theorem is the following:

(8.2) MAIN THEOREM B. *Let* $\tilde{S}$ *be the minimal resolution of the compactification of a Hilbert modular surface* S. *Assume that* A_f^1 *and* A_f^2 *appearing in* (8.1) *are* K_f*-isogenous to one and the same abelian variety* A_f *for each* $f \in \Xi$, *and that* $\mathrm{End}^0(\prod_{f \in \Xi} A_f) \cong \prod_{f \in \Xi} K_f$. *Then* $\mathscr{B}^*(\tilde{S}\times\tilde{S}) = \mathscr{C}^*(\tilde{S}\times\tilde{S})$, *i.e. the Hodge Conjecture holds for* $\tilde{S}\times\tilde{S}$. *Moreover*

$$\dim_{\boldsymbol{Q}} \mathscr{B}^2(\tilde{S}\times\tilde{S})/\mathscr{D}^2(\tilde{S}\times\tilde{S}) = \sum_{f \in \Xi} [K_f : \boldsymbol{Q}] = p_g(\tilde{S}),$$

where $p_g(\tilde{S})$ *denotes the geometric genus of* $\tilde{S}$.

REMARK. Under the assumption of (8.2), the abelian variety $A = \prod_{f \in \Xi} A_f$ is stably non-degenerate. For, since A_f $(f \in \Xi)$ are of type I, and mutually non-isogenous, we have $\dim A = \sum_{f \in \Xi} \mathrm{rdim}\, A_f = \sum_{f \in \Xi} [K_f : \boldsymbol{Q}] = \mathrm{rank}(\mathrm{Hg}(A)_{\boldsymbol{C}})$ (the last equality follows from (8.5) below).

The following two propositions show that the assumption of (8.2) is actually satisfied in some cases.

(8.3) PROPOSITION. *Suppose that the real quadratic field* F *satisfies the following conditions: the class number of* F *is one, the discriminant* D *is a prime congruent to* 1 *modulo* 4, *and all* $f \in \Xi$ *are self-conjugate. Then the assumption of* (8.2) *is satisfied. Therefore for the corresponding Hilbert modular surface* $\tilde{S}$, *we have* $\mathscr{B}^*(\tilde{S}\times\tilde{S}) = \mathscr{C}^*(\tilde{S}\times\tilde{S})$, *and* $\dim_{\boldsymbol{Q}} \mathscr{B}^2(\tilde{S}\times\tilde{S})/\mathscr{D}^2(\tilde{S}\times\tilde{S}) = p_g(\tilde{S})$.

(8.4) PROPOSITION. *If* D *is a prime congruent to* 1 *modulo* 4 *and* $D \leqq 181$, *or* $D = 197$, 269, 293, 317, *then for the corresponding*

Hilbert modular surface $\tilde{S}$, *we have* $\mathscr{B}^*(\tilde{S}\times\tilde{S})=\mathscr{C}^*(\tilde{S}\times\tilde{S})$ *and* $\dim_{\boldsymbol{Q}}\mathscr{B}^2(\tilde{S}\times\tilde{S})/\mathscr{D}^2(\tilde{S}\times\tilde{S})=p_g(\tilde{S})$.

PROOF OF (8.2). Since $\tilde{S}$ is regular (cf. [4, Prop. II. 4]), we have $H^1(\tilde{S}, \boldsymbol{Q})\cong(0)$. Therefore the Künneth decomposition of $H^4(\tilde{S}\times\tilde{S}, \boldsymbol{Q})$ becomes as follows:

$$\begin{aligned} H^4(\tilde{S}\times\tilde{S}, \boldsymbol{Q})\cong &(H^0(\tilde{S}, \boldsymbol{Q})\otimes H^4(\tilde{S}, \boldsymbol{Q})) \\ &\oplus(H^4(\tilde{S}, \boldsymbol{Q})\otimes H^0(\tilde{S}, \boldsymbol{Q})) \\ &\oplus(H^2(\tilde{S}, \boldsymbol{Q})\otimes H^2(\tilde{S}, \boldsymbol{Q})). \end{aligned}$$

