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On holomorphic cusp forms on quaternion unitary groups of degree 2

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A. N. Andrianov proved the functional equation of the L-function associated with a Siegel modular form of degree two. In this article we extend his results to some non-split cases by using adelic language.

§ 0. Introduction

The main purpose of this paper is to prove the meromorphic continuation and the functional equation of the L-function associated with a holomorphic cusp form on a quaternion unitary group of degree 2.

Let k and $\mathfrak{o}$ be a totally real algebraic number field of degree n and its maximal order respectively, and let B be a quaternion algebra over k such that $B_v = B \otimes_k k_v$ is isomorphic to $M_2(\boldsymbol{R})$ at each archimedean place v of k. Denote by $\alpha \mapsto \bar{\alpha}$ $(\alpha \in B)$ the canonical involution and by $\mathfrak{O}$ a maximal order of B. We define an algebraic group G over k by

$$G_k = \left\{ g \in GL_2(B) \,\middle|\, g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \mu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mu(g) \in k^\times \right\},$$

where $g^* = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$ for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(B)$. Let $\mathfrak{A}$ be a two-sided $\mathfrak{O}$-ideal, and for each prime $\mathfrak{p}$ of k we define a maximal open compact subgroup $U_\mathfrak{p}$ of $G_\mathfrak{p}$ by

$$U_\mathfrak{p} = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_\mathfrak{p} \,\middle|\, \alpha, \delta \in \mathfrak{O}_\mathfrak{p},\ \beta \in \mathfrak{A}_\mathfrak{p},\ \gamma \in \mathfrak{A}_\mathfrak{p}^{-1},\ \mu(g) \in \mathfrak{o}_\mathfrak{p}^\times \right\}.$$

Put $U_f = \prod_\mathfrak{p} U_\mathfrak{p}$, where $\mathfrak{p}$ runs through all prime ideals of k. Let $\boldsymbol{l} = (l_1, \cdots, l_n)$ [resp. $\boldsymbol{d} = (d_1, \cdots, d_n)$] be an n-tuple of integers [resp. non-negative integers], and $\rho = \rho_{l,d}$ be a finite dimensional representation of $GL_2(\boldsymbol{C})^n$ determined by $\boldsymbol{l}$ and $\boldsymbol{d}$ ((1-7)). In § 1 we define the space $\mathfrak{S}(\rho, \lambda; U_f)$, which consists of holomorphic cusp forms on G_A of weight ρ, with central character λ, and with respect to U_f. This coincides with the space of so-called Hilbert-Siegel cusp forms when $B=M_2(k)$ and $d_1=\cdots=d_n=0$. For each $F\in\mathfrak{S}(\rho, \lambda; U_f)$ we introduce a function $\varphi^A_{F,\xi}$ ((1-22)), which is called a generalized Whittaker model ([13], [16]). Some local properties of $\varphi^A_{F,\xi}$ will be studied by elementary manner in § 2. In § 3 Main Theorem (Theorem 3-2) is stated and proved. When F is a simultaneous eigen function of the Hecke algebra $\mathscr{H}(G_\mathfrak{p}, U_\mathfrak{p})$ for all primes $\mathfrak{p}$, and ω is an unramified grössencharacter of $k_A^\times$, we define the L-function $Z_F(\omega, s)$ of F, whose $\mathfrak{p}$-part is essentially equal to the denominator of the local Hecke series. Let ω be the trivial character. Our theorem asserts that

$$\zeta_F(1, s)=Z_F(1, s)(d(k)^2\sqrt{N(\mathfrak{D})}/(2\pi)^{2n})^s\prod_{j=1}^{n}\Gamma(s+(d_j+1)/2)\Gamma(s+l_j+(d_j-3)/2)$$

is continued analytically to the whole s-plane as a meromorphic function and it satisfies the functional equation

$$\zeta_F(1, s)=(-1)^{\Sigma l_j}\lambda(\mathfrak{d}_k^2)\Big(\prod_{\mathfrak{p}|\mathfrak{D}}\sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})\Big)\zeta_{F'}(1, 1-s).$$

Here $d(k)$, $\mathfrak{d}_k$, $\mathfrak{D}$, and $N(\mathfrak{D})$ denote the discriminant of k over $\boldsymbol{Q}$, the different ideal of k, the discriminant ideal of B over k, and its norm. F' is the element of $\mathfrak{S}(\rho, \lambda^{-1}; U_f)$ defined by $F'(g)=F(g)\lambda^{-1}(\mu(g))$, and $\sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})$ is the eigen value of some special element in $\mathscr{H}(G_\mathfrak{p}, U_\mathfrak{p})$ for each prime $\mathfrak{p}$ ramifying in B ((1-30)). In the last section § 4, we calculate some examples explicitly in the case $k=\boldsymbol{Q}$ by using Oda's lifting (cf. [22], [14]).

The above theorem has been proved first by A. N. Andrianov (cf. [1], [2]) in the case: $B=M_2(\boldsymbol{Q})$ and $\boldsymbol{d}=0$. V. G. Zhuravlev proved it, under some conditions in the case where the class number of k is 1, $B=M_2(k)$, and $d=0$ ([23]). In [4] T. Arakawa has obtained the above result under some condition for vector valued Siegel modular forms. And for congruence subgroups it has been studied in Evdokimov [6, 7], and Matsuda [12]. But we treat only the full modular cases here. Since it is more convenient and natural for investigating the L-function to take the adelic setting, we adelize Andrianov's results. Our proof of the Euler product expansion (Theorem 3-1) is reduced to the local properties of $\varphi^A_{F,\xi}$ (Theorem 2-1) and rather different from Andrianov's. On the other hand, the proof of the analytic continuation and the functional equation is similar to his argument. When $B=M_2(k)$, I. I. Piatetski-Shapiro has established the above result for more general modular forms by a representation theoretical method ([13], [16]). The author (who has been working independently of Piatetski-Shapiro) hopes that his approach is of separate interest.

The author wishes to express his deepest gratitude to the late Professor Takuro Shintani, who suggested him to reformulate Andrianov's theory in terms of adelic language by investigating $\varphi^{A}_{F,\xi}$, which plays a crucial role in this paper. He would like to express his hearty thanks to Professors Tomoyoshi Ibukiyama and Ki-ichiro Hashimoto, who drew his attention to non-split cases. He also thanks Professors Hideo Shimizu, Shin-ichiro Ihara, Yasutaka Ihara, and Takayuki Oda for their valuable advice and warm encouragement. He is very grateful to the referee for many valuable remarks.

Notations

We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$, respectively, the ring of integers, the rational number field, the real number field, and the complex number field. For an associative ring R with identity element, $R^{\times}$ denotes the group of all invertible elements and $M_m(R)$ the ring of all matrices of size m with coefficients in R. We put $GL_m(R)=M_m(R)^{\times}$. If R is commutative, we denote by $SL_m(R)$ the special linear group of degree m. Let k be a number field and $\mathfrak{o}$ [resp. $\mathfrak{d}_k$] be the ring of integers [resp. the different ideal of k]. For each place v of k, we denote by k_v the v-completion of k, and by $|x|_v$ the module of x for an $x\in k_v^{\times}$. k_A [resp. $k_A^{\times}$] means the adele ring of k [resp. the idele group of k] and for $x=(x_v)\in k_A^{\times}$, put $|x|_A=\prod_v |x_v|_v$. For an algebraic group G defined over k and a field K containing k, we denote by G_K the group of K-rational points of G. We abbreviate G_{k_v} to G_v. We denote by G_A, G_∞, and $G_{A,f}$ the adelized group of G, the infinite part of G_A, and the finite part of G_A, respectively. Similar notations are used for an algebra or a vector space. Each prime ideal $\mathfrak{p}$ of k is identified with the corresponding finite place, and we denote by $\mathfrak{o}_{\mathfrak{p}}$ the ring of integers of $k_{\mathfrak{p}}$. If there is no fear of confusion, the maximal ideal $\mathfrak{p}\mathfrak{o}_{\mathfrak{p}}$ of $\mathfrak{o}_{\mathfrak{p}}$ is written as $\mathfrak{p}$. We denote by $\pi_{\mathfrak{p}}$ a prime element of $k_{\mathfrak{p}}$. When L is an $\mathfrak{o}$-module, put $L_{\mathfrak{p}}=L\otimes_{\mathfrak{o}}\mathfrak{o}_{\mathfrak{p}}$. For a (fractional) ideal $\mathfrak{a}$ of k [resp. $x\in k_{\mathfrak{p}}^{\times}$] we denote its $\mathfrak{p}$-order by $\mathrm{ord}_{\mathfrak{p}}\mathfrak{a}$ [resp. $\mathrm{ord}_{\mathfrak{p}}x$]. When $\lambda=\prod_v\lambda_v$ is an unramified grössencharacter of $k_A^{\times}$, namely a character whose restriction to $k^{\times}\prod_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}}^{\times}$ is trivial, put $\lambda(\mathfrak{a})=\prod_{\mathfrak{p}}\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}\mathfrak{a}})$. For $z\in\boldsymbol{C}$, we put $e[z]=\exp(2\pi\sqrt{-1}z)$. The cardinality of a finite set S is denoted by $\sharp S$ or $|S|$. When K is a finite extension of k, we denote by $\mathrm{Tr}_{K/k}$ [resp. $N_{K/k}$] the trace of K over k [resp. the norm of K over k]. For a quaternion algebra B over k, we denote by $x\mapsto\bar{x}$ $(x\in B)$ the canonical involution of B over k, and put $\mathrm{Tr}_{B/k}(x)=x+\bar{x}$ and $N_{B/k}(x)=x\bar{x}$. We denote by B^- the set of pure quaternions, and for any subset S of B we put $S^-=B^-\cap S$.

§ 1. Definitions of $\mathfrak{S}(\rho, \lambda; U_f)$ and $\varphi^A_{F,\xi}$

1-1. Let k be a totally real algebraic number field of degree n over $\boldsymbol{Q}$, and let B be a quaternion algebra over k; and denote by $\mathfrak{D}$ the discriminant ideal of B over k. We assume that B is unramified at any infinite place of k; so the matrix algebra $M_2(k)$ is included as a special case. We denote by $\infty_1, \cdots, \infty_n$ all infinite places of k. Then by the above assumption on B, $B_{\infty_j}=B\otimes_k k_{\infty_j}$ is isomorphic to $M_2(\boldsymbol{R})$. Fix such an isomorphism once for all, and identify B_{∞_j} with $M_2(\boldsymbol{R})$.

Let G be a linear algebraic group over k defined by

$$G_k=\left\{g\in GL_2(B)\ \middle|\ g^*\begin{pmatrix}0&1\\1&0\end{pmatrix}g=\mu(g)\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \mu(g)\in k^\times\right\}, \tag{1-1}$$

where $g^*=\begin{pmatrix}\bar{\alpha}&\bar{\gamma}\\\bar{\beta}&\bar{\delta}\end{pmatrix}$ for $g=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in M_2(B)$. Then G_{∞_j} is isomorphic to $GSp(2, \boldsymbol{R})=\{g\in GL_4(\boldsymbol{R})\,|\,{}^tgJg=\mu(g)J\}$, where $J=\begin{pmatrix}0&1_2\\-1_2&0\end{pmatrix}$. Denote by G^1 the algebraic subgroup defined by the condition $\mu(g)=1$. Put

$$\begin{aligned}\mathfrak{H}_{j,+}=\Big\{Z_j\in B_{\infty_j}\otimes_{\boldsymbol{R}}\boldsymbol{C}\ \Big|\ &\mathrm{Tr}_{B/k}(Z_j)=0,\\ &(\mathrm{Im}\,Z_j)\begin{pmatrix}0&-1\\1&0\end{pmatrix}\text{ is positive definite}\Big\},\end{aligned} \tag{1-2}$$

$$\mathfrak{H}_{j,-}=\{-Z_j\,|\,Z_j\in\mathfrak{H}_{j,+}\},\quad \mathfrak{H}_j=\mathfrak{H}_{j,+}\cup\mathfrak{H}_{j,-},$$

where $\mathrm{Im}\,Z_j$ means the imaginary part of Z_j. Both $\mathfrak{H}_{j,+}$ and $\mathfrak{H}_{j,-}$ are isomorphic to the Siegel upper half plane of degree 2, and G_{∞_j} acts on $\mathfrak{H}_j$ transitively as a group of holomorphic automorphisms via the mapping

$$Z_j\longmapsto g_j\langle Z_j\rangle=(A_jZ_j+B_j)(C_jZ_j+D_j)^{-1}\quad \text{for}\quad g_j=\begin{pmatrix}A_j&B_j\\C_j&D_j\end{pmatrix}\in G_{\infty_j}. \tag{1-3}$$

Similarly $G^1_{\infty_j}$ acts transitively on $\mathfrak{H}_{j,+}$ and $\mathfrak{H}_{j,-}$. Put

$$(1\text{-}4)\qquad Z_{j,0}=\sqrt{-1}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\in\mathfrak{H}_{j,+},\qquad U_{\infty_j}=\{g_j\in G^1_{\infty_j}\mid g_j\langle Z_{j,0}\rangle=Z_{j,0}\}.$$

The group U_{∞_j}, which is a maximal compact subgroup of $G^1_{\infty_j}$, is isomorphic to the unitary group of degree 2, and $\mathfrak{H}_{j,+}$ is isomorphic to $G^1_{\infty_j}/U_{\infty_j}$. For $g_j=\begin{pmatrix}A_j & B_j\\ C_j & D_j\end{pmatrix}\in G_{\infty_j}$ and $Z_j\in\mathfrak{H}_j$, we define $GL_2(\boldsymbol{C})$-valued holomorphic automorphic factor $J_j(g_j, Z_j)$ on $G_{\infty_j}\times\mathfrak{H}_j$ by

$$(1\text{-}5)\qquad J_j(g_j, Z_j)=C_jZ_j+D_j.$$

Let $\mathfrak{H}$ [resp. $\mathfrak{H}_+$] be the product of $\mathfrak{H}_j$ [resp. $\mathfrak{H}_{j,+}$] $(1\leqq j\leqq n)$. Then $\mathfrak{H}_+$ is a connected component of $\mathfrak{H}$. We put $Z_0=(Z_{1,0},\cdots,Z_{n,0})$ and $U_\infty=U_{\infty_1}\times\cdots\times U_{\infty_n}$. The action of G_∞ on $\mathfrak{H}$ and automorphic factor on $G_\infty\times\mathfrak{H}$ are given componentwise, namely,

$$(1\text{-}6)\qquad \begin{aligned} &g\langle Z\rangle=(g_1\langle Z_1\rangle,\cdots,g_n\langle Z_n\rangle),\\ &J(g, Z)=(J_1(g_1, Z_1),\cdots,J_n(g_n, Z_n))\in GL_2(\boldsymbol{C})^n,\end{aligned}$$

where $g=(g_1,\cdots,g_n)\in G_\infty$ and $Z=(Z_1,\cdots,Z_n)\in\mathfrak{H}$.

For an n-tuple of integers $\boldsymbol{l}=(l_1,\cdots,l_n)$ and an n-tuple of non-negative integers $\boldsymbol{d}=(d_1,\cdots,d_n)$, let $\rho_{\boldsymbol{l},\boldsymbol{d}}$ be a holomorphic irreducible representation of $GL_2(\boldsymbol{C})^n$ defined by

$$(1\text{-}7)\qquad \rho_{\boldsymbol{l},\boldsymbol{d}}(g)=\bigotimes_{j=1}^{n}(\det g_j)^{l_j}\cdot\sigma_{d_j}(g_j),$$

where $g=(g_1,\cdots,g_n)\in GL_2(\boldsymbol{C})^n$ and σ_{d_j} denotes the symmetric tensor representation of $GL_2(\boldsymbol{C})$ of degree d_j. We denote by $V_{\boldsymbol{l},\boldsymbol{d}}$ its representation space. The dimension of $V_{\boldsymbol{l},\boldsymbol{d}}$ is $\prod_{j=1}^n(d_j+1)$. We fix $\boldsymbol{l}$ and $\boldsymbol{d}$ and often omit the indexes $\boldsymbol{l}$, $\boldsymbol{d}$. ρ defines a representation of U_∞ by

$$(1\text{-}8)\qquad U_\infty\ni u=(u_1,\cdots,u_n)\longmapsto\rho_{\boldsymbol{l},\boldsymbol{d}}(J(u, Z_0)).$$

Fix a positive definite hermitian inner product in V such that the above representation of U_∞ becomes unitary.

Now, we fix a maximal order $\mathfrak{O}$ of B and a two-sided $\mathfrak{O}$-ideal $\mathfrak{A}$. Then it is well known that $\mathfrak{A}$ is uniquely written as

$$(1\text{-}9)\qquad \mathfrak{A}=\mathfrak{A}_0\cdot\mathfrak{a},$$

where $\mathfrak{A}_0=\prod_{\mathfrak{p}|\mathfrak{D}}\mathfrak{P}_{\mathfrak{p}}^{e_{\mathfrak{p}}}$ ($\mathfrak{P}_{\mathfrak{p}}$ is the prime ideal of $\mathfrak{O}_{\mathfrak{p}}$), $e_{\mathfrak{p}}=0$ or 1, and $\mathfrak{a}$ is a (fractional) ideal of k. We denote by $\mathfrak{D}_0$ [resp. $\mathfrak{D}_1$] the product of all prime ideals such that $\mathfrak{p}|\mathfrak{D}$ and $e_{\mathfrak{p}}=0$ [resp. $e_{\mathfrak{p}}=1$]. For each prime ideal $\mathfrak{p}$, put

$$U_{\mathfrak{p}}=\left\{g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G_{\mathfrak{p}}\,\middle|\,\alpha,\delta\in\mathfrak{O}_{\mathfrak{p}},\ \beta\in\mathfrak{A}_{\mathfrak{p}},\ \gamma\in\mathfrak{A}_{\mathfrak{p}}^{-1},\ \mu(g)\in\mathfrak{o}_{\mathfrak{p}}^{\times}\right\}, \tag{1-10}$$

where $\mathfrak{O}_{\mathfrak{p}}=\mathfrak{O}\otimes_{\mathfrak{o}}\mathfrak{o}_{\mathfrak{p}}$ and $\mathfrak{A}_{\mathfrak{p}}=\mathfrak{A}\otimes_{\mathfrak{o}}\mathfrak{o}_{\mathfrak{p}}$. Then $U_{\mathfrak{p}}$ is a maximal open compact subgroup of $G_{\mathfrak{p}}$ and $G_{\mathfrak{p}}=P_{\mathfrak{p}}U_{\mathfrak{p}}$, where P is a parabolic subgroup of G defined by $P_k=\left\{g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G_k\,\middle|\,\gamma=0\right\}$. We abbreviate $\prod_{\mathfrak{p}}U_{\mathfrak{p}}$ to U_f, and $U_\infty U_f$ to U.

1-2. Let $\lambda=\prod_v\lambda_v$ be a character of $k_A^{\times}$ whose restriction to $k^{\times}k_\infty^{\times}\prod_{\mathfrak{p}<\infty}\mathfrak{o}_{\mathfrak{p}}^{\times}$ is trivial (i.e., an ideal class character of k). We say that a $V_{l,d}$-valued function F on G_A is holomorphic cusp form of type ($\rho_{l,d}$, λ; U_f) if F satisfies the following three conditions:

(i) $F(\gamma z g u_\infty u_f)=\lambda(z)\rho_{l,d}(J(u_\infty, Z_0))^{-1}F(g)$
for $\forall\gamma\in G_k$, $\forall z\in k_A^{\times}$, $\forall u_\infty\in U_\infty$ and $\forall u_f\in U_f$.

(ii) For any $g=g_\infty g_f$ $(g_\infty\in G_\infty,\ g_f\in G_{A,f})$,

$$\rho(J(g_\infty, Z_0))\prod_{j=1}^{n}|\mu(g_{\infty_j})|^{-(l_j+d_j/2)}F(g_\infty g_f) \tag{1-11}$$

depends only on g_f and $Z=g_\infty\langle Z_0\rangle$, and it is holomorphic on $\mathfrak{H}$ as a function of Z.

(iii) $\displaystyle\int_{N_{1,k}\backslash N_{1,A}}F(ng)dn=0$ for $\forall g\in G_A$,
where N_1 is the unipotent radical of any proper parabolic subgroup of G.

We denote by $\mathfrak{S}(\rho,\lambda;U_f)$ the space of such functions. When B is a matrix algebra $M_2(k)$, it is nothing but the space of Hilbert-Siegel cusp forms. If d_j is odd for some j, then $\mathfrak{S}(\rho_{l,d},\lambda;U_f)=\{0\}$ by (i) and (ii). So let $d_1,\cdots,d_n$ be all even integers hereafter. Note that such cusp form F is bounded on G_A.

For each $g_f\in G_{A,f}$, and $F\in\mathfrak{S}(\rho,\lambda;U_f)$ put

$$\Gamma(g_f)=G_k\cap(G_\infty\times g_fU_fg_f^{-1}), \tag{1-12}$$

and define a function on $\mathfrak{H}$ by

$$F(g_f;Z)=\rho_{l,d}(J(g_\infty, Z_0))\prod_{j=1}^{n}|\mu(g_{\infty_j})|^{-(l_j+d_j/2)}F(g_\infty g_f), \tag{1-13}$$

where g_∞ is an element of G_∞ such that $g_\infty\langle Z_0\rangle = Z$. Then $F(g_f; Z)$ satisfies

$$F(g_f; \gamma\langle Z\rangle) = \rho(J(\gamma, Z)) \prod_{j=1}^{n} |\mu(\gamma)|_{\infty_j}^{-(l_j+d_j/2)} F(g_f; Z) \quad \text{for } \forall\gamma \in \Gamma(g_f). \tag{1-14}$$

We define a lattice $L(g_f)$ in B^- by

$$L(g_f) = \left\{ x \in B^- \,\middle|\, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \Gamma(g_f) \right\}. \tag{1-15}$$

Let $\mathfrak{H}_1$ be any connected component of $\mathfrak{H}$. Then $F(g_f; Z)$ has the following Fourier expansion on $\mathfrak{H}_1$:

$$F(g_f; Z) = \sum_{\xi \in L(g_f)^*} a(g_f; \xi)_{\mathfrak{H}_1} e[\tau(\xi Z)] \quad (Z \in \mathfrak{H}_1), \tag{1-16}$$

where $\tau = \mathrm{Tr}_{k/Q} \circ \mathrm{Tr}_{B/k}$, and $L(g_f)^*$ is the dual lattice of $L(g_f)$ with respect to τ. The Fourier coefficient $a(g_f; \xi)_{\mathfrak{H}_1}$ is given by

$$a(g_f; \xi)_{\mathfrak{H}_1} = \int_{L(g_f)\backslash B_\infty^-} F(g_f; Z) e[-\tau(\xi Z)] d(\mathrm{Re}\, Z) \quad (Z \in \mathfrak{H}_1), \tag{1-17}$$

where $d(\mathrm{Re}\, Z)$ is the Haar measure of B_∞^- normalized so that the total volume of $L(g_f)\backslash B_\infty^-$ is 1. It is easily seen that $a(g_f; \xi)_{\mathfrak{H}_1} = 0$ unless $-\sqrt{-1}\,\xi$ belongs to $\mathfrak{H}_1$.

