J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
31 (1984), 283-286

Existence of good transversal slices to nilpotent orbits in good characteristic

By N. SPALTENSTEIN

(Communicated by N. Iwahori)

Let G be a connected reductive algebraic group defined over an algebraically closed field k and let $\mathfrak{g}$ be its Lie algebra. Recall that any 1-parameter subgroup $\lambda : \boldsymbol{G}_m \to G$ gives a decomposition $\mathfrak{g} = \bigoplus_{i \in \boldsymbol{Z}} \mathfrak{g}_i^\lambda$, where $\mathfrak{g}_i^\lambda = \{x \in \mathfrak{g} \mid \mathrm{Ad}(\lambda(t))x = t^i x \text{ for all } t \in \boldsymbol{G}_m\}$.

THEOREM. *Assume either that G is simple, not of type A, and that the characteristic is good, or that $G = GL_n$. Let $x \in \mathfrak{g}$ be nilpotent and let C be the G-orbit of x in $\mathfrak{g}$. Then there exists a 1-parameter subgroup $\lambda : \boldsymbol{G}_m \to G$ such that:*

(I) $x \in \mathfrak{g}_2^\lambda$.
(II) $T_x(C) \supset \mathfrak{g}_i^\lambda$ *for all* $i > 0$.

REMARKS. 1) A consequence of the hypothesis on G is that the morphism $G \to C$, $g \mapsto \mathrm{Ad}(g)x$ is separable [3].
2) When char(k) is sufficiently large (or 0), the theorem of Jacobson-Morozov gives immediately the required λ.
3) Condition (II) is equivalent to the existence of a subspace $S \subset \mathfrak{g}$ having the following properties:

(II_1) $\mathfrak{g} = S \oplus T_x(C)$.
(II_2) S is $\boldsymbol{G}_m$-stable for the action given by $\mathrm{Ad} \circ \lambda$.
(II_3) $S \subset \bigoplus_{i \leq 0} \mathfrak{g}_i^\lambda$.

If S has these properties, and if G satisfies the hypothesis of the theorem, then the morphism $G \times S \to \mathfrak{g}$, $(g, s) \mapsto \mathrm{Ad}(g)(x+s)$ is smooth in a neighborhood of $(1, 0)$. In view of the homogeneity argument of [5, p. 111], this means that $S+x$ is a transversal slice to C at x in the sense of [5, p. 60].

We review first some results on the classification of nilpotent orbits. They are valid for arbitrary connected reductive groups in good characteristic (Bala-Carter [1], Pommerening [2]).

A nilpotent element $x\in\mathfrak{g}$ is *distinguished* if the connected centre of G is a maximal torus of $C_G(x)$ (this differs slightly from [1]).

Let P be a parabolic subgroup of G. Choose a Borel subgroup $B\subset P$ and a maximal torus $T\subset B$. Let Φ be the root system of G with respect to T, let Δ be the basis of Φ corresponding to B and let $\Phi_P\subset\Phi$ be the set of roots of T in the Lie algebra of P. Define a 1-parameter subgroup $\lambda : \boldsymbol{G}_m\to G'\cap T$, where G' is the derived group of G, by

$$\langle\lambda,\alpha\rangle=\begin{cases}0 & \text{if} \quad \alpha\in\Delta, \quad -\alpha\in\Phi_P\\ 2 & \text{if} \quad \alpha\in\Delta, \quad -\alpha\notin\Phi_P\,.\end{cases}$$

If U is the unipotent radical of P and M is the Levi factor which contains T, then $\mathfrak{u}=\bigoplus_{i>0}\mathfrak{g}_i^\lambda=\bigoplus_{i>0}\mathfrak{g}_{2i}^\lambda$ and $\mathfrak{m}=\mathfrak{g}_0^\lambda$ (we shall always use the corresponding gothic letter to denote the Lie algebra of an algebraic group). The parabolic subgroup P is *distinguished* if $\dim M\cap G'=\dim\mathfrak{g}_2^\lambda$.

The main result of the classification of nilpotent orbits in good characteristic is that for any distinguished nilpotent orbit C in $\mathfrak{g}$, there exists a distinguished parabolic subgroup P of G such that C contains a dense open subset of $\mathfrak{u}$, where U is the unipotent radical of P. Moreover P is unique up to conjugation.

