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Links in some simple flows

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In recent years, several persons are interested in links or knots of closed orbits of flows. J. Birman and R. Williams [BW] studied knots and links of closed orbits of Smale flows on S^3, and J. Franks [F2] calculated the Alexander polynomials of links of closed orbits of non-singular Smale flows on S^3. The author [S] obtained all of the figures of links of some non-singular Morse-Smale flows on S^3. Especially links of closed orbits of non-singular Morse-Smale flows with 3 closed orbits are determined.

In this paper we use methods in [S] to give all of the figures of links of all closed orbits of non-singular Morse-Smale flows on S^3 with 4 or 5 closed orbits. From our results, we conclude that there are different non-singular Morse-Smale flows whose closed orbits make the same link, that is, the correspondence from the equivalence classes of non-singular Morse-Smale flows to the isomorphism classes of links of closed orbits is not injective.

In § 1, we recall some definitions and known results, and state our main theorems; Theorem A and Theorem B, and two Corollaries. We prove Theorem A in § 2 and Theorem B in § 3. Corollaries are proved in § 4.

After our work had done, M. Wada [W] obtained an algorithm to construct links of closed orbits of non-singular Morse-Smale flows on S^3.

The author would like to express his heartly thanks to Professor Itiro Tamura for valuable suggestions and encouragement. He also thanks to the referee for pointing out flaws and errors.

§ 1. Preliminaries and main results.

In this section we recall some definitions and known results, and state our main theorems and corollaries.

For the theory of dynamical systems with which we deal in this paper, see [F3], [M]. [R] is a useful textbook for the theory of low dimensional manifolds.

In this paper we deal with only 3-dimensional manifolds. Thus we assume that a manifold is 3-dimensional unless otherwise stated.

A simple closed curve on the boundary of a solid torus is called a *meridian* if it is not homotopic to a point in the boundary and it bounds a disk in the solid torus. A simple closed curve on the boundary of a solid torus is called a *longitude* if it generates the fundamental group of the solid torus. Even though a solid torus is embedded in S^3, we do not require that a longitude is homologous to zero in the complement of the solid torus in the sphere. We say that a knot K is a *(x, y)-cable* of a knot L if K is a simple closed curve on the boundary of a small tubular neighborhood of L which wraps the neighborhood in the longitudinal direction x times and in the meridinal direction y times, where x and y are coprime integers. In general, $x=0$ is not permitted (see [R] for example). But we permit $x=0$ except for $x=y=0$. A link is called an *iterated torus link* if it is obtained from a trivial link by iterating finitely many times to add (x, y)-cables of a component of the link previously constructed.

A *non-singular Morse-Smale flow* (or an *NMS flow* for short) on a manifold M is a flow which satisfies the following conditions:

(1) There are no singular points, and the non-wandering set consists of a finite number of closed orbits.

(2) The Poincaré map for each closed orbit is hyperbolic.

(3) If c and c' are closed orbits, then the stable manifold of c and the unstable manifold of c' intersect transversely.

The dimension of the unstable bundle of a closed orbit c is called the *index* of c. A closed orbit is called *untwisted* if its unstable bundle is orientable. Otherwise it is called *twisted*. Let $N(2)$ (resp. $N(0)$) denote the number of closed orbits of index 2 (resp. 0), and $N(u)$ (resp. $N(t)$) be the number of untwisted (resp. twisted) closed orbits of index 1.

A link of all closed orbits of an NMS flow is called a *Morse-Smale link*. We say that Morse-Smale links L_1 and L_2 are *isomorphic* if they are isomorphic as links and the corresponding components have the same index and if a component of L_1 is twisted (resp. untwisted) then the corresponding component of L_2 is also twisted (resp. untwisted).

Associated to an NMS flow, we consider a round handle decomposition for M.

DEFINITION 1.1. (a) Let X, Y be manifolds. X is *obtained from Y by attaching a round k-handle* if there are disk bundles E_s^k and E_u^{2-k} over S^1, where the superscript denotes the dimension of the fiber, and an embedding $\theta : \partial E_s^k \times E_u^{2-k} \to \partial Y$ such that $X = Y \bigcup_\theta (E_s^k \oplus E_u^{2-k})$. The total space of $E_s^k \oplus E_u^{2-k}$ is called a *round k-handle*, and the image of the zero-section of $E_s^k \oplus E_u^{2-k}$ is called the *core* of the round handle.

(b) *A round handle decomposition for X* is a filtration

$$R: X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X,$$

where X_0 is a disjoint union of round 0-handles and each X_i is obtained from X_{i-1} by attaching a round handle R_i. Then we use the notation $X_0 \leftarrow R_1 \leftarrow R_2 \leftarrow \cdots \leftarrow R_k$, and $X_l = (X_0 \leftarrow R_1 \leftarrow R_2 \leftarrow \cdots \leftarrow R_l)$ for $l \leqq k$.

D. Asimov and J. Morgan associated an NMS flow with a round handle decomposition as follows.

PROPOSITION 1.2 [A], [M]. *If a manifold M has an NMS flow, then M has a round handle decomposition whose cores are all of the closed orbits of the flow. Conversely, if M has a round handle decomposition, then M has an NMS flow whose closed orbits are all of the cores of round handles.*

REMARKS. (a) For a round handle decomposition $R: X_0 \subset X_1 \subset \cdots \subset X_k = X$, any diffeomorphism F of X induces a new round handle decomposition $F(X_0) \subset F(X_1) \subset \cdots \subset F(X_k) = X$ for X. We denote this round handle decomposition by $F[R]$.

(b) If $X_0 \leftarrow R_1 \leftarrow \cdots \leftarrow R_k$, then we may regard $(R_k \cup R_{k-1} \cup \cdots \cup R_m) \leftarrow R_{m-1} \leftarrow \cdots \leftarrow R_1 \leftarrow R_{0,1} \leftarrow R_{0,2} \leftarrow \cdots \leftarrow R_{0,n}$, where $R_k, \cdots, R_m$ are round 2-handles and $R_{0,1}, \cdots, R_{0,n}$ are round 0-handles such that $X_0 = R_{0,1} \cup \cdots \cup R_{0,n}$, by reversing the direction of the flow associated to the round handle decomposition.

A round handle is called *untwisted* if its core is an untwisted closed orbit. Otherwise it is called *twisted*. We say that round handle decompositions R_1 and R_2 for X are *R-equivalent* if there exists a diffeomorphism F of X such that $F[R_1] = R_2$.

In the following, we consider NMS flows on S^3. In this situation all round 2-handles and all round 0-handles are untwisted, and a round 1-handle H is of the form $H = E_s^1 \oplus E_u^1$ and the part $\partial E_s^1 \times E_u^1$ of ∂H consists of two annuli if H is untwisted, or of an annulus if H is twisted. Each annulus is mapped to a small tubular neighborhood of a circle on the boundary surface of a manifold. Such a circle is called the *attaching circle* of H.

Unless otherwise stated, u, h, s respectively denotes a closed orbit of index 2, 1, 0. We use capital letters U, H, S to denote the round handles whose cores are u, h, s respectively. If we have several closed orbits of the same index (or, round k-handles for same k), we will distinguish them by adding a subscript.