1-3. Let $\chi = \prod_v \chi_v$ be the character of $\boldsymbol{Q}_A$ such that $\chi|_Q = 1$ and $\chi_\infty(x) = e[x]$ for any $x \in \boldsymbol{R}$. For $F \in \mathfrak{S}(\rho, \lambda; U_f)$ and $\xi \in B^-$, put

$$F_\chi(g; \xi) = \int_{B_k^-\backslash B_A^-} F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \chi(-\tau(\xi x)) dx \quad (g \in G_A), \tag{1-18}$$

where $\tau = \mathrm{Tr}_{k/Q} \circ \mathrm{Tr}_{B/k}$ and dx is the Haar measure on B_A^- normalized that the volume of $B_k^-\backslash B_A^-$ is 1. The relation between (1-17) and (1-18) is

$$F_\chi(g_\infty g_f; \xi) = \rho(J(g_\infty, Z_0))^{-1} \prod_{j=1}^{n} |\mu(g_{\infty_j})|^{l_j+d_j/2} \times a(g_f; \xi)_{\mathfrak{H}_1} e[\tau(\xi \cdot g_\infty\langle Z_0\rangle)], \tag{1-19}$$

where $\mathfrak{H}_1 = g_\infty\langle \mathfrak{H}_+\rangle$ and we understand $a(g_f; \xi)_{\mathfrak{H}_1} = 0$ unless $\xi \in L(g_f)^*$. From the definition $F_\chi(g_f; \xi)$ has the following properties:

$$(1\text{-}20)\quad \begin{aligned} &F_\chi(gu_\infty u_f;\xi)=\rho(J(u_\infty,Z_0))^{-1}F_\chi(g;\xi) \qquad \text{for } \forall(u_\infty,u_f)\in U,\\ &F_\chi\left(\begin{pmatrix}\varepsilon\alpha & 0\\ 0 & \alpha\end{pmatrix}g;\xi\right)=F_\chi(g;\varepsilon\alpha^{-1}\xi\alpha) \qquad \text{for } \forall\varepsilon\in k^\times \text{ and } \forall\alpha\in B_k^\times,\\ &F_\chi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g;\xi\right)=\chi(\tau(\xi x))F_\chi(g;\xi) \qquad \text{for } \forall x\in B_A^-,\\ &F\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)=\sum_{\xi\in B_k^-}F_\chi(g;\xi)\chi(\tau(\xi x)) \qquad \text{for } \forall x\in B_A^-,\end{aligned}$$

where the series in the last identity converges absolutely and uniformly in any compact subset of G_A.

Now we introduce a function on G_A, which plays an essential role in this paper. For each $x\in B_A^\times$, we put

$$(1\text{-}21)\qquad \tilde{x}=\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}\in G_A.$$

Let ξ be an element of B_k^- such that ξ^2 is not a square element of k, and set $K=k(\xi)$. Note that $K^\times k_A^\times$ is a closed subgroup of $K_A^\times$ and the quotient group $K^\times k_A^\times\backslash K_A^\times$ is compact. For an idele class character Λ of K such that $\Lambda|k_A^\times=\lambda$, put

$$(1\text{-}22)\qquad \varphi_{F,\xi}^{\Lambda}(g)=\int_{K^\times k_A^\times\backslash K_A^\times}F_\chi(\tilde{\alpha}g;\xi)\Lambda^{-1}(\alpha)d^\times\alpha \qquad (g\in G_A),$$

where $d^\times\alpha$ is the normalized Haar measure of the compact group $K^\times k_A^\times\backslash K_A^\times$.

LEMMA 1-1. *Let F be any non-zero holomorphic cusp form of type $(\rho,\lambda;U_f)$. Then there exists an element $\xi\in B_k^-$ and an idele class character Λ of K satisfying*

(i) $\Lambda|k_A^\times=\lambda$,

(ii) *$\varphi_{F,\xi}^{\Lambda}$ is not zero as a function on G_A.*

PROOF. By (1-18) we can take a $\xi\in B_k^-$ and a $g_1\in G_A$ such that $F_\chi(g_1;\xi)\neq 0$. Note that by the equality (1-19), $N_{B/k}(\xi_{\infty_j})=-\xi^2>0$ $(j=1,\cdots,n)$. Therefore $K=k(\xi)$ is a totally imaginary quadratic extension field over k. The closedness of $K^\times k_A^\times$ in $K_A^\times$ assures that there exists an idele class character Λ_0 of $K_A^\times$ whose restriction to $k_A^\times$ is equal to λ. We define a $V_{l,d}$-valued function f on $K^\times k_A^\times\backslash K_A^\times$, which depends on g_1, ξ and Λ_0, by

$$f(\alpha)=F_\chi(\tilde{\alpha}g_1;\xi)\Lambda_0^{-1}(\alpha) \quad \text{for} \quad \alpha\in K_A^\times.$$

Since f is continuous on $K^\times k_A^\times \backslash K_A^\times$ and not identically zero ($f(1)=F_\chi(g_1;\xi)\neq 0$), its Fourier transform

$$\hat{f}(\Lambda_1)=\int_{K^\times k_A^\times \backslash K_A^\times} f(\alpha)\Lambda_1^{-1}(\alpha)d^\times\alpha,$$

is not identically zero, where Λ_1 is a character of $K^\times k_A^\times \backslash K_A^\times$. Take a Λ_1 such that $\hat{f}(\Lambda_1)\neq 0$. We may regard Λ_1 a character of $K_A^\times$ whose restriction to $K^\times k_A^\times$ is trivial. Put $\Lambda=\Lambda_0\Lambda_1$. Then $\Lambda | k_A^\times=\Lambda_0 | k_A^\times=\lambda$ and $\varphi_{F,\xi}^\Lambda(g_1)=\hat{f}(\Lambda_1)\neq 0$. Q.E.D.

This function $\varphi_{F,\xi}^\Lambda(g)$ has the following properties:

$$\begin{aligned}
&\varphi_{F,\xi}^\Lambda\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)=\chi(\tau(\xi x))\varphi_{F,\xi}^\Lambda(g) \qquad \text{for } \forall x\in B_A^-,\\
\text{(1-23)}\qquad &\varphi_{F,\xi}^\Lambda(\tilde{\alpha}g)=\Lambda(\alpha)\varphi_{F,\xi}^\Lambda(g) \qquad \text{for } \forall\alpha\in K_A^\times,\\
&\varphi_{F,\xi}^\Lambda(gu_\infty u_f)=\rho(J(u_\infty, Z_0))^{-1}\varphi_{F,\xi}^\Lambda(g) \qquad \text{for } \forall(u_\infty, u_f)\in U.
\end{aligned}$$

Denote by $\mathfrak{o}_K$ the maximal order of K. For an integral ideal $\mathfrak{f}$ of k, put

$$\text{(1-24)}\qquad \mathfrak{o}_K(\mathfrak{f})=\mathfrak{o}+\mathfrak{f}\mathfrak{o}_K.$$

Take a character $\Lambda=\prod_v \Lambda_v$ which satisfies the conditions stated in Lemma 1-1. From the continuity of Λ and the unramifiedness of λ, there exists an integral ideal $\mathfrak{f}$ of k such that

$$\text{(1-25)}\qquad \Lambda_\mathfrak{p} | \mathfrak{o}_K(\mathfrak{f})_\mathfrak{p}^\times=1 \quad \text{for any} \quad \mathfrak{p}<\infty.$$

The maximal integral ideal $\mathfrak{f}$ satisfying (1-25) is called the conductor of Λ and written as $\mathfrak{f}_\Lambda$. Using the notations in (1-19), we have

$$\begin{aligned}
\text{(1-26)}\qquad \varphi_{F,\xi}^\Lambda(g_\infty g_f)=&\rho_{l,d}(J(g_\infty, Z_0))^{-1}\prod_{j=1}^n |\mu(g_{\infty_j})|^{l_j+d_j/2}\\
&\times e[\tau(\xi(g_\infty\langle Z_0\rangle))]\\
&\times\pi_\Lambda\left\{r^{-1}\sum_{i=1}^r \Lambda^{-1}(u_i)a(\tilde{u}_i g_f;\xi)_{g_\infty\langle\mathfrak{H}_+\rangle}\right\},
\end{aligned}$$

where $\pi_\Lambda=\int_{K_\infty^1}\rho_{l,d}(\zeta)^{-1}\Lambda_\infty^{-1}(\zeta)d^\times\zeta\in \mathrm{End}(V_{l,d})$, $K_\infty^1=\{u\in K_\infty^\times \mid u\bar{u}=1\}$, and $u_1,\cdots,u_r$ is a complete set of representatives of $K^\times K_\infty^\times\prod_{\mathfrak{p}<\infty}\{x\in \mathfrak{o}_K(\mathfrak{f}_\Lambda)_\mathfrak{p}^\times \mid g_f^{-1}\tilde{x}g_f\in U_f\}\backslash K_A^\times$ such that $u_{i,\infty_j}=1$ $(1\leqq j\leqq n,\ 1\leqq i\leqq r)$.

REMARK 1-1. If $\varphi_{F,\xi}^\Lambda(g_\infty g_f)\neq 0$, then $\varphi_{F,\xi}^\Lambda(g'_\infty g_f)\neq 0$ for any g'_∞ such that $g'_\infty g_\infty^{-1}$ is in the identity component of G_∞. The equality (1-26) shows that for each fixed $g \in G_A$ and $\xi \in B_k^-$, there is only finitely many Λ such that $\varphi_{F,\xi}^{\Lambda}(g) \neq 0$. Hence in the Fourier inversion formula

$$F_\chi(\tilde{u}g;\xi) = \sum_{\Lambda} \varphi_{F,\xi}^{\Lambda}(g)\Lambda(u),$$

the right hand side is a finite sum for a fixed g.

1-4. Fix a right G_A invariant measure $d\dot{g}$ of $G_k k_A^\times \backslash G_A$. We introduce a positive definite hermitian inner product (the Petersson inner product) into $\mathfrak{S}(\rho_{l,d}, \lambda; U_f)$ by

$$\text{(1-27)} \qquad \langle F_1, F_2 \rangle = \int_{G_k k_A^\times \backslash G_A} (F_1(g), F_2(g))d\dot{g} \qquad (F_1, F_2 \in \mathfrak{S}(\rho, \lambda; U_f)),$$

where $(\ ,\ )$ is an inner product in $V_{l,d}$ defined in 1-1. Because of the finiteness of the volume of $G_k k_A^\times \backslash G_A$ and the boundedness of F_i, the integral of the right hand side of (1-27) converges. Equipped with this inner product $\mathfrak{S}(\rho, \lambda; U_f)$ becomes a finite dimensional Hilbert space.

For each prime ideal $\mathfrak{p}$, denote by $\mathscr{H}_\mathfrak{p}$ the (local) Hecke algebra. Namely, it is the space of bi-$U_\mathfrak{p}$-invariant $\boldsymbol{C}$-valued functions on $G_\mathfrak{p}$ with compact support, and forms a $\boldsymbol{C}$-algebra by the convolution product

$$\text{(1-28)} \qquad (\phi_1 * \phi_2)(g) = \int_{G_\mathfrak{p}} \phi_1(gh^{-1})\phi_2(h)dh \quad \text{for} \quad \phi_1, \phi_2 \in \mathscr{H}_\mathfrak{p},$$

where dh is the normalized Haar measure on $G_\mathfrak{p}$. When $\mathfrak{p}$ is unramified in B, we identify $\mathfrak{O}_\mathfrak{p}$ with $M_2(\mathfrak{o}_\mathfrak{p})$, and put

$$c_\mathfrak{p}^{(0)} = \text{the characteristic function of } U_\mathfrak{p}\begin{pmatrix}\pi_\mathfrak{p} & & & \\ & \pi_\mathfrak{p} & & \\ & & \pi_\mathfrak{p} & \\ & & & \pi_\mathfrak{p}\end{pmatrix}U_\mathfrak{p},$$

$$\text{(1-29)} \qquad c_\mathfrak{p}^{(1)} = \text{the characteristic function of } U_\mathfrak{p}\begin{pmatrix}1 & & & \\ & 1 & & \\ & & \pi_\mathfrak{p} & \\ & & & \pi_\mathfrak{p}\end{pmatrix}U_\mathfrak{p},$$

$$c_\mathfrak{p}^{(2)} = \text{the characteristic function of } U_\mathfrak{p}\begin{pmatrix}1 & & & \\ & \pi_\mathfrak{p} & & \\ & & \pi_\mathfrak{p} & \\ & & & \pi_\mathfrak{p}^2\end{pmatrix}U_\mathfrak{p},$$

where $\pi_{\mathfrak{p}}$ is a prime element of $k_{\mathfrak{p}}$. On the other hand, when $\mathfrak{p}$ is ramified in B, we denote by $\Pi_{\mathfrak{p}}$ a prime element of $\mathfrak{O}_{\mathfrak{p}}$, and put

$$
\begin{aligned}
&c_{\mathfrak{p}}^{(0)} = \text{the characteristic function of } U_{\mathfrak{p}}\begin{pmatrix}\Pi_{\mathfrak{p}} & \\ & \Pi_{\mathfrak{p}}\end{pmatrix}U_{\mathfrak{p}}, \\
&c_{\mathfrak{p}}^{(1)} = \text{the characteristic function of } U_{\mathfrak{p}}\begin{pmatrix}1 & \\ & \pi_{\mathfrak{p}}\end{pmatrix}U_{\mathfrak{p}}.
\end{aligned} \tag{1-30}
$$

Then it is well known that (e.g., [17], [19]),

$$
\begin{aligned}
\mathcal{H}_{\mathfrak{p}} &\cong \boldsymbol{C}[c_{\mathfrak{p}}^{(0)}, c_{\mathfrak{p}}^{(0)-1}, c_{\mathfrak{p}}^{(1)}, c_{\mathfrak{p}}^{(2)}] \quad &&\text{if } \mathfrak{p}\nmid\mathfrak{D}, \\
&\cong \boldsymbol{C}[c_{\mathfrak{p}}^{(0)}, c_{\mathfrak{p}}^{(0)-1}, c_{\mathfrak{p}}^{(1)}] &&\text{if } \mathfrak{p}\mid\mathfrak{D},
\end{aligned} \tag{1-31}
$$

and $c_{\mathfrak{p}}^{(i)}$'s are algebraically independent over $\boldsymbol{C}$.

Let $T(\mathfrak{p}^m)$ be the characteristic function of the subset

$$
\left\{g=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in G_{\mathfrak{p}} \,\middle|\, \alpha, \delta \in \mathfrak{O}_{\mathfrak{p}},\ \beta \in \mathfrak{A}_{\mathfrak{p}},\ \gamma \in \mathfrak{A}_{\mathfrak{p}}^{-1},\ \mathrm{ord}_{\mathfrak{p}}(\mu(g))=m\right\}.
$$

Then the following identity holds (cf. [19], [10], [9]).

$$
\sum_{m=0}^{\infty} T(\mathfrak{p}^m)t^m = \tilde{H}_{\mathfrak{p}}(t)/\tilde{Q}_{\mathfrak{p}}(t), \tag{1-32}
$$

where t is an indeterminate and $\tilde{H}_{\mathfrak{p}}(t)$ and $\tilde{Q}_{\mathfrak{p}}(t)$ are polynomials given by (1-33) and (1-34) respectively.

$$
\begin{aligned}
\tilde{H}_{\mathfrak{p}}(t) &= 1-q^2c_{\mathfrak{p}}^{(0)}t^2 \quad &&\text{if } \mathfrak{p}\nmid\mathfrak{D}, \\
&= 1+q^{A_{\mathfrak{p}}}c_{\mathfrak{p}}^{(0)}t &&\text{if } \mathfrak{p}\mid\mathfrak{D},
\end{aligned} \tag{1-33}
$$

$$
\begin{aligned}
\tilde{Q}_{\mathfrak{p}}(t) &= 1-c_{\mathfrak{p}}^{(1)}t+q(c_{\mathfrak{p}}^{(2)}+(q^2+1)c_{\mathfrak{p}}^{(0)})t^2-q^3c_{\mathfrak{p}}^{(0)}c_{\mathfrak{p}}^{(1)}t^3+q^6c_{\mathfrak{p}}^{(0)2}t^4 \\
&\qquad \text{if } \mathfrak{p}\nmid\mathfrak{D}, \\
&= 1-\{c_{\mathfrak{p}}^{(1)}-(q^{A_{\mathfrak{p}}}-1)c_{\mathfrak{p}}^{(0)}\}t+q^3c_{\mathfrak{p}}^{(0)2}t^2 \\
&\qquad \text{if } \mathfrak{p}\mid\mathfrak{D},
\end{aligned} \tag{1-34}
$$

where $q=|\mathfrak{o}/\mathfrak{p}|$ and $A_{\mathfrak{p}}$ means 2 [resp. 1] if $\mathfrak{p}\mid\mathfrak{D}_0$ [resp. $\mathfrak{p}\mid\mathfrak{D}_1$].

The local Hecke algebra $\mathcal{H}_{\mathfrak{p}}$ acts on $\mathfrak{S}(\rho, \lambda; U_f)$ by

$$
(F\mid\phi)(g) = \int_{G_{\mathfrak{p}}} F(gh)\phi(h)dh. \tag{1-35}
$$

Therefore the restricted tensor product $\mathcal{H}_{A,f} = \bigotimes_{\mathfrak{p}<\infty} \mathcal{H}_{\mathfrak{p}}$ acts on $\mathfrak{S}(\rho, \lambda; U_f)$. For $F_i \in \mathfrak{S}(\rho, \lambda; U_f)$ $(i=1, 2)$ and $\phi \in \mathcal{H}_{A,f}$, (1-36) $$\langle F_1|\phi, F_2\rangle=\langle F_1, F_2|\tilde{\phi}\rangle,$$

where $\tilde{\phi}(g)=\overline{\phi(g^{-1})}$ ($\overline{\ \ }$ denotes the complex conjugation). Especially each element of $\mathscr{H}_{A,f}$ is a normal operator with respect to the Petersson inner product (1-27), so $\mathfrak{S}(\rho, \lambda; U_f)$ is spanned by simultaneous eigen functions of $\mathscr{H}_{A,f}$. When F is a simultaneous eigen function, it determines a one-dimensional representation $\sigma_F=\otimes_{\mathfrak{p}<\infty}\sigma_{F,\mathfrak{p}}$ of $\mathscr{H}_{A,f}$ by

(1-37) $$F|\phi=\sigma_F(\phi)F \quad \text{for all } \phi\in\mathscr{H}_{A,f}.$$

§ 2. Some local properties of $\varphi^{\Lambda}_{F,\xi}$

2-1. Notations are the same as in § 1, and throughout this section we fix ξ and Λ satisfying the conditions stated in Lemma 1-1. Put $K=k(\xi)$ and $K_\mathfrak{p}=K\otimes_k k_\mathfrak{p}$ for each prime $\mathfrak{p}$. Let $\mathscr{L}^{\Lambda}_{\xi,\mathfrak{p}}$ be the space of $\boldsymbol{C}$-valued functions on $G_\mathfrak{p}$ satisfying

(2-1)
$$\begin{aligned}
&\text{(i)} \quad \varphi(gu)=\varphi(g) \quad \text{for any } u\in U_\mathfrak{p},\\
&\text{(ii)} \quad \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)=\chi_p(\tau(\xi x))\varphi(g) \quad \text{for any } x\in B^-_\mathfrak{p},\\
&\text{(iii)} \quad \varphi(\tilde{\alpha}g)=\Lambda_\mathfrak{p}(\alpha)\varphi(g) \quad \text{for any } \alpha\in K^\times_\mathfrak{p},
\end{aligned}$$

where χ_p [resp. $\Lambda_\mathfrak{p}$] is the restriction of χ [resp. Λ] to $\boldsymbol{Q}_p$ (p is the prime number divided by $\mathfrak{p}$) [resp. $K^\times_\mathfrak{p}$]. Note that for any fixed $g'\in G_A$ whose $\mathfrak{p}$-part is 1, the function $\varphi^{\Lambda}_{F,\xi}(g'g_\mathfrak{p})$ on $G_\mathfrak{p}$ belongs to $\mathscr{L}^{\Lambda}_{\xi,\mathfrak{p}}\otimes V$.

The local Hecke algebra $\mathscr{H}_\mathfrak{p}$ acts on $\mathscr{L}^{\Lambda}_{\xi,\mathfrak{p}}$ in the same manner as in (1-35); in this section we investigate some properties of eigen functions in $\mathscr{L}^{\Lambda}_{\xi,\mathfrak{p}}$. The main result in this section is Theorem 2-1, which asserts that each eigen space is one dimensional, and in which the generating function will be calculated. Put

(2-2) $$(\mathfrak{A}^-_{0,\mathfrak{p}})'=\{x\in B^-_\mathfrak{p}\,|\,\mathrm{Tr}_{B_\mathfrak{p}/k_\mathfrak{p}}(xy)\in\mathfrak{o}_\mathfrak{p} \text{ for any } y\in\mathfrak{A}^-_{0,\mathfrak{p}}\},$$

where $\mathfrak{A}_0$ is defined in (1-9). We define integers $\nu_\mathfrak{p}$ and $\mu_\mathfrak{p}$ by the conditions:

(2-3) $$\xi=\pi_\mathfrak{p}^{\nu_\mathfrak{p}}\xi_{0,\mathfrak{p}}, \qquad (2\xi_{0,\mathfrak{p}})^2=d_\mathfrak{p}\pi_\mathfrak{p}^{2\mu_\mathfrak{p}},$$

where $\xi_{0,\mathfrak{p}}$ is a primitive element of $(\mathfrak{A}^-_{0,\mathfrak{p}})'$ (i.e., x being an element of $k_\mathfrak{p}$, $x\xi_{0,\mathfrak{p}}$ is in $(\mathfrak{A}^-_{0,\mathfrak{p}})'$ if and only if x is in $\mathfrak{o}_\mathfrak{p}$), and $d_\mathfrak{p}$ is a generator of the discriminant of $K_\mathfrak{p}/k_\mathfrak{p}$.

In 2-2 [resp. 2-3], we shall determine $\cup_{\varphi} \operatorname{supp} \varphi$, where φ runs through all elements of $\mathscr{L}^{A}_{\xi,\mathfrak{p}}$ and $\operatorname{supp}\varphi$ means the support of φ; and describe the action of $\mathscr{H}_{\mathfrak{p}}$ on $\mathscr{L}^{A}_{\xi,\mathfrak{p}}$ explicitly in the case $\mathfrak{p}\nmid\mathfrak{D}$ [resp. $\mathfrak{p}\mid\mathfrak{D}$]. The existence and the uniqueness up to constant multiple of each eigen function in $\mathscr{L}^{A}_{\xi,\mathfrak{p}}$ will be proved in 2-4.

2-2. In this subsection we assume that $\mathfrak{p}$ is a prime ideal of k not dividing $\mathfrak{D}$; so $B_{\mathfrak{p}}\cong M_2(k_{\mathfrak{p}})$ and $\mathfrak{A}_{0,\mathfrak{p}}=\mathfrak{O}_{\mathfrak{p}}$.

LEMMA 2-1.

(i) $\mu_{\mathfrak{p}}\geqq 0$.

(ii) *There exists a $k_{\mathfrak{p}}$-algebra isomorphism $j_{\mathfrak{p}}$ between $B_{\mathfrak{p}}$ and $M_2(k_{\mathfrak{p}})$ satisfying*

$$j_{\mathfrak{p}}(\mathfrak{O}_{\mathfrak{p}})=M_2(\mathfrak{o}_{\mathfrak{p}}) \quad \textit{and} \quad j_{\mathfrak{p}}(\xi_{0,\mathfrak{p}})=\begin{pmatrix} a_0/2 & b_0 \\ 1 & -a_0/2 \end{pmatrix},$$

where $a_0\in\mathfrak{p}^{\mu_{\mathfrak{p}}}$ and $b_0\in\mathfrak{p}^{2\mu_{\mathfrak{p}}}$.

(iii) $$K_{\mathfrak{p}}\cap\mathfrak{O}_{\mathfrak{p}}=\mathfrak{o}_K(\mathfrak{p}^{\mu_{\mathfrak{p}}})_{\mathfrak{p}} =\mathfrak{o}_{\mathfrak{p}}+\mathfrak{o}_{\mathfrak{p}}(\xi_{0,\mathfrak{p}}-a_0/2).$$

PROOF. Take a $k_{\mathfrak{p}}$-algebra isomorphism j' between $B_{\mathfrak{p}}$ and $M_2(k_{\mathfrak{p}})$ such that $j'(\mathfrak{O}_{\mathfrak{p}})=M_2(\mathfrak{o}_{\mathfrak{p}})$. Putting $j'(\xi_{0,\mathfrak{p}})=\begin{pmatrix} a'/2 & b' \\ c' & -a'/2\end{pmatrix}$, the primitivity of $\xi_{0,\mathfrak{p}}$ in $(\mathfrak{A}^{-}_{0,\mathfrak{p}})'$ means that $a', b', c'\in\mathfrak{o}_{\mathfrak{p}}$ and one of these is a unit of $\mathfrak{o}_{\mathfrak{p}}$. Since $(2\cdot\xi_{0,\mathfrak{p}})^2=d_{\mathfrak{p}}\pi_{\mathfrak{p}}^{2\mu_{\mathfrak{p}}}$, we get $a'^2+4b'c'=d_{\mathfrak{p}}\pi_{\mathfrak{p}}^{2\mu_{\mathfrak{p}}}$. First, we shall prove that $\mu_{\mathfrak{p}}$ is non-negative. If $\mathfrak{p}$ does not divide 2, it is obvious because $\operatorname{ord}_{\mathfrak{p}} d_{\mathfrak{p}}=0$ or 1. Suppose that $\mathfrak{p}$ divides 2, and put

$$e=\operatorname{ord}_{\mathfrak{p}} 2, \qquad (2\xi_{0,\mathfrak{p}})^2=\pi_{\mathfrak{p}}^{\beta}\varepsilon \quad (\varepsilon\in\mathfrak{o}_{\mathfrak{p}}^{\times}).$$

When β is even, let t be the maximal integer satisfying

(1°) $0\leqq t\leqq e$.