The first two summands on the right side are generated by algebraic cycles corresponding to (a point)$\times\tilde{S}$, and $\tilde{S}\times$(a point). Hence it suffices to consider Hodge cycles lying in $H^2(\tilde{S}, \boldsymbol{Q})\otimes H^2(\tilde{S}, \boldsymbol{Q})$. For this it suffices to show that every element purely of type (2, 2) in the Hodge structure

$$(H^2(\tilde{S}, \boldsymbol{Q})/\mathscr{B}^1(\tilde{S}))\underset{\boldsymbol{Q}}{\otimes}(H^2(\tilde{S}, \boldsymbol{Q})/\mathscr{B}^1(\tilde{S}))$$

comes from some algebraic cycle on $\tilde{S}\times\tilde{S}$. Here we recall the following:

(8.5) PROPOSITION (cf. [11, Remark, 1.13]). *Notations being as above, we have an isomorphism of homogeneous rational polarized Hodge structures*

$$H^2_{sp}(S, \boldsymbol{Q})\oplus\left(\overset{a}{\oplus}\boldsymbol{Q}(-1)\right)\cong H^2(\tilde{S}, \boldsymbol{Q})\oplus\left(\overset{b}{\oplus}\boldsymbol{Q}(-1)\right)$$

for some non-negative integers a, b *where* $\boldsymbol{Q}(-1)$ *is the Hodge structure of Tate of weight two.*

In particular, we have

$$H^2_{sp}(S, \boldsymbol{Q})/\mathscr{B}^1(H^2_{sp}(S, \boldsymbol{Q}))\cong H^2(\tilde{S}, \boldsymbol{Q})/\mathscr{B}^1(H^2(\tilde{S}, \boldsymbol{Q})).$$

Hence, combining (7.2) and Lefschetz's theorem, we see that it suffices to prove the (2, 2)-part of $H^2_{sp}(S, \boldsymbol{Q})/\mathscr{B}^1(H^2_{sp}(S, \boldsymbol{Q}))\otimes_{\boldsymbol{Q}}H^2_{sp}(S, \boldsymbol{Q})/\mathscr{B}^1(H^2_{sp}(S, \boldsymbol{Q}))$ is generated by the images of algebraic cycles.

Let us consider the (2, 2)-part of the Hodge structure $H_f\otimes_{\boldsymbol{Q}}H_f$ (cf. (8.1)). Note that $H^1(A_f, \boldsymbol{Q})$ has a natural action of the Hodge group $\mathrm{Hg}(A_f)$ of the abelian variety A_f, which is defined by the Hodge structure of $H^1(A_f, \boldsymbol{Q})$. On $H^1(A_f, \boldsymbol{Q})$ the action of $\mathrm{Hg}(A_f)$ commutes with the action of K_f, since $K_f \cong \mathrm{End}^0 A_f \cong \mathrm{End}_{\mathrm{Hg}(A_f)} H^1(A_f, \boldsymbol{Q})$ (see (1.3) for the latter equality). Hence, if we put $V_f = H^1(A_f, \boldsymbol{Q})$, the Hodge structure of $(V_f \otimes_{K_f} V_f) \otimes_{\boldsymbol{Q}} (V_f \otimes_{K_f} V_f)$ is described by the tensor product of the action of $\mathrm{Hg}(A_f)$. We compute $\dim_{\boldsymbol{Q}} [(V_f \otimes_{K_f} V_f) \otimes_{\boldsymbol{Q}} (V_f \otimes_{K_f} V_f)]^{\mathrm{Hg}(A_f)}$. This is equal to