We recall some results by J. Franks and the author.

PROPOSITION 1.3 [F1]. *Any NMS flow on S^3 satisfies the following* *inequalities on the number of closed orbits.*

(a) $N(0) \geqq 1$ *and* $N(2) \geqq 1$.

(b) $N(u) \geqq N(0)-1$ *and* $N(u) \geqq N(2)-1$.

Let an NMS flow on S^3 be given.

LEMMA 1.4 [S]. *Let U be a solid torus in S^3 such that the flow is outwardly transverse to the boundary ∂U of U. Let H be an untwisted round* 1*-handle with core h, which is attached to U. Then one of the following holds.*

(A) *The boundary of the resulting manifold $(U \leftarrow H)$ is the disjoint union of two tori.*

(B) *$(U \leftarrow H)$ is a solid torus, in which U and h are put trivially.*

LEMMA 1.5 [S]. *Let U be as in Lemma* 1.4. *Let H be a twisted round* 1*-handle with core h, which is attached to U. Let $S = S^3 - (U \leftarrow H)$. Then $(U \leftarrow H)$ is diffeomorphic to one of the following.*

(A) *A solid torus.*

(B) *The exterior of a* (2, *odd*)*-torus knot. In this case S is the tubular neighborhood of the knot, and U is an unknotted solid torus in S^3. (See Figure* 1.1.)

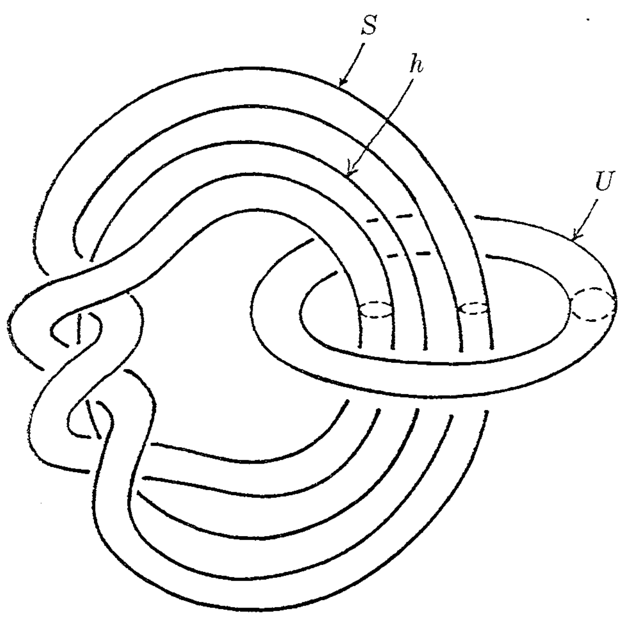


Figure 1.1.

THEOREM 1.6 [S]. *Suppose that a non-singular Morse-Smale flow on* S^3 *has a single closed orbit* h_0 *of index* 2, *and a single closed orbit* h_{n+1} *of index* 0, *and* n *closed orbits* $h_1, h_2, \cdots, h_n$ *of index* 1.

(A) *If all of the closed orbits of index* 1 *are untwisted, then the link consisting of all closed orbits is trivial.*

(B) *If all of the closed orbits of index* 1 *are twisted, then by reordering* $h_1, h_2, \cdots, h_n$ *appropriately, we find* k $(0 \leqq k \leqq n)$ *such that*

(a) h_k *and* h_{k+1} *make the Hopf link,*

(b) *for any* $i<k$, h_i *is a* (2, odd)*-cable of* h_{i+1},

and

(c) *for any* $j>k$, h_{j+1} *is a* (2, odd)*-cable of* h_j.

Before we state our main theorems, we explain the notation.

Notation. In List A and List B, 2 (resp. u, t, 0) represents a closed orbit of index 2 (resp. an untwisted closed orbit of index 1, a twisted closed orbit of index 1, a closed orbit of index 0).

THEOREM A. *The collection of all Morse-Smale links with* 4 *components on* S^3 *coincides with List* A.

THEOREM B. *The collection of all Morse-Smale links with* 5 *components on* S^3 *coincides with List* B.

REMARK. Some links appear in List B repeatedly. But we allow this duplication to make the list simple.

From our main results, we conclude the following corollaries:

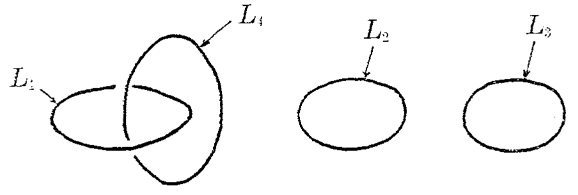
COROLLARY C. *Even if Morse-Smale links are isomorphic, round handle decompositions associated to them are not necessarily R-equivalent.*

COROLLARY D. *Any Morse-Smale link with at most* 5 *components is an iterated torus link.*

REMARK. There exists a Morse-Smale link with 6 components on S^3 which is not an iterated torus link.

List A.

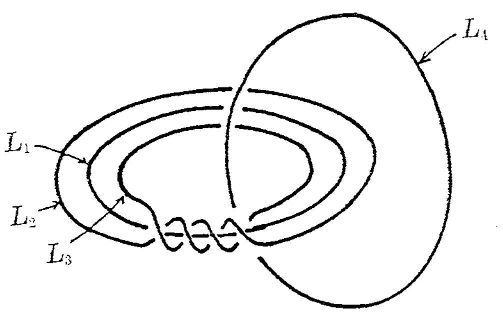
[1]



where

	L_1	L_2	L_3	L_4
(a)	2	2	u	0
(b)	0	0	u	2
(c)	2	u	0	t
(d)	0	u	2	t

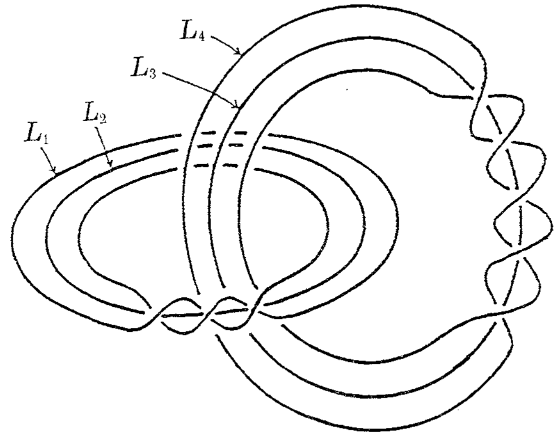
[2]



where L_2 and L_3 are parallel (x, y)-cables of L_1, and

	L_1	L_2	L_3	L_4	x	y
(a)	2	2	u	0	arbitrary	
(b)	2	u	0	2	arbitrary	
(c)	0	0	u	2	arbitrary	
(d)	0	u	2	0	arbitrary	