(2°) There exists an x in $\mathfrak{o}_{\mathfrak{p}}^{\times}$ such that $x^2-\varepsilon\in\mathfrak{p}^{2t}$.

Since 1 and $\pi_{\mathfrak{p}}^{-t}(x+2\pi_{\mathfrak{p}}^{-\beta/2}\xi_{0,\mathfrak{p}})$ span $\mathfrak{o}_{K,\mathfrak{p}}$ over $\mathfrak{o}_{\mathfrak{p}}$, we obtain that $\operatorname{ord}_{\mathfrak{p}} d_{\mathfrak{p}}=2(e-t)$ and $\mu_{\mathfrak{p}}=\beta/2-e+t$. To see that $\mu_{\mathfrak{p}}\geqq 0$, we may assume $\beta/2<e$. In this case, $a'\in\mathfrak{p}^{\beta/2}$ and $a'^2/\pi_{\mathfrak{p}}^{\beta}-\varepsilon\in\mathfrak{p}^{2(e-\beta/2)}$; from the choice of t (by (2°)), we have $\mu_{\mathfrak{p}}\geqq 0$. On the other hand, when β is odd, 1 and $2\xi_{0,\mathfrak{p}}\pi_{\mathfrak{p}}^{(-\beta+1)/2}$ span $\mathfrak{o}_{K,\mathfrak{p}}$ over $\mathfrak{o}_{\mathfrak{p}}$, and $\operatorname{ord}_{\mathfrak{p}} d_{\mathfrak{p}}=2e+1$ and $\mu_{\mathfrak{p}}=\beta-e$. It is clear that $\mu_{\mathfrak{p}}\geqq 0$. Secondly, we shall prove (ii). By the primitivity of $\xi_{0,\mathfrak{p}}$, there exists a U_1 in $GL_2(\mathfrak{o}_{\mathfrak{p}})$ such that

$$U_1^{-1}j'(\xi_{0,\mathfrak{p}})U_1=\begin{pmatrix} a''/2 & b'' \\ 1 & -a''/2 \end{pmatrix}.$$

We can take two elements a_1 and $b_1 \in \mathfrak{o}_\mathfrak{p}$ such that $a_1^2+4b_1=d_\mathfrak{p}$. Then, $U_2=\begin{pmatrix} 1 & (a''-a_1\pi_\mathfrak{p}^{\mu_\mathfrak{p}})/2 \\ 0 & 1 \end{pmatrix}$ is in $GL_2(\mathfrak{o}_\mathfrak{p})$ and

$$U_2^{-1}U_1^{-1}j'(\xi_{0,\mathfrak{p}})U_1U_2=\begin{pmatrix} a_1\pi_\mathfrak{p}^{\mu_\mathfrak{p}}/2 & b_1\pi_\mathfrak{p}^{2\mu_\mathfrak{p}} \\ 1 & -a_1\pi_\mathfrak{p}^{\mu_\mathfrak{p}} \end{pmatrix}.$$

Thus the isomorphism $j_\mathfrak{p}$: $j_\mathfrak{p}(X)=U_2^{-1}U_1^{-1}j'(X)U_1U_2$ $(X\in B_\mathfrak{p})$ has the required properties. Finally, (iii) is checked easily by using an $\mathfrak{o}_\mathfrak{p}$-basis of $\mathfrak{o}_{K,\mathfrak{p}}$. Q.E.D.

From now on we fix such an isomorphism $j_\mathfrak{p}$, and identify $\mathfrak{O}_\mathfrak{p}$ with $M_2(\mathfrak{o}_\mathfrak{p})$ through it. Put

$$S_\mathfrak{p}=\left\{\begin{pmatrix} u \\ v \end{pmatrix}\in\mathfrak{o}_\mathfrak{p}^2 \;\middle|\; u \text{ or } v \text{ is in } \mathfrak{o}_\mathfrak{p}^\times\right\}. \tag{2-4}$$

For each non-negative integer m, we introduce an equivalence relation $\underset{m}{\simeq}$ into $S_\mathfrak{p}$:

$$\begin{pmatrix} u \\ v \end{pmatrix}\underset{m}{\simeq}\begin{pmatrix} u' \\ v' \end{pmatrix}\Longleftrightarrow \text{there exists an } \alpha \text{ in } \mathfrak{o}_\mathfrak{p}^\times \text{ such that } \alpha u-u'\in\mathfrak{p}^m \text{ and } \alpha v-v'\in\mathfrak{p}^m, \tag{2-5}$$

and denote by $S_\mathfrak{p}/\underset{m}{\simeq}$ the set of equivalence classes. For $\begin{pmatrix} u \\ v \end{pmatrix}\in S_\mathfrak{p}$, set

$$f_0\begin{pmatrix} u \\ v \end{pmatrix}=u^2-a_0\pi_\mathfrak{p}^{-\mu_\mathfrak{p}}uv-b_0\pi_\mathfrak{p}^{-2\mu_\mathfrak{p}}v^2. \tag{2-6}$$

The following two lemmata are easily shown (cf. § 2.3 in [2])

LEMMA 2-2.

(i) *Let X be an element of $S_\mathfrak{p}/\underset{m}{\simeq}$. Then there exists a representative $\begin{pmatrix} u \\ v \end{pmatrix}$ of X in $S_\mathfrak{p}$ satisfying*

$$0\leqq \operatorname{ord}_\mathfrak{p} f_0\begin{pmatrix} u \\ v \end{pmatrix}\leqq m. \tag{2-7}$$

Moreover $\operatorname{ord}_\mathfrak{p} f_0\begin{pmatrix} u \\ v \end{pmatrix}$ *is independent of the choice of* $\begin{pmatrix} u \\ v \end{pmatrix}$ *satisfying*

(2-7) (*we denote it by* $\mathrm{Ord}(X)$).

(ii) *For* $m\geqq 1$,

$\#\{X\in S_{\mathfrak{p}}/\underset{m}{\simeq}\,|\,\mathrm{Ord}(X)=m\}$
$=2$ *if* $K_{\mathfrak{p}}\cong k_{\mathfrak{p}}\oplus k_{\mathfrak{p}}$,
$=0$ *if* $K_{\mathfrak{p}}$ *is an unramified extension field,*
$=1$ *if* $K_{\mathfrak{p}}$ *is ramified and* $m=1$,
$=0$ *if* $K_{\mathfrak{p}}$ *is ramified and* $m\geqq 2$.

LEMMA 2-3. *Let* $V=\begin{pmatrix}u_1 & v_1\\ u_2 & v_2\end{pmatrix}$ *be an element of* $GL_2(k_{\mathfrak{p}})$ *and* β *be an integer. Assume that* $\det V\in\mathfrak{o}_{\mathfrak{p}}^{\times}$ *and* $-u_1v_1+a_0u_2v_1+b_0u_2v_2$ *belongs to* $\mathfrak{p}^{\rho-\beta}$, *where* $\rho=\mathrm{ord}_{\mathfrak{p}}(u_1^2-a_0u_1u_2-b_0u_2^2)$. *Then there exists a* V' *in* $GL_2(\mathfrak{o}_{\mathfrak{p}})$ *such that*

$$V\begin{pmatrix}1 & \\ & \pi_{\mathfrak{p}}^{\beta}\end{pmatrix}=(u_1+u_2(\xi_{0,\mathfrak{p}}-a_0/2))\begin{pmatrix}1 & \\ & \pi_{\mathfrak{p}}^{\beta-\rho}\end{pmatrix}V'.$$

LEMMA 2-4.

$$B_{\mathfrak{p}}^{\times}=GL_2(k_{\mathfrak{p}})=\coprod_{m\geqq 0}K_{\mathfrak{p}}^{\times}\begin{pmatrix}1 & 0\\ 0 & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m}\end{pmatrix}\mathfrak{O}_{\mathfrak{p}}^{\times}\quad (\textit{disjoint}).$$

PROOF. Although this is a well-known property as the locally principality of lattices in a quadratic extension (cf. Proposition 1 in [11]), we give here a proof for the sake of convenience. We shall only prove that $B_{\mathfrak{p}}^{\times}$ is the union of $K_{\mathfrak{p}}^{\times}\begin{pmatrix}1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m}\end{pmatrix}\mathfrak{O}_{\mathfrak{p}}^{\times}$ $(m=0, 1, \cdots)$, because the disjointness is easily checked. Take any element g of $B_{\mathfrak{p}}^{\times}$ and put $g_1=\begin{pmatrix}1 & 0\\ 0 & \pi_{\mathfrak{p}}^{\mu_{\mathfrak{p}}}\end{pmatrix}g$. By the elementary divisor theory, there is a $z_1\in k_{\mathfrak{p}}^{\times}$ and $m\geqq 0$, such that $g_1\in z_1\mathfrak{O}_{\mathfrak{p}}^{\times}\begin{pmatrix}1 & 0\\ 0 & \pi_{\mathfrak{p}}^{m}\end{pmatrix}\mathfrak{O}_{\mathfrak{p}}^{\times}$. As there is nothing to prove in the case $m=0$, we assume $m\geqq 1$. Note that when $U=\begin{pmatrix}u_1 & v_1\\ u_2 & v_2\end{pmatrix}\in\mathfrak{O}_{\mathfrak{p}}^{\times}$ and $\begin{pmatrix}u_1'\\ u_2'\end{pmatrix}\in S_{\mathfrak{p}}$,

$$(2\text{-}8)\qquad \begin{pmatrix}u_1\\ u_2\end{pmatrix}\underset{m}{\simeq}\begin{pmatrix}u_1'\\ u_2'\end{pmatrix}\Longleftrightarrow U'=\begin{pmatrix}u_1' & v_1\\ u_2' & v_2\end{pmatrix}\in\mathfrak{O}_{\mathfrak{p}}^{\times}\ \text{ and }\ U'\begin{pmatrix}1 & \\ & \pi_{\mathfrak{p}}^{m}\end{pmatrix}\in U\begin{pmatrix}1 & \\ & \pi_{\mathfrak{p}}^{m}\end{pmatrix}\mathfrak{O}_{\mathfrak{p}}^{\times}.$$

Thus, by Lemma 2-2, there exists a $U=\begin{pmatrix}u_1 & v_1\\ u_2 & v_2\end{pmatrix}\in\mathfrak{O}_{\mathfrak{p}}^{\times}$ such that $g_1\in z_1U\begin{pmatrix}1 & 0\\ 0 & \pi_{\mathfrak{p}}^{m}\end{pmatrix}\mathfrak{O}_{\mathfrak{p}}^{\times}$ and $\mathrm{ord}_{\mathfrak{p}}f_0\begin{pmatrix}u_1\\ u_2\end{pmatrix}\leqq m$; so $g\in z_1\begin{pmatrix}u_1 & \pi_{\mathfrak{p}}^{\mu_{\mathfrak{p}}}v_1\\ \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}}u_2 & v_2\end{pmatrix}\begin{pmatrix}1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m}\end{pmatrix}\mathfrak{O}_{\mathfrak{p}}^{\times}$. By Lemma 2-3, we know that g belongs to $K_{\mathfrak{p}}^{\times}\begin{pmatrix}1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m-\rho}\end{pmatrix}\mathfrak{O}_{\mathfrak{p}}^{\times}$, where

$\rho=\operatorname{ord}_{\mathfrak{p}} f_0\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Q.E.D.

LEMMA 2-5. *Let* $c_{\mathfrak{p}}=\operatorname{ord}_{\mathfrak{p}} \mathfrak{f}_\Lambda$, $\varphi \in \mathscr{L}_{\xi,\mathfrak{p}}^\Lambda$ *and* $t \in k_{\mathfrak{p}}^\times$. *Then*

$$\varphi\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\widetilde{\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m} \end{pmatrix}}\right)=0 \quad \textit{for} \quad 0 \leqq m < c_{\mathfrak{p}}.$$

PROOF. Let $c_{\mathfrak{p}} \geqq 1$. Note that 1 and $\pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}}(\xi_{0,\mathfrak{p}}-a_0/2)$ span $\mathfrak{o}_{K,\mathfrak{p}}$ over $\mathfrak{o}_{\mathfrak{p}}$. From the definition of $\mathfrak{f}_\Lambda$, there exists a $z=x+\pi_{\mathfrak{p}}^{c_{\mathfrak{p}}-1-\mu_{\mathfrak{p}}}(\xi_{0,\mathfrak{p}}-a_0/2)y$ in $\mathfrak{o}_K(\mathfrak{p}^{c_{\mathfrak{p}}-1})_{\mathfrak{p}}^\times$ such that $\Lambda_{\mathfrak{p}}(z) \neq 1$. When x is a unit, we may assume that $x=1$ because the restriction of $\Lambda_{\mathfrak{p}}$ to $k_{\mathfrak{p}}^\times$ is unramified. Put $u=\pi_{\mathfrak{p}}^{c_{\mathfrak{p}}-1-m}y$, where $0 \leqq m < c_{\mathfrak{p}}$. By Lemma 2-3 there exists a $V \in \mathfrak{O}_{\mathfrak{p}}^\times$ such that

$$\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}=z\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m} \end{pmatrix}V.$$

So we obtain

$$\varphi\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\widetilde{\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m} \end{pmatrix}}\right)=\Lambda_{\mathfrak{p}}(z)\varphi\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\widetilde{\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m} \end{pmatrix}}\right),$$

which implies that $\varphi\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\widetilde{\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+m} \end{pmatrix}}\right)=0$. If $x \in \mathfrak{p}$, then $c_{\mathfrak{p}}$ must be 1 and y must be in $\mathfrak{o}_{\mathfrak{p}}^\times$; so the only possible m is 0. Applying the same argument as above to

$$\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}} \end{pmatrix}\begin{pmatrix} x & y^{-1} \\ y & 0 \end{pmatrix}=\begin{pmatrix} x & \pi_{\mathfrak{p}}^{\mu_{\mathfrak{p}}}y^{-1} \\ \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}}y & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}} \end{pmatrix},$$

we obtain $\varphi\left(\begin{pmatrix} t & \\ & 1 \end{pmatrix}\widetilde{\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}} \end{pmatrix}}\right)=0$. Q.E.D.

Put

$$\text{(2-9)} \qquad h_{\mathfrak{p}}(m)=\begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}+c_{\mathfrak{p}}+m} \end{pmatrix} \in B_{\mathfrak{p}}^\times \qquad \text{for} \quad m \in \boldsymbol{Z},$$

$$g_{\mathfrak{p}}(m, l)=\begin{pmatrix} t_{0,\mathfrak{p}}\pi_{\mathfrak{p}}^{-\alpha_{\mathfrak{p}}+m+l} & \\ & 1 \end{pmatrix}\widetilde{h_{\mathfrak{p}}(m)} \in G_{\mathfrak{p}} \qquad \text{for} \quad m, l \in \boldsymbol{Z},$$

where $t_{0,\mathfrak{p}}$ is a generator of $\mathfrak{d}_k^{-1}\mathfrak{f}_\Lambda\pi_{\mathfrak{p}}^{-(\nu_{\mathfrak{p}}+\mu_{\mathfrak{p}})}\mathfrak{o}_{\mathfrak{p}}$, $\alpha_{\mathfrak{p}}=\operatorname{ord}_{\mathfrak{p}}\mathfrak{a}$, $c_{\mathfrak{p}}=\operatorname{ord}_{\mathfrak{p}}\mathfrak{f}_\Lambda$, and $\mu_{\mathfrak{p}}$, $\nu_{\mathfrak{p}}$ are defined in (2-3). For any $l \in \boldsymbol{Z}$, we put $g_{\mathfrak{p}}(l)=g_{\mathfrak{p}}(0, l)$.

PROPOSITION 2-1. *Notations being as above, we have*

$$\bigcup_{\varphi} \operatorname{supp} \varphi = \coprod_{\substack{m \geqq 0 \\ l \geqq 0}} N_{\mathfrak{p}} \widetilde{K_{\mathfrak{p}}^{\times}} g_{\mathfrak{p}}(m, l) U_{\mathfrak{p}} \qquad (\textit{disjoint}),$$

where φ *runs through all elements of* $\mathscr{L}_{\xi,\mathfrak{p}}^{\Lambda}$, *and*

$$N_{\mathfrak{p}} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G_{\mathfrak{p}} \,\middle|\, x \in B_{\mathfrak{p}}^{-} \right\}.$$

PROOF. Put

$$X = \bigcup_{\varphi} \operatorname{supp} \varphi \quad \text{and} \quad Y = \bigcup_{\substack{m \geqq 0 \\ l \geqq 0}} N_{\mathfrak{p}} \widetilde{K_{\mathfrak{p}}^{\times}} g_{\mathfrak{p}}(m, l) U_{\mathfrak{p}}.$$

By using the Iwasawa decomposition, Lemma 2-4, and Lemma 2-5, we have

$$X \subset \bigcup_{\substack{m \geqq 0 \\ t \in k_{\mathfrak{p}}^{\times}}} N_{\mathfrak{p}} \widetilde{K_{\mathfrak{p}}^{\times}} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \widetilde{h_{\mathfrak{p}}(m)} U_{\mathfrak{p}}.$$

For any $x \in B_{\mathfrak{p}}^{-}$, $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \widetilde{h_{\mathfrak{p}}(m)} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & th_{\mathfrak{p}}(m)xh_{\mathfrak{p}}(m)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \widetilde{h_{\mathfrak{p}}(m)}$. So, if $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \widetilde{h_{\mathfrak{p}}(m)} \in X$, then $\chi_p(\tau(\xi th_{\mathfrak{p}}(m)xh_{\mathfrak{p}}(m)^{-1}))$ must be 1 for any $x \in \mathfrak{A}_{\mathfrak{p}}^{-}$; namely $\pi_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}} t_{0,\mathfrak{p}}^{-1} \pi_{\mathfrak{p}}^{-m} t \in \mathfrak{o}_{\mathfrak{p}}$; this means that $X \subset Y$. As it is easy to see that the right hand side of Y is disjoint union, it remains to prove that $X \supset Y$. For each $m \geqq 0$ and $l \geqq 0$, put

$$\varphi_{m,l}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tilde{z} g_{\mathfrak{p}}(m', l') u \right) = \begin{cases} \chi_p(\tau(\xi x)) \Lambda_{\mathfrak{p}}(z) & \text{if} \quad (m', l') = (m, l), \\ 0 & \text{otherwise}, \end{cases}$$

where $x \in B_{\mathfrak{p}}^{-}$, $z \in K_{\mathfrak{p}}^{\times}$, $u \in U_{\mathfrak{p}}$, and $m', l' \geqq 0$. If $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tilde{z} g_{\mathfrak{p}}(m, l) = \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \tilde{z}' g_{\mathfrak{p}}(m, l) u'$, then $\chi_p(\tau(\xi x)) = \chi_p(\tau(\xi x'))$ and $\Lambda_{\mathfrak{p}}(z) = \Lambda_{\mathfrak{p}}(z')$; so $\varphi_{m,l}$ is well-defined. Since $\varphi_{m,l}$ is in $\mathscr{L}_{\xi,\mathfrak{p}}^{\Lambda}$ and its support is $N_{\mathfrak{p}} \widetilde{K_{\mathfrak{p}}^{\times}} g_{\mathfrak{p}}(m, l) U_{\mathfrak{p}}$, our assertion has been verified. Q.E.D.

Let φ be an element of $\mathscr{L}_{\xi,\mathfrak{p}}^{\Lambda}$. Then φ is a simultaneous eigen function of $\mathscr{H}_{\mathfrak{p}}$ if and only if φ is a common eigen function of $c_{\mathfrak{p}}^{(0)}$, $c_{\mathfrak{p}}^{(1)}$ and $c_{\mathfrak{p}}^{(2)}$ (see (1-31)). Take a system $\left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\}$ of representatives of $S_{\mathfrak{p}} / \underset{1}{\sim}$, and for each $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ choose an element $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in $S_{\mathfrak{p}}$ such that $V = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ is in $\mathfrak{O}_{\mathfrak{p}}^{\times}$. Denote by $L(\mathfrak{p})$ the set of such V's; then $\mathfrak{O}_{\mathfrak{p}}^{\times} \begin{pmatrix} 1 & \\ & \pi_{\mathfrak{p}} \end{pmatrix} \mathfrak{O}_{\mathfrak{p}}^{\times} =$

$\coprod_{V\in L(\mathfrak{p})} V\begin{pmatrix}1 & \\ & \pi_{\mathfrak{p}}\end{pmatrix}\mathfrak{O}_{\mathfrak{p}}^{\times}$ (disjoint union). We recall the right $U_{\mathfrak{p}}$-coset decomposition of $\operatorname{supp} c_{\mathfrak{p}}^{(i)}$ ($i=0, 1, 2$).

$$\text{(2-10)}\qquad \operatorname{supp} c_{\mathfrak{p}}^{(0)}=\begin{pmatrix}\pi_{\mathfrak{p}} & & & \\ & \pi_{\mathfrak{p}} & & \\ & & \pi_{\mathfrak{p}} & \\ & & & \pi_{\mathfrak{p}}\end{pmatrix}U_{\mathfrak{p}},$$

$$\text{(2-11)}\qquad \operatorname{supp} c_{\mathfrak{p}}^{(1)}=\coprod_{\substack{X=\begin{pmatrix}x & y\\ z & -x\end{pmatrix}\\ x,y,z\in\mathfrak{o}/\mathfrak{p}}}\left(\begin{array}{cc|c}\pi_{\mathfrak{p}} & & \pi_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}X\\ & \pi_{\mathfrak{p}} & \\ \hline & & \begin{matrix}1 & \\ & 1\end{matrix}\end{array}\right)U_{\mathfrak{p}}$$

$$\coprod\begin{pmatrix}1 & & & \\ & 1 & & \\ & & \pi_{\mathfrak{p}} & \\ & & & \pi_{\mathfrak{p}}\end{pmatrix}U_{\mathfrak{p}}$$

$$\coprod_{\substack{V\in L(\mathfrak{p})\\ X=\begin{pmatrix}0 & 0\\ x & 0\end{pmatrix},\, x\in\mathfrak{o}/\mathfrak{p}}}\tilde{V}\left(\begin{array}{cc|c}1 & & \pi_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}X\\ & \pi_{\mathfrak{p}} & \\ \hline & & \begin{matrix}1 & \\ & \pi_{\mathfrak{p}}\end{matrix}\end{array}\right)U_{\mathfrak{p}},$$

$$\text{(2-12)}\qquad \operatorname{supp} c_{\mathfrak{p}}^{(2)}=\coprod_{\substack{V\in L(\mathfrak{p})\\ X=\begin{pmatrix}x & 0\\ z & -\pi_{\mathfrak{p}}x\end{pmatrix}\\ x\in\mathfrak{o}/\mathfrak{p},\, z\in\mathfrak{o}/\mathfrak{p}^2}}\tilde{V}\left(\begin{array}{cc|c}\pi_{\mathfrak{p}} & & \pi_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}X\\ & \pi_{\mathfrak{p}}^2 & \\ \hline & & \begin{matrix}1 & \\ & \pi_{\mathfrak{p}}\end{matrix}\end{array}\right)U_{\mathfrak{p}}$$

$$\coprod_{V\in L(\mathfrak{p})}\tilde{V}\begin{pmatrix}1 & & & \\ & \pi_{\mathfrak{p}} & & \\ & & \pi_{\mathfrak{p}} & \\ & & & \pi_{\mathfrak{p}}^2\end{pmatrix}U_{\mathfrak{p}}$$

$$\coprod_{X=\begin{pmatrix}x & y\\ z & -x\end{pmatrix}}{}'\left(\begin{array}{cc|c}\pi_{\mathfrak{p}} & & \pi_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}X\\ & \pi_{\mathfrak{p}} & \\ \hline & & \begin{matrix}\pi_{\mathfrak{p}} & \\ & \pi_{\mathfrak{p}}\end{matrix}\end{array}\right)U_{\mathfrak{p}},$$

where in the last union in (2-12), (x, y, z) runs through the set

$\{(x, y, z)\in(\mathfrak{o}/\mathfrak{p})^3 \mid x^2+yz\in\mathfrak{p}$, one of x, y, z is a unit$\}$.