It was shown by Richardson [4] that in such a situation P acts transitively on $C\cap\mathfrak{u}$. If λ and M are chosen as above and $x\in C\cap\mathfrak{u}$, the U-orbit of x is dense in $x+\left(\bigoplus_{i\geq 2}\mathfrak{g}_{2i}^\lambda\right)$, for dimension reasons. As U is unipotent it is also closed, hence equal to $x+\left(\bigoplus_{i\geq 2}\mathfrak{g}_{2i}^\lambda\right)$. In particular C meets $\mathfrak{g}_2^\lambda$. If $x\in C\cap\mathfrak{g}_2^\lambda$, it is easily shown that λ, viewed as a 1-parameter subgroup in G, is unique up to conjugation by $C_G^0(x)$.

If the nilpotent element x of $\mathfrak{g}$ is not distinguished, let S be a maximal torus of $C_G(x)$ and let $H=C_G(S)$. Then x is distinguished in $\mathfrak{h}$, and we get as above a 1-parameter subgroup $\lambda : \boldsymbol{G}_m\to H'\subset G$ (with $x\in\mathfrak{h}_2^\lambda\subset\mathfrak{g}_2^\lambda$), where H' is the derived group of H. If λ is viewed as a 1-parameter subgroup of G, it is unique up to conjugation by $C_G^0(x)$ and λ determines the orbit of x.

PROOF OF THE THEOREM. Let $x\in\mathfrak{g}$ be nilpotent and choose λ as above. Let $G_{\boldsymbol{Z}}$ be a split reductive group scheme over $\boldsymbol{Z}$ which gives G by extension of scalars. Let $T_{\boldsymbol{Z}}$ be a maximal torus of $G_{\boldsymbol{Z}}$. We can assume that λ comes from a 1-parameter subgroup $\lambda_{\boldsymbol{Z}} : (\boldsymbol{G}_m)_{\boldsymbol{Z}}\to T_{\boldsymbol{Z}}$. We get also a Lie algebra $\mathfrak{g}_{\boldsymbol{Z}}$ over $\boldsymbol{Z}$, with a grading $\mathfrak{g}_{\boldsymbol{Z}}=\bigoplus_{i\in\boldsymbol{Z}}(\mathfrak{g}_{\boldsymbol{Z}})_i^\lambda$, such that $\mathfrak{g}=\mathfrak{g}_{\boldsymbol{Z}}\otimes k$ and $\mathfrak{g}_i^\lambda=(\mathfrak{g}_{\boldsymbol{Z}})_i^\lambda\otimes k$. If A is any commutative ring, let $\mathfrak{g}_A=\mathfrak{g}_{\boldsymbol{Z}}\otimes A$, $(\mathfrak{g}_A)_i^\lambda=(\mathfrak{g}_{\boldsymbol{Z}})_i^\lambda\otimes A$.

By extension of scalars from $\boldsymbol{Z}$ to A we get also a group scheme G_A and a one parameter subgroup λ_A. If B is an A-algebra and $y_A \in \mathfrak{g}_A$, let $y_B = y_A \otimes 1_B \in \mathfrak{g}_B$.

Let now K be the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$, and let A be a valuation ring of K with maximal ideal $\mathfrak{m}$ and residual characteristic equal to char(k). We identify $A/\mathfrak{m}$ with a subfield of k.

We can certainly find an element $x_A \in (\mathfrak{g}_A)_2^{\lambda}$ such that

i) x_C lies in the orbit of $\mathfrak{g}_C$ corresponding to λ_C;
ii) x_k lies in the orbit of $\mathfrak{g}_k = \mathfrak{g}$ corresponding to $\lambda_k = \lambda$.