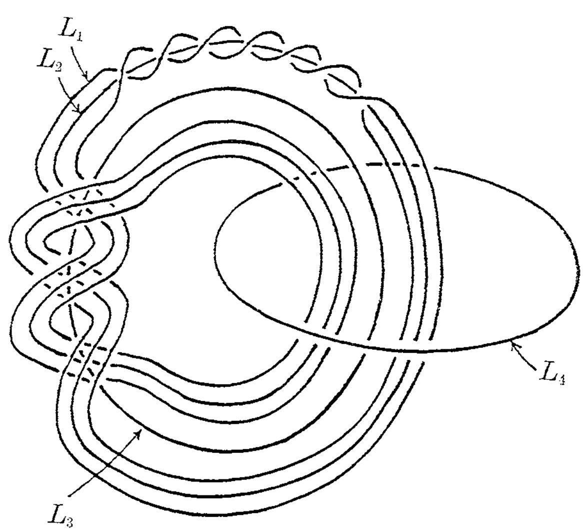
[3]



where L_1 (resp. L_4) is a (2, odd)-cable of L_2 (resp. L_3), and

L_1	L_2	L_3	L_4
2	t	t	0

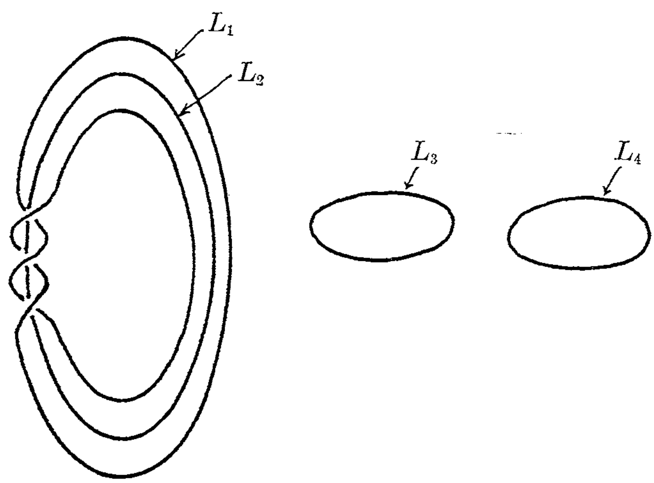
[4]



where L_i is a (2, odd)-cable of L_{i+1} ($i=1, 2$), and

	L_1	L_2	L_3	L_4
(a)	2	t	t	0
(b)	0	t	t	2

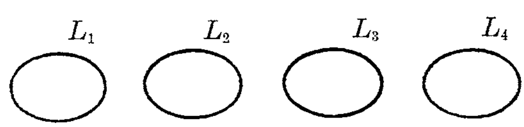
[5]



where L_1 is a (2, odd)-cable of L_2, and

	L_1	L_2	L_3	L_4
(a)	2	t	u	0
(b)	0	t	u	2

[6]

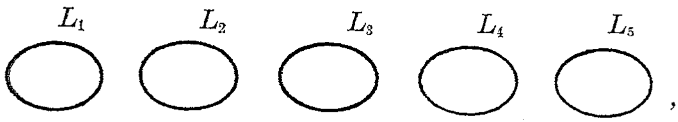


where

L_1	L_2	L_3	L_4
2	u	u	0

List B.

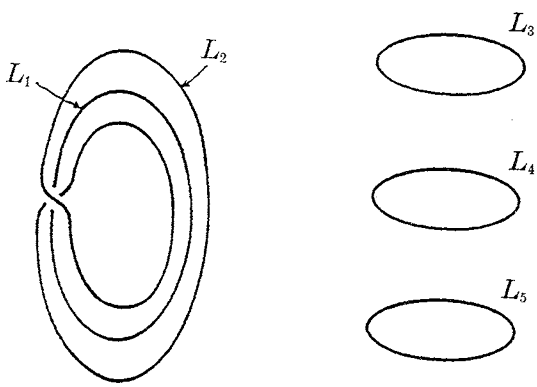
[1]



where

	L_1	L_2	L_3	L_4	L_5
(a)	2	2	u	u	0
(b)	2	u	u	t	0
(c)	2	u	u	u	0
(d)	0	0	u	u	2

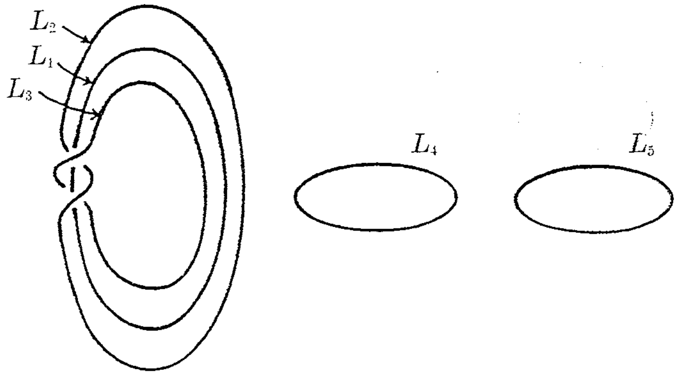
[2]



where L_2 is a (x, y)-cable of L_1, and

	L_1	L_2	L_3	L_4	L_5	x	y
(a)	t	2	u	u	0	2	odd
(b)	t	0	u	u	2	2	odd
(c)	2	0	2	u	u	0	1
(d)	2	t	0	u	u	0	1
(e)	0	t	2	u	u	0	1
(f)	0	2	0	u	u	0	1
(g)	0	t	2	u	u	0	1
(h)	2	t	0	u	u	0	1

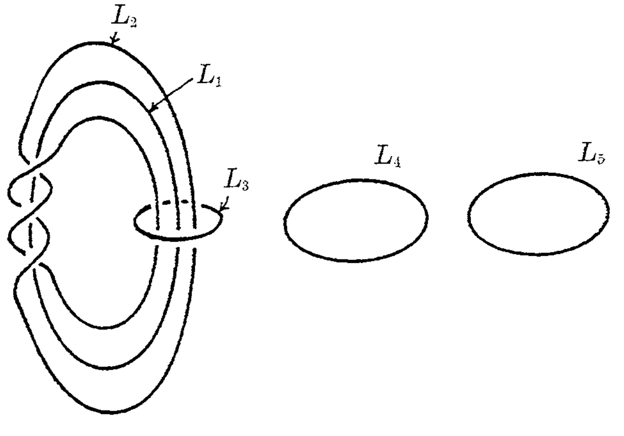
[3]



where L_2 and L_3 are parallel (x, y)-cables of L_1 and

	L_1	L_2	L_3	L_4	L_5	x	y
(a)	2	u	0	2	u	arbitrary	
(b)	2	2	u	u	0	arbitrary	
(c)	0	u	2	0	u	arbitrary	
(d)	0	0	u	u	2	arbitrary	
(e)	0	2	u	u	2	arbitrary	
(f)	2	0	u	u	0	arbitrary	