We denote by $\left(\frac{K}{\mathfrak{p}}\right)$ the Legendre symbol, i.e., it equals to -1, 0, or 1 according as $\mathfrak{p}$ remains prime in K, ramifies in K, or splits in K. Unless $\left(\frac{K}{\mathfrak{p}}\right)=-1$, we can take an element $\varpi_{K_\mathfrak{p}}$ of $K_\mathfrak{p}$ such that $N_{K_\mathfrak{p}/k_\mathfrak{p}}(\varpi_{K_\mathfrak{p}})\in\pi_\mathfrak{p}\mathfrak{o}_\mathfrak{p}^\times$. When $c_\mathfrak{p}=0$, namely $\Lambda_\mathfrak{p}$ is unramified, put

$$\varepsilon_\mathfrak{p}=\begin{cases}0 & \text{if } \left(\frac{K}{\mathfrak{p}}\right)=-1,\\ \Lambda_\mathfrak{p}(\varpi_{K_\mathfrak{p}}) & \text{if } \left(\frac{K}{\mathfrak{p}}\right)=0,\\ \Lambda_\mathfrak{p}(\varpi_{K_\mathfrak{p}})+\Lambda_\mathfrak{p}(\pi_\mathfrak{p}\varpi_{K_\mathfrak{p}}^{-1}) & \text{if } \left(\frac{K}{\mathfrak{p}}\right)=1.\end{cases} \tag{2-13}$$

Note that $\varepsilon_\mathfrak{p}$ is independent of the choice of $\varpi_{K_\mathfrak{p}}$.

LEMMA 2-6. *Let $t\in k_\mathfrak{p}^\times$, $m\geqq 0$ and $\varphi\in\mathscr{L}_{\xi,\mathfrak{p}}^1$. Then*

$$\sum_{V\in L(\mathfrak{p})}\varphi\left(\begin{pmatrix}t&0\\0&1\end{pmatrix}\widetilde{h_\mathfrak{p}(m)V\begin{pmatrix}1&\\&\pi_\mathfrak{p}\end{pmatrix}}\right)$$
$$=q\varphi\left(\begin{pmatrix}t&0\\0&1\end{pmatrix}\widetilde{h_\mathfrak{p}(m+1)}\right)+\Lambda_\mathfrak{p}(\pi_\mathfrak{p})\varphi\left(\begin{pmatrix}t&0\\0&1\end{pmatrix}\widetilde{h_\mathfrak{p}(m-1)}\right)\qquad \text{if}\quad m+c_\mathfrak{p}\geqq 1,$$
$$=\left(q-\left(\frac{K}{\mathfrak{p}}\right)\right)\varphi\left(\begin{pmatrix}t&0\\0&1\end{pmatrix}\widetilde{h_\mathfrak{p}(m+1)}\right)+\varepsilon_\mathfrak{p}\varphi\left(\begin{pmatrix}t&0\\0&1\end{pmatrix}\widetilde{h_\mathfrak{p}(m)}\right)\qquad \text{if}\quad m=c_\mathfrak{p}=0,$$

where $q=|\mathfrak{o}/\mathfrak{p}|$ and $c_\mathfrak{p}=\operatorname{ord}_\mathfrak{p}\mathfrak{f}_\Lambda$.

PROOF. First we suppose that $m+c_\mathfrak{p}\geqq 1$. We may take $\{V_s, V' \mid s\in\mathfrak{o}/\mathfrak{p}\}$ as $L(\mathfrak{p})$, where $V_s=\begin{pmatrix}1&0\\s&1\end{pmatrix}$ and $V'=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. Applying Lemma 2-3 to $h_\mathfrak{p}(m)V_s\begin{pmatrix}1&\\&\pi_\mathfrak{p}\end{pmatrix}=\begin{pmatrix}1&0\\\pi_\mathfrak{p}^{-\mu_\mathfrak{p}+c_\mathfrak{p}+m}s&1\end{pmatrix}\begin{pmatrix}1&\\&\pi_\mathfrak{p}^{-\mu_\mathfrak{p}+c_\mathfrak{p}+m+1}\end{pmatrix}$, we obtain

$$h_\mathfrak{p}(m)V_s\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\in\{1+\pi_\mathfrak{p}^{c_\mathfrak{p}+m}s\pi_\mathfrak{p}^{-\mu_\mathfrak{p}}(\xi_{0,\mathfrak{p}}-a_0/2)\}h_\mathfrak{p}(m+1)\mathfrak{O}_\mathfrak{p}^\times.$$

Since $1+\pi_\mathfrak{p}^{c_\mathfrak{p}+m-\mu_\mathfrak{p}}s(\xi_{0,\mathfrak{p}}-a_0/2)$ is in $\mathfrak{o}_K(\mathfrak{p}^{c_\mathfrak{p}})_\mathfrak{p}^\times$ and $h_\mathfrak{p}(m)V'\begin{pmatrix}1&\\&\pi_\mathfrak{p}\end{pmatrix}$ is in $\pi_\mathfrak{p}\begin{pmatrix}1&\\&\pi_\mathfrak{p}^{-\mu_\mathfrak{p}+c_\mathfrak{p}+m-1}\end{pmatrix}\mathfrak{O}_\mathfrak{p}^\times$, our assertion is proved. Secondly, we suppose that $m=c_\mathfrak{p}=0$. As above, we know that the contribution of $V=\begin{pmatrix}u_1&v_1\\u_2&v_2\end{pmatrix}$ such that $f_0\begin{pmatrix}u_1\\u_2\end{pmatrix}\in\mathfrak{o}_\mathfrak{p}^\times$ is $\varphi\left(\begin{pmatrix}t&0\\0&1\end{pmatrix}\widetilde{h_\mathfrak{p}(m+1)}\right)$; by Lemma 2-2 (ii) and Lemma 2-3, the contribution of the remaining $\left(1+\left(\frac{K}{\mathfrak{p}}\right)\right)$ elements is $\varepsilon_{\mathfrak{p}}\varphi\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\widetilde{h_{\mathfrak{p}}(m)}\right)$. Q.E.D.

Let φ be an element of $\mathscr{L}^{A}_{\varepsilon,\mathfrak{p}}$, and put

$$a(m, l)=\begin{cases}\varphi(g_{\mathfrak{p}}(m, l)) & \text{if } m\geqq 0 \text{ and } l\geqq 0, \\ 0 & \text{otherwise.}\end{cases} \tag{2-14}$$

PROPOSITION 2-2. *Notations being as above, for* $m, l\geqq 0$ *we have*

$$(\varphi\,|\,c_{\mathfrak{p}}^{(0)})(g_{\mathfrak{p}}(m, l))=\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m, l),$$

$$\begin{aligned}(\varphi\,|\,c_{\mathfrak{p}}^{(1)})(g_{\mathfrak{p}}(m, l))&=q^3a(m, l+1)+q\left(q-\delta_m\left(\frac{K}{\mathfrak{p}}\right)\right)a(m+1, l-1)\\ &\quad+\delta_m q\varepsilon_{\mathfrak{p}}a(m, l)+q\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m-1, l+1)\\ &\quad+\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m, l-1) \qquad \textit{if} \quad c_{\mathfrak{p}}=0,\\ &=q^3a(m, l+1)+q^2a(m+1, l-1)\\ &\quad+q\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m-1, l+1)+\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m, l-1)\\ &\qquad\qquad \textit{if} \quad c_{\mathfrak{p}}\geqq 1,\end{aligned}$$

$$\begin{aligned}(\varphi\,|\,c_{\mathfrak{p}}^{(2)})(g_{\mathfrak{p}}(m, l))&=q^3\left(q-\delta_m\left(\frac{K}{\mathfrak{p}}\right)\right)a(m+1, l)+\delta_m q^3\varepsilon_{\mathfrak{p}}a(m, l+1)\\ &\quad+q^3\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m-1, l+2)\\ &\quad+\left\{q^2-1-\delta_l q^2+\delta_l\delta_m q\left(\frac{K}{\mathfrak{p}}\right)\right\}\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m, l)\\ &\quad+\left(q-\delta_m\left(\frac{K}{\mathfrak{p}}\right)\right)\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m+1, l-2)\\ &\quad+\delta_m\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})\varepsilon_{\mathfrak{p}}a(m, l-1)+\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})^2a(m-1, l)\\ &\qquad\qquad \textit{if} \quad c_{\mathfrak{p}}=0,\\ &=q^4a(m+1, l)+q^3\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m-1, l+2)\\ &\quad+(q^2-1-\delta_l q^2)\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m, l)\\ &\quad+q\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})a(m+1, l-2)+\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})^2a(m-1, l)\\ &\qquad\qquad \textit{if} \quad c_{\mathfrak{p}}\geqq 1,\end{aligned}$$

where $q=|\mathfrak{o}/\mathfrak{p}|$ *and* $\delta_m=1$ *or* 0 *according whether* $m=0$ *or not.*

This proposition follows easily from (2-10)-(2-12) and Lemma 2-6.

2-3. In this subsection $\mathfrak{p}$ denotes a prime ideal of k dividing $\mathfrak{D}$.

Let K_0 be the unique unramified quadratic extension field of $k_\mathfrak{p}$. We realize $B_\mathfrak{p}$ as a cyclic algebra $(K_0, \pi_\mathfrak{p})$; i.e., $B_\mathfrak{p}=K_0+K_0\Pi_\mathfrak{p}$, $\Pi_\mathfrak{p}^2=\pi_\mathfrak{p}$, $\bar{\Pi}_\mathfrak{p}=-\Pi_\mathfrak{p}$, and $\Pi_\mathfrak{p}X\Pi_\mathfrak{p}^{-1}=\bar{X}$ for any $X\in K_0$ ($\Pi_\mathfrak{p}$ is a prime element of the division quaternion algebra $B_\mathfrak{p}$). We denote by $\mathfrak{O}_0$ the maximal order of K_0; so $\mathfrak{O}_\mathfrak{p}=\mathfrak{O}_0+\mathfrak{O}_0\Pi_\mathfrak{p}$ is the maximal order of $B_\mathfrak{p}$ and $\mathfrak{P}_\mathfrak{p}=\pi_\mathfrak{p}\mathfrak{O}_0+\mathfrak{O}_0\Pi_\mathfrak{p}$ is the maximal two-sided ideal of $\mathfrak{O}_\mathfrak{p}$. Take an element ι of $\mathfrak{O}_0^\times$ such that $\bar{\iota}=-\iota$.

LEMMA 2-7. *Let $\xi_{0,\mathfrak{p}}$ be a primitive element of $(\mathfrak{A}_{0,\mathfrak{p}}^-)'$. When $\mathfrak{A}_{0,\mathfrak{p}}=\mathfrak{O}_\mathfrak{p}$ [resp. $\mathfrak{A}_{0,\mathfrak{p}}=\mathfrak{P}_\mathfrak{p}$], $\xi_{0,\mathfrak{p}}$ is written as*

$$(2\text{-}15)\qquad \xi_{0,\mathfrak{p}}=(\iota/2)X+Y\Pi_\mathfrak{p}^{-1}\qquad [\textit{resp. } \xi_{0,\mathfrak{p}}=(2\pi_\mathfrak{p})^{-1}\iota X+Y\Pi_\mathfrak{p}^{-1}],$$

where $X\in\mathfrak{o}_\mathfrak{p}$, $Y\in\mathfrak{O}_0$ and one of them is a unit of $\mathfrak{O}_0$. If $K_\mathfrak{p}=k_\mathfrak{p}(\xi_{0,\mathfrak{p}})$ is unramified over $k_\mathfrak{p}$, then $Y\in\pi_\mathfrak{p}\mathfrak{O}_0$ [resp. $X\in\mathfrak{o}_\mathfrak{p}^\times$] and $\mu_\mathfrak{p}=0$ [resp. $\mu_\mathfrak{p}=-1$]. If $K_\mathfrak{p}=k_\mathfrak{p}(\xi_{0,\mathfrak{p}})$ is ramified over $k_\mathfrak{p}$, then $Y\in\mathfrak{O}_0^\times$ [resp. $X\in\mathfrak{p}$] and $\mu_\mathfrak{p}=-1$. Here, $\mu_\mathfrak{p}$ is defined in (2-3).

PROOF. Assume that $\mathfrak{A}_{0,\mathfrak{p}}=\mathfrak{O}_\mathfrak{p}$. It is clear that $\xi_{0,\mathfrak{p}}$ is written as in (2-15). Put $(2\xi_{0,\mathfrak{p}})^2=\iota^2X^2+4\pi_\mathfrak{p}^{-1}Y\bar{Y}=\pi_\mathfrak{p}^\beta\varepsilon$ ($\varepsilon\in\mathfrak{o}_\mathfrak{p}^\times$). If $Y\in\pi_\mathfrak{p}\mathfrak{O}_0$, then $(2\xi_{0,\mathfrak{p}})^2\equiv\iota^2X^2 \pmod{4\mathfrak{p}}$. As $X\in\mathfrak{o}_\mathfrak{p}^\times$, $K_\mathfrak{p}$ is unramified and $\beta=0$, $\mu_\mathfrak{p}=0$. Let Y be a unit of $\mathfrak{O}_0$. When $\mathfrak{p}\nmid 2$, our assertion is trivial. Thus we may suppose that $\mathfrak{p}|2$ and use the same notations as in the proof of Lemma 2-1 (i). If β is odd (i.e., $X\in\mathfrak{p}^e$), $\beta=2e-1$. This means that $K_\mathfrak{p}$ is ramified and $\mu_\mathfrak{p}=-1$. If β is even (i.e., $X\notin\mathfrak{p}^e$), we use the following two sublemmata, which are easily seen.

SUBLEMMA 1. *There exist elements a and v of $\mathfrak{o}_\mathfrak{p}^\times$ such that $a^{-2}\iota^2=1+\pi_\mathfrak{p}^{2e}v$.*

SUBLEMMA 2. *Put $\varepsilon_1=1+\pi_\mathfrak{p}^{2m-1}u$ with $u\in\mathfrak{o}_\mathfrak{p}^\times$ and $1\leqq m\leqq e$. Then $k_\mathfrak{p}(\sqrt{\varepsilon_1})$ is ramified and the $\mathfrak{p}$-order of the discriminant of $k_\mathfrak{p}(\sqrt{\varepsilon_p})/k_\mathfrak{p}$ is* $2(e-m+1)$.

Put $2=\pi_\mathfrak{p}^e w$ and $X=\pi_\mathfrak{p}^{\beta/2}X'$. From Sublemma 1 we obtain

$$(2\xi_{0,\mathfrak{p}})^2=(aX)^2\{1+\pi_\mathfrak{p}^{2e-\beta-1}(\pi_\mathfrak{p}^{\beta+1}v+N_{B_\mathfrak{p}/k_\mathfrak{p}}(wa^{-1}X'^{-1}Y))\},$$

where a and v are given as above. It follows from Sublemma 2 that $K_\mathfrak{p}$ is ramified and $\mu_\mathfrak{p}=-1$. The other case $\mathfrak{A}_{0,\mathfrak{p}}=\mathfrak{P}_\mathfrak{p}$ is treated quite similarly. Q.E.D.

Since $B_\mathfrak{p}$ is a division quaternion algebra,

$$B_{\mathfrak{p}}^{\times}=\begin{cases} K_{\mathfrak{p}}^{\times}\mathfrak{O}_{\mathfrak{p}}^{\times} & \text{if } K_{\mathfrak{p}} \text{ is ramified,} \\ K_{\mathfrak{p}}^{\times}\mathfrak{O}_{\mathfrak{p}}^{\times} \coprod K_{\mathfrak{p}}^{\times}\Pi_{\mathfrak{p}}\mathfrak{O}_{\mathfrak{p}}^{\times} & \text{if } K_{\mathfrak{p}} \text{ is unramified,} \end{cases} \tag{2-16}$$

and

$$\mathfrak{O}_{\mathfrak{p}}\cap K_{\mathfrak{p}}=\mathfrak{o}_{K,\mathfrak{p}}. \tag{2-17}$$

From the Iwasawa decomposition and the definition of $\mathscr{L}_{\xi,\mathfrak{p}}^{\Lambda}$ (see (2-1)), we may assume that $\Lambda_{\mathfrak{p}}|\mathfrak{o}_{K,\mathfrak{p}}^{\times}=1$, namely, $\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{f}_{\Lambda}=0$. Put

$$g_{\mathfrak{p}}(l)=\begin{pmatrix} t_{0,\mathfrak{p}}\pi_{\mathfrak{p}}^{-\alpha_{\mathfrak{p}}+l} & 0 \\ 0 & 1 \end{pmatrix}\in G_{\mathfrak{p}} \quad \text{for} \quad l\in \boldsymbol{Z}, \tag{2-18}$$

where $t_{0,\mathfrak{p}}$ is a generator of $\mathfrak{d}_k^{-1}\pi_{\mathfrak{p}}^{-\nu_{\mathfrak{p}}}\mathfrak{o}_{\mathfrak{p}}$, and $\alpha_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{a}$. By using the same argument of Proposition 2-1, we obtain

PROPOSITION 2-3.

$$\begin{aligned} \bigcup_{\varphi} \operatorname{supp}\varphi &= \coprod_{l\geqq 0} N_{\mathfrak{p}}\tilde{K}_{\mathfrak{p}}^{\times}g_{\mathfrak{p}}(l)U_{\mathfrak{p}} && \textit{if } K_{\mathfrak{p}} \textit{ is ramified,} \\ &= \coprod_{l\geqq 0} (N_{\mathfrak{p}}\tilde{K}_{\mathfrak{p}}^{\times}g_{\mathfrak{p}}(l) \coprod N_{\mathfrak{p}}\tilde{K}_{\mathfrak{p}}^{\times}\tilde{\Pi}_{\mathfrak{p}}g_{\mathfrak{p}}(l)U_{\mathfrak{p}}) \\ & && \textit{if } K_{\mathfrak{p}} \textit{ is unramified,} \end{aligned}$$

where φ *runs through* $\mathscr{L}_{\xi,\mathfrak{p}}^{\Lambda}$ *and* $N_{\mathfrak{p}}=\left\{\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\middle|\, x\in B_{\mathfrak{p}}^{-}\right\}$.

The Hecke algebra $\mathscr{H}_{\mathfrak{p}}$ is generated by $c_{\mathfrak{p}}^{(0)}$ and $c_{\mathfrak{p}}^{(1)}$ (see (1-31)). We recall the right $U_{\mathfrak{p}}$-coset decomposition of $c_{\mathfrak{p}}^{(i)}$ $(i=0, 1)$.

$$\operatorname{supp} c_{\mathfrak{p}}^{(0)}=\begin{pmatrix} \Pi_{\mathfrak{p}} & 0 \\ 0 & \Pi_{\mathfrak{p}}\end{pmatrix}U_{\mathfrak{p}}, \tag{2-19}$$

$$\begin{aligned} \operatorname{supp} c_{\mathfrak{p}}^{(1)} = & \coprod_{X\in \mathfrak{A}_{0,\mathfrak{p}}^{-}/\pi_{\mathfrak{p}}\mathfrak{A}_{0,\mathfrak{p}}^{-}} \begin{pmatrix} \pi_{\mathfrak{p}} & \pi_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}X \\ 0 & 1\end{pmatrix}U_{\mathfrak{p}} \coprod \begin{pmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{p}}\end{pmatrix}U_{\mathfrak{p}} \\ & \coprod_{Y\in \{(\pi_{\mathfrak{p}}^{-1}\mathfrak{A}_{0,\mathfrak{p}})^{-}-\mathfrak{A}_{0,\mathfrak{p}}^{-}\}/\mathfrak{A}_{0,\mathfrak{p}}^{-}} \begin{pmatrix} \Pi_{\mathfrak{p}} & \pi_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}\Pi_{\mathfrak{p}}Y \\ 0 & \Pi_{\mathfrak{p}}\end{pmatrix}U_{\mathfrak{p}} \end{aligned} \tag{2-20}$$

where $\alpha_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{a}$ (see (1-9)).

Let φ be a non-zero eigen function of $c_{\mathfrak{p}}^{(0)}$ with eigen value $\sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})$. Then from the definition of $\mathscr{L}_{\xi,\mathfrak{p}}^{\Lambda}$ ((2-1)), $\sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})$ must satisfy the condition:

$$\begin{aligned} \sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})^2 &= \Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}) && \text{if } K_{\mathfrak{p}} \text{ is unramified,} \\ \sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(0)}) &= \Lambda_{\mathfrak{p}}(\varpi_{K_{\mathfrak{p}}}) && \text{if } K_{\mathfrak{p}} \text{ is ramified,} \end{aligned} \tag{2-21}$$

where $\varpi_{K_\mathfrak{p}}$ denotes a prime element of $K_\mathfrak{p}$. Put for $l \in \boldsymbol{Z}$,

$$a(l)=\begin{cases}\varphi(g_\mathfrak{p}(l)) & \text{if } l\geqq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{2-22}$$

PROPOSITION 2-4. *Let φ be an eigen function of $c_\mathfrak{p}^{(0)}$ with eigen value $\sigma_\mathfrak{p}(c_\mathfrak{p}^{(0)})$. Then for $l\geqq 0$ we have*

$$\begin{aligned}\varphi\,|\,c_\mathfrak{p}^{(1)}(g_\mathfrak{p}(l)) &= q^3a(l+1)+\sigma_\mathfrak{p}(c_\mathfrak{p}^{(0)})\left\{q^2-1-q^2\delta_l\delta\left(\left(\frac{K}{\mathfrak{p}}\right)=0\right)\right\}a(l)\\ &\quad +\Lambda_\mathfrak{p}(\pi_\mathfrak{p})a(l-1) \qquad \textit{if } \mathfrak{A}_{0,\mathfrak{p}}=\mathfrak{O}_\mathfrak{p},\\ &= q^3a(l+1)+\sigma_\mathfrak{p}(c_\mathfrak{p}^{(0)})\left\{q-1-q\delta_l\delta\left(\left(\frac{K}{\mathfrak{p}}\right)=-1\right)\right\}a(l)\\ &\quad +\Lambda_\mathfrak{p}(\pi_\mathfrak{p})a(l-1) \qquad \textit{if } \mathfrak{A}_{0,\mathfrak{p}}=\mathfrak{P}_\mathfrak{p},\end{aligned}$$

where $q=|\mathfrak{o}/\mathfrak{p}|$, δ_l means 1 or 0 according as $l=0$ or not, and $\delta((*))$ means 1 if $(*)$ is satisfied and 0 otherwise.