$$\begin{aligned}
&\dim_{\boldsymbol{C}}\left(\left[\left(V_f \underset{K_f}{\otimes} V_f\right) \underset{\boldsymbol{Q}}{\otimes} \left(V_f \underset{K_f}{\otimes} V_f\right)\right]^{\mathrm{Hg}(A_f)} \underset{\boldsymbol{Q}}{\otimes} \boldsymbol{C}\right) \\
&= \dim_{\boldsymbol{C}}\left[\left(\left(V_f \underset{K_f}{\otimes} V_f\right) \underset{\boldsymbol{Q}}{\otimes} \left(V_f \underset{K_f}{\otimes} V_f\right)\right) \underset{\boldsymbol{Q}}{\otimes} \boldsymbol{C}\right]^{\mathrm{Hg}(A_f) \otimes_{\boldsymbol{Q}} \boldsymbol{C}} \\
&= \dim_{\boldsymbol{C}}\left[\left(\left(V_f \underset{K_f}{\otimes} V_f\right) \underset{\boldsymbol{Q}}{\otimes} \left(V_f \underset{K_f}{\otimes} V_f\right)\right) \underset{\boldsymbol{Q}}{\otimes} \boldsymbol{C}\right]^{\mathscr{L}ie\,(\mathrm{Hg}(A_f)_{\boldsymbol{C}})}.
\end{aligned}$$

Here we denote by $\mathscr{L}ie(\mathrm{Hg}(A_f)_{\boldsymbol{C}})$ the Lie algebra of the algebraic group $\mathrm{Hg}(A_f) \otimes_{\boldsymbol{Q}} \boldsymbol{C}$. Put $\mathrm{Hom}(K_f, \boldsymbol{C}) = \{\sigma_1, \cdots, \sigma_{d_f}\}$ $(d_f = [K_f : \boldsymbol{Q}])$, then

$$\begin{aligned}
&\left(\left(V_f \underset{K_f}{\otimes} V_f\right) \underset{\boldsymbol{Q}}{\otimes} \left(V_f \underset{K_f}{\otimes} V_f\right)\right) \underset{\boldsymbol{Q}}{\otimes} \boldsymbol{C} \\
&\cong \left(\left(V_f \underset{K_f}{\otimes} V_f\right) \underset{\boldsymbol{Q}}{\otimes} \boldsymbol{C}\right) \underset{\boldsymbol{C}}{\otimes} \left(\left(V_f \underset{K_f}{\otimes} V_f\right) \underset{\boldsymbol{Q}}{\otimes} \boldsymbol{C}\right) \\
&\cong \left[\bigoplus_{\sigma_i} \left(\left(V_f \underset{K_f}{\otimes} V_f\right) \underset{K_f, \sigma_i}{\otimes} \boldsymbol{C}\right)\right] \underset{\boldsymbol{C}}{\otimes} \left[\bigoplus_{\sigma_j} \left(\left(V_f \underset{K_f}{\otimes} V_f\right) \underset{K_f, \sigma_j}{\otimes} \boldsymbol{C}\right)\right] \\
&\cong \bigoplus_{i,j} \left(\left(V_f \underset{K_f, \sigma_i}{\otimes} \boldsymbol{C}\right) \underset{\boldsymbol{C}}{\otimes} \left(V_f \underset{K_f, \sigma_i}{\otimes} \boldsymbol{C}\right) \underset{\boldsymbol{C}}{\otimes} \left(V_f \underset{K_f, \sigma_j}{\otimes} \boldsymbol{C}\right) \otimes \left(V_f \underset{K_f, \sigma_j}{\otimes} \boldsymbol{C}\right)\right).
\end{aligned}$$

On the other hand, we have the following:

(8.6) PROPOSITION. $\mathrm{Hg}(A_f) \cong SL_2(K_f)$ *and the natural representation of* $\mathrm{Hg}(A_f)$ *in* V_f *is equivalent to the natural representation of* $SL_2(K_f)$ *in* $K_f \oplus K_f$.

PROOF OF (8.6). Since we have $\mathrm{Hg}(A_f)_{\boldsymbol{C}} \cong \prod_{i=1}^{d_f} SL_2(\boldsymbol{C}) \cong SL_2(K_f) \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ (cf. [2]), it suffices to show that $\mathrm{Hg}(A_f)$ is contained in $SL_2(K_f)$. We already know that $\mathrm{Hg}(A_f)$ acts K_f-linearly on V_f. Recall that $\mathrm{Hg}(A_f)$ preserves the Riemann form I on V_f, which is $\boldsymbol{Q}$-bilinear and skew-symmetric (cf. §1). But it is known that the $\boldsymbol{Q}$-bilinear form I is induced from a K_f-bilinear form I on V_f, i.e., $I = \mathrm{Tr}_{K_f/\boldsymbol{Q}} \circ I$ (cf. [11, Ch. II, 6.2]). From this and the non-degeneracy of $\mathrm{Tr}_{K_f/\boldsymbol{Q}}$ follows that $\mathrm{Hg}(A_f)$ is contained in $Sp(V_f) \cong SL_2(K_f)$. This proves (8.6).