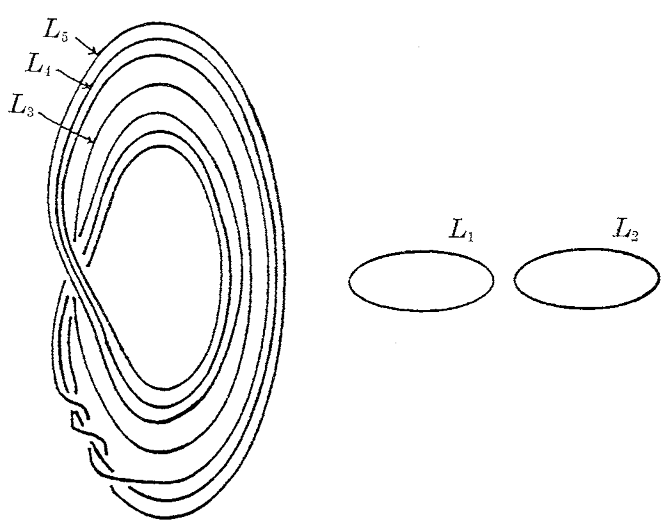
[4]



where L_2 is a (x, y)-cable of L_1 and

	L_1	L_2	L_3	L_4	L_5	x	y
(a)	t	2	0	2	u	2	odd
(b)	t	0	2	2	u	2	odd
(c)	2	u	0	2	u	arbitrary	
(d)	2	u	2	0	u	arbitrary	
(e)	t	0	t	2	u	2	odd
(f)	t	t	0	2	u	2	odd
(g)	t	2	t	0	u	2	odd
(h)	t	t	2	0	u	2	odd
(i)	t	0	2	0	u	2	odd
(j)	t	2	0	0	u	2	odd
(k)	0	u	2	0	u	arbitrary	
(l)	0	u	0	2	u	arbitrary	

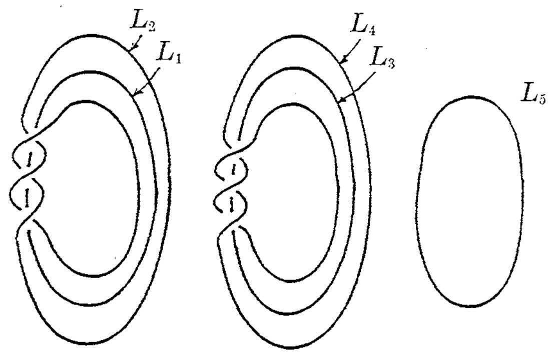
[5]



where L_{i+1} is a (2, odd)-cable of L_i for $i=3, 4$, and

	L_1	L_2	L_3	L_4	L_5
(a)	2	u	t	t	0
(b)	0	u	t	t	2

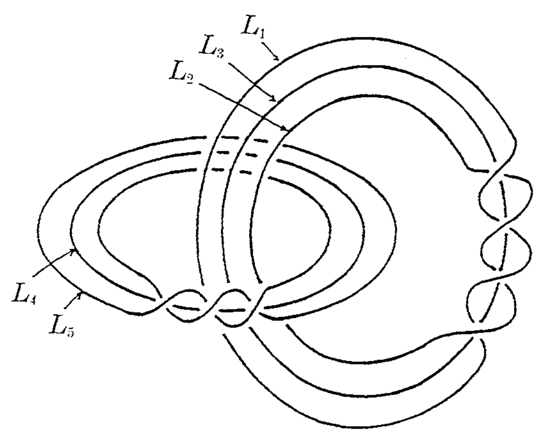
[6]



where L_2 is a (x, y)-cable of L_1 and L_4 is a (z, w)-cable of L_3, and

	L_1	L_2	L_3	L_4	L_5	x	y	z	w
(a)	2	0	2	0	u	0	1	0	1
(b)	t	2	t	0	u	0	1	0	1
(c)	0	2	t	2	u	0	1	2	odd
(d)	t	0	t	2	u	0	1	2	odd
(e)	t	2	t	0	u	0	1	2	odd
(f)	2	0	t	0	u	0	1	2	odd
(g)	t	0	t	2	u	2	odd	2	odd
(h)	0	2	t	2	u	0	1	0	1
(i)	2	0	t	0	u	0	1	0	1

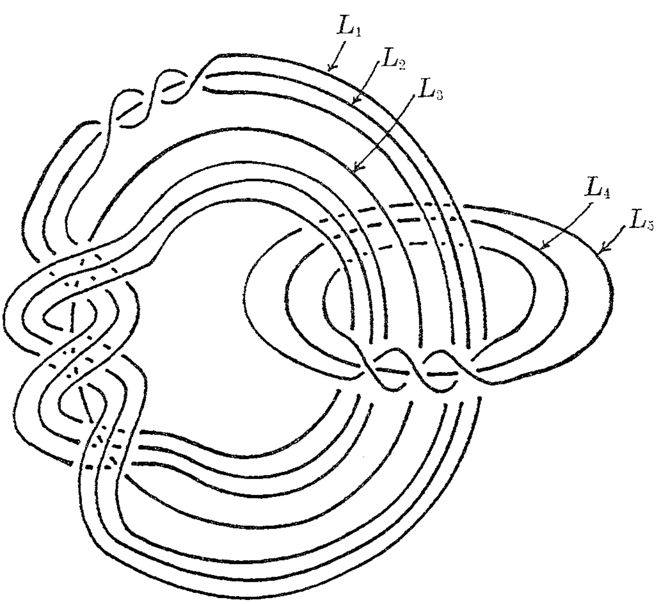
[7]



where L_1 and L_2 are parallel (x, y)-cables of L_3 and L_5 is a $(2, \text{odd})$-cable of L_4, and

	L_1	L_2	L_3	L_4	L_5	x	y
(a)	2	u	2	t	0	arbitrary	
(b)	u	0	2	t	2	arbitrary	
(c)	0	u	0	t	2	arbitrary	
(d)	u	2	0	t	0	arbitrary	
(e)	u	2	0	t	2	arbitrary	
(f)	u	0	2	t	0	arbitrary	

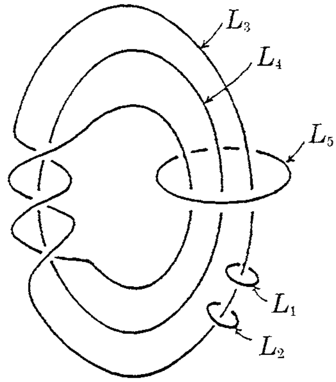
[8]



where L_1 (resp. L_2, L_5) is a $(2, \text{odd})$-cable of L_2 (resp. L_3, L_4), and

	L_1	L_2	L_3	L_4	L_5
(a)	2	t	t	t	0
(b)	0	t	t	t	2

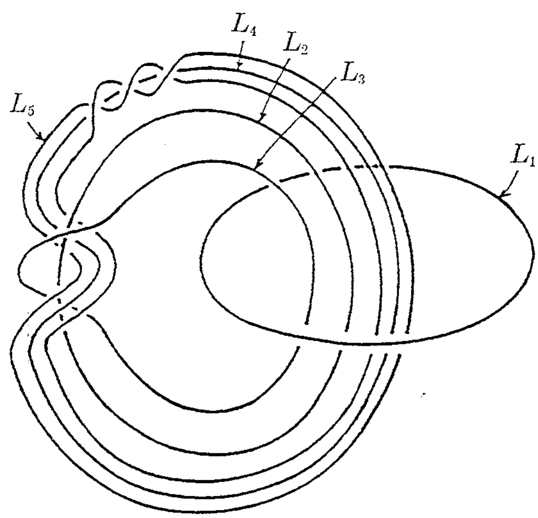
[9]



where L_1 and L_2 are parallel (x, y)-cables of L_3 and L_3 is a $(2, \text{odd})$-cable of L_4, and