PROOF. Put $t=t_{0,\mathfrak{p}}\pi_\mathfrak{p}^{-\alpha_\mathfrak{p}+l}$. Then by (2-20),

$$\begin{aligned}\sum_X \varphi\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\pi_\mathfrak{p} & \pi_\mathfrak{p}^{\alpha_\mathfrak{p}}X\\ 0 & 1\end{pmatrix}\right) &= \sum_X \chi_p(\tau(t_{0,\mathfrak{p}}\pi_\mathfrak{p}^l\xi X))\varphi\left(\begin{pmatrix}\pi_\mathfrak{p}t & 0\\ 0 & 1\end{pmatrix}\right)\\ &= q^3a(l+1),\end{aligned}$$

and

$$\begin{aligned}&\sum_Y \varphi\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Pi_\mathfrak{p} & \pi_\mathfrak{p}^{\alpha_\mathfrak{p}}\Pi_\mathfrak{p}Y\\ & \Pi_\mathfrak{p}\end{pmatrix}\right)\\ &= \sum_Y \chi_p(\tau(t_{0,\mathfrak{p}}\pi_\mathfrak{p}^l\xi\Pi_\mathfrak{p}Y\Pi_\mathfrak{p}^{-1}))\varphi\left(\begin{pmatrix}\Pi_\mathfrak{p}t & \\ & \Pi_\mathfrak{p}\end{pmatrix}\right)\\ &= \sigma_\mathfrak{p}(c_\mathfrak{p}^{(0)})a(l)\sum_Y \chi_p(\tau(t_{0,\mathfrak{p}}\pi_\mathfrak{p}^l\xi\Pi_\mathfrak{p}Y\Pi_\mathfrak{p}^{-1})).\end{aligned}$$

If $l\geqq 1$, the summation of the right hand side is equal to $q^{A_\mathfrak{p}}-1$ ($A_\mathfrak{p}$ is defined in (1-34)). Suppose that $l=0$ and $\mathfrak{A}_{0,\mathfrak{p}}=\mathfrak{O}_\mathfrak{p}$. Writing $\xi_{0,\mathfrak{p}}=\iota X_0/2+Y_0\Pi_\mathfrak{p}^{-1}$ as in Lemma 2-7,

$$\sum_Y \chi_p(\tau(t_{0,\mathfrak{p}}\pi_\mathfrak{p}^l\xi\Pi_\mathfrak{p}Y\Pi_\mathfrak{p}^{-1})) = \sum_{Z\in(\mathfrak{O}_0-\pi_\mathfrak{p}\mathfrak{O}_0)/\pi_\mathfrak{p}\mathfrak{O}_0} \chi_p(\tau(\pi_\mathfrak{p}^{\nu_\mathfrak{p}-1}Y_0t_{0,\mathfrak{p}}Z)).$$

It is equal to -1 if $Y_0\in\mathfrak{O}_\mathfrak{p}^\times$ and to q^2-1 if $Y_0\in\pi_\mathfrak{p}\mathfrak{O}_\mathfrak{p}$. By Lemma 2-7, our assertion is proved. The other case is treated quite similarly. Q.E.D.

2-4. Let α, β, A_1, A_2, A_3, A_4, and A_5 be given complex numbers such that $A_3=A_1A_2$ and $A_1(A_1+A_4)\neq 0$. We consider the following $\boldsymbol{C}$-valued recursion formula: for non-negative integers m and l,

$$\begin{aligned}
\alpha a(m, l) &= a(m, l+1)+(A_1+\delta_m A_4)a(m+1, l-1) \\
&\quad +\delta_m A_5 a(m, l)+A_2 a(m-1, l+1)+A_3 a(m, l-1), \\
\beta a(m, l) &= (A_1+\delta_m A_4)a(m+1, l)+\delta_m A_5 a(m, l+1) \\
&\quad +A_2 a(m-1, l+2)+(2A_3-\delta_l A_3-\delta_l\delta_m A_2A_4)a(m, l) \\
&\quad +A_3(A_1+\delta_m A_4)a(m+1, l-2)+A_2A_3 a(m-1, l) \\
&\quad +\delta_m A_3A_5 a(m, l-1),
\end{aligned} \tag{2-23}$$

where we put $a(m', l')=0$ if m' or l' is negative.

PROPOSITION 2-5. *The recursion formula* (2-23) *has solutions and each solution* $\{a(m, l)\}$ *is determined uniquely by* $a(0, 0)$. *The generating function of* $a(m, l)$ *is*

$$\sum_{m,l=0}^{\infty} a(m, l)x^m y^l = a(0, 0)\frac{H(x, y)}{P(x)Q(y)},$$

where x *and* y *are indeterminates and the polynomials appearing in the right hand side are given as follows.*

$$\begin{aligned}
P(x) &= 1-(\beta-2A_1A_2)A_1^{-1}x+(\alpha^2-2\beta+2A_1A_2)A_1^{-1}A_2x^2 \\
&\quad -(\beta-2A_1A_2)A_1^{-1}A_2^2x^3+A_2^4x^4,
\end{aligned}$$

$$Q(y)=1-\alpha y+\beta y^2-\alpha A_3y^3+A_3^2y^4,$$

$$\begin{aligned}
H(x, y) &= (1+A_2A_3xy^2)(M_1(x)(1+A_2x)+A_2A_5A_1^{-1}\alpha x^2) \\
&\quad -A_2xy\{\alpha M_1(x)-A_5M_2(x)\}-A_5P(x)y-A_2A_4P(x)y^2,
\end{aligned}$$

$$M_1(x)=1-A_1^{-1}(A_1+A_4)^{-1}(A_1A_5\alpha+A_4\beta-A_1A_5^2-2A_1A_2A_4)x+A_1^{-1}A_2^2A_4x^2,$$
$$M_2(x)=1+A_1^{-1}(A_1A_2-\beta)x+A_1^{-1}A_2(A_1A_2-\beta)x^2+A_2^3x^3.$$

Especially,

$$\sum_{l=0}^{\infty} a(0, l)y^l=a(0, 0)(1-A_5y-A_2A_4y^2)/Q(y).$$

This assertion is checked by direct calculation. Since it is too tedious to reproduce the proof here, we omit it.

Let $\sigma_\mathfrak{p}$ be a homomorphism from $\mathscr{H}_\mathfrak{p}$ into $\boldsymbol{C}$. We define a subspace of $\mathscr{L}^A_{\xi,\mathfrak{p}}$ by

$$\mathscr{L}^A_{\xi,\mathfrak{p}}(\sigma_\mathfrak{p})=\{\varphi\in\mathscr{L}^A_{\xi,\mathfrak{p}}\mid \varphi|\phi=\sigma_\mathfrak{p}(\phi)\varphi \text{ for any } \phi\in\mathscr{H}_\mathfrak{p}\}. \tag{2-24}$$

We may assume that $\sigma_\mathfrak{p}$ satisfies the following conditions.

$$\begin{aligned} &\sigma_\mathfrak{p}(c_\mathfrak{p}^{(0)})=\Lambda_\mathfrak{p}(\pi_\mathfrak{p}) && \text{if } \mathfrak{p}\nmid\mathfrak{D}, \\ &\sigma_\mathfrak{p}(c_\mathfrak{p}^{(0)})=\Lambda_\mathfrak{p}(\varpi_{K\mathfrak{p}}) && \text{if } \mathfrak{p}\mid\mathfrak{D} \text{ and } \left(\frac{K}{\mathfrak{p}}\right)=0, \\ &\sigma_\mathfrak{p}(c_\mathfrak{p}^{(0)})^2=\Lambda_\mathfrak{p}(\pi_\mathfrak{p}) && \text{if } \mathfrak{p}\mid\mathfrak{D} \text{ and } \left(\frac{K}{\mathfrak{p}}\right)=-1. \end{aligned} \tag{2-25}$$

THEOREM 2-1. *Let $\sigma_\mathfrak{p}$ be a homomorphism from $\mathscr{H}_\mathfrak{p}$ into $\boldsymbol{C}$ satisfying* (2-25). *Then $\mathscr{L}^A_{\xi,\mathfrak{p}}(\sigma_\mathfrak{p})$ is one-dimensional, and each element φ of $\mathscr{L}^A_{\xi,\mathfrak{p}}(\sigma_\mathfrak{p})$ is determined by the value at $g_\mathfrak{p}(0)$ (see* (2-9) *and* (2-18)). *Moreover, the following identity holds.*

(i) *Assume* $\mathfrak{p}\nmid\mathfrak{D}$.

$$\sum_{l=0}^{\infty}\varphi(g_\mathfrak{p}(l))y^l=\varphi(g_\mathfrak{p}(0))H_\mathfrak{p}(y)/Q_\mathfrak{p}(y).$$

Here

$$\begin{aligned} Q_\mathfrak{p}(y)=1&-q^{-3}\sigma_\mathfrak{p}(c_\mathfrak{p}^{(1)})y+\{\sigma_\mathfrak{p}(c_\mathfrak{p}^{(2)})+\Lambda_\mathfrak{p}(\pi_\mathfrak{p})(q^2+1)\}q^{-5}y^2 \\ &-q^{-6}\sigma_\mathfrak{p}(c_\mathfrak{p}^{(1)})\Lambda_\mathfrak{p}(\pi_\mathfrak{p})y^3+\Lambda_\mathfrak{p}(\pi_\mathfrak{p})^2q^{-6}y^4, \end{aligned}$$

$$H_\mathfrak{p}(y)=\begin{cases} 1 & \textit{if } c_\mathfrak{p}\geqq 1, \\ 1-\Lambda_\mathfrak{p}(\pi_\mathfrak{p})q^{-4}y^2 & \textit{if } c_\mathfrak{p}=0 \textit{ and } \left(\frac{K}{\mathfrak{p}}\right)=-1, \\ 1-\Lambda_\mathfrak{p}(\varpi_{K_\mathfrak{p}})q^{-2}y & \textit{if } c_\mathfrak{p}=0 \textit{ and } \left(\frac{K}{\mathfrak{p}}\right)=0, \\ (1-\Lambda_\mathfrak{p}(\varpi_{K_\mathfrak{p}})q^{-2}y)(1-\Lambda_\mathfrak{p}(\pi_\mathfrak{p}\varpi_{K_\mathfrak{p}}^{-1})q^{-2}y) & \\ & \textit{if } c_\mathfrak{p}=0 \textit{ and } \left(\frac{K}{\mathfrak{p}}\right)=1. \end{cases}$$

(ii) *Assume* $\mathfrak{p}\mid\mathfrak{D}$.

$$\sum_{l=0}^{\infty}\varphi(g_\mathfrak{p}(l))y^l=\varphi(g_\mathfrak{p}(0))H_\mathfrak{p}(y)/Q_\mathfrak{p}(y).$$

Here

$$Q_{\mathfrak{p}}(y)=1-\{\sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(1)})-(q^{A_{\mathfrak{p}}}-1)\sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})\}q^{-3}y+\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})q^{-3}y^2,$$

$$H_{\mathfrak{p}}(y)=\begin{cases}1+q^{-1}\sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})y & \textit{if } \mathfrak{p}|\mathfrak{D}_0 \textit{ and } \mathfrak{p} \textit{ ramifies in } K,\\ 1 & \textit{if } \mathfrak{p}|\mathfrak{D}_0 \textit{ and } \mathfrak{p} \textit{ remains prime in } K,\\ 1 & \textit{if } \mathfrak{p}|\mathfrak{D}_1 \textit{ and } \mathfrak{p} \textit{ ramifies in } K,\\ 1+q^{-2}\sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})y & \textit{if } \mathfrak{p}|\mathfrak{D}_1 \textit{ and } \mathfrak{p} \textit{ remains prime in } K.\end{cases}$$

PROOF. When $\mathfrak{p}$ does not divide $\mathfrak{D}$, $\varphi(g_{\mathfrak{p}}(m, 1))$ satisfies the recursion formula (2-23) with

$$\alpha=\sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(1)})q^{-3}, \qquad \beta=\{\sigma_{\mathfrak{p}}(c_{\mathfrak{p}}^{(2)})+\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})(q^2+1)\}q^{-5}, \quad A_1=q^{-1},$$

$$A_2=\Lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})q^{-2}, \qquad A_4=\begin{cases}-\left(\dfrac{K}{\mathfrak{p}}\right)q^{-2} & \text{if } c_{\mathfrak{p}}=0,\\ 0 & \text{if } c_{\mathfrak{p}}\geqq 1,\end{cases}$$

$$A_5=\begin{cases}\varepsilon_{\mathfrak{p}}q^{-2} & \text{if } c_{\mathfrak{p}}=0,\\ 0 & \text{if } c_{\mathfrak{p}}\geqq 1,\end{cases}$$

where $\varepsilon_{\mathfrak{p}}$ is defined in (2-13). Hence our statement is merely a corollary of Proposition 2-5. When $\mathfrak{p}|\mathfrak{D}$, our statement is clear from Proposition 2-4. Q.E.D.

We denote by $\varphi_{\sigma_{\mathfrak{p}}}$ the element of $\mathscr{L}_{\xi,\mathfrak{p}}^{\Lambda}(\sigma_{\mathfrak{p}})$, whose value at $g_{\mathfrak{p}}(0)$ is 1, and we call it the normalized function in $\mathscr{L}_{\xi,\mathfrak{p}}^{\Lambda}(\sigma_{\mathfrak{p}})$.

§ 3. Functional equation of the L-function

3-1. Let F be a non-zero element of $\mathfrak{S}(\rho_{l,d}, \lambda; U_f)$, which is a simultaneous eigen function of the Hecke algebra $\mathscr{H}_{A,f}=\otimes_{\mathfrak{p}<\infty}\mathscr{H}_{\mathfrak{p}}$. We denote by $\sigma_F=\otimes_{\mathfrak{p}<\infty}\sigma_{F,\mathfrak{p}}$ the one dimensional representation of $\mathscr{H}_{A,f}$ determined by F. By Lemma 1-2, there exist an element ξ of B^- and an idele class character Λ of $K_A^\times$ such that $\Lambda|k_A^\times=\lambda$ and $\varphi_{F,\xi}^{\Lambda}\neq 0$. We fix such F, ξ, and Λ. Put

$$g_0=\prod_{\mathfrak{p}<\infty} g_{\mathfrak{p}}(0)\in G_{A,f}, \tag{3-1}$$

where $g_{\mathfrak{p}}(0)$ is defined in (2-9) and (2-18). The next proposition is a direct consequence of Theorem 2-1. Note that the right hand side is essentially a finite product for each fixed $g_\infty g_f\in G_A$.

PROPOSITION 3-1. *For any $g_\infty\in G_\infty$ and $g_f=(g_{f,\mathfrak{p}})\in G_{A,f}$, we have*

$$\varphi^{A}_{F,\xi}(g_\infty g_f)=\prod_{\mathfrak{p}<\infty}\varphi_{\sigma_{F,\mathfrak{p}}}(g_{f,\mathfrak{p}})\cdot\varphi^{A}_{F,\xi}(g_\infty g_0),$$

where $\varphi_{\sigma_{F,\mathfrak{p}}}$ *is the normalized function in* $\mathscr{L}^{A}_{\xi,\mathfrak{p}}(\sigma_{F,\mathfrak{p}})$.

Since F is bounded on G_A, there exists a positive constant C, not depending on $\mathfrak{p}$, such that

$$\text{(3-2)}\qquad \begin{aligned}&|\sigma_{F,\mathfrak{p}}(c^{(1)}_{\mathfrak{p}})|\leqq Cq^{3} &&\text{for all}\quad \mathfrak{p}<\infty,\\ &|\sigma_{F,\mathfrak{p}}(c^{(2)}_{\mathfrak{p}})|\leqq Cq^{4} &&\text{for all}\quad \mathfrak{p}\ (\mathfrak{p}\nmid\mathfrak{D}).\end{aligned}$$

Take an unramified unitary grössencharacter $\omega=\prod_v\omega_v$ of $k^\times_A$, namely ω is a unitary character which is trivial on $k^\times\prod_{\mathfrak{p}<\infty}\mathfrak{o}^\times_{\mathfrak{p}}$. For each $t\in \boldsymbol{R}^\times_+$ call $z(t)$ the idele (z_v) such that $z_{\mathfrak{p}}=1$ for every finite place $\mathfrak{p}$ and $z_{\infty_j}=t$ for $1\leqq j\leqq n$. Then $k^\times_A$ is the direct product of $z(\boldsymbol{R}^\times_+)$ and k^1_A, where k^1_A is the subgroup of $k^\times_A$ defined by $|z|_A=1$ (cf. Ch. IV-4 in [21]). Hereafter we assume that

$$\text{(3-3)}\qquad \omega\quad\text{is trivial on}\quad z(\boldsymbol{R}^\times_+).$$

This assumption does not lose generality for our purpose. Put

$$\text{(3-4)}\qquad \begin{aligned}Z_F(\omega,s)=&\prod_{\mathfrak{p}<\infty}Q_{\mathfrak{p},F}(\omega_{\mathfrak{p}}(\pi_{\mathfrak{p}})|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{s-3/2})^{-1}\\ &\prod_{\mathfrak{p}|\mathfrak{D}}(1-\sigma_{F,\mathfrak{p}}(c^{(0)}_{\mathfrak{p}})\omega_{\mathfrak{p}}(\pi_{\mathfrak{p}})|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{s+1/2})^{-1}.\end{aligned}$$

Here $Q_{\mathfrak{p},F}(y)$ is a polynomial in y given by

$$\begin{aligned}Q_{\mathfrak{p},F}(y)&=1-\sigma_{F,\mathfrak{p}}(c^{(1)}_{\mathfrak{p}})q^{-3}y+(\sigma_{F,\mathfrak{p}}(c^{(2)}_{\mathfrak{p}})+\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})(q^2+1))q^{-5}y^2\\ &\quad-\sigma_{F,\mathfrak{p}}(c^{(1)}_{\mathfrak{p}})\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})q^{-6}y^3+\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})^2q^{-6}y^4 &&\text{if}\quad \mathfrak{p}\nmid\mathfrak{D},\\ &=1-\{\sigma_{F,\mathfrak{p}}(c^{(1)}_{\mathfrak{p}})-(q^{A_{\mathfrak{p}}}-1)\sigma_{F,\mathfrak{p}}(c^{(0)}_{\mathfrak{p}})\}q^{-3}y\\ &\quad+\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})q^{-3}y^2 &&\text{if}\quad \mathfrak{p}|\mathfrak{D},\end{aligned}$$

where $q=|\mathfrak{o}/\mathfrak{p}|$ and $A_{\mathfrak{p}}$ is defined in (1-34). By (3-2), the infinite product (3-4) converges absolutely and uniformly on some right half-plane, so $Z_F(\omega,s)$ determines a holomorphic function there.

We normalize the Haar measure $d^\times t=\prod_v d^\times t_v$ on $k^\times_A$ by the conditions

$$\int_{\mathfrak{o}^\times_{\mathfrak{p}}}d^\times t_{\mathfrak{p}}=1\quad\text{and}\quad d^\times t_{\infty_j}=\frac{dt_{\infty_j}}{|t_{\infty_j}|_{\infty_j}},$$

where dt_{∞_j} is the usual Lebesgue measure on $k_{\infty_j}=\boldsymbol{R}$. For $g_\infty\in G_\infty$, we put

$$\Phi^{\Lambda}_{F,\xi}(g_\infty;\omega,s)=\int_{k_A^\times}\varphi^{\Lambda}_{F,\xi}\left(\begin{pmatrix}t&0\\0&1\end{pmatrix}g_\infty g_0\right)\omega(t)|t|_A^{s-3/2}d^\times t. \tag{3-5}$$

This integral converges absolutely and uniformly on some right half-plane. For any grössencharacter $\Lambda'=\prod_w \Lambda'_w$ of $K_A^\times$, the L-function of Λ' is defined by

$$L_K(\Lambda',s)=\prod_{\mathfrak{P}}(1-\Lambda'_{\mathfrak{P}}(\varpi_{\mathfrak{P}})|\varpi_{\mathfrak{P}}|_{\mathfrak{P}}^s)^{-1}, \tag{3-6}$$

where $\mathfrak{P}$ runs through all prime ideals of K such that $\Lambda'_{\mathfrak{P}}$ is unramified, and $\varpi_{\mathfrak{P}}$ denotes a prime element of $K_{\mathfrak{P}}$.

THEOREM 3-1. *Let $g_\infty=\begin{pmatrix}*&0\\0&*\end{pmatrix}\in G_\infty$ and assume that $-\mu(g_\infty)\sqrt{-1}\xi\in\mathfrak{H}_+$. Then, in some right half-plane, the following identity holds.*

$$\begin{aligned}\Phi^{\Lambda}_{F,\xi}(g_\infty;\omega,s)=&\prod_{j=1}^{n}e^{2\pi\alpha_j}\frac{\Gamma(s+s_j+l_j+(d_j-3)/2)}{(2\pi\alpha_j)^{s+s_j+l_j+(d_j-3)/2}}\\&\times\prod_{\substack{\mathfrak{p}|\mathfrak{D}_0\\ \mathfrak{p}|\mathfrak{d}(K/k)}}(1+\sigma_{F,\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})\omega_{\mathfrak{p}}(\pi_{\mathfrak{p}})|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{s-1/2})\\&\times\prod_{\substack{\mathfrak{p}|\mathfrak{D}_0\\ \mathfrak{p}\nmid\mathfrak{d}(K/k)}}(1+\sigma_{F,\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})\omega_{\mathfrak{p}}(\pi_{\mathfrak{p}})|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{s+1/2})^{-1}\\&\times L_K(\Lambda_1,s+1/2)^{-1}\times Z_F(\omega,s)\times\varphi^{\Lambda}_{F,\xi}(g_\infty g_0).\end{aligned}$$

Here $\mathfrak{d}(K/k)$ denotes the discriminant ideal of K/k and

$$\alpha_j=\mathrm{Tr}_{B_{\infty_j}/k_{\infty_j}}(-\sqrt{-1}\,\xi(g_{\infty_j}\langle Z_{j,0}\rangle))\qquad(1\leqq j\leqq n), \tag{3-7}$$

$$\omega_{\infty_j}(x)=x^{s_j}\qquad \textit{for any } x\in\boldsymbol{R}_+^\times, \tag{3-8}$$

$$\Lambda_1(z)=\Lambda(\bar{z})\omega(z\bar{z})\qquad \textit{for any } z\in K_A^\times. \tag{3-9}$$

PROOF. Proposition 3-1 asserts that

$$\begin{aligned}\Phi^{\Lambda}_{F,\xi}(g_\infty;\omega,s)=&\prod_{\mathfrak{p}<\infty}\int_{k_{\mathfrak{p}}^\times}\varphi_{\sigma_{F,\mathfrak{p}}}\left(\begin{pmatrix}t_{\mathfrak{p}}&\\&1\end{pmatrix}g_{0,\mathfrak{p}}\right)\omega_{\mathfrak{p}}(t_{\mathfrak{p}})|t_{\mathfrak{p}}|_{\mathfrak{p}}^{s-3/2}d^\times t_{\mathfrak{p}}\\&\times\int_{k_\infty^\times}\varphi^{\Lambda}_{F,\xi}\left(\begin{pmatrix}t_\infty&\\&1\end{pmatrix}g_\infty g_0\right)\omega_\infty(t_\infty)|t_\infty|_\infty^{s-3/2}d^\times t_\infty.\end{aligned}$$

By (1-26), for any $t_\infty\in k_\infty^\times$,

$$\varphi^{\Delta}_{F,\xi}\left(\begin{pmatrix} t_\infty & \\ & 1\end{pmatrix} g_\infty g_0\right)=\begin{cases}\left(\prod_{j=1}^{n}|t_{\infty j}|_{\infty j}^{l_j+d_j/2}\right)e[\tau(t_\infty\xi(g_\infty\langle Z_0\rangle))]v \\ \qquad\qquad \text{if} \quad t_{\infty j}>0 \text{ for all } j, \\ 0 \qquad\qquad \text{otherwise,}\end{cases}$$

where v is a constant vector in $V_{l,d}$ not depending on t_∞. On the other hand, we have

$$\int_{k_\mathfrak{p}^\times}\varphi_{\sigma_F,\mathfrak{p}}\left(\begin{pmatrix} t_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix} g_{0,\mathfrak{p}}\right)\omega_\mathfrak{p}(t_\mathfrak{p})|t_\mathfrak{p}|_\mathfrak{p}^{s-3/2}d^\times t_\mathfrak{p}$$
$$=H_{\sigma_{F,\mathfrak{p}}}(\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s-3/2})/Q_{\sigma_{F,\mathfrak{p}}}(\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s-3/2}),$$

where $H_{\sigma_{F,\mathfrak{p}}}(y)$ and $Q_{\sigma_{F,\mathfrak{p}}}(y)$ are the polynomials defined in Theorem 2-1. Let $l_{K_\mathfrak{p}}(\Lambda_{1,\mathfrak{p}}, s+1/2)$ denote the $\mathfrak{p}$-part of $L_K(\Lambda_1, s+1/2)$. If $\mathfrak{p}\nmid\mathfrak{D}$, it is clear that $H_{\sigma_{F,\mathfrak{p}}}(\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s-3/2})=l_{K_\mathfrak{p}}(\Lambda_{1,\mathfrak{p}}, s+1/2)^{-1}$. We assume that $\mathfrak{p}|\mathfrak{D}_0$. If $\mathfrak{p}$ ramifies in K, then $H_{\sigma_{F,\mathfrak{p}}}(\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s-3/2})=1+\sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s-1/2}$ and $l_{K_\mathfrak{p}}(\Lambda_{1,\mathfrak{p}}, s+1/2)^{-1}=1-\sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s+1/2}$, since $\Lambda_\mathfrak{p}(\varpi_{K_\mathfrak{p}})=\sigma_{K,\mathfrak{p}}(c_\mathfrak{p}^{(0)})$ (cf. (2-25)). If $\mathfrak{p}$ remains prime in K, then $H_{\sigma_{F,\mathfrak{p}}}(\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s-3/2})=1$ and

$$l_{K_\mathfrak{p}}(\Lambda_{1,\mathfrak{p}}, s+1/2)^{-1}=(1+\sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s+1/2})$$
$$\times(1-\sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s+1/2}).$$

the case $\mathfrak{p}|\mathfrak{D}_1$ is treated similarly. Q.E.D.