Now we put $V_f \otimes_{K_f, \sigma_i} \boldsymbol{C} = V_i$ for $1 \leqq i \leqq d_f$. Then $\dim_{\boldsymbol{C}} V_i = 2$ and the representation of $SL_2(K_f)\otimes_Q C$ in $V_f\otimes_Q C$ is equivalent to the direct sum of the natural representations, i.e., the representation of $SL_2(C)\times\cdots\times SL_2(C)$ (d_f times) in $V_1\oplus\cdots\oplus V_{d_f}$. Hence we obtain an isomorphism

$$\left[\left(\left(V_f\underset{K_f}{\otimes}V_f\right)\underset{Q}{\otimes}\left(V_f\underset{K_f}{\otimes}V_f\right)\right)\underset{Q}{\otimes}C\right]^{\mathscr{Lie}(\mathrm{Hg}(A_f)_C)}$$
$$\cong\left(\bigoplus_{i=1}^{d_f}[V_i\otimes V_i\otimes V_i\otimes V_i]^{\mathfrak{sl}_2(C)}\right)$$
$$\oplus\left(\bigoplus_{i\neq j}[V_i\otimes V_i]^{\mathfrak{sl}_2(C)}\otimes[V_j\otimes V_j]^{\mathfrak{sl}_2(C)}\right).$$

Note that we have $(H_f\otimes_Q C)^{1,1}$ (=the (1, 1)-part of $H_f\otimes_Q C)\cong\bigoplus_i[V_i\otimes V_i]^{\mathfrak{sl}_2(C)}$ by the previous observation.

(8.7) LEMMA. *Let e_i (resp. $\bar{e}_i$) denote the standard basis $\begin{pmatrix}1\\0\end{pmatrix}$ (resp. $\begin{pmatrix}0\\1\end{pmatrix}$) of $V_i\cong C^2$ with natural action of the Lie algebra $\mathfrak{sl}_2(C)$. Then*

$$[V_i\otimes V_i]^{\mathfrak{sl}_2(C)}=\langle e_i\otimes\bar{e}_i-\bar{e}_i\otimes e_i\rangle_C,$$
$$[V_i\otimes V_i\otimes V_i\otimes V_i]^{\mathfrak{sl}_2(C)}$$
$$=\langle e_i\otimes\bar{e}_i\otimes e_i\otimes\bar{e}_i-e_i\otimes\bar{e}_i\otimes\bar{e}_i\otimes e_i-\bar{e}_i\otimes e_i\otimes e_i\otimes\bar{e}_i+\bar{e}_i\otimes e_i\otimes\bar{e}_i\otimes e_i,$$
$$e_i\otimes e_i\otimes\bar{e}_i\otimes\bar{e}_i-e_i\otimes\bar{e}_i\otimes\bar{e}_i\otimes e_i-\bar{e}_i\otimes e_i\otimes e_i\otimes\bar{e}_i+\bar{e}_i\otimes\bar{e}_i\otimes e_i\otimes e_i\rangle_C.$$

PROOF OF (8.7). This follows from a straightforward computation.

Combining the above results, we obtain the following:

(8.8) COROLLARY. $\dim_Q\mathscr{B}^2(H_f\otimes_Q H_f)=d_f^2+d_f$.