	L_1	L_2	L_3	L_4	L_5	x	y
(a)	2	u	2	t	0	arbitrary	
(b)	0	u	2	t	2	arbitrary	
(c)	2	u	0	t	2	arbitrary	
(d)	0	u	0	t	2	arbitrary	
(e)	2	u	0	t	0	arbitrary	
(f)	0	u	2	t	0	arbitrary	

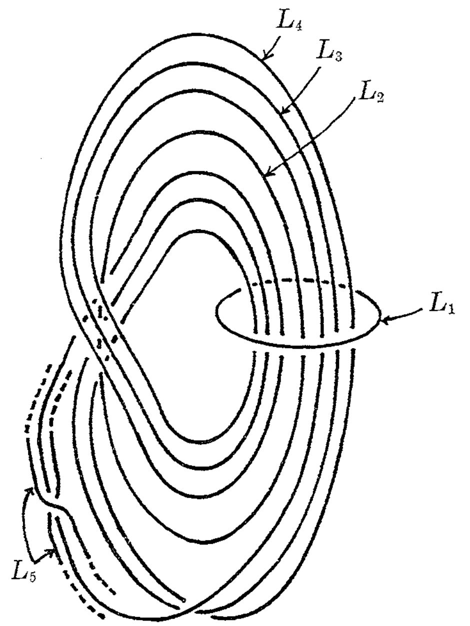
[10]



where L_3 and L_4 are parallel (x, y)-cables of L_2 and L_5 is a $(2, \text{odd})$-cable of L_4, and

	L_1	L_2	L_3	L_4	L_5	x	y
(a)	2	2	u	t	0	arbitrary	
(b)	0	2	u	t	2	arbitrary	
(c)	0	0	u	t	2	arbitrary	
(d)	2	0	u	t	0	arbitrary	

[11]



where L_i is a $(2, \text{odd})$-cable of L_{i-1} $(i=3, 4, 5)$, and

	L_1	L_2	L_3	L_4	L_5
(a)	2	t	t	t	0
(b)	0	t	t	t	2

§ 2. Proof of Theorem A.

In this section we prove Theorem A. By Proposition 1.3, we have four cases as follows:

(I) $N(0)=2,\ N(u)=N(2)=1$, or
$N(0)=N(u)=1,\ N(2)=2$.
(In this case, $N(t)=0$.)
(II) $N(0)=N(2)=1,\ N(u)=0,\ N(t)=2$.
(III) $N(0)=N(2)=1,\ N(u)=1,\ N(t)=1$.
(IV) $N(0)=N(2)=1,\ N(u)=2,\ N(t)=0$.

In case (I), we consider the case $N(0)=2,\ N(u)=N(2)=1$. The other case is reduced to this case by reversing the flow-direction. We have four closed orbits u_1, u_2, h, s and four round handles U_1, U_2, H, S. To make S^3 by attaching these round handles, H should be attached to both of U_1 and U_2. We denote the attaching circle of H on ∂U_j by K_j. Let $[K_j]=a_j[m_j]+b_j[l_j]$ in $H_1(\partial U_j\,;Z)$, where m_j (resp. l_j) is the meridian (resp. a longitude) of U_j. We may assume $b_j\geqq 0$, and $-b_j/2<a_j<b_j/2$ if $b_j>0$ by choosing a longitude appropriately.

If $[K_1]=[K_2]=0$, then $\partial((U_1\cup U_2)\leftarrow H)$ has two connected components. It is impossible to attach S to it so that $S^3=((U_1\cup U_2)\leftarrow H\leftarrow S)$.

Assume that one of $[K_j]$'s, say $[K_1]$, is trivial, and the other is non-trivial. Then $H_1(((U_1\cup U_2)\leftarrow H)\,;Z)\cong Z\oplus Z/b_2$. Since $((U_1\cup U_2)\leftarrow H)=S^3-S$ is a knot exterior in S^3, $H_1(((U_1\cup U_2)\leftarrow H)\,;Z)\cong Z$. Thus $b_2=1$, and $((U_1\cup U_2)\leftarrow H)$ is a solid torus. (See Figure 2.1.). Thus we obtain List A [1] (a). By reversing the flow-direction, we obtain List A [1] (b).

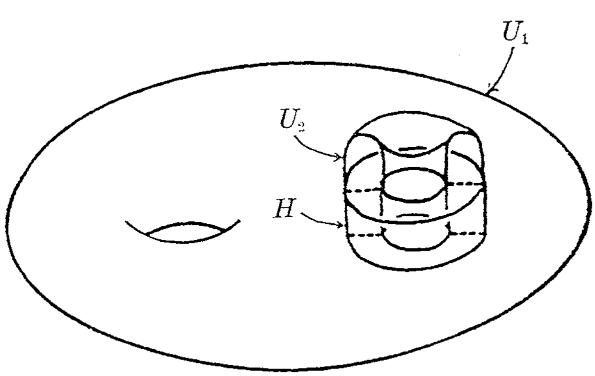


Figure 2.1.

Assume that $[K_1]\neq 0$ and $[K_2]\neq 0$. Then we have five subcases:

(i) If $b_1=0$, then $a_1=\pm 1$. Hence $H_1(((U_1\cup U_2)\leftarrow H);Z)\cong Z\oplus Z/b_2$. Thus $b_2=1$ and $a_2=0$. Thus we obtain List A [2] (a) with $x=0$, $y=1$. (See Figure 2.2.)

(ii) If $b_2=0$, then we obtain List A [2] (a) with $x=0$, $y=1$.

(iii) If $b_1=1$, then $a_1=0$. Thus $((U_1\cup U_2)\leftarrow H)$ is a solid torus as in Figure 2.3, in which u_1 and h are (x,y)-cables of u_2. (x, y are arbitrary coprime integers.) Hence we obtain List A [2] (a).

(iv) If $b_2=1$, then we also obtain List A [2] (a) as in (iii).

(v) Assume $b_1>1$ and $b_2>1$. Then we have

$$\pi_1(((U_1\cup U_2)\leftarrow H))=\langle u_1, u_2\,;\, u_1^{b_1}=u_1^{b_2}\rangle\,,$$

where we use the same symbols as those of closed orbits to denote elements of the fundamental group represented by those closed orbits. $\pi_1(\partial((U_1\cup U_2)\leftarrow H))$ is generated by $u_1^{-x_1}\cdot u_2^{x_2}$ and $u_1^{b_1}$, where x_j is the integer such that $0<x_j<b_j$ and $a_jx_j\equiv -1 \pmod{b_j}$. Let $\psi:\partial((U_1\cup U_2)\leftarrow H)\to\partial S$ be the inverse of the attaching map of S to $((U_1\cup U_2)\leftarrow H)$. The map ψ_* induced from ψ on the fundamental groups is represented as follows:

$$\psi_*(u_1^{b_1})=m_s^{\alpha}l_s^{\beta} \quad\text{and}$$

$$\psi_*(u_1^{-x_1}\cdot u_2^{x_2})=m_s^{\gamma}l_s^{\delta}\,,$$

where m_s (resp. l_s) is the meridian (resp. a longitude) of S, and $\alpha\delta-\beta\gamma=\pm 1$. Then we have

$$\pi_1(((U_1\cup U_2)\leftarrow H\leftarrow S)=\langle u_1, u_2\,;\, u_1^{b_1}=u_2^{b_2},\, u_1^{\delta b_1}(u_1^{-x_1}\cdot u_2^{x_2})^{-\beta}=1\rangle\,.$$