The rest of this section will be devoted to proving the next theorem.

THEOREM 3-2 (MAIN THEOREM). *Let $F\in\mathfrak{S}(\rho_{l,d},\lambda; U_f)$ be a simultaneous eigen function of Hecke algebra $\mathscr{H}_{A,f}=\otimes_{\mathfrak{p}<\infty}\mathscr{H}_\mathfrak{p}$, $\sigma_F=\otimes_{\mathfrak{p}<\infty}\sigma_{F,\mathfrak{p}}$ be the one dimensional representation of $\mathscr{H}_{A,f}$ determined by F, and ω be an unramified unitary grössencharacter of k satisfying* (3-3). *Put*

$$\zeta_F(\omega, s)=\prod_{j=1}^{n}\Gamma(s+s_j+(d_j+1)/2)\Gamma(s+s_j+l_j+(d_j-3)/2)$$
$$\times(d(k)^2N(\mathfrak{D})^{1/2}/(2\pi)^{2n})^s Z_F(\omega, s).$$

Then $\prod_{\mathfrak{p}|\mathfrak{D}_0}(1+\sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})\omega_\mathfrak{p}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s-1/2})\times\zeta_F(\omega, s)$ is continued to the whole complex plane as a meromorphic function. This function is holomorphic, except possible simple poles at $s=3/2$ and $-1/2$, and unless $d_1=\cdots=d_n=0$ and $\lambda\omega^2=1$, it is an entire function. Furthermore, $\zeta_F(\omega, s)$ satisfies the functional equation:

$$\zeta_F(\omega, s)=(-1)^{\sum_{j=1}^n l_j}(\lambda\omega^2)(\mathfrak{d}_k^2)\omega(\mathfrak{D})\prod_{\mathfrak{p}|\mathfrak{D}}\sigma_{F,\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})\zeta_{F'}(\omega^{-1}, 1-s).$$

Here, $F'(g)=F(g)\lambda^{-1}(\mu(g))$, which is an element of $\mathfrak{S}(\rho_{l,d}, \lambda^{-1}; U_f)$ and also a simultaneous eigen function of $\mathscr{H}_{A,f}$, and $\mathfrak{d}_k$ denotes the different ideal of k over $\boldsymbol{Q}$, and s_j $(1\leqq j\leqq n)$ is a complex number given in (3-8).

REMARK 3-1. Assume that $F_\chi(g;\xi)\neq 0$. Then the first statement of the above theorem still holds if we replace

$$\prod_{\mathfrak{p}|\mathfrak{D}_0}(1+\sigma_{F,\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})\omega_{\mathfrak{p}}(\pi_{\mathfrak{p}})|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{s-1/2})\times\zeta_F(\omega, s)$$

by

$$\prod_{\mathfrak{p}'}(1+\sigma_{F,\mathfrak{p}'}(c_{\mathfrak{p}'}^{(0)})\omega_{\mathfrak{p}'}(\pi_{\mathfrak{p}'})|\pi_{\mathfrak{p}'}|_{\mathfrak{p}'}^{s-1/2})\times\zeta_F(\omega, s),$$

where $\mathfrak{p}'$ runs through all prime ideals dividing $\mathfrak{D}_0$ and the discriminant of $k(\xi)$ over k.

3-2. We take an element η of $B^\times$ such that

(3-10) $$\mathrm{Tr}_{B/k}(\eta)=0 \quad \text{and} \quad \mathrm{Tr}_{B/k}(\xi\eta)=0,$$

and fix this η once and for all. Then $B=K+K\eta$ and $x\eta=\eta\bar{x}$ for any $x\in K=k(\xi)$. We introduce an algebraic group G' defined over k by

(3-11) $$G'_k=\{g\in GL_2(K)|\ \det g\in k^\times\}.$$

The mapping

(3-12) $$\psi: G'\longrightarrow G, \qquad \psi\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\alpha & \beta\eta\\ \eta^{-1}\gamma & \bar{\delta}\end{pmatrix},$$

determines an algebraic group homomorphism defined over k. Note that

(3-13) $$\xi_{\infty_j}^2<0 \quad \text{and} \quad \eta_{\infty_j}^2>0 \qquad (1\leqq j\leqq n).$$

We write as

$$\xi_{\infty_j}=\begin{pmatrix}a_j/2 & b_j\\ c_j & -a_j/2\end{pmatrix}$$

under the identification of B_{∞_j} and $M_2(\boldsymbol{R})$, and put

(3-14) $$\kappa_{\infty_j}=\xi_{\infty_j}.$$

Put

(3-15) $$R_{\infty_j}=\begin{pmatrix}\sqrt{\eta_{\infty_j}^2}\,h_{\infty_j} & 0\\ 0 & h_{\infty_j}\end{pmatrix}\in G_{\infty_j}\quad\text{and}\quad R_\infty=\prod_{j=1}^n R_{\infty_j}\in G_\infty,$$

where

$$h_{\infty_j}=\begin{pmatrix}\sqrt{-(2\xi_{\infty_j})^2} & a_j\\ 0 & 2c_j\end{pmatrix}\in B_{\infty_j}^\times.$$

Obviously $-\mu(R_\infty)\sqrt{-1}\,\xi$ belongs to $\mathfrak{H}_+$ and

(3-16) $$\alpha_j=\mathrm{Tr}_{B/k}(-\sqrt{-1}\,\xi R_{\infty_j}\langle Z_{j,0}\rangle)=\sqrt{-(2\xi_{\infty_j})^2\eta_{\infty_j}^2}.$$

Put

(3-17) $$M_{\infty_j}=\{g\in G'_{\infty_j}\mid \det g=1,\ {}^t\bar{g}g=1\},\quad\text{and}$$
$$M_\infty=\prod_{j=1}^n M_{\infty_j}.$$

M_{∞_j} is isomorphic to the special unitary group of degree two. We can easily check that

(3-18) $$R_\infty^{-1}\psi(m_\infty)R_\infty\in U_\infty\qquad\text{for any }m_\infty\in M_\infty.$$

We define a representation $\tilde{\rho}$ of M_∞ in $V_{l,d}$ by

(3-19) $$\tilde{\rho}(m_\infty)=\rho_{l,d}(J(R_\infty^{-1}\psi(m_\infty)R_\infty,\ Z_0))\quad(m_\infty\in M_\infty).$$

This representation does not depend on l and is equivalent to a unitary representation $\otimes_{j=1}^n\sigma_{d_j}$ of $SU(2)^n$.

Let $\mathfrak{p}$ be a prime ideal and write ξ as in (2-3). When $\mathfrak{p}\nmid\mathfrak{D}$ and $\eta_\mathfrak{p}=\begin{pmatrix}x_\mathfrak{p} & -(a_0x_\mathfrak{p}+b_0y_\mathfrak{p})\\ y_\mathfrak{p} & -x_\mathfrak{p}\end{pmatrix}$ through the identification of $B_\mathfrak{p}$ and $M_2(k_\mathfrak{p})$ stated in Lemma 2-1, we put

(3-20) $$\kappa_\mathfrak{p}=\begin{pmatrix}x_\mathfrak{p} & b_0y_\mathfrak{p}\\ y_\mathfrak{p} & x_\mathfrak{p}-a_0y_\mathfrak{p}\end{pmatrix}\in K_\mathfrak{p}^\times,$$

where $\xi_{0,\mathfrak{p}}=\begin{pmatrix}a_0/2 & b_0\\ 1 & -a_0/2\end{pmatrix}$. So $\kappa_\mathfrak{p}^{-1}\eta_\mathfrak{p}=\begin{pmatrix}1 & -a_0\\ 0 & -1\end{pmatrix}$ is a unit of $\mathfrak{O}_\mathfrak{p}$. When $\mathfrak{p}\mid\mathfrak{D}$, put

(3-21) $$\kappa_\mathfrak{p}=\begin{cases}\varpi_{K_\mathfrak{p}}^{e_\mathfrak{p}} & \text{if }\ \mathfrak{p}\mid\mathfrak{d}(K/k)\text{ and }\mathfrak{p}\mid\mathfrak{D}_0,\\ \varpi_{K_\mathfrak{p}}^{e_\mathfrak{p}-1} & \text{if }\ \mathfrak{p}\mid\mathfrak{d}(K/k)\text{ and }\mathfrak{p}\mid\mathfrak{D}_1,\\ \varpi_{K_\mathfrak{p}}^{(e_\mathfrak{p}-1)/2} & \text{if }\ \mathfrak{p}\nmid\mathfrak{d}(K/k),\end{cases}$$

where $\mathfrak{d}(K/k)$ is the descriminant of K over k, $e_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}\,\eta^2$, and $\varpi_{K_{\mathfrak{p}}}$ denotes a prime element of $K_{\mathfrak{p}}$. Note that if $\mathfrak{p}\nmid\mathfrak{d}(K/k)$ then $e_{\mathfrak{p}}$ is odd. From the above definition of $\kappa_{\mathfrak{p}}$ for $\mathfrak{p}\,|\,\mathfrak{D}$, we have

$$\text{(3-22)}\qquad \mathrm{ord}_{\mathfrak{P}_{\mathfrak{p}}}\,\kappa_{\mathfrak{p}}^{-1}\eta=\begin{cases}0 & \text{if } \mathfrak{p}\,|\,\mathfrak{d}(K/k) \text{ and } \mathfrak{p}\,|\,\mathfrak{D}_0,\\ 1 & \text{otherwise.}\end{cases}$$

Put

$$\text{(3-23)}\qquad \begin{aligned} &R_{\mathfrak{p}}=\begin{pmatrix} t_{0,\mathfrak{p}}^{-1} & 0\\ 0 & 1\end{pmatrix} g_{\mathfrak{p}}(0), \qquad R_f=\prod_{\mathfrak{p}<\infty} R_{\mathfrak{p}},\\ &R=R_\infty R_f, \qquad t_0=\prod_{\mathfrak{p}<\infty} t_{0,\mathfrak{p}}\end{aligned}$$

where $t_{0,\mathfrak{p}}$ is defined after (2-9) and (2-18). Let $\mathfrak{D}(\xi)$ be the product of all prime ideals dividing $\mathfrak{D}_0$ and not dividing $\mathfrak{d}(K/k)$. For each prime $\mathfrak{p}$, we put

$$\text{(3-24)}\qquad \begin{aligned} M_{\mathfrak{p}}&=\begin{pmatrix}1 & \\ & \kappa_{\mathfrak{p}}\end{pmatrix}\left\{g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G'_{\mathfrak{p}}\,\middle|\, \begin{matrix}\alpha,\beta,\gamma,\delta\in\mathfrak{o}_{K,\mathfrak{p}}\\ \det g\in\mathfrak{o}_{\mathfrak{p}}^{\times}\end{matrix}\right\}\begin{pmatrix}1 & \\ & \kappa_{\mathfrak{p}}^{-1}\end{pmatrix},\\ M'_{\mathfrak{p}}&=\begin{pmatrix}1 & \\ & \kappa_{\mathfrak{p}}\end{pmatrix}\left\{g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G'_{\mathfrak{p}}\,\middle|\, \begin{matrix}\alpha,\beta,\gamma,\delta\in\mathfrak{o}_K(\mathfrak{f}_{\Delta})_{\mathfrak{p}}\\ \det g\in\mathfrak{o}_{\mathfrak{p}}^{\times}\end{matrix}\right\}\begin{pmatrix}1 & \\ & \kappa_{\mathfrak{p}}^{-1}\end{pmatrix},\\ M''_{\mathfrak{p}}&=\begin{pmatrix}1 & \\ & \kappa_{\mathfrak{p}}\end{pmatrix}\left\{g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G'_{\mathfrak{p}}\,\middle|\, \begin{matrix}\alpha,\beta,\gamma,\delta\in\mathfrak{o}_K(\mathfrak{f}_{\Delta})_{\mathfrak{p}}\\ \det g\in\mathfrak{o}_{\mathfrak{p}}^{\times},\ \gamma\in\mathfrak{D}(\xi)\mathfrak{o}_K(\mathfrak{f}_{\Delta})_{\mathfrak{p}}\end{matrix}\right\}\begin{pmatrix}1 & \\ & \kappa_{\mathfrak{p}}^{-1}\end{pmatrix}.\end{aligned}$$

These are all open compact subgroups of $G'_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ is a maximal compact subgroup of $G'_{\mathfrak{p}}$. Put

$$\text{(3-25)}\qquad M=\prod_v M_v,\qquad M'=\prod_v M'_v,\quad\text{and}\quad M''=\prod_v M''_v,$$

with $M'_{\infty_j}=M''_{\infty_j}=M_{\infty_j}$ for each j.

Let us define a function $\tilde{F}$ on G'_A by

$$\text{(3-26)}\qquad \tilde{F}(g)=F(\psi(g)R)\qquad (g\in G'_A).$$

Then it satisfies

$$\text{(3-27)}\qquad \tilde{F}(\gamma g m_\infty m''_f)=\tilde{\rho}(m_\infty)^{-1}\tilde{F}(g)\quad\text{for}\quad {}^\forall\gamma\in G'_k,\ g\in G'_A,\ m_\infty m''_f\in M''.$$

For any character σ of $K_\infty^1=\{u\in K_\infty\,|\,u\bar{u}=1\}$, put

$$\text{(3-28)}\qquad V_\sigma=\left\{v\in V\,\middle|\,\tilde{\rho}\begin{pmatrix}\bar{u} & \\ & u\end{pmatrix}v=\sigma(u)v\ \text{ for all }\ u\in K_\infty^1\right\}.$$

Since $\left\{\begin{pmatrix}\bar{u} & \\ & u\end{pmatrix}\middle| u\in K_\infty^1\right\}$ is commutative, V is decomposed as $V=\bigoplus_\sigma V_\sigma$; we denote by P_σ the projection $V\to V_\sigma$. Then P_σ commutes with all $\tilde{\rho}\begin{pmatrix}\bar{u} & \\ & u\end{pmatrix}$ $(u\in K_\infty^1)$. For any character Ω of $K_A^\times$, we abbreviate $V_{\Omega|K_\infty^1}$ [resp. $P_{\Omega|K_\infty^1}$] to V_Ω [resp. P_Ω]. Note that for any $g_f\in G_{A,f}$ and $t\in k_A^\times$, $\varphi_{F,\xi}^{A}\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}R_\infty g_f\right)$ is in V_A.

Assume that $\mathrm{Re}(s)$ is sufficiently large, and put

$$\begin{aligned}(3\text{-}29)\qquad A_{F,\xi}^{A}(\omega, s)&=\int_{k_A^\times}\varphi_{F,\xi}^{A}\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}R\right)\omega(t)\,|t|_A^{s-3/2}d^\times t\\ &=\omega(t_0)\,|t_0|_A^{s-3/2}\Phi_{F,\xi}^{A}(R_\infty;\,\omega,\, s).\end{aligned}$$

Since

$$\sum_{\alpha\in k^\times}\varphi_{F,\xi}^{A}\left(\begin{pmatrix}\alpha t & \\ & 1\end{pmatrix}R\right)=\int_{K^\times k_A^\times\backslash K_A^\times}\left\{\sum_{\alpha\in k^\times}F_\chi\left(\tilde{u}\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}R;\,\alpha\xi\right)\right\}\Lambda(u)^{-1}d^\times u,$$

and

$$\sum_{\alpha\in k^\times}F_\chi(\tilde{u}g;\,\alpha\xi)=\int_{K\backslash K_A}F\left(\begin{pmatrix}1 & x\eta\\ 0 & 1\end{pmatrix}\tilde{u}g\right)dx\qquad \text{for } g\in G_A,$$

we have

$$\begin{aligned}(3\text{-}30)\qquad A_{F,\xi}^{A}(\omega, s)=&\int_{k^\times\backslash k_A^\times}\int_{K^\times k_A^\times\backslash K_A^\times}\int_{K\backslash K_A}\tilde{F}\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}tu & \\ & \bar{u}\end{pmatrix}\right)\\ &\times\Lambda(u)^{-1}\omega(t)\,|t|_A^{s-3/2}dx\,d^\times u\,d^\times t.\end{aligned}$$

We define an algebraic subgroup B' of G' by

$$(3\text{-}31)\qquad B_k'=\left\{b'=\begin{pmatrix}\alpha & x\\ 0 & \beta\end{pmatrix}\middle|\ \alpha\beta\in k^\times,\ x\in K\right\}.$$

For $b'=\begin{pmatrix}\alpha & x\\ 0 & \beta\end{pmatrix}\in B'$, we put $\beta(b')=\beta$ and $t(b')=\alpha\bar{\beta}^{-1}$. Taking a suitable right B_A'-invariant measure $d_r b'$ on $B_k'k_A^\times\backslash B_A'$, we obtain

$$(3\text{-}32)\qquad A_{F,\xi}^{A}(\omega, s)=\int_{B_k'k_A^\times\backslash B_A'}\tilde{F}(b')\Lambda^{-1}(\overline{\beta(b')})|t(b')|_A^{s-3/2}\omega(t(b'))d_r b'.$$

We denote by X the characteristic function of $B_A'M''$; namely,

$X(g)=1$ or 0 according whether g belongs to $B'_A M''$ or not. For any $g\in B'_A M''$, we put $g=b(g)m(g)$ $(b(g)\in B'_A,\ m(g)=m_\infty(g)m_f(g)\in M'')$, $\beta(g)=\beta(b(g))$ and $t(g)=t(b(g))$. Since $B'_A M''$ is an open subset of G'_A and M'' is compact, the integral representation (3-32) is transformed into the form

$$\text{(3-33)}\qquad \begin{aligned} A^{A}_{F,\xi}(\omega, s) &= \int_{B'_k k^\times_A\backslash G'_A} P_A X(g)\omega(\det g)\Lambda_1^{-1}(\beta(g)) \\ &\qquad \times |t(g)|_A^{s+1/2}\tilde{\rho}(m_\infty(g))\tilde{F}(g)d\dot{g} \\ &= \int_{G'_k k^\times_A\backslash G'_A} \Big\{ \sum_{\gamma\in B'_k\backslash G'_k} X(\gamma g)\Lambda_1^{-1}(\beta(\gamma g)) \\ &\qquad \times |t(\gamma g)|_A^{s+1/2} P_A\tilde{\rho}(m_\infty(\gamma g))\Big\}\omega(\det g)\tilde{F}(g)d\dot{g}, \end{aligned}$$

where $\Lambda_1(z)=\Lambda(\bar{z})\omega(z\bar{z})$ for $z\in K^\times_A$ and $d\dot{g}$ is a suitable invariant measure on $G'_k k^\times_A\backslash G'_A$. Note that the integrand in (3-33) is well-defined.

3-3. First we quote some results on Eisenstein series from Godement [8] with a slight modification. Let $W=K\oplus K$ and view it as a vector space over k. We denote by $\mathscr{S}(W_A, \tilde{\rho})$ the space of End(V)-valued Schwartz-Bruhat functions φ on W_A satisfying

$$\text{(3-34)}\qquad \varphi(xm_\infty)=\varphi(x)\tilde{\rho}(m_\infty) \qquad ({}^\forall m_\infty\in M_\infty).$$

For $\varphi\in\mathscr{S}(W_A, \tilde{\rho})$ and $g\in G'_A$, put

$$\text{(3-35)}\qquad L^{\Lambda_1}_{\varphi}(g, s)=|\det g|_A^{s+1/2}\int_{K^\times_A}\Lambda_1(t)\,|t\bar{t}|_A^{s+1/2}\varphi(t(0, 1)g)d^\times t,$$

where Λ_1 is a grössencharacter of $K^\times_A$ defined in (3-9), and the Haar measure $d^\times t$ on $K^\times_A$ is normalized as below;

$$d^\times t=\prod_v d^\times t_v, \qquad \int_{o_K(\mathfrak{p}^{e_\mathfrak{p}})^\times_\mathfrak{p}} d^\times t_\mathfrak{p}=1,$$
$$d^\times t_{\infty_j}=|dt_{\infty_j}\wedge d\bar{t}_{\infty_j}|/|t_{\infty_j}\bar{t}_{\infty_j}|,$$

where $2^{-1}|dt_{\infty_j}\wedge d\bar{t}_{\infty_j}|$ is the usual Lebesgue measure on $K_{\infty_j}=\boldsymbol{C}$. The integral in (3-35) converges absolutely in $\mathrm{Re}(s)>1/2$. If $g=\begin{pmatrix} t\bar{\beta} & * \\ 0 & \beta\end{pmatrix}m$ $(m=m_\infty m_f\in M)$, then

$$\text{(3-36)}\qquad L^{\Lambda_1}_{\varphi}(g, s)=\Lambda_1^{-1}(\beta)\,|t|_A^{s+1/2}L^{\Lambda_1}_{\varphi}(m_f, s)\tilde{\rho}(m_\infty).$$

Note that

$$L_{\varphi}^{\Lambda_1}(g_f, s)=L_{\varphi}^{\Lambda_1}(g_f, s)P_{\Lambda_1^{-1}} \qquad \text{for} \quad {}^{\forall}g_f \in G'_{A,f}. \tag{3-37}$$

We define the Fourier transform φ^* of φ by

$$\varphi^*(x, y)=\int_{W_A} \varphi(u, v)\chi(\mathrm{Tr}_{K/Q}(vx-uy))dudv, \tag{3-38}$$

where du and dv are the self dual Haar measures on K_A with respect to $\chi\circ\mathrm{Tr}_{K/Q}$. Put

$$E_{\varphi}^{\Lambda_1}(g, s)=\sum_{\gamma\in B'_k\backslash G'_k} L_{\varphi}^{\Lambda_1}(\gamma g, s). \tag{3-39}$$

For any fixed $g\in G'_A$, it converges absolutely on some right half plane. For $g\in G'_A$, $t\in K_A^{\times}$ and $\varphi\in\mathscr{S}(W_A, \tilde{\rho})$, we put

$$\theta_{\varphi}(t, g)=\sum_{0\neq v\in W_k} \varphi(tvg). \tag{3-40}$$

As in [8], using Poisson's summation formula we obtain that

$$\theta_{\varphi}(t, g)+\varphi(0)=\frac{1}{|N(t)\det(g)|_A^2}\{\theta_{\varphi^*}(t^{-1}(\det g)^{-1}, g)+\varphi^*(0)\}.$$

Put

$$E_{\varphi}^{\Lambda_1}(g, s)^+=|\det g|_A^{s+1/2}\int_{K^{\times}\backslash K_A^{\times}, |t\bar{t}|_A\geqq|\det g|_A^{-1}} \theta_{\varphi}(t, g)\Lambda_1(t)|t\bar{t}|_A^{s+1/2}d^{\times}t. \tag{3-41}$$

This integral converges absolutely for any s, thus, $E_{\varphi}^{\Lambda_1}(g, s)^+$ determines an entire function of s. Since

$$E_{\varphi}^{\Lambda_1}(g, s)=|\det g|_A^{s+1/2}\int_{K^{\times}\backslash K_A^{\times}} \theta_{\varphi}(t, g)\Lambda_1(t)|t\bar{t}|_A^{s+1/2}d^{\times}t, \tag{3-42}$$

we obtain

$$\begin{aligned} E_{\varphi}^{\Lambda_1}(g, s)=&E_{\varphi}^{\Lambda_1}(g, s)^+ + \Lambda_1^{-1}(\det g)E_{\varphi^*}^{\Lambda_1^{-1}}(g, 1-s)^+ \\ &-\delta(\Lambda_1=1)c_0\{\varphi^*(0)/(-s+3/2)+\varphi(0)/(s+1/2)\}, \end{aligned} \tag{3-43}$$

where c_0 is a positive constant and $\delta(\Lambda_1=1)$ means 1 or 0 according as $\Lambda_1=1$ or not. Therefore $E_{\varphi}^{\Lambda_1}(g, s)$ is continued to a meromorphic function on $\boldsymbol{C}$ and it is holomorphic except possible simple poles at $s=-1/2$ and $3/2$.