Now we have

$$\left(H^2_{sp}(S,Q)\underset{Q}{\otimes}H^2_{sp}(S,Q)\right)^{(2,2)}\cong\left(\bigoplus_{f\in B}\left(H_f\underset{Q}{\otimes}H_f\right)^{(2,2)}\right)\oplus\left(\bigoplus_{f\neq g}\left(H_f\underset{Q}{\otimes}H_g\right)^{(2,2)}\right).$$

But since A_f and A_g are not isogenous for $f\neq g$ by the assumption of (8.2), we see that $\mathrm{Hg}(A_f\times A_g)\cong\mathrm{Hg}(A_f)\times\mathrm{Hg}(A_g)$ and that the representation of $\mathrm{Hg}(A_f\times A_g)$ in $H^1(A_f\times A_g)$ is equivalent to the direct sum representation of $\mathrm{Hg}(A_f)\times\mathrm{Hg}(A_g)$ in $H^1(A_f,Q)\oplus H^1(A_g,Q)$ (cf. §1), so if $f\neq g$, then

$$\left(\left(V_f\underset{K_f}{\otimes}V_f\right)\underset{Q}{\otimes}\left(V_g\underset{K_g}{\otimes}V_g\right)\right)^{(2,2)}\cong\left[V_f\underset{K_f}{\otimes}V_f\right]^{\mathrm{Hg}(A_f)}\underset{Q}{\otimes}\left[V_g\underset{K_g}{\otimes}V_g\right]^{\mathrm{Hg}(A_g)}.$$

Since cycles in $[V_f \otimes_{K_f} V_f]^{\mathrm{Hg}(A_f)}$ $(f \in \Xi)$ are algebraic, we reduce the problem to show that $\bigoplus_{f \in \Xi}(H_f \otimes_Q H_f)^{(2,2)} = \bigoplus_{f \in \Xi}[H_f \otimes_Q H_f]^{\mathrm{Hg}(A_f)}$ consists of algebraic cycles.

First note that every algebraic correspondence arising from Hecke operator is orthogonal to $[H_f]^{\mathrm{Hg}(A_f)} \otimes [H_g]^{\mathrm{Hg}(A_g)}$ with respect to the intersection form if $f \neq g$, for a Hecke operator maps H_f to H_f which is orthogonal to H_g.

Now we consider a map $\tilde{\iota}$ of $H \times H$ to $H \times H$ defined by $\tilde{\iota}(z_1, z_2) = (z_2, z_1)$ for $(z_1, z_2) \in H \times H$. Since $\tilde{\iota}$ is contained in the normalizer of $SL_2(O_F)$ in the group of diffeomorphisms of $H \times H$, we have an involutive automorphism $\iota: \tilde{S} \to \tilde{S}$. Hence it defines an algebraic correspondence on $\tilde{S}$. We denote by ι^* the endomorphism of the Hodge structure H_f induced by ι. We know that ι^* commutes with the action of Hecke operators under the assumption of (8.2) (cf. [11, 7.7, 9.4, 9.5]), hence it maps H_f to H_f. So a Hecke operator composed with ι^* gives an algebraic correspondence which is orthogonal to $[H_f]^{\mathrm{Hg}(A_f)} \otimes [H_g]^{\mathrm{Hg}(A_g)}$ if $f \neq g$ by the same reason as above.

Hence we are reduced to show that when T runs through the set of Hecke operators the algebraic correspondences arising from T and $T \circ \iota^*$ generate the space $\bigoplus_{f \in \Xi}[H_f \otimes_Q H_f]^{\mathrm{Hg}(A_f)}$. Here we recall the for any embedding $\sigma_i: K_f \to \boldsymbol{C}$, ι^* acts on $H_f \otimes_{K_f, \sigma_i} \boldsymbol{C}$ by

$$\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

for some basis (cf. [loc. cit., Prop. 8.10]). On the other hand, Hecke operators acts on $H_f \otimes_{K_f, \sigma_i} \boldsymbol{C}$ by scalar action. Hence we see that $\bigoplus_{f \in \Xi}(K_f \oplus K_f \circ \iota^*)$ generates a $2p_g$-dimensional subspace of $\bigoplus_{f \in \Xi}[H_f \otimes_Q H_f]^{\mathrm{Hg}(A_f)}$, but the latter is of dimension $2p_g$ by (8.7), so we are done.

The second assertion in (8.2) follows from the above computation and the equality $\sum_{f \in \Xi}[K_f: \boldsymbol{Q}] = p_g$. Thus our theorem (8.2) is proved.