Since $((U_1\cup U_2)\leftarrow H\leftarrow S)\cong S^3$, this group should be trivial. Hence, by an argument similar to that in the proof of Lemma 2.2 in [S], $|\beta|=1$. Thus, by changing a longitude appropriately, we may assume that $\psi_*(u_1^{b_1})=l_s$. This means tnat H is attached to S along two closed curves on ∂S homo-

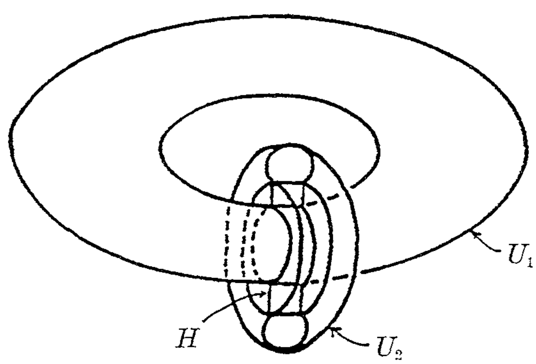


Figure 2.2.

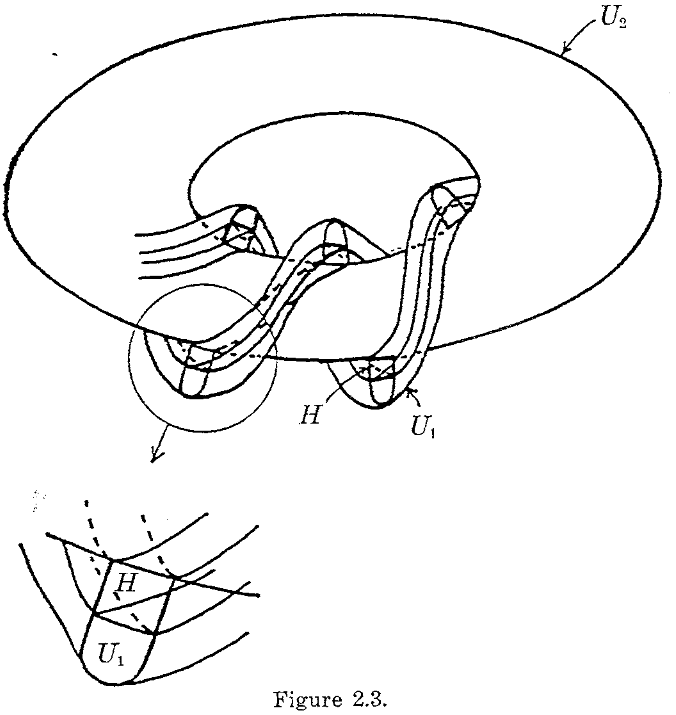


Figure 2.3.

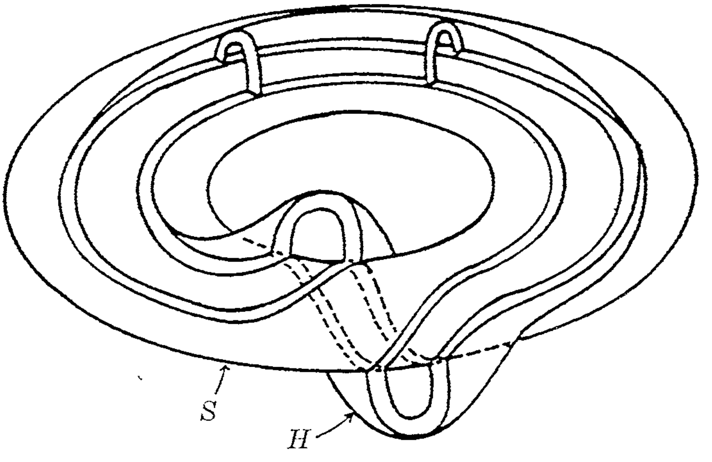


Figure 2.4.

topic to the longitude of S. Thus $(S\leftarrow H)=S^1\times S^1\times I$, where I is a closed interval. (See Figure 2.4.) A connected component of $\partial(S\leftarrow H)$ bounds a solid torus U_1 or U_2, say U_1. Thus $((S\leftarrow H)\leftarrow U_1)$ is also a solid torus, whose center circle is u_1 and both of s and h are parallel (x, y)-cables of u_1 in the solid torus. Hence we obtain List A [2] (b).

By reversing the flow-direction, we obtain List A [2] (c) (d).

In case (II), we apply Theorem 1.6 and obtain List A [3], [4].

In case (III), we have u, s, h_1, h_2 and U, S, H_1, H_2. We may assume that $U\leftarrow H_1\leftarrow H_2\leftarrow S$. We consider the case that H_1 is twisted. The other case is reduced to this case by reversing the flow-direction. By Lemma 1.5, $(U\leftarrow H_1)$ is diffeomorphic to a solid torus or the exterior of a (2, odd)-torus knot. In both cases, $\partial(U\leftarrow H_1)$ is a torus. Thus $\partial(S\leftarrow H_2)$ is also a torus. By Lemma 1.4, this implies that $(S\leftarrow H_2)$ is also a solid torus. Gluing $(U\leftarrow H_1)$ and $(S\leftarrow H_2)$, we obtain List A [1] (c), [5] (a). By reversing the flow-direction, we obtain List A [1] (d), [5] (b).

In case (IV), we obtain List A [6], by Theorem 1.6.

Conversely, any link in List A is a set of all closed orbits of an NMS flow on S^3.

The proof of Theorem A is completed. □

§ 3. Proof of Theorem B.

In this section we prove Theorem B. By Proposition 1.3, we have three cases as follows:

(I) $N(0)=2$, $N(2)=2$, $N(u)=1$.
(II) $N(0)=2$, $N(2)=1$, $N(u)=j$ $(j=1, 2)$, or
$N(0)=1$, $N(2)=2$, $N(u)=j$ $(j=1, 2)$.
(III) $N(0)=1$, $N(2)=1$, $N(u)=j$ $(j=1, 2, 3)$.