Now we shall use this formula (3-43) to prove the theorem. Let

$\varphi_\mathfrak{p}$ be the characteristic function of $\kappa_\mathfrak{p}\mathfrak{o}_K(\mathfrak{f}_\Lambda)_\mathfrak{p}\oplus\mathfrak{o}_K(\mathfrak{f}_\Lambda)_\mathfrak{p}$, and

$$\varphi_\infty(x_\infty)=\left(\prod_{j=1}^{n} t_j^{d_j}e^{-2\pi t_j^2}\right)\tilde{\rho}(m_\infty), \tag{3-44}$$

where $x_\infty=t\,(0,1)m_\infty\,(t=(t_1,\cdots,t_n),\ t_j\geqq 0,\ m_\infty\in M_\infty)$. Then $\varphi=\prod_v\varphi_v$ belongs to $\mathscr{S}(W_A,\tilde{\rho})$ and $L_\varphi^{\Lambda_1}(g,s)$ has Euler product expansion, namely, if we put

$$\begin{aligned} l_{\varphi_\mathfrak{p}}^{\Lambda_1}(g_\mathfrak{p},s)&=|\det g_\mathfrak{p}|_\mathfrak{p}^{s+1/2}\int_{K_\mathfrak{p}^\times}\varphi_\mathfrak{p}(t_\mathfrak{p}(0,1)g_\mathfrak{p})\Lambda_{1,\mathfrak{p}}(t_\mathfrak{p})|t_\mathfrak{p}\bar{t}_\mathfrak{p}|_\mathfrak{p}^{s+1/2}d^\times t_\mathfrak{p},\\ l_{\varphi_\infty}^{\Lambda_1}(g_\infty,s)&=\prod_{j=1}^{n}|\det g_{\infty_j}|_{\infty_j}^{s+1/2}\int_{K_\infty^\times}\varphi_\infty(t_\infty(0,1)g_\infty)\Lambda_{1,\infty}(t_\infty)|t_\infty\bar{t}_\infty|_\infty^{s+1/2}d^\times t_\infty\,, \end{aligned}\tag{3-45}$$

then

$$L_\varphi^{\Lambda_1}(g,s)=\left\{\prod_{\mathfrak{p}<\infty} l_{\varphi_\mathfrak{p}}^{\Lambda_1,\mathfrak{p}}(g_\mathfrak{p},s)\right\}\times l_{\varphi_\infty}^{\Lambda_1,\infty}(g_\infty,s). \tag{3-46}$$

Let us calculate local factors (3-45). Because of the usual Iwasawa decomposition, we may assume $g_v=m_v\in M_v$.

LEMMA 3-1. (i) *Let* $c_\mathfrak{p}=0$ (*i.e.*, $\Lambda_\mathfrak{p}$ *is unramified*). *Then*

$$l_{\varphi_\mathfrak{p}}^{\Lambda_1,\mathfrak{p}}(m_\mathfrak{p},s)=\begin{cases}(1-\Lambda_{1,\mathfrak{p}}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{2s+1})^{-1} & \textit{if}\ \left(\frac{K}{\mathfrak{p}}\right)=-1,\\ (1-\Lambda_{1,\mathfrak{p}}(\varpi_{K_\mathfrak{p}})|\pi_\mathfrak{p}|_\mathfrak{p}^{s+1/2})^{-1} & \textit{if}\ \left(\frac{K}{\mathfrak{p}}\right)=0,\\ (1-\Lambda_{1,\mathfrak{p}}(\varpi_{K_\mathfrak{p}})|\pi_\mathfrak{p}|_\mathfrak{p}^{s+1/2})^{-1}(1-\Lambda_{1,\mathfrak{p}}(\pi_\mathfrak{p}\varpi_{K_\mathfrak{p}}^{-1})|\pi_\mathfrak{p}|_\mathfrak{p}^{s+1/2})^{-1} & \\ & \textit{if}\ \left(\frac{K}{\mathfrak{p}}\right)=1,\end{cases}$$

where $\varpi_{K_\mathfrak{p}}$ *denotes a prime element of* $K_\mathfrak{p}$.
(ii) *Let* $c_\mathfrak{p}>0$. *Then*

$$l_{\varphi_\mathfrak{p}}^{\Lambda_1,\mathfrak{p}}(m_\mathfrak{p},s)=\begin{cases}0 & \textit{if}\quad m_\mathfrak{p}\notin B'_\mathfrak{p}M'_\mathfrak{p}\\ \Lambda_{1,\mathfrak{p}}^{-1}(\beta_1) & \textit{if}\quad m_\mathfrak{p}=\begin{pmatrix}\alpha_1 & *\\ 0 & \beta_1\end{pmatrix}m'_\mathfrak{p}\quad \textit{with}\quad m'_\mathfrak{p}\in M'_\mathfrak{p}.\end{cases}$$

PROOF. As (i) is well-known, we shall prove only (ii).

$$\begin{aligned} l_{\varphi_\mathfrak{p}}^{\Lambda_1,\mathfrak{p}}(m_\mathfrak{p},s)&=\int_{K_\mathfrak{p}^\times\cap\mathfrak{o}_{K,\mathfrak{p}}-\mathfrak{o}_{K,\mathfrak{p}}^\times}\varphi_\mathfrak{p}(t(0,1)m_\mathfrak{p})\Lambda_{1,\mathfrak{p}}(t)|N(t)|_\mathfrak{p}^{s+1/2}d^\times t\\ &\quad+\int_{\mathfrak{o}_{K,\mathfrak{p}}^\times}\varphi_\mathfrak{p}(t(0,1)m_\mathfrak{p})\Lambda_{1,\mathfrak{p}}(t)d^\times t. \end{aligned}\tag{3-47}$$

Put $m_\mathfrak{p}=\begin{pmatrix}1 & \\ & \kappa_\mathfrak{p}\end{pmatrix}\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & \\ & \kappa_\mathfrak{p}^{-1}\end{pmatrix}$. Then $(0, 1)m_\mathfrak{p}=(\kappa_\mathfrak{p}\gamma, \delta)$. First assume that $t\in K_\mathfrak{p}^\times\cap\mathfrak{o}_{K,\mathfrak{p}}-\mathfrak{o}_{K,\mathfrak{p}}^\times$ and $t\gamma\in\mathfrak{o}_K(\mathfrak{p}^{e_\mathfrak{p}})_\mathfrak{p}$. Then for any ε of $\mathfrak{o}_K(\mathfrak{p}^{e_\mathfrak{p}-1})_\mathfrak{p}^\times$, $\varepsilon t\gamma$ is in $\mathfrak{o}_K(\mathfrak{p}^{e_\mathfrak{p}})_\mathfrak{p}$. Thus, from the non-triviality of $\Lambda_{1,\mathfrak{p}}$ on $\mathfrak{o}_K(\mathfrak{p}^{e_\mathfrak{p}-1})_\mathfrak{p}^\times$, we know that the first integral in (3-47) vanishes. Secondly assume that $t\in\mathfrak{o}_{K,\mathfrak{p}}^\times$ and $t(0, 1)m_\mathfrak{p}\in\operatorname{supp}\varphi_\mathfrak{p}$. Then as easily seen, $m_\mathfrak{p}$ must be in $B_\mathfrak{p}'M_\mathfrak{p}'$. Hence, $l_{\varphi_\mathfrak{p}}^{\Lambda_1,\mathfrak{p}}(m_\mathfrak{p}, s)=0$ unless $m_\mathfrak{p}\in B_\mathfrak{p}'M_\mathfrak{p}'$. We may assume $m_\mathfrak{p}\in M_\mathfrak{p}'$. If $\gamma, \delta, t\gamma, t\delta\in\mathfrak{o}_K(\mathfrak{p}^{e_\mathfrak{p}})_\mathfrak{p}$ and $t\in\mathfrak{o}_{K,\mathfrak{p}}^\times$, then $t\in\mathfrak{o}_K(\mathfrak{p}^{e_\mathfrak{p}})_\mathfrak{p}^\times$. Thus the second integral in (3-47) is 1 for any $m_\mathfrak{p}\in M_\mathfrak{p}'$. Q.E.D.

Note that

(3-48) $$l_{\varphi_\infty}^{\Lambda_1,\infty}(1, s) = (2\pi)^n\prod_{j=1}^n\frac{\Gamma(s+s_j+(d_j+1)/2)}{(2\pi)^{s+s_j+(d_j+1)/2}}P_{\Lambda_1^{-1}},$$

where s_j is defined in (3-8). Put

(3-49) $$\begin{aligned}B_{F,\xi}^{\Lambda}(\omega, s)=&\prod_{\mathfrak{p}\mid\mathfrak{D}(\xi)}(1+\sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})^{-1}\omega_\mathfrak{p}^{-1}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{-(s+1/2)})\\&\times\prod_{j=1}^n(2\pi)^{1-(s+s_j+(d_j+1)/2)}\Gamma(s+s_j+(d_j+1)/2)\\&\times L_K(\Lambda_1, s+1/2)\times A_{F,\xi}^{\Lambda}(\omega, s).\end{aligned}$$

LEMMA 3-2.

$$B_{F,\xi}^{\Lambda}(\omega, s)=\int_{G_k'k_A^\times\backslash G_A'}\omega(\det g)E_\varphi^{\Lambda_1}(g, s)\tilde{F}(g)d\dot{g}.$$

PROOF. For each prime $\mathfrak{p}$ dividing $\mathfrak{D}(\xi)$, we put

$$\tau_\mathfrak{p}=\begin{pmatrix}1 & \\ & \kappa_\mathfrak{p}\end{pmatrix}\begin{pmatrix}0 & \pi_\mathfrak{p}^{-1} \\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & \\ & \kappa_\mathfrak{p}^{-1}\end{pmatrix}.$$

Then by (3-45), we have for $m_\mathfrak{p}\in M_\mathfrak{p}'=M_\mathfrak{p}$,

(3-50) $$l_{\varphi_\mathfrak{p}}^{\Lambda_1,\mathfrak{p}}(m_\mathfrak{p}\tau_\mathfrak{p}, s)=\begin{cases}|\pi_\mathfrak{p}|_\mathfrak{p}^{-(s+1/2)}l_{\varphi_\mathfrak{p}}^{\Lambda_1,\mathfrak{p}}(m_\mathfrak{p}, s) & \text{if } m_\mathfrak{p}\in M_\mathfrak{p}''\\ \Lambda_{1,\mathfrak{p}}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{s+1/2}l_{\varphi_\mathfrak{p}}^{\Lambda_1,\mathfrak{p}}(m_\mathfrak{p}, s) & \text{if } m_\mathfrak{p}\notin M_\mathfrak{p}''.\end{cases}$$

We put

$$L'(g, s)=\sum_{P\subset\tilde{\mathfrak{D}}(\xi)}(-1)^{\sharp P}\prod_{\mathfrak{p}\in P}\Lambda_{1,\mathfrak{p}}^{-1}(\pi_\mathfrak{p})|\pi_\mathfrak{p}|_\mathfrak{p}^{-(s+1/2)}L_\varphi^{\Lambda_1}(g\prod_{\mathfrak{p}\in P}\tau_\mathfrak{p}, s),$$

where $\tilde{\mathfrak{D}}(\xi)$ denotes the set of all prime ideals dividing $\mathfrak{D}(\xi)$, and P runs through all subsets of $\tilde{\mathfrak{D}}(\xi)$. By (3-50) and (3-36), for $g=\begin{pmatrix}\alpha & * \\ 0 & \beta\end{pmatrix}m$

$(m \in M')$, we have

$$\begin{aligned} L'(g, s) = & \prod_{\mathfrak{p} \mid \mathfrak{D}(\xi)} (1 - \Lambda_1^{-1}(\pi_\mathfrak{p}) |\pi_\mathfrak{p}|_\mathfrak{p}^{-2s-1}) \Lambda_1^{-1}(\beta) |\alpha \bar{\beta}^{-1}|_A^{s+1/2} \\ & \times \prod_{j=1}^{n} (2\pi)^{1-(s+s_j+(d_j+1)/2)} \Gamma(s+s_j+(d_j+1)/2) \\ & \times L_K(\Lambda_1, s+1/2) P_{\Lambda_1^{-1}} \tilde{\rho}(m_\infty) \quad \text{if} \quad m \in M'', \\ = & \, 0 \qquad \text{otherwise.} \end{aligned}$$

Using the integral representation (3-33) of $A_{F,\xi}^{\Lambda}(\omega, s)$, we obtain

$$\begin{aligned} B_{F,\xi}^{\Lambda}(\omega, s) = & \prod_{\mathfrak{p} \mid \mathfrak{D}(\xi)} (1 + \sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})^{-1} \omega_\mathfrak{p}^{-1}(\pi_\mathfrak{p}) |\pi_\mathfrak{p}|_\mathfrak{p}^{-(s+1/2)}) \\ & \times (1 - \sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})^{-2} \omega_\mathfrak{p}^{-2}(\pi_\mathfrak{p}) |\pi_\mathfrak{p}|_\mathfrak{p}^{-2s-1})^{-1} \\ & \times \int_{G_k' k_A^\times \backslash G_A'} \omega(\det g) \sum_{\gamma \in B_k' \backslash G_k'} L'(\gamma g, s) \tilde{F}(g) d\dot{g} \\ = & \prod_{\mathfrak{p} \mid \mathfrak{D}(\xi)} (1 - \sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})^{-1} \omega_\mathfrak{p}^{-1}(\pi_\mathfrak{p}) |\pi_\mathfrak{p}|_\mathfrak{p}^{-(s+1/2)})^{-1} \\ & \times \sum_{P \subset \tilde{\mathfrak{D}}(\xi)} (-1)^{\# P} \prod_{\mathfrak{p} \in P} \sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})^{-2} \omega_\mathfrak{p}^{-2}(\pi_\mathfrak{p}) |\pi_\mathfrak{p}|_\mathfrak{p}^{-(s+1/2)} \\ & \times \int_{G_k' k_A^\times \backslash G_A'} \omega(\det g) \sum_{\gamma \in B_k' \backslash G_k'} L_\varphi^{\Lambda_1}(\gamma g \prod_{\mathfrak{p} \in P} \tau_\mathfrak{p}, s) \tilde{F}(g) d\dot{g}. \end{aligned}$$

Here we have used the fact that $P_{\Lambda_1^{-1}} = P_\Lambda$ and if $\mathfrak{p} \mid \mathfrak{D}(\xi)$, $\sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})^2 = \Lambda_\mathfrak{p}(\pi_\mathfrak{p})$. Since $R_\mathfrak{p}^{-1} \psi(\tau_\mathfrak{p}^{-1}) R_\mathfrak{p} \in \operatorname{supp} c_\mathfrak{p}^{(0)}$, transforming g to $g \prod_{\mathfrak{p} \in P} \tau_\mathfrak{p}^{-1}$ in the last integral, it equals

$$\int_{G_k' k_A^\times \backslash G_A'} \prod_{\mathfrak{p} \in P} \omega_\mathfrak{p}(\pi_\mathfrak{p}) \sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)}) \omega(\det g) E_\varphi^{\Lambda_1}(g, s) \tilde{F}(g) d\dot{g}.$$

Therefore we have

$$\begin{aligned} B_{F,\xi}^{\Lambda}(\omega, s) = & \prod_{\mathfrak{p} \mid \mathfrak{D}(\xi)} (1 - \sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})^{-1} \omega_\mathfrak{p}^{-1}(\pi_\mathfrak{p}) |\pi_\mathfrak{p}|_\mathfrak{p}^{-(s+1/2)})^{-1} \\ & \times \sum_{P \subset \tilde{\mathfrak{D}}(\xi)} (-1)^{\# P} \prod_{\mathfrak{p} \in P} \sigma_{F,\mathfrak{p}}(c_\mathfrak{p}^{(0)})^{-1} \omega_\mathfrak{p}^{-1}(\pi_\mathfrak{p}) |\pi_\mathfrak{p}|_\mathfrak{p}^{-(s+1/2)} \\ & \times \int_{G_k' k_A^\times \backslash G_A'} \omega(\det g) E_\varphi^{\Lambda_1}(g, s) \tilde{F}(g) d\dot{g}. \qquad \text{Q.E.D.} \end{aligned}$$

Using the formula (3-43), we have

(3-51)
$$\begin{aligned} & \int_{G_k' k_A^\times \backslash G_A'} \omega(\det g) E_\varphi^{\Lambda_1}(g, s) \tilde{F}(g) d\dot{g} \\ = & \int_{G_k' k_A^\times \backslash G_A'} \omega(\det g) E_\varphi^{\Lambda_1}(g, s)^+ \tilde{F}(g) dg + \int_{G_k' k_A^\times \backslash G_A'} (\omega \Lambda_1^{-1})(\det g) \end{aligned}$$

$$\times E_{\varphi^*}^{A_1^{-1}}(g,\ 1-s)^+\tilde{F}(g)d\dot{g}-\delta(A_1=1)c_0\Big\{\varphi^*(0)/(-s+3/2)$$

$$\times\int\omega(\det g)\tilde{F}(g)d\dot{g}+\varphi(0)/(s+1/2)\int\omega(\det g)\tilde{F}(g)\,d\dot{g}\Big\}.$$

Note that, s being in a fixed compact subset of $\boldsymbol{C}$, $|E_{\varphi}^{A_1}(g,\ s)^+\tilde{F}(g)|$ and $|E_{\varphi^*}^{A_1^{-1}}(g,\ 1-s)^+\tilde{F}(g)|$ are both bounded on G'_A. As $F(g)$ is also bounded, from the finiteness of the volume of $G'_k k'_A\backslash G'_A$, the integral representation (3-51) gives a meromorphic continuation of $B_{F,\xi}^{A}(\omega,\ s)$ to the whole complex plane. This function is holomorphic, except possible simple poles at $s=3/2$ and $-1/2$. If A_1 is not trivial or V is not one-dimensional, then $B_{F,\xi}^{A}(\omega,\ s)$ is an entire function. By Theorem 3-1 and (3-49), we have

$$B_{F,\xi}^{A}(\omega,\ s)=c_{F,\xi}^{A}(\omega)c_1c_2^s\times\prod_{\substack{\mathfrak{p}|\mathfrak{D}_0\\ \mathfrak{p}|\mathfrak{d}(K/k)}}(1+\sigma_{F,\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})\omega_{\mathfrak{p}}(\pi_{\mathfrak{p}})|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{s-1/2})\times\zeta_F(\omega,\ s)\times\varphi_{F,\xi}^{A}(R_\infty g_0). \tag{3-52}$$

Here

$$c_{F,\xi}^{A}(\omega)=\prod_{j=1}^{n}\alpha_j^{-s_j}\prod_{\mathfrak{p}|\mathfrak{D}(\xi)}\sigma_{F,\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})^{-1}\omega_{\mathfrak{p}}^{-1}(\pi_{\mathfrak{p}})\times\omega(t_0),$$

$$c_1=(2\pi)^{2n-\sum_{j=1}^{n}(l_j+d_j)}|t_0|_A^{-3/2}\prod_{\mathfrak{p}|\mathfrak{D}(\xi)}|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{-1/2}\prod_{j=1}^{n}\alpha_j^{(3-d_j)/2-l_j}e^{2\pi\alpha_j},$$

$$c_2=|t_0|_A d(k)^{-2}N(\mathfrak{D})^{-1/2}\prod_{j=1}^{n}\alpha_j^{-1}\prod_{\mathfrak{p}|\mathfrak{D}(\xi)}|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{-1}.$$

Therefore $\prod_{\mathfrak{p}|\mathfrak{D}_0\mathfrak{D}(\xi)^{-1}}(1+\sigma_{F,\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})\omega_{\mathfrak{p}}(\pi_{\mathfrak{p}})|\pi_{\mathfrak{p}}|_{\mathfrak{p}}^{s-1/2})\times\zeta_F(\omega,\ s)$ is meromorphically continued and the first half of Theorem 3-2 has been proved.

3-4. In this subsection we calculate the Fourier transform φ^* of φ, and prove the functional equation of $\zeta_F(\omega,\ s)$. For each prime $\mathfrak{p}$, put

$$\mathfrak{o}_K(\mathfrak{f}_A)_{\mathfrak{p}}^{\perp}=\{x\in K_{\mathfrak{p}}|\ \mathrm{Tr}_{K/k}(vx)\in\mathfrak{d}_{k,\mathfrak{p}}^{-1}\quad\text{for all}\quad v\in\mathfrak{o}_K(\mathfrak{f}_A)_{\mathfrak{p}}\}, \tag{3-53}$$

and

$$V_{\mathfrak{p}}=(2\xi)^{-1}\pi_{\mathfrak{p}}^{\nu_{\mathfrak{p}}+\mu_{\mathfrak{p}}-c_{\mathfrak{p}}-\delta_{\mathfrak{p}}}, \tag{3-54}$$

where $\nu_{\mathfrak{p}}$ and $\mu_{\mathfrak{p}}$ are defined in (2-3), $\delta_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{d}_k$ and $c_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{f}_A$. Then it is easily checked from Lemma 2-7 that

$$\mathfrak{o}_K(\mathfrak{f}_A)_{\mathfrak{p}}^{\perp} = V_{\mathfrak{p}}\mathfrak{o}_K(\mathfrak{f}_A)_{\mathfrak{p}}. \tag{3-55}$$

LEMMA 3-3.

$$\varphi^*(x) = (-1)^{(d_1+\cdots+d_n)/2}|N(\gamma_f)|_A\tilde{\rho}(j)\varphi(\gamma_f x),$$

where $\gamma_f = \prod_{\mathfrak{p}<\infty}\gamma_{\mathfrak{p}} \in K_{A,f}^{\times}$, $\gamma_{\mathfrak{p}} = \kappa_{\mathfrak{p}}V_{\mathfrak{p}}^{-1}$, $\kappa_{\mathfrak{p}}$ *is defined in* (3-20) *and* (3-21), *and* $j = \left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \cdots, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) \in M_\infty$.

PROOF. We must check only

$$\varphi_\infty^*(x) = (-1)^{(d_1+\cdots+d_n)/2}\tilde{\rho}(j)\varphi_\infty(x) \quad (x \in W_\infty).$$

Using the fact that $\tilde{\rho}$ is equivalent to $\otimes\sigma_{d_j}$, we can verify it quite elementarily. Note that all d_j are even. Q.E.D.