PROOF OF (8.3). In case D is a prime congruent to 1 modulo 4 with $D \leqq 23$, the corresponding surface $\tilde{S}$ is rational by [4], hence the assertion follows from the equality $\mathscr{B}^1(\tilde{S}) = H^2(\tilde{S}, \boldsymbol{Q})$ and the Künneth formula. In particular, $\dim_Q \mathscr{B}^2(\tilde{S} \times \tilde{S})/\mathscr{D}^2(\tilde{S} \times \tilde{S}) = 0 = p_g(\tilde{S})$. In case D is a prime $\geqq 29$ congruent to 1 modulo 4, it is known that any self-conjugate primitive form f is obtained by the Doi-Naganuma lifting of some primitive form $h(f) \in S_2(\Gamma_0(D), \varepsilon_D)$, where ε_D is defined by the

Legendre symbol $\left(\frac{D}{-}\right)$ (cf. [11, § 18]), and that the abelian varieties A_f^1 and A_f^2 are K_f-isogenous to an abelian variety $B_{h(f)}$ attached to $h(f)$ (cf. [loc. cit., Th. 17.2]). Moreover, it follows from [loc. cit., Prop. 13.5] that the abelian variety $\prod_{f\in\Xi} A_f^1$ is isogenous to $\prod_{h\in\Xi'} B_h$, where Ξ' is a maximal subset of the set of all primitive forms of $S_2(\Gamma_0(D), \varepsilon_D)$ such that any two elements of Ξ' are not companions mutually. Thus we see by our theorem (8.2) that it suffices to show the following proposition whose proof is due to Ribet:

(8.9) PROPOSITION. *If the discriminant D is a prime congruent to* 1 *modulo* 4, *then*

$$\mathrm{End}^0\Big(\prod_{h\in\Xi'} B_h\Big)\cong \prod_{h\in\Xi'} \mathrm{End}^0 B_h\cong \prod_{f\in\Xi} K_f. \tag{8.9.1}$$

PROOF OF (8.9). In the terminology of Ribet [13], (8.9.1) is equivalent to the statement:

(8.9.2) any primitive form of level D and character ε_D has no twist nor inner twist.

Namely,

(8.9.3) if h is a primitive form of level D and character ε_D and $h\otimes\chi$ is, up to finitely many Euler factors, equal to f again of this type for a character χ, then χ is either the trivial character or ε_D, hence f is either h or the complex conjugate of h.

This is proved as follows. Since "Nebentypus" character of $h\otimes\chi$ is $\varepsilon_D\chi^2$, we know that χ is quadratic (or trivial). If we look at the l-adic representations attached to f and h, we obtain one set of representations from the other by twisting by χ. These representations can be ramified only at D and at l, hence χ can be ramified only at D and at l. Choosing two different primes l, we see that χ is in fact ramified only at D. Therefore χ is either the trivial character or ε_D. This completes the proof of (8.9).

PROOF OF (8.4). It is shown in [11, § 18] that the assumption of Proposition (8.3) is satisfied for D appearing in the statement of (8.4). Thus (8.4) is proved.

REMARK. 1) Hirzebruch and Van de Ven [4] proved that the Hilbert modular surface $\tilde{S}$ is

i) an elliptic $K3$-surface if $D=29, 37, 41$,

ii) an honestly elliptic surface if $D=53, 61, 73$,

iii) a surface of general type if D is a prime $\geqq 89$ congruent to 1 modulo 4.

Hence our result gives some examples of algebraic surface X, which is $K3$, elliptic or of general type, such that the Hodge Conjecture holds for its power $X\times X$.

2) Shioda pointed out the in case i) above the minimal model of $\tilde{S}$ is dominated by the Kummer surface $\mathrm{Km}(E\times E)$ associated to the power $E\times E$ of an elliptic curve E and that the validity of the Hodge Conjecture for the power $\tilde{S}\times\tilde{S}$ is implied by that for the power E^4.

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Department of Mathematical Science
College of Science and Engineering
Tokyo Denki University
Hatoyama-cho, Hikigun
Saitama
350-03 Japan