In case (I), we have close orbits u_1, u_2, h, s_1, s_2 and round handles U_1, U_2, H, S_1, S_2 corresponding to them. To make S^3 from these round handles, H should be attached to both of U_1 and U_2. Let K_j be the attaching circle of H to U_j. $((U_1\cup U_2)\leftarrow H)$ is the exterior of a 2-component link $\{s_1, s_2\}$, and thus $\partial((U_1\cup U_2)\leftarrow H)$ should have two connected components. This implies that both K_1 and K_2 are trivial on $\partial(U_1\cup U_2)$. Hence $((U_1\cup U_2)\leftarrow H)=(S^1\times D^2)\sharp(S^1\times D^2)$ in S^3. A 2-sphere C, which is used to make the connected sum, bounds 3-balls on both sides in S^3 and separates the boundary components of $((U_1\cup U_2)\leftarrow H)$. Let D_1, D_2 be solid tori in S^3 obtained from $((U_1\cup U_2)\leftarrow H)$ by cutting along C and pasting 3-balls. D_1 and one of S_1 and S_2 make S^3. Thus they are standard solid tori in S^3. The same holds for D_2. Consequently the round handles are put in S^3 as in Figure 3.1, and

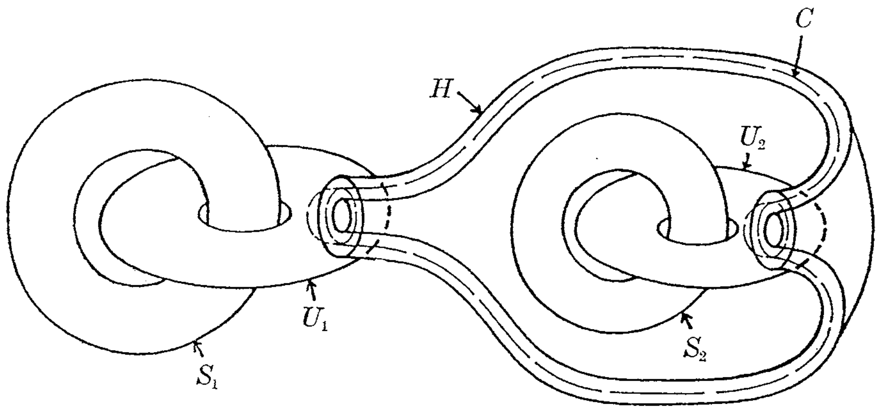


Figure 3.1.

we obtain List B [6] (a).

In case (II), we consider the case $N(0)=1$, $N(2)=2$. By reversing the flow-direction, the other case is similar. We have closed orbits u_1, u_2, h_1, h_2, s and round handles U_1, U_2, H_1, H_2, S and we may assume that $(U_1 \cup U_2) \leftarrow H_1 \leftarrow H_2 \leftarrow S$. We have two subcases:

(Subcase 1. $N(u)=1$.) Assume first that H_1 is untwisted. Then H_1 should be attached to both U_1 and U_2. Let K_j be the attaching circle of H_1 to U_j, and let $[K_j]=a_j[m_j]+b_j[l_j]$ in $H_1(\partial U_j\,; Z)$ as above. Since $(S \leftarrow H_2)$ is a solid torus or the exterior of a knot by Lemma 1.5, $H_1(((U_1 \cup U_2) \leftarrow H_1)\,; Z) \cong Z$. Thus the same argument as in case (I) of § 2 implies that either one of K_1 and K_2 is trivial and the other is non-trivial, or both K_1 and K_2 are non-trivial. In the former case we can prove that the non-trivial attaching circle is a longitude; hence $((U_1 \cup U_2) \leftarrow H_1)$ is a solid torus as in Figure 2.1. We may regard this solid torus as a round 0-handle. Consequently this case reduces to Theorem 1.6 with 3 closed orbits, and we obtain List B [4] (a) (b). In the latter case we have

(i) $b_1 \leqq 1$ or $b_2 \leqq 1$, or

(ii) $b_1 > 1$ and $b_2 > 1$.

In case (i), $((U_1 \cup U_2) \leftarrow H_1)$ is a solid torus and we obtain List B [7] (a), [9] (a). In case (ii), since $H_1(((U_1 \cup U_2) \leftarrow H_1)\,; Z) \cong Z$, b_1 and b_2 are coprime integers, and $\pi_1(((U_1 \cup U_2) \leftarrow H_1)) \cong \langle u_1, u_2\,; u_1^{b_1}=u_2^{b_2} \rangle$. Hence $((U_1 \cup U_2) \leftarrow H_1)$ is the exterior of the (b_1, b_2)-torus knot. (See [R] p. 54.) Thus $(S \leftarrow H_2)$ is a solid torus. Consequently this case reduces to (I) of § 2, and we obtain List B [10] (a).

Secondly we assume that H_2 is untwisted. Then we may assume that

$U_1 \leftarrow H_1$. If $(U_1 \leftarrow H_1)$ is a solid torus then this case reduces to (I) of § 2, and we obtain List B [4] (a), [6] (c), [7] (b) (e), [10] (b). If $(U_1 \leftarrow H_1)$ is not a solid torus then $S^3-(U_1 \leftarrow H_1)=((S \leftarrow H_2) \leftarrow U_2)$ is a solid torus. To investigate how S, H_2, U_2 are located in a solid torus, we consider that the solid torus is a standard solid torus in S^3. Then the complement of the solid torus is regarded as a round 0-handle. We obtain the link of s, h_2, u_2 in a solid torus by deleting L_1 or L_2 in List A [1] (a), or L_1 in List A [2] (a), (b), or L_4 in List A [2] (b), or L_2 in List A [2] (a) when $x=1$ or $y=1$. By embedding the solid torus as a tubular neighborhood of a (2, odd)-torus knot, we obtain List B [4] (b), [6] (h), [9] (b) (c).

(Subcase 2. $N(u)=2$.) Assume first that H_1 is attached to one of U_1 and U_2, say U_1. To obtain S^3 by gluing $(U_1 \leftarrow H_1)$, U_2, H_2, and S, $\partial(U_1 \leftarrow H_1)$ should be connected. Hence $(U_1 \leftarrow H_1)$ is a solid torus by Lemma 1.4, and this case reduces to (I) of § 2. We obtain List B [1] (a), [2] (c), [3] (a) (e), [4] (c). Assume that H_1 is attached to both of U_1 and U_2. Then $\partial((U_1 \cup U_2) \leftarrow H_1)$ should be connected. Hence $\partial(S \leftarrow H_2)$ should be connected. Thus $(S \leftarrow H_2)$ is a solid torus by Lemma 1.4, and this case also reduces to (I) of § 2. We obtain List B [1] (a), [3] (b), [4] (d).

In case (III), we have five closed orbits u, h_1, h_2, h_3, s and round handles U, H_1, H_2, H_3, S corresponding to them respectively. By reordering h_1, h_2, h_3 if necessary, we may assume that $U \leftarrow H_1 \leftarrow H_2 \leftarrow H_3 \leftarrow S$.

Case 1. If $N(u)=0$, we obtain List B [8] (a) (b), [11] (a) (b) by Theorem 1.6.

Case 2. If $N(u)=1$, we have three subcases:

(Subcase 1. H_1 is untwisted.) By Lemma 1.4, either $(U \leftarrow H_1)$ is a solid torus or $\partial(U \leftarrow H_1)$ has two connected components. In the latter case we can choose a simple closed curve in S^3 which intersects each component of $\partial(U \leftarrow H_1)$ at one point; a contradiction. Thus $(U \leftarrow H_1)$ is a solid torus, in which u and h_1 are put trivially. Now this case reduces to (II) in § 2. Consequently we obtain List B [4] (e) (f), [5] (a).

(Subcase 2. H_2 is untwisted.) If either $(U \leftarrow H_1)$ or $(S \leftarrow H_3)$ is a solid torus then this case is reduced to (III) in § 2, and we obtain List B [6] (d) (e) (g). Suppose that neither $(U \leftarrow H_1)$ nor $(S \leftarrow H_3)$ is a solid torus. By Lemma 1.5, $(U \leftarrow H_1)$ is the exterior of a (2, odd)-torus knot. Thus $S'=S^3-(U \leftarrow H_1)=((S \leftarrow H_3) \leftarrow H_2)$ is a solid torus. We consider how H_2 is embedded in the solid torus S'. Let K_1 and K_2 be the attaching circles of H_2 on $\partial(U \leftarrow H_1)=\partial S'$. Then we can prove easily that one of K_1 and K_2, say K_1, is trivial in $\partial S'$ and the other is non-trivial. Since K_1 and K_2 are homotopic in $H_2 \subset S'$ and $K_1 \cong 0$ in S', K_2 is the meridian of S'. Thus $S'-H_2$ is a cube with a (knotted) hole. On the other hand, by Lemma 1.5, $S'-H_2=$

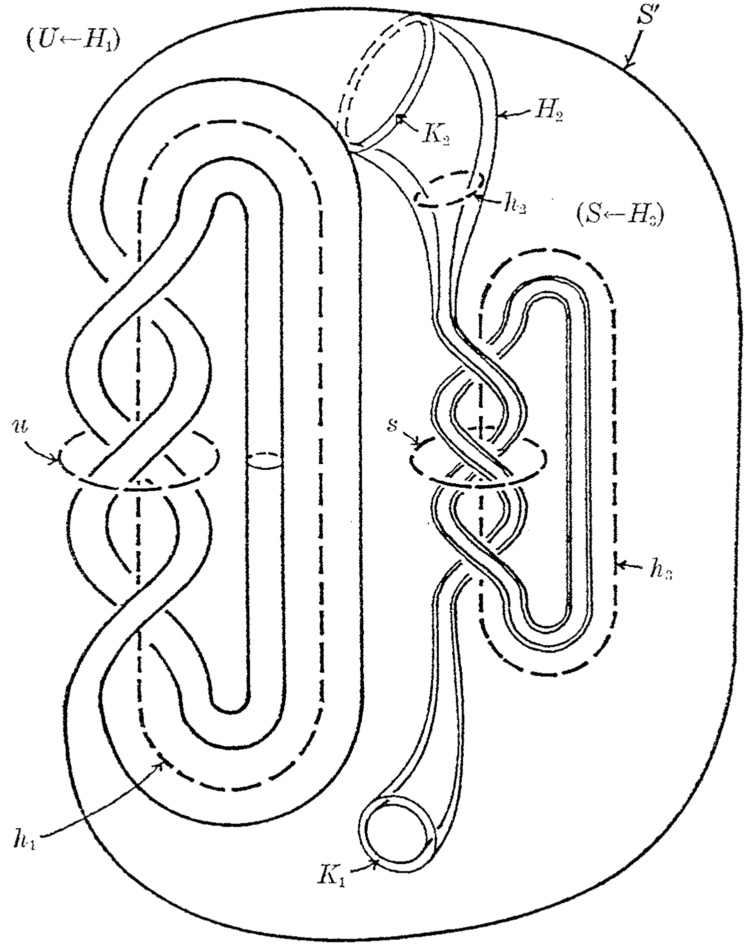


Figure 3.2.

$(S\leftarrow H_3)$ is the exterior of a (2, odd)-torus knot. Thus H_2 is embedded in S' as in Figure 3.2. Consequently we obtain List B [6] (b).

(Subcase 3. H_3 is untwisted.) We can apply subcase 1 by reversing the flow-direction, and we get List B [4] (g) (h), [5] (b).

Case 3. Assume that $N(u)=2$.

If H_1 is twisted then $\partial(U\leftarrow H_1)$ is connected, and thus $\partial((S\leftarrow H_3)\leftarrow H_2)$ is connected. Hence, if $\partial(S\leftarrow H_3)$ is not connected, we can choose a simple closed curve and a torus which intersect at one point; a contradiction. Thus $\partial(S\leftarrow H_3)$ is connected. That is, $(S\leftarrow H_3)$ is a solid torus, and $(S\leftarrow H_3\leftarrow H_2)$

is also a solid torus in which s, h_3, h_2 are put trivially. Hence this is reduced to Theorem 1.6, and we obtain List B [2] (a) (h).

If H_2 is twisted, then both of $(U \leftarrow H_1)$ and $(S \leftarrow H_3)$ are solid tori. Hence this is reduced to Theorem 1.6, and we obtain List B [1] (b).

If H_3 is twisted, then this case is the case that H_1 is twisted with the flow-direction reversed. Hence we obtain List B [2] (b) (g).

Case 4. If $N(u)=3$, then we obtain List B [1] (c) by Theorem 1.6.

The rest of List B is obtained by reversing the flow-direction in all of the cases above.

Conversely any link in List B is the link of closed orbits of an NMS flow.

The proof of Theorem B is completed. □

§ 4. Proof of corollaries.

In this section we prove corollaries.

PROOF OF COROLLARY C. We give two round handle decompositions, each of which consists of a round 0-handle, 2 twisted round 1-handles, an untwisted round 1-handle and a round 2-handle. Consider two round handle decompositions consisting of 4 round handles corresponding to List A [3]. In one of which, let L_4 be a $(2, p_1)$-cable of L_3 and, in the other, let L_4 be a $(2, p_2)$-cable of L_3, where p_1 and p_2 are different odd integers. By embedding a solid torus consisting of a round 2-handle and an untwisted round 1-handle as (B) of Lemma 1.4 to the tubular neighborhood of L_4 in each of the round handle decompositions, we obtain desired two round handle decompositions, which are not R-equivalent, but the links corresponding to which are isomorphic. This implies Corollary C. □

PROOF OF COROLLARY D. Immediate from List A and List B. □

References

[A] Asimov, D., Round handles and non-singular Morse-Smale flows, Ann. of Math. **102** (1975), 55-64.

[BW] Birman, J. and R. F. Williams, Knotted periodic orbits in dynamical systems, I: Lorenz's equations, Topology **22** (1983), 47-82; II: Knot holders for fibered knots, preprint.

[F1] Franks, J. M., The periodic structure of non-singular Morse-Smale flows, Comment. Math. Helv. **53** (1978), 279-294.

[F2] Franks, J. M., Knots, links, and symbolic dynamics, Ann. of Math. **113** (1981), 529-552.

[F3] Franks, J. M., Homology and dynamical systems, C. B. M. S. Regional Conf. Series in Math., 49, Amer. Math. Soc., Providence, R. I., 1982.

[M] Morgan, J., Non-singular Morse-Smale flows in 3-dimensional manifolds, Topology 18 (1978), 41-53.

[R] Rolfsen, D., Knots and links, Publish or perish, Berkeley, 1976.

[S] Sasano, K., Links of closed orbits of non-singular Morse-Smale flows, to appear in Proc. Amer. Math. Soc.

[W] Wada, M., Links which consist of the closed orbits of non-singular Morse-Smale flows on 3-spheres, Preprint.

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