From this lemma, we get

$$E_{\varphi^*}^{\Lambda_1}(g, s) = (-1)^{(\Sigma d_j)/2}|N(\gamma_f)|_A^{-s+1/2}\Lambda_1^{-1}(\gamma_f)\tilde{\rho}(j)E_{\varphi}^{\Lambda_1}(g, s).$$

Therefore

$$\begin{aligned} B_{F,\xi}^{\Lambda}(\omega, s) = (-1)^{(\Sigma d_j)/2}&|N(\gamma_f)|_A^{s-1/2}\Lambda_1(\gamma_f)\tilde{\rho}(j)\\ &\times\Big\{\int_{G_k' k_A^{\times}\backslash G_A'}\omega(\det g)E_{\varphi^*}^{\Lambda_1}(g, s)^+\tilde{F}(g)d\dot{g}\\ &+\int_{G_k' k_A^{\times}\backslash G_A'}(\omega\Lambda_1^{-1})(\det g)E_{\varphi}^{\Lambda_1^{-1}}(g, 1-s)^+\tilde{F}(g)d\dot{g}\\ &-\delta(\Lambda_1=1)c_0\Big(\varphi(0)/(-s+3/2)\int\omega(\det g)\tilde{F}(g)d\dot{g}\\ &+\varphi^*(0)/(s+1/2)\int\omega(\det g)\tilde{F}(g)d\dot{g}\Big)\Big\}. \end{aligned} \tag{3-56}$$

Since $\tilde{F}'(g) = \Lambda^{-1}(\mu(R)\det g)\tilde{F}(g)$, we obtain

$$\begin{aligned} B_{F,\xi}^{\Lambda}(\omega, s) = (-1)^{(\Sigma d_j)/2}&\Lambda_1(\gamma_f)|N(\gamma_f)|_A^{s-1/2}\tilde{\rho}(j)\lambda(\mu(R))\\ &\times\Big\{\int(\omega^{-1}\Lambda_1)(\det g)E_{\varphi^*}^{\Lambda_1}(g, s)^+\tilde{F}'(g)d\dot{g}\\ &+\int\omega^{-1}(\det g)E_{\varphi}^{\Lambda_1^{-1}}(g, 1-s)^+\tilde{F}'(g)d\dot{g}\\ &-\delta(\Lambda_1^{-1}=1)c_0\Big(\varphi(0)/(-s+3/2)\int\omega^{-1}(\det g)\tilde{F}'(g)d\dot{g}\\ &+\varphi^*(0)/(s+1/2)\int\omega^{-1}(\det g)\tilde{F}'(g)d\dot{g}\Big)\Big\}. \end{aligned} \tag{3-57}$$

Comparing this with (3-51), we get the functional equation

$$B^{\Lambda}_{F,\xi}(\omega, s)=(-1)^{(\Sigma d_j)/2}\Lambda_1(\gamma_f)|N(\gamma_f)|_A^{s-1/2}\tilde{\rho}(j) \times\lambda(\mu(R))B^{\Lambda^{-1}}_{F',\xi}(\omega^{-1}, 1-s). \tag{3-58}$$

Finally we shall rewrite this equation in terms of $\zeta_F(\omega, s)$.

LEMMA 3-4.

$$\tilde{\rho}(j)\varphi^{\Lambda^{-1}}_{F',\xi}(R_\infty g_0)=\lambda^{-1}(\mu(R_\infty g_0))\Lambda_f(\kappa_f)\Lambda_\infty(\xi_\infty)(-1)^{\Sigma(l_j+d_j/2)} \times\prod_{\mathfrak{p}|\mathfrak{D}_1\mathfrak{D}(\xi)}\sigma_{F,\mathfrak{p}}(c^{(0)}_{\mathfrak{p}})\varphi^{\Lambda}_{F,\xi}(R_\infty g_0).$$

PROOF. Put $S=R_\infty g_0$.

$$\begin{aligned}\varphi^{\Lambda^{-1}}_{F',\xi}(S)&=\int_{K^\times k_A^\times\backslash K_A^\times}F'_\chi(\tilde{u}S;\ \xi)\Lambda(u)d^\times u\\&=\lambda^{-1}(\mu(S))\int_{K^\times k_A^\times\backslash K_A^\times}F_\chi(\tilde{u}S;\ \xi)\Lambda^{-1}(\bar{u})d^\times u.\end{aligned}$$

By transforming $u\mapsto\bar{u}=\eta^{-1}u\eta$,

$$\begin{aligned}\varphi^{\Lambda^{-1}}_{F',\xi}(S)&=\lambda^{-1}(\mu(S))\int_{K^\times k_A^\times\backslash K_A^\times}F_\chi\left(\widetilde{u\eta}\begin{pmatrix}-1&\\&1\end{pmatrix}S;\ \xi\right)\Lambda^{-1}(u)d^\times u\\&=\lambda^{-1}(\mu(S))\Lambda(\kappa)\int F_\chi(\tilde{u}SW;\ \xi)\Lambda^{-1}(u)d^\times u,\end{aligned}\tag{3-59}$$

where $W=S^{-1}\widetilde{\kappa^{-1}\eta}\begin{pmatrix}-1&\\&1\end{pmatrix}S$ $(\kappa=\prod_v\kappa_v)$. If $\mathfrak{p}$ does not divide $\mathfrak{D}$, $W_\mathfrak{p}$ is in $U_\mathfrak{p}$, and if $\mathfrak{p}$ divides $\mathfrak{D}$, the $\mathfrak{p}$-component of $\kappa^{-1}\eta$ belongs to $\prod_\mathfrak{p}\mathfrak{O}_\mathfrak{p}^\times$ or $\mathfrak{O}_\mathfrak{p}^\times$ according as $\mathfrak{p}|\mathfrak{D}_1\mathfrak{D}(\xi)$ or $\mathfrak{p}|\mathfrak{D}_0\mathfrak{D}(\xi)^{-1}$. So we have

$$F_\chi(\tilde{u}SW;\ \xi)=\prod_{\mathfrak{p}|\mathfrak{D}_1\mathfrak{D}(\xi)}\sigma_{F,\mathfrak{p}}(c^{(0)}_\mathfrak{p})F_\chi(\tilde{u}SW_\infty;\ \xi).$$

By simple computation we know that

$$W_\infty\langle Z_0\rangle=Z_0,$$

and

$$\tilde{\rho}(j)=(-1)^{\Sigma l_j+d_j/2}\rho_{l,d}(J(\mu(W_\infty)^{-1/2}W_\infty,\ Z_0))^{-1}.$$

Thus our assertion is verified. Q.E.D.

Recall that $\varphi^{\Lambda}_{F,\xi}(R_\infty g_0)\neq 0$. By (3-52) and (3-58), we have

$$\zeta_F(\omega, s)=(-1)^{(\Sigma l_j)}\prod_{\mathfrak{p}|\mathfrak{D}}\sigma_{F,\mathfrak{p}}(c_{\mathfrak{p}}^{(0)})\times\varepsilon \times(c_2^2|N(\gamma_f)|_A^{-1}\prod_{\mathfrak{p}|\mathfrak{D}_0\mathfrak{D}(\xi)^{-1}}|\pi_{\mathfrak{p}}|_{\mathfrak{p}})^{-s+1/2} \times\zeta_{F'}(\omega^{-1}, 1-s), \tag{3-60}$$

where $\varepsilon=\prod_{\mathfrak{p}|\mathfrak{D}_0\mathfrak{D}(\xi)^{-1}}(\lambda\omega)(\pi_{\mathfrak{p}}^{-1})\prod_{\mathfrak{p}|\mathfrak{D}(\xi)}(\lambda\omega^2)(\pi_{\mathfrak{p}})\times(\Lambda\omega)(\kappa_f\bar{\kappa}_f)\Lambda_f(\bar{V}_f^{-1})\omega(V_f^{-1}\bar{V}_f^{-1})$ $\times\Lambda_\infty(\xi_\infty)\omega_\infty(-(2\xi)^2\eta^2)(\lambda\omega^2)(t_0^{-1})$. We can easily check that $c_2^2|N(\gamma_f)|_A^{-1}$ $\times\prod_{\mathfrak{p}|\mathfrak{D}_0\mathfrak{D}(\xi)^{-1}}|\pi_{\mathfrak{p}}|_{\mathfrak{p}}=1$ and $\varepsilon=(\lambda\omega^2)(\mathfrak{d}_k^2)\omega(\mathfrak{D})$. Therefore Theorem 3-2 is proved completely.

§ 4. Examples by Oda lifting

4-1. In this section we give some examples of cusp forms on a quaternion unitary group of degree 2 over $\boldsymbol{Q}$ by using Oda's lifting ([14]). First, we describe the action of Hecke operators on the space of cusp forms of half-integral weight at various cusps. We use the same notations as in [20].

Let N be an odd square free integer and κ be a positive odd integer, and we put $M=4N$. For a positive divisor Δ of N, we define a Dirichlet character (modulo M) χ_Δ by

$$\chi_\Delta(m)=\left(\frac{\Delta}{m}\right). \tag{4-1}$$

For any $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(M)$, we put $\chi_\Delta(\gamma)=\chi_\Delta(d)$. We denote by $S_\kappa(M, \chi_\Delta)$ the space of holomorphic cusp forms of weight $\kappa/2$, with respect to $\Gamma_0(M)$ and with character χ_Δ.

By the assumption on M, the equivalence classes of cusps under the action of $\Gamma_0(M)$ are bijectively corresponding to positive divisors of M. For each divisor M_1 of M, put $M_2=M/M_1$, $d_{M_1}=(M_1, M_2)$ (the greatest common divisor of M_1 and M_2), $w_{M_1}=M_2/d_{M_1}$, and $N_i=M_i/(4, M_i)$ $(i=1, 2)$. We take a pair of integers (α, β) so that

$$\alpha M_1+\beta w_{M_1}=1, \tag{4-2}$$

and put

$$A_{M_1}=\begin{pmatrix} w_{M_1} & -\alpha \\ M_1 & \beta \end{pmatrix}, \quad \text{and} \quad A_{M_1}^*=(A_{M_1}, \sqrt{M_1z+\beta}). \tag{4-3}$$

Let f be an element of $S_\kappa(M, \chi_\Delta)$. The Fourier expansion of f at the cusp corresponding to M_1 is given as follows:

$$f\|[A_{M_1}^{*-1}]_\kappa(z) = \sum_{\substack{u>0 \\ (u/d_{M_1})-r_{M_1}\in Z}} a_f^{[M_1]}(u)e\left[\frac{u}{M_2}z\right], \tag{4-4}$$

where

$$r_{M_1} = \begin{cases} 0 & \text{if} \quad d_{M_1}=1, \\ \dfrac{1}{4} & \text{if} \quad d_{M_1}=2 \quad \text{and} \quad \Delta N_2 \equiv \kappa \pmod{4}, \\ \dfrac{3}{4} & \text{if} \quad d_{M_1}=2 \quad \text{and} \quad \Delta N_2 \equiv -\kappa \pmod{4}. \end{cases}$$

Note that the Fourier coefficients $a_f^{[M_1]}(u)$ are independent of the choice of α and β. For a prime p, the Hecke operator $T_{\kappa,\chi_\Delta}^M(p^2)$ acting on $S_\kappa(M, \chi_\Delta)$ is defined in [20]. For an odd integer m, we put $\varepsilon_m=1$ [resp. $\sqrt{-1}$] if $m\equiv 1 \pmod{4}$ [resp. $m\equiv 3 \pmod{4}$].

PROPOSITION 4-1. *Let f be an element of $S_\kappa(M, \chi_\Delta)$ and p be a prime. Put $g=f|T_{\kappa,\chi_\Delta}^M(p^2)$. The Fourier coefficients of g at the cusp corresponding to M_1 is given as follows.*

(i) *When p does not divide M,*

$$a_g^{[M_1]}(u) = a_f^{[M_1]}(p^2u) + p^{\kappa-2}a_f^{[M_1]}(u/p^2) + p^{(\kappa-3)/2}\chi_\Delta(p)(\varepsilon_p)^{\kappa-1}\left(\frac{M_2u}{p}\right)a_f^{[M_1]}(u)\,.$$

(ii) *Assume that p divides N. When $p|N_1$,*

$$a_g^{[M_1]}(u) = a_f^{[M_1]}(p^2u).$$

When $p\nmid N_1$,

$$\begin{aligned} a_g^{[M_1]}(u) &= \varepsilon_p^\kappa\left(\frac{M_1N_2\Delta/p}{p}\right)\frac{p\delta(p|2u)-1}{p}a_f^{[pM_1]}(pu) \\ &\quad + p^{(\kappa-3)/2}\varepsilon_p\left(\frac{M_1d_{M_1}u/p}{p}\right)a_f^{[pM_1]}(u/p) \\ &\quad + p^{\kappa-2}a_f^{[M_1]}(u/p^2) \qquad \text{if} \quad p\nmid\Delta, \\ &= \sqrt{p}^{\,-1}\varepsilon_p^{\kappa-1}\left(\frac{\Delta N_2d_{M_1}u/p^2}{p}\right)a_f^{[pM_1]}(pu) \end{aligned}$$

$$+p^{-2+\kappa/2}\{p\delta(p^2|2u)-1\}a_f^{[pM_1]}(u/p)$$
$$+p^{\kappa-2}a_f^{[M_1]}(u/p^2) \qquad if \quad p|\Delta,$$

where $\delta((*))$ *means* 1 *or* 0 *according as the condition* $(*)$ *is satisfied or not.*

(iii) *Assume that* $p=2$. *Then for any positive divisor* N_1 *of* N,

$$a_g^{[4N_1]}(u)=a_f^{[4N_1]}(4u),$$
$$a_g^{[2N_1]}(u)=\left(\frac{2}{\Delta_2 N_2}\right)a_f^{[4N_1]}(2u),$$
$$a_g^{[N_1]}(u)=\frac{1}{4}\alpha_{N_1}(u)a_f^{[4N_1]}(u)+2^{\kappa-2}a_f^{[N_1]}(u/4)$$
$$+2^{(\kappa-4)/2}e\left[\frac{uN_1}{8}\right]\left(\frac{2}{\Delta_1 N_1}\right)a_f^{[2N_1]}(u/2),$$

where $\Delta_i=(\Delta, N_i)$ $(i=1, 2)$ *and*

$$\alpha_{N_1}(u)=e\left[\frac{uN_1}{4}\right]+\left(\sqrt{-1}\right)^{\kappa}\left(\frac{-1}{\Delta N}\right)e\left[-\frac{uN_1}{4}\right].$$

PROOF. We shall prove only (ii) in the case $p\nmid N_1$ (cf. [5; Lemma 2]). For j and $l\in \boldsymbol{Z}$, put

$$\xi_j^*=\left(\begin{pmatrix}1 & j\\ 0 & p^2\end{pmatrix}, \sqrt{p}\right),\ \eta_l^*=\left(\begin{pmatrix}p & l\\ 0 & p\end{pmatrix}, 1\right) \quad\text{and}\quad \sigma^*=\left(\begin{pmatrix}p^2 & 0\\ 0 & 1\end{pmatrix}, \sqrt{p^{-1}}\right).$$

Fix α and β satisfying (4-2). If $\beta-M_1j\in \boldsymbol{Z}_p^\times$, there exists a $u\in \boldsymbol{Z}$ such that $u\equiv 0 \pmod{4M_2p^{-1}}$ and $(\beta-M_1j)u\equiv\alpha+w_{M_1}j \pmod{p^2}$. Thus

$$\xi_j^* A_{M_1}^{*-1}=\gamma_j^* A_{pM_1}^{*-1}\xi_u^*\left(1,\ \varepsilon_p^{-1}\left(\frac{M_1N_2/p}{p}\right)\right), \tag{4-5}$$

where γ_j is a suitable element of $\Gamma_0(M)$ and it satisfies

$$\chi_\Delta(\gamma_j)=\begin{cases}\left(\dfrac{\Delta}{p}\right) & \text{if} \quad p\nmid\Delta,\\[2ex] \left(\dfrac{-\Delta N_1u/(4M_2)}{p}\right) & \text{if} \quad p|\Delta.\end{cases}$$

If $\beta-M_1j=p\beta_j$ $(\beta_j\in \boldsymbol{Z}_p^\times)$, there exists a $u\in \boldsymbol{Z}$ such that $u\equiv 0 \pmod{4M_2/p}$ and $u\beta_j\equiv\alpha+w_{M_1}j \pmod{p}$. Thus

(4-6) $$\xi_j^* A_{M_1}^{*-1} = \gamma_j^* A_{pM_1}^{*-1} \eta_u^* \left(1, \left(\frac{pN_1u/(4M_2)}{p}\right)\right)$$

where γ_j is a suitable element of $\Gamma_0(M)$ and it satisfies

$$\chi_\Delta(\gamma_j) = \begin{cases} 1 & \text{if } p \nmid \Delta, \\ \left(\frac{pN_1u/(4M_2)}{p}\right) & \text{if } p \mid \Delta. \end{cases}$$

If $\beta \equiv M_1 j \pmod{p^2}$, then

(4-7) $$\xi_j^* A_{M_1}^{*-1} = \gamma_j^* A_{M_1}^{*-1} \sigma^*,$$

where γ_j is a suitable element of $\Gamma_0(M)$ satisfying $\chi_\Delta(\gamma_j) = 1$. Using (4-5)-(4-7), we obtain the required result easily. The remaining cases are treated similarly. Q.E.D.

4-2. Let B be an indefinite quaternion algebra over $\boldsymbol{Q}$, D its discriminant, and $\mathfrak{O}$ a maximal order of B. Let G and G^1 have the same meaning as in §1. For any positive divisor D_1 of D, we take a unique two-sided $\mathfrak{O}$-ideal $\mathfrak{A}$ such that $N_{B/Q}(\mathfrak{A}) = (D_1)$. Let $\Gamma_{\mathfrak{A}}$ be the intersection of G_Q^1 and $\left\{\begin{pmatrix} \mathfrak{O} & \mathfrak{A} \\ \mathfrak{A}^{-1} & \mathfrak{O} \end{pmatrix}\right\}$. We denote by $\mathfrak{S}_l(\Gamma_{\mathfrak{A}})$, for a positive integer l, the space of holomorphic function f on $\mathfrak{H}_+$ such that

(4-8)
(i) $f(\gamma\langle Z\rangle) = N(J(\gamma, Z))^l f(Z)$ for all $\gamma \in \Gamma_{\mathfrak{A}}$,
(ii) $f(g\langle Z_0\rangle) N(J(g, Z_0))^{-l}$ is bounded on G_∞^1.

Here Z_0 and $J(g, Z)$ have the same meanings as in §1. Since $G_A = \boldsymbol{Q}_A^\times G_Q G_\infty^1 U_f$, this space is identified with $\mathfrak{S}(\rho_{l,0}, 1; U_f)$ through

(4-9) $$f \longmapsto F_f\colon F_f(\gamma\zeta u) = N(J(\zeta, Z_0))^{-l} f(\zeta\langle Z_0\rangle),$$

for $\forall\gamma \in G_Q$, $\forall\zeta \in G_\infty^1$ and $\forall u \in U_f$.

For each positive integer m we define Hecke operator $T_l(m)$ acting on $\mathfrak{S}_l(\Gamma_{\mathfrak{A}})$ by

(4-10) $$(T_l(m)f)(Z) = m^{2l-3} \sum_{\sigma \in \Gamma_{\mathfrak{A}} \backslash S_m} N(J(\sigma, Z))^{-l} f(\sigma\langle Z\rangle),$$

where $S_m = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_Q \,\middle|\, \alpha, \delta \in \mathfrak{O}, \beta \in \mathfrak{A}, \gamma \in \mathfrak{A}^{-1}, \mu(g) = m \right\}$.

Put

(4-11) $$M = 4D/(2, D_0) = 4N, \quad \Delta = D_0/(2, D_0), \quad D = D_0 D_1.$$

We assume that D_1 is odd (so, N is also odd). For any positive divisor N_2 of N, put $N_1N_2=N$ and

$$\begin{aligned} \psi(N_2)&=\mathrm{sgn}(\tau)(-1)^{n(N_1)}\sqrt{\Delta_2N_2}\,\varepsilon_{\Delta_2N_2}^{-1}\left(\frac{\Delta_1N_1}{\Delta_2N_2}\right), \\ \psi(2N_2)&=|\tau|(-1)^{n(N_1)+1}\sqrt{2\Delta_2N_2}\,\varepsilon_{\Delta_2N_2}\left(\frac{\Delta_1N_1}{\Delta_2N_2}\right)\left(\frac{-2}{\Delta_1}\right), \\ \psi(4N_2)&=|\tau|(-1)^{n(N_1)}\sqrt{2\Delta_2N_2}\,e\left[-\frac{1}{8}\right]\varepsilon_{\Delta_1N_1}\left(\frac{\Delta_2N_2}{\Delta_1N_1}\right), \end{aligned} \tag{4-12}$$

where τ is 1 [resp. -2] if D_0 is odd [resp. even], $\Delta=\Delta_1\Delta_2$ ($\Delta_1|N_1$, $\Delta_2|N_2$), and $n(N_1)$ denotes the number of primes which divide N_1. For a rational number u such that $ud_{M_1}^{-1}M_2^{-1}-r_{M_1}=t\in \boldsymbol{Z}$, we define $\varepsilon_{M_1}(u/M_2)$ by

$$\varepsilon_{M_1}\left(\frac{u}{M_2}\right)=\begin{cases}1 & \text{if } d_{M_1}=1, \\ (-1)^t & \text{if } d_{M_1}=2.\end{cases} \tag{4-13}$$

PROPOSITION 4-2. *Assume that* D_1 *is odd. Let* l *be an even integer* ($\geqq 6$), f *be an element of* $S_{2l-1}(M,\chi_\Delta)$ *with Fourier expansion in* (4-4). *For each* $\xi\in(\mathfrak{A}^-)^*$, *put*

$$C_f(\xi)=\sum_{\substack{r>0\\ r|\xi}} r^{l-1}\sum_{\substack{M=M_1M_2\\ \frac{m}{d_{M_1}M_2r^2}-r_{M_1}\in \boldsymbol{Z}}} \frac{\overline{\psi(M_2)}\varepsilon_{M_1}\left(\dfrac{m/r^2}{M_2}\right)}{d_{M_1}} M_2^{l-3/2}a_f^{[M_1]}(m/(r^2M_2)),$$

where $m=-\Delta(2D_1\xi)^2$ *and* ψ *is defined in* (4-12). *Then*

$$J(f)(Z)=\sum_{\substack{\xi\in(\mathfrak{A}^-)^*\\ -\sqrt{-1}\xi\in\mathfrak{H}_+}} C_f(\xi)e[\mathrm{Tr}(\xi Z)]$$

belongs to $\mathfrak{S}_l(\Gamma_{\mathfrak{A}})$.

Note that G^1 is isogenous to $SO(2,3)$. In [14], T. Oda has constructed holomorphic cusp forms on $SO(2,q)$, and when q is even, their Fourier coefficients are calculated by using so-called Zagier identity ([14; Corollary of Theorem 5]). When q is odd, similar formula holds (this is mentioned in [15; p. 336]). Though the value $\psi(M_2)$ is not calculated explicitly in [14], we can evaluate it in our case, and our assertion follows.

PROPOSITION 4-3. *Assume that* $T^M_{2l-1,\chi_\Delta}(p^2)f=\omega_pf$ *and put* $F=J(f)$.

(i) *If p does not divide M, then*

$$T_l(p)F=(\omega_p+p^{l-1}+p^{l-2})F,$$
$$(T_l(p)^2-T_l(p^2))F=\{p^{l-2}(p+1)\omega_p+p^{2l-4}+2p^{2l-3}\}F.$$

(ii) *If p is an odd prime dividing D, or if $p=2$ and D_0 is even, then*

$$T_l(p)F=\{\omega_p+p^{l-3+A_p}+p^{2l-3}\omega_p^{-1}\}F,$$

where A_p means 2 or 1 according as $p|D_0$ or $p|D_1$.

From Proposition 4-1, we know that $\omega_p\neq 0$ if $p|M$. We can prove this proposition by direct calculation using Proposition 4-1, Proposition 4-2 and the definition of ψ (cf. formulae (13)-(16) in [3]). So we omit the proof.

REMARK 4-1. Assume that p is an odd prime dividing D_1 and f belongs to $S_{2l-1}(M/p, \chi_\Delta)$, which is a subspace of $S_{2l-1}(M, \chi_\Delta)$. If $T^{M/p}_{2l-1,\chi_\Delta}(p^2)f=\omega'_p f$, then

$$T_l(p)J(f)=(\omega'_p+p^{l-2})J(f).$$

REMARK 4-2. Assume that $p|\Delta$ and f belong to $S_{2l-1}(M/p, \chi_{\Delta/p})$. Then $g(z)=f(pz)$ is an element of $S_{2l-1}(M, \chi_\Delta)$ (see [18]). We can easily check that $J(g)=0$.

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