We can clearly replace x by x_k. We have:

$$\dim_k[x, \mathfrak{g}] = \dim_k[x_k, \mathfrak{g}_k] \leqq \operatorname{rank}_A[x_A, \mathfrak{g}_A] = \dim_C[x_C, \mathfrak{g}_C]. \tag{1}$$

Since the Lie algebra of $C_{G_k}(x_k)$ (resp. $C_{G_C}(x_C)$) is $\mathfrak{c}_{\mathfrak{g}_k}(x_k)$ (resp. $\mathfrak{c}_{\mathfrak{g}_C}(x_C)$) (see Remark 1), we have therefore

$$\dim C_{G_k}(x_k) \geqq \dim C_{G_C}(x_C). \tag{2}$$

We shall show that we have actually equality. Assuming this, we get also equality in (1). Using row and column operations, we can find a pair of basis of $\mathfrak{g}_A$ with respect to which $\operatorname{ad}(x_A)$ has a matrix of the form $\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}$, where I is an identity matrix and all the coefficients of M are in $\mathfrak{m}$. As we have equality in (1), we must have $M=0$, and $[x_A, \mathfrak{g}_A]$ is a direct factor of $\mathfrak{g}_A$. We know already by the Jacobson-Morozov theorem that $(\mathfrak{g}_C)_i^{\lambda} \subset [x_C, \mathfrak{g}_C]$ for every $i>0$. It follows that $(\mathfrak{g}_A)_i^{\lambda} \subset [x_A, \mathfrak{g}_A]$, for $i>0$, and therefore $(\mathfrak{g}_k)_i^{\lambda} \subset [x_k, \mathfrak{g}_k]$ for $i>0$, as desired.

It remains to check that we have equality in (2). Let N be the number of positive roots of G. If q is a prime-power, there are exactly q^{2N} nilpotent elements in $\mathfrak{g}_{F_q}$ [7]. Suppose that the characteristic of $\boldsymbol{F}_q$ is good, and let $n_\lambda(q)$ be the number of nilpotent elements of $\mathfrak{g}_{F_q}$ which belong to the orbit in $\mathfrak{g}_{\bar{F}_q}$ corresponding to λ. Then

$$\sum_{\lambda} n_\lambda(q) = q^{2N} \tag{3}$$

where the summation is over a set of representatives of the 1-parameter subgroups considered by Bala-Carter. It can be shown that $n_\lambda(q)$ is of the form $q^{-\delta}P_\lambda(q)$, where $P_\lambda \in \boldsymbol{Q}[X]$ depends only on λ and $\delta \in \boldsymbol{Z}$ is such that $\deg(P_\lambda) - \delta$ is the dimension of the orbit in $\mathfrak{g}_{\bar{F}_q}$ corresponding to λ [6]. We can normalize P_λ in such a way that $\delta=0$ if char$(\boldsymbol{F}_q)$ is large enough. Then (3) gives also

$$\sum_{\lambda} P_\lambda(q) = q^{2N}. \tag{4}$$

By (2), we have $\delta \geqq 0$. Comparing (3) and (4), we find therefore that we must have equality in (2). This proves the theorem.

REMARK. One can show that there exists $x_Z \in (\mathfrak{g}_Z)_2^\lambda$ such that for every algebraically closed field K of good characteristic the element $x_K \in \mathfrak{g}_K$ lies in the orbit corresponding to λ_K. If R is the ring generated over $\boldsymbol{Z}$ by the inverses of the bad primes, one finds similarly that $[x_R, \mathfrak{g}_R]$ is a direct factor of $\mathfrak{g}_R$.

References

[1] Bala, P. and R. W. Carter, Classes of unipotent elements in simple algebraic groups, Math. Proc. Cambridge Philos. Soc. **79** (1976), 401-425 and **80** (1976), 1-18.

[2] Pommerening, K., Ueber die unipotenten Klassen reduktiver Gruppen II, J. Algebra **65** (1980), 373-398.

[3] Richardson, R. W., Conjugacy classes in Lie algebras and algebraic groups, Ann. of Math. **86** (1967), 1-15.

[4] Richardson, R. W., Conjugacy classes in parabolic subgroups of semisimple algebraic groups, Bull. London Math. Soc. **6** (1974), 21-24.

[5] Slodowy, P., Simple singularities and simple algebraic groups, Lecture Notes in Math. 815, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[6] Spaltenstein, N., Nilpotent orbits of exceptional Lie algebras over algebraically closed fields of bad characteristic, To appear in J. Austral. Math. Soc.

[7] Springer, T. A., The Steinberg function of a finite Lie algebra, Invent. Math. 58 (1980), 211-216.

(Received September 16, 1983)

Forschungsinstitut für Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland